Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **23** (1997), 203-210

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR PARTIAL DIFFERENTIAL-FUNCTIONAL EQUATIONS OF THE FIRST ORDER WITH DEVIATING ARGUMENT OF THE DERIVATIVE OF UNKNOWN FUNCTION

A. Augustynowicz

Communicated by Il. D. Iliev

Abstract. We consider the existence and uniqueness problem for partial differential-functional equations of the first order with the initial condition for which the right-hand side depends on the derivative of unknown function with deviating argument.

1. Introduction. We consider the following Cauchy problem

$$\text{(1)} \qquad \begin{cases} D_x z(x,y) = f(x,y,z(\cdot),D_y z(\alpha(x,y),\beta(x))), & (x,y)\in E \\ z(0,y) = v(y), & y\in[0,b], \end{cases}$$

where $E=[0,a]\times[0,b]$, $a,b>0$, $f\in C(E\times C(E;\mathcal{R}^n)\times\mathcal{R}^n;\mathcal{R}^n)$, $v\in C([0,b];\mathcal{R}^n)$, $\alpha\in C(E;[0,a])$, $\beta\in C([0,a];[0,b])$ and $D_p g$ means the partial derivative of g with respect to p.

1991 *Mathematics Subject Classification:* 35F25

Key words: differential-functional equations, partial of the first order, existence and uniqueness, deviating argument of derivative

The problem $D_x z(x,y) = f(x,y,z(\cdot), D_y z(x,y))$, $z(0,y) = v(y)$ has been investigated intensively in the literature (see [5], [6], [7] for references). But there are only few results concerning equations with $D_y z$ depending on a deviating argument (see [1]–[3], [8]–[10]). In the present paper we consider such an equation. But notice that the deviating argument we consider is of peculiar type and it is not a generalization of not deviating argument (x,y).

The proof of our result is based on Bielecki method of changing norm (see [4]).

2. Notations and assumptions. We denote $E_x = \{(s,y) \in E : 0 \le s \le x\}$, $C = C(E;\mathcal{R}^n)$,

$$\|z\|_{0,x} = \sup\{|z(s,y)| : (s,y) \in E_x\},$$

$$\|z\|_K = \sup\{|z(x,y)|e^{-Kx} : (x,y) \in E\}$$

for $z \in C$, $x \in [0,a]$, $K \in \mathcal{R}$, where $|\cdot|$ means a fixed norm in $\mathcal{R}^n$. Notice that $\|\cdot\|_K$ is a norm equivalent to the supremum norm and C is a Banach space with this norm.

$D_q f$ means the partial derivative of f with respect to the fourth argument

We need the following assumptions.

(H_1) The function f satisfies the following Volterra condition: for all $(x,y) \in E$, $u,v \in C$, $q \in \mathcal{R}^n$ if $u(s,y) = v(s,y)$ for $(s,y) \in E_x$, then $f(x,y,u(\cdot),q) = f(x,y,v(\cdot),q)$.

(H_2) The functions f, $D_y f$, $D_q f$, α, $D_y \alpha$, β, v, $D_y v$ are continuous and there exists $L > 0$ such that

$$|f(x,y,p,q) - f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

$$|D_y f(x,y,p,q) - D_y f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

$$|D_q f(x,y,p,q) - D_q f(x,y,P,Q)| \le L(\|p-P\|_{0,x} + |q-Q|),$$

for $(x,y) \in E$, $p,P \in C$, $q,Q \in \mathcal{R}^n$.

(H_3) $\alpha(x,y) \le x$ for $(x,y) \in E$ and

$$l = \sup\{|D_y \alpha(x,y) D_q f(x,y,p,q)| : (x,y) \in E,\ p \in C,\ q \in \mathcal{R}^n\} < 1$$

Notice that (H_3) implies that the growth of $f(x, y, p, \cdot)$ is at most linear except for $D_y\alpha(x, y) = 0$.

3. The result.

Now we state

Theorem. *Suppose that assumptions* (H_1), (H_2), (H_3) *are satisfied, then there exists exactly one solution* $\hat{z}$ *of* (1) *such that* $D_x D_y \hat{z}$ *and* $D_y D_x \hat{z}$ *exist and they are continuous functions.*

Proof. Let us define

$$F(U,u)(x,y) = \int_0^x \int_0^y u(s,t)dtds + v(y) + \int_0^x f\Big(s, 0, U, v'(\beta(s)) + \int_0^{\alpha(s,y)} u(\sigma, \beta(s))\, d\sigma\Big) ds$$

$$G(U,u)(x,y) = D_y f\Big(x, y, U, v'(\beta(x)) + \int_0^{\alpha(x,y)} u(s, \beta(x))\, ds\Big) + D_y\alpha(x,y) D_q f\Big(x, y, U, v'(\beta(x)) + \int_0^{\alpha(x,y)} u(s, \beta(x))\, ds\Big) u(\alpha(x,y), \beta(x))$$

for $U, u \in C$, $(x, y) \in E$. We prove that there exists exactly one pair $(\hat{U}, \hat{u})$ of continuous functions, which are solutions of the equations

$$U = F(U,u) \quad \text{and} \quad u = G(U,u). \tag{2}$$

Let $K > L$ be fixed. For every U_1, U_2, $u \in C$ assumption (H_2) gives

$$|F(U_1,u)(x,y) - F(U_2,u)(x,y)| \le \int_0^x L\|U_1 - U_2\|_{0,s} ds \le$$

$$\le L \int_0^x \|U_1 - U_2\|_K \, e^{Ks} ds \le \frac{L}{K} e^{Ks} \|U_1 - U_2\|_K,$$

so

$$\|F(U_1,u) - F(U_2,u)\|_K \le \frac{L}{K} \|U_1 - U_2\|_K.$$

Therefore $F(\cdot,u)$ is a contraction and for each $u\in C$ there exists exactly one fixed point $U(u)$ of $F(\cdot,u)$. Moreover

$$\begin{aligned}
&|U(u_1)(x,y)-U(u_2)(x,y)| = |F(U(u_1),u_1)(x,y)-F(U(u_2),u_2)(x,y)| \le\\
&\le |F(U(u_1),u_1)(x,y)-F(U(u_2),u_1)(x,y)|+\\
&\qquad\qquad +|F(U(u_2),u_1)(x,y)-F(U(u_2),u_2)(x,y)| \le\\
&\le \frac{L}{K}\|U(u_1)-U(u_2)\|_K\, e^{Kx} + \int_0^x\int_0^y |u_1(s,t)-u_2(s,t)|\,ds\,dt+\\
&\qquad\qquad +L\int_0^x\int_0^{\alpha(s,t)} |u_1(\sigma,\beta(s))-u_2(\sigma,\beta(s))|\,d\sigma\,ds \le\\
&\le \frac{L}{K}\|U(u_1)-U(u_2)\|_K\, e^{Kx} + b\int_0^x \|u_1-u_2\|_K\, e^{Ks}ds+\\
&\qquad\qquad +L\int_0^x\int_0^s \|u_1-u_2\|_K\, e^{K\sigma}d\sigma ds \le\\
&\le (\frac{L}{K}\|U(u_1)-U(u_2)\|_K + (\frac{b}{K}+\frac{aL}{K})\|u_1-u_2\|_K)e^{Kx},
\end{aligned}$$

hence

$$\|U(u_1)-U(u_2)\|_K \le l_K\|u_1-u_2\|_K,$$

where $l_K=(K-L)^{-1}(b+aL)$. For sufficiently large K let us define

$$W_K=\{u\in C:\ \|u\|_K\le M\},$$

where

$$M=(1-(Ll_K+\frac{L}{K}+l))^{-1}P,\quad P=1+\max_{(x,y)\in E}|D_yf(x,y,U(\Theta),v'(\beta(x)))|$$

and $\Theta(x,y)=0$ for $(x,y)\in E$. Denote also $G_U(u)=G(U(u),u)$. We prove that $G_U(W_K)\subset W_K$ for $K>0$ such that $Ll_K+\dfrac{L}{K}+l<1$.

If $u\in W_K$ then

$$|G(U(u),u)(x,y)| \le$$

$$\le \Big|D_y f\Big(x,y,U(u),v'(\beta(x))+\int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big) - D_y f\Big(x,y,U(\Theta),v'(\beta(x))\Big)\Big| +$$
$$+|D_y f\Big(x,y,U(\Theta),v'(\beta(x))\Big)| + l|u(\alpha(x,y),\beta(x))| \le$$

$$\le L\|U(u)-U(\Theta)\|_{0,x} + L\int_0^{\alpha(x,y)} |u(s,\beta(x))|\,ds + P + l|u(\alpha(x,y),\beta(x))| \le$$
$$\le (L\|U(u)-U(\Theta)\|_K + \frac{L}{K}\|u\|_K + P + l\|u\|_K)e^{Kx} \le$$
$$\le (Ll_K + \frac{L}{K} + l)\|u\|_K\, e^{Kx} + Pe^{Kx},$$

therefore

$$\|G_U(u)\|_K \le (Ll_K + \frac{L}{K} + l)M + P = M.$$

We have just showed that $G_U(W_K) \subset W_K$ for sufficiently large K. We prove that G_U is a contraction on W_K with respect to the norm $\|\cdot\|_J$ for sufficiently large J and K.

For $u,\bar u \in W_K,\ c = \max_{(x,y)\in E}|D_y\alpha(x,y)|$ we have

$$|G(U(u),u)(x,y) - G(U(\bar u),\bar u)(x,y)| \le$$

$$\le L\|U(u)-U(\bar u)\|_{0,x} + L\int_0^{\alpha(x,y)} |u(s,\beta(x)) - \bar u(s,\beta(x))|\,ds +$$

$$+|D_y\alpha(x,y)|\Big|D_q f\Big(x,y,U(u),v'(\beta(x)) + \int_0^{\alpha(x,y)} u(s,\beta(x))\,ds\Big) -$$

$$-D_q f\Big(x,y,U(\bar u),v'(\beta(x)) + \int_0^{\alpha(x,y)} \bar u(s,\beta(x))\,ds\Big)\Big||u(\alpha(x,y),\beta(x))|$$

$$+|D_y\alpha(x,y)||D_q f\Big(x,y,U(\bar u),v'(\beta(x)) + \int_0^{\alpha(x,y)} \bar u(s,\beta(x))\,ds\Big)| \times$$

$$\times|u(\alpha(x,y),\beta(x)) - \bar u(\alpha(x,y),\beta(x))| \le$$

$$\leq L\|U(u)-U(\bar u)\|_J\, e^{Jx} + \frac{L}{J}\|u-\bar u\|_J\, e^{Jx} +$$

$$+cL\Big(\|U(u)-U(\bar u)\|_{0,x} + \int\limits_0^{\alpha(x,y)} |u(s,\beta(x)) - \bar u(s,\beta(x))|\, ds\Big) Me^{Kx} +$$

$$+l\|u-\bar u\|_J\, e^{Jx}$$

$$\leq (Ll_J + \frac{L}{J} + cL(l_J + \frac{1}{J})Me^{Kx} + l)e^{Jx}\|u-\bar u\|_J$$

so

$$\|G_U(u) - G_U(\bar u)\|_J \leq (Ll_J + \frac{L}{J} + cLM(l_J + \frac{1}{J})Me^{Ka} + l)\|u-\bar u\|_J.$$

Now it is clear that the operator G_U is contractive in W_K if K and J are sufficiently large and this operator has exactly one fixed point $\hat u$ in W_K. Since every fixed point $\bar u$ of G_U in C satisfies $|\bar u(0,y)| \leq P-1$ for $y \in [0,b]$ and $M > P$, then there exists K such that $\bar u \in W_K$. We get from the above that the function $\hat u$ is the unique fixed point of G_U in C. Of course the pair $(U(\hat u), \hat u)$ is the unique solution of (2). Now we demonstrate that $V = U(\hat u)$ is a solution of (1). From the definition of the operators F and G we get

$$V(0,y) = v(y),$$

$$D_yV(x,y) = \int\limits_0^x \hat u(s,y)\, ds + v'(y),$$

$$\hat u(x,y) = G(V,\hat u)(x,y) = \frac{d}{dy} f(x,y,V,v'(\beta(x)) + \int\limits_0^{\alpha(x,y)} \hat u(s,\beta(x))\, ds) =$$

$$= \frac{d}{dy} f(x,y,V,D_yV(\alpha(x,y),\beta(x))),$$

$$D_xV(x,y) = \int\limits_0^y \hat u(x,t)\, dt + f(x,0,V,D_yV(\alpha(x,0),\beta(x))) =$$

$$= f(x,y,V,D_yV(\alpha(x,y),\beta(x))),$$

so V is a solution of (1) and it is obvious that $D_xD_yV = D_yD_xV = \hat{u}$, hence these derivatives are continuous. On the other hand, if z is a solution of (1) and $D_xD_yz = D_yD_xz$ then differentiating equation (1) with respect to y we obtain

$$\begin{aligned} D_yD_xz(x,y) &= D_yf(x,y,z,D_yz(\alpha(x,y),\beta(x))) + \\ &+ D_qf(x,y,z,D_yz(\alpha(x,y),\beta(x)))\, D_xD_yz(\alpha(x,y),\beta(x))\, D_y\alpha(x,y) \end{aligned}$$

and

$$D_yz(x,y) = \int_0^x D_xD_yz(s,y)\,ds + v'(y),$$

hence $D_xD_yz = G(z, D_xD_yz)$ and it is easy to verify that $z = F(z, D_xD_yz)$, so $z = U(D_xD_yz)$ and $D_xD_yz = \hat{u}$. We get that z is a solution of (1) if and only if (z, D_xD_yz) is a solution of (2). The proof is complete.

Remark. Under analogous assumptions we can prove a similar result for equation (1) with $y = (y_1, y_2, \ldots, y_k) \in \mathcal{R}^k$.

REFERENCES

[1] A. Augustynowicz, H. Leszczyński. On the existence of analytic solutions to the Cauchy problem for first-order partial differential equations with retarded variables. *Comment. Math.*, to appear.

[2] A. Augustynowicz, H. Leszczyński. On x-analytic solutions to the Cauchy problem for partial differential equations with retarded variables. *Z. Anal. Adwendungen* **15**, *2* (1996), 345-356.

[3] A. Augustynowicz, H. Leszczyński. Periodic solutions to the Cauchy problem for PDEs with retarded variables. Preprint.

[4] A. Bielecki. Une remarque sur la methode de Banach-Cacciopoli-Tikhonov dans la theorie des equations differentielles ordinaires. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys.* **4** (1956), 261-264.

[5] T. Człapiński. On the existence of generalized solutions of nonlinear first order partial differential-functional equations in two independent variables. *Czechoslovak Math. J.* **41** (1991), 490-506.

[6] T. Człapiński, Z. Kamont. Generalized solutions of quasi-linear hyperbolic systems of partial differential-functional equations. *J. Math. Anal. Appl.*, **172** (1993), 353-370.

[7] D. Jaruszewska-Walczak. Existence of solutions of first order partial differential-functional equations. *Boll. Un. Mat. Ital. B (7)* **4-B** (1990), 57-82.

[8] H. Leszczyński. Fundamental solutions to linear first-order equations with a delay at derivatives. *Boll. Un. Mat. Ital. A (7)* **10-A** (1996), 363-375.

[9] W. Walter. Functional differential equations of the Cauchy-Kovalevsky type. *Aequationes Math.* **28** (1985), 102-113.

[10] T. Yamanaka, H. Tamaki. Cauchy-Kovalevska theorem for functional partial differential equations. *Comment. Math. Univ. St. Paul.* **29**, *1* (1980), 55-64.

Institute of Mathematics
Gdańsk University
ul.Wita Stwosza 57
80-952 Gdańsk
Poland
e-mail: `antek@ksinet.univ.gda.pl`

Received March 10, 1992
Revised November 4, 1996