Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **23** (1997), 243-254

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

OSCILLATION THEOREMS FOR SECOND ORDER SUBLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Communicated by J.-P. Françoise

Abstract. Oscillation criteria are given for the second order sublinear non-autonomous differential equation.

$$(r(t)\psi(x)x'(t))' + q(t)g(x(t)) = \phi(t).$$

These criteria extends and improves earlier oscillation criteria of Kamenev, Kura, Philos and Wong. Oscillation criteria are also given for second order sublinear damped non-autonomous differential equations.

1. Introduction. In this paper we consider the second order nonlinear non-autonomous differential equation

$$(E_1) \qquad (r(t)\psi(x)x'(t))' + q(t)g(x(t)) = \phi(t), \ \ ('= \frac{d}{dt}), \ \ t \geq t_0$$

or, more generally, of the form

$$(E_2) \qquad (r(t)\psi(x)x'(t))' + p(t)x'(t) + q(t)g(x) = \phi(t),$$

where $g, \psi \in C(\mathbb{R})$ with $xg(x) > 0$, $g'(x) \geq 0$ for all $x \neq 0$, and $\psi(x) > 0$ for all $x \in \mathbb{R}$. ψ/g satisfies the sublinear condition

1991 *Mathematics Subject Classification:* 34C15
Key words: second order differential equation, oscillation

$$(c_1) \qquad \int_{\pm 0}^{\pm\varepsilon} \frac{\psi(u)}{g(u)} du < \infty, \quad t \geq t_0.$$

$p, q, r, \phi \in C[t_0, \infty)$ with $r(t) > 0$ for all $t \in [t_0, \infty)$.

Throughout this paper, we restrict our attention only to continuable solutions of equation (E_1) (or (E_2)), i.e., those exist and can be continued on some interval of the form $[t_0, \infty)$, where $t_0 \geq 0$ may depend on the particular solution. Such a solution is called oscillatory if the set $\{t : x(t) = 0\}$ is unbounded. Equation (E_1) ((E_2)) is said to be oscillatory if all its solutions are oscillatory. The oscillation problem for second order nonlinear differential equations is of particular interest and, therefore, it is the subject of investigation by many authors. Many physical systems are modelled by second order nonlinear ordinary differential equations. The prototype of equation (E_1) is the so-called Emden-Fowler equation

$$(E_3) \qquad x''(t) + q(t)|x(t)|^\alpha \operatorname{sgn} x(t) = 0, \quad \alpha > 0.$$

Equation (E_3) arises in the study of gas dynamics and fluid mechanics. Also, this equation appears in the study of relativistic mechanics, nuclear physics and in the study of chemically reacting systems. The study of the Emden-Fowler equation originates from earlier theories concerning gaseous dynamics in astrophysics around the turn of the century. For more details for this equation we refer to the paper by Wong [10], and to the survey article by Ŝevelo [9] for a detailed account of second order nonlinear oscillation and its physical motivation.

Oscillation criteria for equation (E_3) in the sublinear case, i.e., $0 < \alpha < 1$, when $q(t)$ is allowed to assume negative values for arbitrary large values of t, have received more attention in recent years since the early work of Belohorec [1]. We list here some of the more important oscillation criteria for (E_3) for easy reference; where $\sigma(t)$ in the following criteria, if any, stands for a non-negative increasing concave function.

(I) $\displaystyle\int_0^\infty t^\beta q(t)dt = \infty$, for some $\beta \in [0, \alpha]$ (Belohorec [1])

(II) $\displaystyle\lim_{t\to\infty} \sup \frac{1}{t} \int_{t_0}^t \int_{t_0}^s q(u)duds = \infty$, (Kamenev [4])

(III) $\displaystyle\lim_{t\to\infty} \sup \frac{1}{t^n} \int_{t_0}^t (t-s)^n q(s)ds = \infty$, for some $n > 1$ (Kamenev [5])

(IV) $\displaystyle\lim_{t\to\infty} \sup \frac{1}{t} \int_{t_0}^t \int_{t_0}^s u^\beta q(u)duds = \infty$, for some $\beta \in [0, \alpha]$, (Kura [6])

(V) $$\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} \sigma(s) q(s) ds = \infty,\ n \geq 2,$$ (Philos [8])

(VI) $$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \sigma^{\alpha}(u) q(u) du ds = \infty,$$ (Kwong and Wong [7])

(VII) $$\limsup_{t\to\infty} \frac{1}{t^{n}} \int_{t_0}^{t} (t-s)^{n} \sigma^{\alpha}(s) q(s) ds = \infty,$$ for some $n \geq 1$ (Chen [2] and Wong [11])

Wong [12] showed recently an extension of Kamenev result (II) to a more general equation (E_1), with $r(t) \equiv 1$, $\psi(x) \equiv 1$, and $\phi(t) \equiv 0$ if $g'(x) \int_0^{x(t)} \frac{du}{g(u)} \geq \frac{1}{c}$ for $x \neq 0$ and,

(VIII) $$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} \sigma^{\lambda}(u) q(u) du ds = \infty,$$

for $\lambda = \frac{1}{1+c} < 1$, then equation (E_1) is oscillatory.

Remarks.

1. We note that the requirement $\lambda < 1$ is essential in the proof of Wong's result [12].
2. Kura criteria (IV) unifies and considerably improves the results of Belohorec (I) and Kamenev (II). Also, condition (V) has the result of Kura (IV) as a particular case, (VII) includes (V).

The purpose of this paper is to extend condition (VII) for a abroad class of second order nonlinear equations of the type (E_1) (and (E_2)) (Theorems 1 and 2). Also, we present new oscillation criteria for (E_1) and (E_2) (Theorems 3 and 4).

2. Main results. We prove the following theorem for the case when $\phi \equiv 0$ in (E_1).

Theorem 1. *Let n be an integer with $n > 1$ and ρ be a positive and twice continuously differentiable function on the interval $[t_0, \infty)$ with*

(c_2) $$\rho'(t) \geq 0 \quad \textit{and} \quad \rho''(t) \leq 0 \quad \textit{for all} \quad t \in [t_0, \infty).$$

Equation (E_1) is oscillatory if (c_1) holds, and

(c_3) $$\gamma(t) = \frac{r'(t)}{r(t)} - 2\lambda \rho^{\lambda-1}(t) \rho'(t) \geq 0 \quad \textit{and} \quad \gamma'(t) \leq 0 \quad \textit{for all} \quad t \in [t_0, \infty),$$

$$(c_4) \qquad \lim_{t\to\infty} \sup \frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\frac{\rho^\lambda(s)q(s)}{r(s)}ds=\infty, \quad \textit{for} \ \ \lambda\in[0,1].$$

P r o o f. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x for $t\geq t_0$, of (E_1). Without loss of generality, we suppose that $x(t)>0$ for all $t\geq t_0$. We define

$$w(t)=\rho^\lambda(t)\int_0^{x(t)}\frac{\psi(u)}{g(u)}du, \quad t\geq t_0.$$

This and equation (E_1) imply

$$\begin{aligned} w''(t) &= \rho^\lambda(t)\left[\frac{(\psi(x)x'(t))'}{g(x)}-\frac{\psi(x)g'(x)x'^2(t)}{g^2(x)}+\frac{2\lambda\rho^{\lambda-1}(t)\psi(x)x'(t)}{g(x)}\right] \\ &\quad +\lambda(\rho^{\lambda-1}(t)\rho''(t)+(\lambda-1)\rho^{\lambda-2}(t)\rho'^2(t))\int_0^{x(t)}\frac{\psi(u)}{g(u)}du \\ &\leq -\frac{\rho^\lambda(t)q(t)}{r(t)}-\frac{\gamma(t)\psi(x)x'(t)}{g(x)}, \quad \text{for all } t\geq t_0. \end{aligned}$$

Thus, we have for $t\geq t_0$

$$\int_{t_0}^{t}(t-s)^n\frac{\rho^\lambda(s)q(s)}{r(s)}ds\leq-\int_{t_0}^{t}(t-s)^n w''(s)ds-\int_{t_0}^{t}\frac{(t-s)^n\gamma(s)\psi(x(s))x'(s)ds}{g(x(s))}$$

$$=(t-t_0)^n w'(t_0)-n(n-1)\int_{t_0}^{t}(t-s)^{n-2}w(s)ds-\int_{t_0}^{t}\frac{(t-s)^n\gamma(s)\psi(x(s))x'(s)ds}{g(x(s))}.$$

But, by using the Bonnet theorem, for a fixed $t\geq t_0$ and some $\xi\in[t_0,t]$, we have

$$\begin{aligned} -\int_{t_0}^{t}\frac{(t-s)^n\gamma(s)\psi(x(s))x'(s)ds}{g(x(s))} &= -\gamma(t_0)(t-t_0)^n\int_{t_0}^{\xi}\frac{\psi(x(s))}{g(x(s))}x'(s)ds \\ &= -\gamma(t_0)(t-t_0)^n\int_{x(t_0)}^{x(\xi)}\frac{\psi(u)}{g(u)}du \\ &= \gamma(t_0)(t-t_0)^n\int_{x(\xi)}^{x(t_0)}\frac{\psi(u)}{g(u)}du. \end{aligned}$$

Since

$$\int_{x(\xi)}^{x(t_0)}\frac{\psi(u)}{g(u)}du<\begin{cases} 0, & \text{if } \ x(\xi)>x(t_0) \\ \displaystyle\int_{+0}^{x(t_0)}\frac{\psi(u)}{g(u)}du, & \text{if } \ x(\xi)\leq x(t_0), \end{cases}$$

and $\gamma(t_0)$ is positive for all $t \geq t_0$, we have

$$-\int_{t_0}^{t} \frac{(t-s)^n \gamma(s)\psi(x(s))x'(s)}{g(x(s))} ds \leq k_1(t-t_0)^n,$$

where $k_1 = \gamma(t_0) \int_{+0}^{x(t_0)} \frac{\psi(u)}{g(u)} du.$

Hence, for every $t \geq t_0$, we derive

$$\int_{t_0}^{t} (t-s)^n \cdot \frac{\rho^\lambda(s)q(s)}{r(s)} ds \leq (t-t_0)^n w'(t_0) + k_1(t-t_0)^n - n(n-1) \int_{t_0}^{t} (t-s)^{n-2} w(s) ds.$$

Since $n > 1$, the integral in the right hand side of the above inequality exists for every $t \geq t_0$ and is nonnegative.

Therefore,

$$\int_{t_0}^{t} (t-s)^n \cdot \frac{\rho^\lambda(s)q(s)}{r(s)} ds \leq (t-t_0)^n w'(t_0) + k_1(t-t_0)^n.$$

Dividing by t^n and then taking the upper limit, we get the desirable contradiction. This completes the proof. □

Theorem 2. *Suppose that* (c_1), (c_2) *and* (c_3) *hold. Furthermore, we suppose that*

$$(c_5) \qquad p(t) \geq 0 \quad \text{and} \quad \left(\frac{p(t)\rho^\lambda(t)}{r(t)}\right)' \leq 0 \quad \text{for all} \quad t \in [t_0, \infty)$$

$$(c_6) \qquad \limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \frac{\rho^\lambda(s)}{r(s)} (q(s) - \beta|\phi(s)|) ds = \infty,$$

for $n > 1$ *and* $0 \leq \lambda \leq 1$, $\beta = \frac{1}{g(c)}$, *where* $c = \inf_{t\geq t_0} x(t)$, *for* $x > 0$, $c = \sup_{t\geq t_0} x(t)$, *for* $x < 0$. *Then equation* (E_2) *is oscillatory.*

P r o o f. Assume, that there exists a nonoscillatory solution x, for $t \geq t_0$ of (E_2) and $x(t) > 0$ for $t \geq t_0$. Define $w(t)$ as in Theorem 1, then

$$w''(t) \leq -\frac{\rho^\lambda(t)q(t)}{r(t)} + \frac{\beta\rho^\lambda|\phi(t)|}{r(t)} - \frac{\gamma(t)\psi(x)x'(t)}{g(x)} - \frac{p(t)\rho^\lambda(t)x'(t)}{r(t)g(x)},$$

hence for all $t \geq t_0$ we have

$$\int_{t_0}^{t} \frac{(t-s)^n \rho^\lambda(s)}{r(s)}(q(s)-\beta|\psi(s)|)ds \leq -\int_{t_0}^{t}(t-s)^n w''(s)ds-$$

$$-\int_{t_0}^{t} \frac{(t-s)^n \gamma(s)\psi(x(s))x'(s)}{g(x(s))}ds - \int_{t_0}^{t} \frac{(t-s)^n p(s)\rho^\lambda(s)x'(s)}{r(s)g(x(s))}ds.$$

But, by the Bonnet theorem, for a fixed $t \geq t_0$ and for some $\xi \in [t_0, t]$ we have

$$\begin{aligned} -\int_{t_0}^{t} \frac{(t-s)^n p(s)\rho^\lambda(s)x'(s)}{r(s)g(x(s))}ds &= -\left(\frac{\rho^\lambda(t_0)p(t_0)}{r(t_0)}\right)(t-t_0)^n \int_{t_0}^{\xi} \frac{x'(s)}{g(x(s))}ds \\ &= -\left(\frac{\rho^\lambda(t_0)p(t_0)}{r(t_0)}\right)(t-t_0)^n \int_{x(\xi)}^{x(t_0)} \frac{du}{g(u)}. \end{aligned}$$

Since

$$\int_{x(\xi)}^{x(t_0)} \frac{du}{g(u)} < \begin{cases} 0, & \text{if } x(\xi) > x(t_0) \\ \displaystyle\int_{+0}^{x(t_0)} \frac{du}{g(u)}, & \text{if } x(\xi) \leq x(t_0) \end{cases}$$

and $\left(\frac{\rho^\lambda(t_0)p(t_0)}{r(t_0)}\right) \geq 0$ for all $t \geq t_0$, we have

$$-\int_{t_0}^{t}(t-s)^n \cdot \frac{p(s)\rho^\lambda(s)x'(s)}{r(s)g(x(s))}ds \leq k_2(t-t_0)^n,$$

where

$$k_2 = \frac{\rho^\lambda(t_0)p(t_0)}{r(t_0)} - \int_{+0}^{x(t_0)} \frac{du}{g(u)}.$$

The rest of the proof can be carried out as for Theorem 1. □

Remark. If $\psi(x) \equiv 1$ or $\psi(x)$ is bounded then a condition on the sign of the damping coefficient can be removed.

Corollary. *If condition (c_6) in Theorem 2 is replaced by*

(c_7)
$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} \frac{\rho^\lambda(s)(t-s)^n q(s)}{r(s)}ds = \infty,$$

and

(c_8)
$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} \frac{\rho^\lambda(s)(t-s)^n}{r(s)}|\phi(s)|ds < \infty$$

then equation (E_2) *is oscillatory.*

Remark. This result improves our result in [3].

In the following result, no sign condition is assumed on $p(t)$.

Theorem 3. *Assume, in addition to* (c_1), (c_2) *and* (c_3) *that*

$$(c_9) \qquad g'(x) \geq \frac{k}{\psi(x)} > 0 \quad \text{for all} \quad x \neq 0,$$

$$(c_{10}) \qquad \lim_{t\to\infty} \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\psi(s)| \right) ds = \infty,$$

$$(c_{11}) \qquad \lim_{t\to\infty} \sup \int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \frac{\rho(s)}{r(s)} \left(q(u) - \frac{p^2(u)}{4kr(u)} - \beta|\phi(u)| \right) duds = \infty.$$

Then equation (E_2) *is oscillatory.*

Proof. Let x be a nonoscillatory solution on an interval $[t_0, \infty)$ of the differential equation (E_2). Without loss of generality, we assume that $x(t) > 0$ for all $t \geq t_0$. We define $w(t)$ for $t \geq t_0$ as follows

$$w(t) = \rho(t) \int_0^{x(t)} \frac{\psi(u)}{g(u)} du,$$

then

$$(1) \qquad w'(t) = \frac{\rho(t)\psi(x)x'(t)}{g(x)} + \rho'(t) \int_0^{x(t)} \frac{\psi(u)}{g(u)} du,$$

and

$$\begin{aligned} w''(t) = \quad & \rho(t) \left[\frac{(\psi(x)x'(t))'}{g(x)} - \frac{\psi(x)g'(x)x'^2(t)}{g^2(x)} \right] + \frac{2\rho'(t)\psi(x)x'(t)}{g(x)} \\ + \quad & \rho''(t) \int_0^{x(t)} \frac{\psi(u)}{g(u)} du. \end{aligned}$$

This and equation(E_2) imply

$$w''(t) \quad \leq \quad -\frac{\rho(t)q(t)}{r(t)} + \frac{\beta|\phi(t)|\rho(t)}{r(t)} - \frac{\gamma(t)\psi(x)x'(t)}{g(x)} - \frac{\rho(t)p(t)x'(t)}{r(t)g(x)}$$

$$-\frac{\rho(t)\psi(x)g'(x)x'^2(t)}{g^2(x)}$$

$$= -\frac{\rho(t)q(t)}{r(t)} + \frac{\beta|\phi(t)|\rho(t)}{r(t)} - \frac{\gamma(t)\psi(x)x'(t)}{g(x)} + \frac{\rho(t)p^2(t)}{4r^2(t)\psi(x)g'(x)}$$

$$-\left[\sqrt{\rho(t)\psi(x)g'(x)}\cdot\frac{x'(t)}{g(x)} + \frac{\sqrt{\rho(t)}p(t)}{2r(t)\sqrt{\psi(x)g'(x)}}\right]^2$$

$$\leq -\frac{\rho(t)q(t)}{r(t)} + \frac{\beta|\phi(t)|\rho(t)}{r(t)} - \frac{\gamma(t)\psi(x)x'(t)}{g(x)} + \frac{\rho(t)p^2(t)}{4kr^2(t)}.$$

Hence, for all $t \geq t_0$

$$w'(t) \leq w'(t_0) - \int_{t_0}^{t}\frac{\rho(s)}{r(s)}\left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\phi(s)|\right)ds - \int_{t_0}^{t}\frac{\gamma(s)\psi(x(s))x'(s)ds}{g(x(s))}.$$

This and (1) imply that

$$\frac{\rho(t)\psi(x)x'(t)}{g(x)} \leq w'(t_0) - \int_{t_0}^{t}\frac{\rho(s)}{r(s)}\left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\phi(s)|\right)ds$$

$$-\int_{t_0}^{t}\frac{\gamma(s)\psi(x(s))x'(s)}{g(x(s))}ds$$

$$\leq M - \int_{t_0}^{t}\frac{\rho(s)}{r(s)}\left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\phi(s)|\right)ds,$$

$M = w'(t_0) + k_1$, where k_1 is as in Theorem 1.
Condition (c_{10}) implies for large values of t, that

$$\int_{t_0}^{t}\frac{\rho(s)}{r(s)}\left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\phi(s)|\right)ds \geq 2M.$$

Hence,

$$\frac{\rho(t)\psi(x)x'(t)}{g(x)} \leq -\frac{1}{2}\int_{t_0}^{t}\frac{\rho(s)}{r(s)}\left(q(s) - \frac{p^2(s)}{4kr(s)} - \beta|\phi(s)|\right)ds$$

therefore, for all $t \geq t_0$

$$\int_0^{x(t)}\frac{\psi(u)}{g(u)}du \leq \int_0^{x(t)}\frac{\psi(u)}{g(u)}du$$

$$-\frac{1}{2}\int_{t_0}^{t}\frac{1}{\rho(s)}\int_{t_0}^{s}\frac{\rho(u)}{r(u)}\left(q(u)-\frac{p^2(u)}{4kr(u)}-\beta|\phi(u)|\right)duds.$$

Consequently $\int_0^{x(t)}\frac{\psi(u)}{g(u)}du \to -\infty$ as $t \to -\infty$, contradicting the fact $\int_0^{x(t)}\frac{\psi(u)}{g(u)} \geq 0$, this completes the proof. □

Theorem 4. *If, in addition to* (c_1), *we assume that*

(c_{12}) $$\psi(x)g'(x) \geq 0 \quad \textit{for all} \quad x \neq 0,$$

(c_{13}) *There exists a continuously differentiable function* $\rho : [t_0,\infty) \to (0,\infty)$ *such that*
(i) $\rho'(t) \leq 0$ *for all* $t \in [t_0,\infty)$
or
(ii) $\rho'(t) \geq 0$ *and* $(\rho'(t)r(t))' \leq 0$ *for all* $t \in [t_0,\infty)$,

(c_{14}) $$\limsup_{t\to\infty}\left(\int_{t_0}^{t}\frac{\rho(s)}{r(s)}ds\right)^{-1}\int_{t_0}^{t}\frac{\rho(s)}{r(s)}\int_{t_0}^{s}(q(u)-\beta|\phi(u)|)duds = \infty,$$

where β *is as in Theorem* 2, *then equation* (E_1) *is oscillatory.*

P r o o f. Let x be a nonoscillatory solution, for $t \geq t_0$ of equation (E_1), and suppose that $x(t) > 0$ for all $t \geq t_0$. We define

$$w(t) = \frac{r(t)\psi(x)x'(t)}{g(x)}, \quad \text{for all} \quad t \geq t_0,$$

then

$$w'(t) + q(t) - \beta|\phi(t)| + \frac{r(t)\psi(x)g'(x)x'^2(t)}{g^2(x)} \leq 0.$$

Therefore, for all $t \geq t_0$

$$\begin{aligned}\frac{r(t)\psi(x)x'(t)}{g(x)} + \int_{t_0}^{t}(q(s)-\beta|\phi(s)|)ds \quad &\leq \quad w(t_0)\\ -\int_{t_0}^{t}\frac{r(s)\psi(x(s))g'(x(s))x'^2(s)}{g^2(x(s))}ds \quad &\leq \quad w(t_0).\end{aligned}$$

Multiplying by $\frac{\rho(t)}{r(t)}$ and then integrating from t_0 to t

$$\int_{t_0}^{t} \frac{\rho(s)\psi(x(s))}{g(x(s))} x'(s)ds + \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} (q(u) - \beta|\phi(u)|)duds \leq w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds$$

or

$$\rho(t) \int_{0}^{x(t)} \frac{\psi(u)}{g(u)} du - \int_{t_0}^{t} \rho'(s) \int_{0}^{x(s)} \frac{\psi(u)}{g(u)} duds + \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} (q(u) - \beta|\phi(u)|)duds$$
$$\leq w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds + \rho(t_0) \int_{0}^{x(t_0)} \frac{\psi(u)}{g(u)} du.$$

Write, for convenience,

$$G(t) = \int_{0}^{x(t)} \frac{\psi(u)}{g(u)} du, \quad F(t) = \int_{t_0}^{t} \rho'(s) \int_{0}^{x(s)} \frac{\psi(u)}{g(u)} duds = \int_{t_0}^{t} \rho'(s)G(s)ds$$

hence we have

$$\begin{aligned} &\rho(t)G(t) - F(t) + \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} (q(u) - \beta|\phi(u)|)duds \\ &\quad \leq w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds + \rho(t_0) \int_{0}^{x(t_0)} \frac{\psi(u)}{g(u)} du. \end{aligned} \tag{2}$$

Now, we consider two cases:

Case 1. $\rho'(t) \leq 0$, for all $t \geq t_0$. Thus, $\rho(t)G(t) - F(t) > 0$, and then, we have from (2)

$$\int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} (q(u) - \beta|\phi(u)|)duds \leq w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds + \rho(t_0) \int_{0}^{x(t)} \frac{\psi(u)}{g(u)} du.$$

Dividing by $\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds$ and then taking the upper limit we get a contradiction.

Case 2. $\rho'(t) \geq 0$ and $(\rho'(t)r(t))' \leq 0$, for $t \geq 0$. Thus, if $\rho(t)G(t) - F(t) > 0$, then we have a contradiction as in Case 1.

Suppose, there exists $T \geq t_0$ such that $\rho(t)G(t) - F(t) \leq 0$ for all $t \geq T$. Thus, we have $G(t)$ is bounded above, say by $A > 0$. For, $\rho(t)G(t) - F(t) \leq 0$ implies, $\rho(t)F'(t) \leq \rho'(t)F(t)$, then $\left(\frac{\rho(t)}{F(t)}\right)' \geq 0$, for all $t \geq T$. Thus, $\frac{\rho(t)}{F(t)} \geq \frac{\rho(T)}{F(T)}$, $t \geq T$.

Hence,

$$\rho(t)G(t) \le F(t) \le \frac{F(T)\rho(t)}{\rho(T)}, \quad t \ge T$$

implies,

$$0 < G(t) \le \frac{F(T)}{\rho(T)} = A, \quad \text{for all} \quad t \ge T.$$

This and (2) imply that

$$\begin{aligned} &\int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} (q(u) - \beta|\phi(u)|) du ds \\ (3) \qquad &\le w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds + \rho(t_0) \int_{0}^{x(t_0)} \frac{\psi(u)}{g(u)} du + F(t) \\ &\le w(t_0) \int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds + \rho(t_0) \int_{0}^{x(t_0)} \frac{\psi(u)}{g(u)} du + A\rho(t). \end{aligned}$$

Now, since $(\rho'(t)r(t))' \le 0$, $\lim_{t\to\infty} \frac{\rho(t)}{\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds}$ does exist and is finite. For, if $\lim_{t\to\infty} \rho(t) < \infty$, then the conclusion is true; on the other hand, if $\lim_{t\to\infty} \rho(t) = \infty$, then $\lim_{t\to\infty} \frac{\rho(t)}{\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds} = \lim_{t\to\infty} \frac{r(t)\rho'(t)}{\rho(t)} = 0.$

Therefore, dividing both sides of (3) by $\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds$ and then taking the upper limit we have a contradiction. This completes the proof. □

Corollary. *If condition* (c_{14}) *in Theorem* 4 *is replaced by*

$$\lim_{t\to\infty} \sup \left(\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds \right)^{-1} \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t_0}^{s} q(u) du ds = \infty,$$

and

$$\lim_{t\to\infty} \sup \left(\int_{t_0}^{t} \frac{\rho(s)}{r(s)} ds \right)^{-1} \int_{t_0}^{t} \frac{\rho(s)}{r(s)} \int_{t}^{s} |\phi(u)| du ds < \infty$$

then the conclusion of Theorem 4 *is true.*

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Received October 6, 1995