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UNIFORM EBERLEIN COMPACTA AND UNIFORMLY GÂTEAUX SMOOTH NORMS

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Abstract. It is shown that the dual unit ball B_{X^*} of a Banach space X^* in its weak star topology is a uniform Eberlein compact if and only if X admits a uniformly Gâteaux smooth norm and X is a subspace of a weakly compactly generated space. The bidual unit ball $B_{X^{**}}$ of a Banach space X^{**} in its weak star topology is a uniform Eberlein compact if and only if X admits a weakly uniformly rotund norm. In this case X admits a locally uniformly rotund and Fréchet differentiable norm. An Eberlein compact K is a uniform Eberlein compact if and only if $C(K)$ admits a uniformly Gâteaux differentiable norm.

A compact space K is called a *uniform Eberlein compact* if K is homeomorphic to a weakly compact set in a Hilbert space endowed with its weak topology. If the Hilbert space in this definition is replaced with $c_0(\Gamma)$ for some Γ, then we speak of an *Eberlein compact.* The notion of a uniform Eberlein compact

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was introduced by Y. Benyamini and T. Starbird in [4] and further studied by S. Argyros, Y. Benyamini, V. Farmaki, M. E. Rudin, M. Wage and others (see e.g. [2], [3]). The aim of this note is to study the relationship of the existence of uniformly Gâteaux smooth norms on Banach spaces and the uniform Eberlein property of their dual balls in their weak star topology.

The notation used in this note is standard. In particular, the unit ball of a Banach space X is denoted by B_X i.e. $B_X = \{x \in X; \|x\| \leq 1\}$ and the unit sphere of X is $S_X = \{x \in X; \|x\| = 1\}$. The dual unit ball of X^* is $B_{X^*} = \{x^* \in X^*; \sup_{x \in B_X} x^*(x) \leq 1\}$ and the dual unit sphere of X^* is $S_{X^*} = \{x^* \in X^*; \sup_{x \in B_X} x^*(x) = 1\}$.

Recall that a Banach space X is weakly compactly generated if there is a weakly compact set $K \subset X$ such that X is the closed linear span of K. A norm $\|\cdot\|$ on a Banach space X is called *weakly uniformly rotund* if $x_n - y_n \to 0$ weakly, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in the unit ball B_X of X and $\|x_n + y_n\| \to 2$. The norm $\|\cdot\|$ is called *locally uniformly rotund* if $x_n \to x$ in norm whenever $x, x_n \in B_X$ and $\|x + x_n\| \to 2$. The norm $\|\cdot\|$ is *uniformly Gâteaux smooth* or *uniformly Gâteaux differentiable* if for every $h \in S_X$ and every $\epsilon > 0$ there is $\delta > 0$ such that

$$\frac{1}{\tau}(\|x + \tau h\| + \|x - \tau h\| - 2) < \epsilon$$

whenever $0 < \tau < \delta$ and $x \in S_X$.

All notions used and not explained in this note can be found e.g. in [5], [6], [7] or [10].

The main result in this note is the following theorem.

Theorem 1. *Let X be a Banach space. Then the dual unit ball B_{X^*} of X^* in its weak star topology is a uniform Eberlein compact if and only if X is a subspace of a weakly compactly generated Banach space and X admits an equivalent uniformly Gâteaux differentiable norm.*

If X is itself a dual space, the requirement on X to be a subspace of a weakly compactly generated space in Theorem 1 can be dropped. Namely, we obtain the following result.

Theorem 2. *Let X be a Banach space that is isomorphic to a dual space. Then X admits an equivalent uniformly Gâteaux differentiable norm if and only if the dual unit ball B_{X^*} of X^* endowed with its weak star topology is a uniform Eberlein compact.*

From Theorem 2 we obtain the following corollary.

Corollary 3. *A Banach space X admits an equivalent weakly uniformly rotund norm if and only if the bidual unit ball $B_{X^{**}}$ of X^{**} in its weak star topology is a uniform Eberlein compact. Every Banach space with weakly uniformly rotund norm admits an equivalent norm that is locally uniformly rotund and Fréchet differentiable.*

For spaces of continuous functions we have the following theorem.

Theorem 4. *Let K be an Eberlein compact. Then K is a uniform Eberlein compact if and only if $C(K)$ admits an equivalent uniformly Gâteaux differentiable norm.*

The proofs of these results depend on the following three lemmas.

The first one is a variant of the result of S. Troyanski in [19].

Let $(X, \|\cdot\|)$ be a Banach space. For $x, y \in X$ we write $x \perp y$ if $\|y + tx\| \geq \|y\|$ for all $t \in \mathbb{R}$.

Lemma 5. *Let $\|\cdot\|$ be a uniformly Gâteaux smooth norm on a Banach space X. Then for every $\epsilon > 0$ there are sets $S_i^\epsilon \subset S_X$, $i \in \mathbb{N}$, such that $\bigcup_{i=1}^{\infty} S_i^\epsilon = S_X$ and*

$$\|x_1 + \ldots + x_i\| < \epsilon i$$

whenever $x_1, \ldots, x_i \in S_i^\epsilon$ and $x_{j+1} \perp \mathrm{sp}\{x_1, \ldots, x_j\}$, $j = 1, \ldots, i-1$.

Proof. Let $\epsilon > 0$ and $i \in \mathbb{N}$. If $\epsilon i \leq 2$, put $S_i^\epsilon = \emptyset$. Otherwise let S_i^ϵ be the set of all $x \in S_X$ such that for every $y \in S_X$, with $x \perp y$, and every $\tau \in \left(0, \dfrac{2}{\epsilon i - 2}\right)$,

$$\frac{1}{\tau}(\|y + \tau x\| - 1) < \frac{\epsilon}{2}.$$

The uniform Gâteaux smoothness and the orthogonality guarantee that $S_X = \bigcup_{i=1}^{\infty} S_i^{\epsilon}$.
To see this, note that for $x, y \in S_X$, $x \perp y$, $(\|y+\tau x\|-1) \leq (\|y+\tau x\|-1+\|y-\tau x\|-1)$ and use the definition of uniform Gâteaux differentiability of $\|\cdot\|$ as stated above.

Let $\epsilon > 0$ and $i \in \mathbb{N}$ be such that $\epsilon i > 2$ and choose $x_1, \dots, x_i \in S_i^{\epsilon}$ as in Lemma 5. Put $v_j = x_1 + \ldots + x_j$, $j = 1, \dots, i$. We shall show by induction that

$$\|v_j\| < \frac{\epsilon}{2}(i+j), \quad j = 1, \dots, i.$$

Clearly, this is true for $j = 1$. Assume it holds for $j < i$.

If $\|v_j\| > \frac{\epsilon i}{2} - 1$, then $\|v_j\|^{-1} < \frac{2}{\epsilon i - 2}$ and thus

$$\begin{aligned} \|v_{j+1}\| &= \|v_j + x_{j+1}\| = \|v_j\| \left\| \frac{v_j}{\|v_j\|} + \frac{1}{\|v_j\|} x_{j+1} \right\| \\ &< \|v_j\| \left(1 + \frac{\epsilon}{2} \frac{1}{\|v_j\|}\right) < \frac{\epsilon}{2}(i+j) + \frac{\epsilon}{2} = \frac{\epsilon}{2}(i+j+1). \end{aligned}$$

If $\|v_j\| \leq \frac{\epsilon i}{2} - 1$, then

$$\|v_{j+1}\| = \|v_j + x_{j+1}\| \leq \frac{\epsilon i}{2} - 1 + 1 = \frac{\epsilon i}{2} < \frac{\epsilon}{2}(i+j+1).$$

In particular, for $j = i$ we have $\|v_i\| < \frac{\epsilon}{2}(i+i) = \epsilon i$. $\square$

The next lemma is a result of Y. Benyamini, M. Rudin and M. Wage in [3].

Lemma 6 [3]. *A compact space K is a uniform Eberlein compact if and only if it admits a family $\mathcal{U}$ of open F_σ sets such that*

(i) $\mathcal{U}$ separates the points of K, i.e., whenever $x, y \in K$ are distinct, then $\operatorname{card}(\{x, y\} \cap U) = 1$ *for some $U \in \mathcal{U}$, and*

(ii) There exist $\kappa : \mathbb{N} \to \mathbb{N}$ and a decomposition $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ such that for every $x \in K$ and every $n \in \mathbb{N}$

$$\operatorname{ord}(x, \mathcal{U}_n) := \operatorname{card}\{U \in \mathcal{U}_n : \ U \ni x\} \leq \kappa(n).$$

Let $(X,\|\cdot\|)$ be a nonseparable Banach space and let μ be the first ordinal of cardinality equal to the density character of X (i.e, the smallest cardinality of a dense subset in X). A *projectional resolution of the identity* (P.R.I. in short) on $(X,\|\cdot\|)$ is a family $\{P_\gamma :\ 0 \le \gamma \le \mu\}$ of linear projections on X such that $P_0 \equiv 0$, P_μ is the identity mapping, and for all $0 < \gamma \le \mu$ the following hold

(i) $\|P_\gamma\| = 1$,

(ii) $\operatorname{dens} P_\gamma X \le \max(\aleph_0, \operatorname{card}\gamma)$,

(iii) $P_\gamma(P_\beta) = P_\beta(P_\gamma) = P_\beta \quad \text{if} \quad 0 \le \beta \le \gamma$,

(iv) $\bigcup\limits_{\beta<\gamma} P_{\beta+1}X$ is norm dense in $P_\gamma X$.

Lemma 7. *Let a Banach space X be a subspace of a weakly compactly generated Banach space. Assume that the norm $\|\cdot\|$ of X is uniformly Gâteaux differentiable. Then the dual unit ball B_{X^*} of X^* endowed with the weak* topology is a uniform Eberlein compact.*

Proof. Let $I_n = (a_n, b_n),\ n \in \mathbb{N}$, be an enumeration of all open intervals, with rational endpoints, belonging either to $(0,2)$ or to $(-2,0)$. We will prove the following statement:

For every $n, i \in \mathbb{N}$ there exist sets $\Delta_i^n \subset S_X$ such that, if

$$\mathcal{U}_i^n = \{U_x^n :\ x \in \Delta_i^n\},$$

where

$$U_x^n = \{x^* \in B_{X^*} :\ \langle x^*, x\rangle \in I_n\},$$

then the family $\bigcup\limits_{n,i=1}^{\infty} \mathcal{U}_i^n$ separates the points of B_{X^*} and for every $x^* \in B_{X^*}$ and every $n, i \in \mathbb{N}$, we have

$$\operatorname{ord}(x^*, \mathcal{U}_i^n) \le i.$$

Note that each set U_x^n is open and F_σ in (B_{X^*}, w^*). Thus the above implies, by Lemma 6, that (B_{X^*}, w^*) is a uniform Eberlein compact.

Assume first that X is separable. Let $\{x_i :\ i \in \mathbb{N}\}$ be dense in the unit sphere S_X of X. Then it is enough to put $\Delta_i^n = \{x_i\},\ n, i \in \mathbb{N}$.

Now let an uncountable cardinal $\aleph$ be given and assume that our statement was already proved for every space whose density character is less than $\aleph$. Let the space X have the density character equal to $\aleph$. Let $\{P_\gamma : \gamma \in [0,\mu)\}$ be a P.R.I. on X with respect to the norm $\|\cdot\|$ ([11], see e.g. [5] or [7]). For $\gamma \in [0,\mu)$, denote by $Q_\gamma = P_{\gamma+1} - P_\gamma$. Then every subspace $Q_\gamma X$ satisfies the assumptions of our lemma and moreover the density character of $Q_\gamma(X)$ is less than $\aleph$. Hence, by the induction assumption, for every $n, i \in \mathbb{N}$ there exists a set ${}^\gamma\Delta_i^n \subset S_{Q_\gamma X}$ such that, when denoting by

$$^\gamma\mathcal{U}_i^n = \{{}^\gamma U_y^n : y \in {}^\gamma\Delta_i^n\},$$

where

$$^\gamma U_y^n = \{y^* \in B_{(Q_\gamma X)^*} : \langle y^*, y\rangle \in I_n\},$$

the family $\bigcup\limits_{n,i=1}^{\infty} {}^\gamma\mathcal{U}_i^n$ separates the points of the unit ball $B_{(Q_\gamma X)^*}$ and for every $y^* \in B_{(Q_\gamma X)^*}$ and every $n, i \in \mathbb{N}$

$$\mathrm{ord}(y^*, {}^\gamma\mathcal{U}_i^n) \le i.$$

For $\epsilon > 0$ let S_l^ϵ, $l \in \mathbb{N}$, be the sets from Lemma 5. For $n \in \mathbb{N}$ define

$$\epsilon_n = \begin{cases} a_n & \text{if} \quad I_n = (a_n, b_n) \subset (0,2) \\ -b_n & \text{if} \quad I_n = (a_n, b_n) \subset (-2,0). \end{cases}$$

For $\gamma \in [0,\mu)$ and for $n, i, l \in \mathbb{N}$ put

$$^\gamma\mathcal{U}_{i,l}^n = \{U_x^n : x \in {}^\gamma\Delta_i^n \cap S_l^{\epsilon_n}\},$$

where

$$U_x^n = \{x^* \in B_{X^*} : \langle x^*, x\rangle \in I_n\}.$$

Put also

$$\mathcal{U}_{i,l}^n = \bigcup_{\gamma\in[0,\mu)} {}^\gamma\mathcal{U}_{i,l}^n, \quad \Delta_{i,l}^n = \bigcup_{\gamma\in[0,\mu)} {}^\gamma\Delta_i^n \cap S_l^{\epsilon_n}, \quad n, i, l \in \mathbb{N}.$$

We claim that the family $\bigcup\{\mathcal{U}_{i,l}^n : n, i, l \in \mathbb{N}\}$ separates the points of B_{X^*}. To see this, take distinct $x_1^*, x_2^* \in B_{X^*}$. We find $\gamma \in [0,\mu)$ so that $x^*_{1|Q_\gamma X} \ne x^*_{2|Q_\gamma X}$; this is possible since $\bigcup\limits_{\gamma<\mu} Q_\gamma X$ is linearly dense in X. By the induction assumption, there exist $n, i \in \mathbb{N}$ and $U \in {}^{\gamma}\mathcal{U}_i^n$ such that $\mathrm{card}(\{x^*_{1|Q_\gamma X}, x^*_{2|Q_\gamma X}\} \cap U) = 1$. We know that $U = \{y^* \in B_{(Q_\gamma X)^*} : \ \langle y^*, y\rangle \in I_n\}$, with some $y \in {}^{\gamma}\Delta_i^n$. Then $\langle x_j^*, y\rangle = \langle x^*_{j|Q_\gamma X}, y\rangle$, $j = 1, 2$, and so $\mathrm{card}(\{x_1^*, x_2^*\} \cap U_y^n) = 1$. Now it remains to observe that $U_y^n \in {}^{\gamma}\mathcal{U}_{i,l}^n \subset \mathcal{U}_{i,l}^n$ for some $l \in \mathbb{N}$, since $\bigcup\limits_{l=1}^{\infty} S_l^{\epsilon_n} = S_X$.

We will now show that (ii) in Lemma 6 is satisfied. To this end fix any $x^* \in B_{X^*}$ and any $n, i, l \in \mathbb{N}$. For every $\gamma \in [0, \mu)$ we have

$$\begin{aligned}
\mathrm{ord}(x^*, {}^{\gamma}\mathcal{U}_{i,l}^n) &= \#\{U_x^n : \ x \in {}^{\gamma}\Delta_i^n \cap S_l^{\epsilon_n}, \ U_x^n \ni x^*\} \\
&\le \#\{U_x^n : \ x \in {}^{\gamma}\Delta_i^n, \ \langle x^*, x\rangle \in I_n\} \\
&= \#\{{}^{\gamma}U_x^n : \ x \in {}^{\gamma}\Delta_i^n, \ \langle {x^*}_{|Q_\gamma X}, x\rangle \in I_n\} \\
&= \#\{U \in {}^{\gamma}\mathcal{U}_i^n : \ U \ni {x^*}_{|Q_\gamma X}\} \\
&= \mathrm{ord}({x^*}_{|Q_\gamma X}, {}^{\gamma}\mathcal{U}_i^n) \le i.
\end{aligned}$$

Assume that there are $\gamma_1 < \cdots < \gamma_l < \mu$ such that $\mathrm{ord}(x^*, {}^{\gamma_j}\mathcal{U}_{i,l}^n) > 0$. For $j = 1, \ldots, l$ we find $x_j \in {}^{\gamma_j}\Delta_i^n \cap S_l^{\epsilon_n}$ so that $x^* \in U_{x_j}^n$. If $\epsilon_n = a_n$, then $\langle x^*, x_j\rangle > \epsilon_n$ and so

$$\|x_1 + \cdots + x_l\| \ge \langle x^*, x_1 + \cdots + x_l\rangle > l\epsilon_n.$$

If $\epsilon_n = -b_n$, we have $\langle -x^*, x_j\rangle > \epsilon_n$ and so

$$\|x_1 + \cdots + x_l\| \ge \langle -x^*, x_1 + \cdots + x_l\rangle > l\epsilon_n.$$

Note that $x_{j+1} \perp \mathrm{sp}\{x_1, \ldots, x_j\}$, $j = 0, \ldots \ell - 1$. Indeed, if $\alpha_1, \ldots \alpha_j, t \in \mathbb{R}$, then $\|\alpha_1 x_1 + \cdots + \alpha_j x_j + t x_{j+1}\| \ge \|P_{\gamma_j}(\alpha_1 x_1 + \cdots + \alpha_j x_j + t x_{j+1})\| = \|P_{\gamma_j}(\alpha_1 x_1 + \cdots + \alpha_j x_j)\| = \|\alpha_1 x_1 + \cdots + \alpha_j x_j\|$.

Since, moreover, $x_j \in S_l^{\epsilon_n}$, $j = 1, \ldots, \ell$, from Lemma 5 we have

$$\|x_1 + \cdots + x_l\| < l\epsilon_n,$$

a contradiction. Therefore

$$\mathrm{ord}(x^*, \mathcal{U}_{i,l}^n) < il$$

for every $x^* \in B_{X^*}$.

By enumerating the set $\mathbb{N} \times \mathbb{N}$ by elements of $\mathbb{N}$ we get from the families $\mathcal{U}_{i,l}^n$, $\Delta_{i,l}^n$, $i, l \in \mathbb{N}$, new families $\mathcal{U}_i^n$, Δ_i^n, $i \in \mathbb{N}$, such that $\bigcup_{n,i=1}^{\infty} \mathcal{U}_i^n$ separates the points of B_{X^*} and

$$\operatorname{ord}(x^*, \mathcal{U}_i^n) \leq \kappa(i)$$

for every $x^* \in B_{X^*}$, where $\kappa : \mathbb{N} \to \mathbb{N}$. Finally, by adding some empty families and by repeating some of the families $\mathcal{U}_i^n$, if necessary, we can arrange things such that

$$\operatorname{ord}(x^*, \mathcal{U}_i^n) \leq i$$

for all $x^* \in B_{X^*}$ and all $n, i \in \mathbb{N}$. Now it suffices to enumerate the system $\mathcal{U}_i^n$, $n, i \in \mathbb{N}$, by positive integers and to apply Lemma 6. □

Proof of Theorem 1. Let X be a Banach space that admits an equivalent uniformly Gâteaux differentiable norm and is a subspace of a weakly compactly generated Banach space. Then the dual unit ball B_{X^*} of X^* in its weak star topology is a uniform Eberlein compact by Lemma 7. On the other hand, assume that the dual unit ball B_{X^*} of a Banach space X in its weak star topology is a uniform Eberlein compact. Then a Hilbert space can be mapped linearly and continuously onto a dense subset of $C(B_{X^*})$ by [3, Theorem 3.2] (see e.g [10]). Thus $C(B_{X^*})$ admits an equivalent uniformly Gâteaux differentiable norm by [5, Theorem II.6.8(ii)]. As X is isometric to a subspace of $C(B_{X^*})$, we obtain that X admits an equivalent uniformly Gâteaux differentiable norm. Since B_{X^*} in its weak star topology is Eberlein compact (use the canonical map of a Hilbert space $\ell_2(\Gamma)$ into $c_0(\Gamma)$), the space $C(B_{X^*})$ is weakly compactly generated by a result of D. Amir and J. Lindenstrauss (see e.g. [5], [7] or [10]). As X is isometric to $C(B_{X^*})$, X is a subspace of a weakly compactly generated Banach space. The proof of Theorem 1 is finished. □

Proof of Theorem 4. Let K be an Eberlein compact such that $C(K)$ admits an equivalent uniformly Gâteaux differentiable norm. As K is an Eberlein compact, $C(K)$ is weakly compactly generated by the result of D. Amir and J. Lindenstrauss (see e.g. [5], [7] or [10]). Thus $B_{(C(K))^*}$ in its weak star topology is a uniform Eberlein compact by Theorem 1. Since K is homeomorphic to a subspace of $B_{(C(K))^*}$ in its weak star topology, we have that K is a uniform Eberlein compact. On the other hand, if K is a uniform Eberlein compact, then there is a continuous linear map of a Hilbert space onto a dense subset of $C(K)$ by [3, Theorem 3.2] (see e.g. [10]). Thus $C(K)$ admits an equivalent uniformly Gâteaux differentiable norm by [5, Theorem II.6.8 (ii)]. □

P r o o f o f T h e o r e m 2. Assume that X admits an equivalent uniformly Gâteaux differentiable norm and that X is isomorphic to a Banach space Y^*. If X is separable, then B_{X^*} in its weak star topology is easily seen to be a uniform Eberlein compact as there is a one-to-one bounded linear weak star-weak continuous map of X^* into $\ell_2(\mathbb{N})$ (see e.g. [10, Chapter 3, Ex. 24]). So, assume that X is not separable. The space Y^* admits an equivalent uniformly Gâteaux differentiable norm $\|\cdot\|$. By Šmulyan's duality lemma (see e.g. [5, Chapter II]), the dual norm $\|\cdot\|^*$ of $\|\cdot\|$ on Y^{**} is weakly star uniformly rotund (i.e. $f_n - g_n \to 0$ in the weak star topology whenever f_n and g_n are norm one elements of Y^{**} such that $\|f_n + g_n\|^* \to 2$). Thus the restriction of $\|\cdot\|^*$ to Y is weakly uniformly rotund. The dual norm of this restricted norm $\|\cdot\|^*$ is uniformly Gâteaux differentiable on Y^* by Šmuljan's duality lemma (see e.g. [5] or [10]). Since Y is weakly uniformly rotund, Y is an Asplund space (i.e. each separable subspace of Y has separable dual) by [12]. By [8] and [15], Y^* has P.R.I. such that $(P_{\gamma+1} - P_\gamma)Y^*$ are isometric to duals of Asplund spaces, so the induction argument can be used to finish the proof that $B_{Y^{**}}$ in its weak star topology is a uniform Eberlein compact along the lines of the proof of Lemma 7. Since X is isomophic to Y^*, B_{X^*} in its weak star topology is a uniform Eberlein compact as well. On the other hand, if B_{X^*} is a uniform Eberlein compact, then X admits an equivalent uniformly Gâteaux differentiable norm and is a subspace of a weakly compactly generated Banach space as shown in the proof of Theorem 1. □

P r o o f o f C o r o l l a r y 3. If X has an equivalent weakly uniformly rotund norm $\|\cdot\|$, then the dual norm of $\|\cdot\|$ on X^* is uniformly Gâteaux differentiable by Šmulyan's duality lemma (see e.g. [5]). Then the bidual unit ball of X^{**} is a uniform Eberlein compact in its weak star topology by Theorem 2. If, on the other hand, the bidual unit ball $B_{X^{**}}$ is a uniform Eberlein compact in its weak star topology, then X^* is a subspace of a weakly compactly generated Banach space and admits an equivalent uniformly Gâteaux differentiable norm by Theorem 1. Thus X admits an equivalent weakly uniformly rotund norm (see the proof of Theorem 2). If X^* is a subspace of a weakly compactly generated space, then X admits an equivalent norm that is locally uniformly rotund and Fréchet differentiable (see e.g. [5, Chapter VII]). □

Remarks. H. P. Rosenthal constructed in [17] a probability measure μ such that the space $L_1(\mu)$ (which is weakly compactly generated and admits an equivalent uniformly Gâteaux differentiable norm by [5, Theorem II.6.8 (ii)]) contains a subspace $X_{\mathcal{R}}$ with unconditional basis that is not weakly compactly generated. Thus $X_{\mathcal{R}}$ is an example of a Banach space with unconditional basis and uniformly Gâteaux differentiable norm that is not weakly compactly generated. We do not know of any Banach space with uniformly Gâteaux differentiable norm that is not a subspace of a weakly compactly generated space. From the results in [18] and [19] it follows that the space T constructed in [11] is an example of a nonseparable Banach space that admits an equivalent weakly uniformly rotund norm and yet it does not admit any bounded linear one to one operator into any Hilbert space. Finally, let us mention a related problem whether an Asplund space admits an equivalent locally uniformly rotund norm if it admits an equivalent weakly locally uniformly rotund norm (for definitions see e.g. [5]).

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