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TOWARD CLEMENS' CONJECTURE IN DEGREES BETWEEN 10 AND 24

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Abstract. We introduce and study a likely condition that implies the following form of Clemens' conjecture in degrees d between 10 and 24: given a general quintic threefold F in complex $\mathbf{P}^4$, the Hilbert scheme of rational, smooth and irreducible curves C of degree d on F is finite, nonempty, and reduced; moreover, each C is embedded in F with balanced normal sheaf $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and in $\mathbf{P}^4$ with maximal rank.

1. Introduction. Ten years ago, Clemens posed a conjecture about the rational curves on a general quintic threefold F in complex $\mathbf{P}^4$. At once, S. Katz [8] considered the conjecture in the following form: *the Hilbert scheme of rational, smooth and irreducible curves C of degree d on F is finite, nonempty and reduced; in fact, each curve is embedded with balanced normal sheaf* $\mathcal{O}(-1) \oplus$

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$\mathcal{O}(-1)$. Katz proved this statement for $d \leq 7$. Recently, Nijsse [10] and the authors [7] independently proved the statement for $d \leq 9$ by developing Katz's argument. In the present paper, we will discuss the possibility of developing Katz's approach further, especially in the range $10 \leq d \leq 24$. Notably, we'll focus on a likely condition on a certain closed subset I'_d of the incidence scheme I_d of all C and F. In Section 2, we'll derive some consequences from the condition, including the above form of Clemens' conjecture for $d \leq 24$. In Section 3, we'll discuss some evidence supporting the condition.

For $d \leq 9$, a stronger statement holds: the incidence scheme I_d is reduced and irreducible of dimension 125. In fact, Katz proved that, for any d, if I_d is irreducible, then the above form of Clemens' conjecture is true. Katz established the irreducibility of I_d for $d \leq 7$, and Nijsse and the authors established it, via different arguments, for $d = 8, 9$. Moreover, when I_d is irreducible, then, on a general F, each C has several significant additional properties; see [7, Corollary 2.5]. For example, each C has maximal rank in $\mathbf{P}^4$; that is, for every $k \geq 1$, the natural restriction map,

$$H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(k)) \to H^0(C, \mathcal{O}_C(k)),$$

is either surjective or injective (or both). These additional properties play an important role in the authors' work in [7] on Clemens' full conjecture, which is discussed briefly below.

On the other hand, I_d is reducible for $d \geq 12$, according to Proposition (3.2) below. In fact, I_d always has at least one component of dimension 125 dominating the space $\mathbf{P}^{125}$ of all quintic threefolds F; see Lemma (2.4). This component was constructed, more or less explicitly, for infinitely many d by Clemens [2, Theorem 1.27, p. 26], and for all d by Katz [8, p. 153] (who observed that the general case follows via Clemens' deformation-theoretic argument from an existence result of Mori's). We will see in Section 3 that I_d contains some special subsets, which do not dominate $\mathbf{P}^{125}$. One of them has dimension $2d + 101$ for $d \geq 10$, so yields a second component of I_d for $d \geq 12$. For $d = 10, 11$ it is unclear whether I_d is irreducible or not.

Clemens' conjecture is, of course, no less likely to be true. In fact, in the above form, it is implied, for $d \leq 24$, along with the other consequences of irreducibility, by a likely weaker condition. This condition concerns another component of I_d, which exists when $d \leq 24$. We'll call it the *principal component*, and denote it by $\overline{I}_{d,0}$. It is constructed as follows. In the space M_d of all C, form the open subset $M_{d,0}$ where $H^1(\mathcal{I}_C(5))$ vanishes; here $\mathcal{I}_C$ denotes the ideal of C in $\mathbf{P}^4$. Form the preimage $I_{d,0}$ in I_d of $M_{d,0}$. Then $\overline{I}_{d,0}$ is simply the closure of $I_{d,0}$. For $d \leq 24$, we expect that $\overline{I}_{d,0}$ is equal to the Clemens–Katz component. In fact, we expect that $\overline{I}_{d,0}$ is the only component of I_d that dominates $\mathbf{P}^{125}$. In other words, we expect that, if $I'_d := I_d - I_{d,0}$, then I'_d *does not dominate* $\mathbf{P}^{125}$. This, finally, is our proposed weaker condition for $d \leq 24$. One reasonable way to try to prove it is to look for a decomposition of I'_d into manageable pieces, each of which can be shown not to dominate $\mathbf{P}^{125}$. On the other hand, for $d \geq 25$, the preimage $I_{d,0}$ is empty, and so the geometry of I_d is radically different in this range.

Clemens [3, p. 639] strengthened his conjecture, after Katz's work, by adding these two assertions: *all the rational, reduced and irreducible curves on a general F are smooth and mutually disjoint; and the number n_d of curves of degree d is divisible by* $5^3 \cdot d$. These assertions are not completely true. Vainsencher proved that F contains 17,601,000 six-nodal quintic plane curves. Ellingsrud and Strømme and, independently, Candelas, De la Ossa, Green, and Parkes found that n_3 is equal to 371,206,375, which is divisible by 5^3, but not 3. In fact, in their landmark work introducing mirror symmetry, Candelas et. al. developed an algorithm that produces, for any d, a number, which they conjecture is equal to n_d. Afterwards, Kontsevich gave a somewhat different algorithm, which, he conjectured, also gives the n_d. Although Kontsevich too was inspired by mathematical physics, his treatment is more algebraic-geometric. Moreover, its numbers clearly count both smooth and nodal curves, which must be connected, but may be reducible. However, the authors [7, Theorems. 3.1 and 4.1] proved that the only singular, reduced and irreducible, rational curve of degree at most 9 on F is a six-nodal plane quintic and that there is on F no pair of intersecting rational, reduced and irreducible curves whose degrees total at most 9; thus the enumerative significance of Kontsevich's numbers is established in degree at most 9. The case of degree at least 10 remains open.

Throughout this paper, we use the following notation, which has already been introduced informally:

(a) M_d denotes the open subscheme of the Hilbert scheme of $\mathbf{P}^4$ parametrizing the rational, smooth and irreducible curves C of degree d;

(b) $\mathbf{P}^{125}$ denotes the projective space parametrizing the quintic threefolds F in $\mathbf{P}^4$;

(c) I_d denotes the "incidence" subscheme of $M_d \times \mathbf{P}^{125}$ of pairs (C, F) such that $C \subset F$;

(d) $M_{d,0}$ denotes the subset of M_d parametrizing the curves C such that $h^1(\mathcal{I}_C(5)) = 0$ where $\mathcal{I}_C$ denotes the ideal of C in $\mathbf{P}^4$;

(e) $I_{d,0}$ denotes the preimage in I_d of $M_{d,0}$;

(f) $\overline{I}_{d,0}$ denotes the closure of $I_{d,0}$;

(g) I'_d denotes the complement, $I_d - I_{d,0}$.

2. The principal component. In this section, we'll derive some consequences from the (likely) condition that the (closed) set I'_d does not dominate the space $\mathbf{P}^{125}$ of quintic threefolds. Our main result is Theorem (2.7); it asserts that this condition implies Katz's form of Clemens' conjecture. The theorem will be derived from Proposition (2.5), which asserts this: if I'_d doesn't dominate $\mathbf{P}^{125}$, then its complement $I_{d,0}$ does; in fact, then the closure $\overline{I}_{d,0}$ is the one and only component that does. We'll call $\overline{I}_{d,0}$ the *principal component* of I_d. We'll also prove Proposition (2.2), which asserts that, if I'_d doesn't dominate $\mathbf{P}^{125}$, then, on a general quintic threefold F, the rational curves C possess certain significant properties; for example, each C has maximal rank in $\mathbf{P}^4$.

Lemma 2.1. *If $d \le 24$, then $I_{d,0}$ is smooth, irreducible, and of dimension* 125*; moreover, it dominates $M_{d,0}$, it's open in I_d, and its closure $\overline{I}_{d,0}$ is a component. If $d \ge 25$, then $I_{d,0}$ is empty.*

P r o o f. It is well known that M_d is smooth of dimension $5d + 1$ and is irreducible. Moreover, these properties are not hard to establish. Indeed, fix $C \in M_d$. The restricted Euler sequence,

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 5} \to \mathcal{T}_{\mathbf{P}^4}|C \to 0, \tag{2.1}$$

yields $H^1(\mathcal{T}_{\mathbf{P}^4}|C) = 0$. So the sequence of tangent and normal sheaves,

$$0 \to \mathcal{T}_C \to \mathcal{T}_{\mathbf{P}^4}|C \to \mathcal{N}_{C/\mathbf{P}^4} \to 0, \tag{2.2}$$

yields $H^1(\mathcal{N}_{C/\mathbf{P}^4}) = 0$. Hence, by the standard theory of the Hilbert scheme, M_d is smooth at C of dimension $h^0(\mathcal{N}_{C/\mathbf{P}^4})$, and the latter number can be found using the same two exact sequences. Finally, M_d is irreducible as it's the image of an open subset of the space of parametrized maps from $\mathbf{P}^1$ to $\mathbf{P}^4$, and this space of maps is just the space of 5-tuples of homogeneous polynomials of degree d in two variables.

Again, fix $C \in M_d$. Then $C \in M_{d,0}$ if and only if the natural map,

$$H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(5)) \to H^0(C, \mathcal{O}_C(5)), \tag{2.3}$$

is surjective, thanks to the long exact cohomology sequence. Hence, if $d \le 7$, then $M_{d,0} = M_d$. Indeed, the surjectivity of (2.3) is obvious if C is a line, a conic, or a twisted cubic. If $4 \le d \le 7$, then C cannot lie in plane, and the surjectivity holds by the theorem on p. 492 of [4]. If $8 \le d \le 25$, then $M_{d,0}$ is nonempty; indeed, if $C \in M_d$ is general, then the surjectivity of (2.3) holds by the maximal-rank theorem [1, Theorem 1, p. 215], because the source and target have dimensions 126 and $5d + 1$. If $d \ge 26$, then surjectivity cannot hold, and so $M_{d,0}$ is empty.

The subset $M_{d,0}$ of M_d is open for any d. Indeed, let $\mathbf{C}$ be the universal curve in $\mathbf{P}^4 \times M_d$, and $\mathcal{I}_{\mathbf{C}}$ its ideal. Then $\mathcal{I}_{\mathbf{C}}$ is flat over M_d. Hence $h^1(\mathcal{I}_{\mathbf{C}}(5))$ is upper semi-continuous. Therefore, $M_{d,0}$ is open. Hence, its preimage $I_{d,0}$ is open in I_d, and its closure will be a component provided $I_{d,0}$ is nonempty and irreducible.

Let $C \in M_{d,0}$. Then, by definition, $H^1(\mathcal{I}_C(5))$ vanishes. Hence the direct image $\mathcal{Q}$ of $\mathcal{I}_{\mathbf{C}}(5)$ is locally free on $M_{d,0}$, and its formation commutes with base change to the fibers. Hence $I_{d,0}$ is equal to $\mathbf{P}(\mathcal{Q}^*|M_{d,0})$. Now, for $d \leq 25$, since (2.3) is surjective, $H^0(\mathcal{I}_C(5))$ has dimension $125 - 5d$. Hence, $H^0(\mathcal{I}_C(5))$ is zero for $d = 25$, and is nonzero for $d \leq 24$. For $d \leq 24$, therefore, $I_{d,0}$ is smooth, irreducible, of dimension 125, and dominates $M_{d,0}$. Thus the lemma is proved. □

Proposition 2.2. *Assume that I'_d does not dominate $\mathbf{P}^{125}$. Let F be a general quintic threefold in $\mathbf{P}^4$, and let C be a rational, smooth and irreducible curve of degree d at most* 24 *on F.*

(1) *Then C is embedded in $\mathbf{P}^4$ with maximal rank.*

(2) *Form the restriction to C of the twisted sheaf of differentials of $\mathbf{P}^4$. Then this locally free sheaf has* generic *splitting type; namely, if $d = 4n+m$ where $0 \leq m < 4$, then*

$$\Omega^1_{\mathbf{P}^4}(1)|C = \mathcal{O}_C(-n-1)^m \oplus \mathcal{O}_C(-n)^{4-m}.$$

(3) *If $d \leq 4$, then C is a rational normal curve of degree d, and if $d \geq 4$, then C spans $\mathbf{P}^4$.*

(4) *If $d = 1$, then C is* 1*-regular; if $2 \leq d \leq 4$, then C is* 2*-regular; if $5 \leq d \leq 7$, then C is* 3*-regular; if $8 \leq d \leq 11$, then C is* 4*-regular; if $15 \leq d \leq 17$, then C is* 5*-regular; and if $18 \leq d \leq 24$, then C is* 6*-regular.*

Proof. This result was proved in [7, Corollary 2.5] (without the hypothesis on I'_d) for $d \leq 9$. For $10 \leq d \leq 24$, the proof is similar. First, observe that, since F is general, C does not lie in any given proper closed subset N of M_d. Indeed, the preimage of N in I_d consists of two parts, the part in I'_d and that in $I_{d,0}$. Neither part dominates $\mathbf{P}^{125}$: the first doesn't by hypothesis, and the second doesn't by virtue of Lemma 2.1, which implies that this part has dimension at most 124.

To prove (2), apply the observation above to the subset N of M_d of curves without the asserted splitting type; N is a proper closed subset by a theorem of Verdier's [12, Theorem, p.139] (see also [11, Theorem 1, p. 181]). To prove (3), apply the observation above to the subset N of M_d of curves not spanning $\mathbf{P}^4$; here N is proper if $d \geq 4$, because, clearly, $\dim N \leq 4d + 4$ whereas $\dim M_d = 5d + 1$. To prove (1), apply the observation above to the subset N of M_d of curves that either don't span $\mathbf{P}^4$ or aren't of maximal rank; here N is proper if $d \geq 4$ by (3) and by the maximal-rank theorem [1, Theorem 1, p. 215]. Finally, (1) implies (4) by virtue of the long exact sequence of cohomology extending the map (2.3).

Lemma 2.3. *Let $(C, F) \in I_d$, and assume that F is smooth along C. Then the following conditions are equivalent:*

(i) At (C, F), the incidence scheme I_d is smooth of dimension 125, and the differential $d\beta$ of the projection $\beta\colon I_d \to \mathbf{P}^{125}$ is surjective.

(ii) At C, the Hilbert scheme of F is reduced of dimension 0.

(iii) The normal sheaf $\mathcal{N}_{C/F}$ has a balanced decomposition,

$$\mathcal{N}_{C/F} = \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

If any one of these conditions obtains, then (C, F) lies on a unique component of I_d, which is generically reduced, has dimension 125, and dominates $\mathbf{P}^{125}$.

Proof. It is necessary and sufficient for (i) to hold that, at (C, F), the fiber of β be smooth of dimension 0 and that β be flat. However, in any event, I_d is simply an open subscheme of the relative Hilbert scheme $\mathrm{Hilb}_{\mathbf{F}/\mathbf{P}^{125}}$ where $\mathbf{F}$ is the universal family of quintics. Hence (i) implies (ii). Moreover, (i) is implied by (iii), because, by the standard theory of the relative Hilbert scheme, when $H^1(\mathcal{N}_{C/F})$ vanishes, then $\mathrm{Hilb}_{\mathbf{F}/\mathbf{P}^{125}}$ is smooth over $\mathbf{P}^{125}$ with $H^0(\mathcal{N}_{C/F})$ as fiber dimension.

It is also part of standard theory that $H^0(\mathcal{N}_{C/F})$ is equal to the Zariski tangent space to the Hilbert scheme of F at the point C; hence (ii) holds if and only if $H^0(\mathcal{N}_{C/F})$ vanishes. Now, it is easy to see that $\mathcal{N}_{C/F}$ has as determinant $\mathcal{O}_{\mathbf{P}^1}(-2)$. Indeed, the sequences (2.1) and (2.2) show that $\mathcal{N}_{C/\mathbf{P}^4}$ has as determinant $\mathcal{O}_{\mathbf{P}^1}(5d-2)$. Then the sequence of normal sheaves,

$$0 \to \mathcal{N}_{F/\mathbf{P}^4} \to \mathcal{N}_{C/\mathbf{P}^4} \to \mathcal{N}_{C/F} \to 0$$

yields the determinant of $\mathcal{N}_{C/F}$. Now, $\mathcal{N}_{C/F} = \mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b)$ for some a and b. Hence $a + b = -2$. Hence $H^0(\mathcal{N}_{C/F})$ vanishes if and only if both a and b are -1. Therefore, (ii) and (iii) are equivalent to each other, whence also to (i).

Suppose one of the conditions (i), (ii) or (iii) obtains; then all three do. Hence (i) implies that (C, F) lies in the smooth locus of I_d, so in a unique component, which is reduced at (C, F). Moreover, (i) implies that this component has dimension 125, and that the projection β onto $\mathbf{P}^{125}$ is smooth at (C, F), so open on a neighborhood of it. Therefore, the component of (C, F) dominates $\mathbf{P}^{125}$, and the proof is complete. □

Lemma 2.4. *The incidence scheme I_d always has at least one component that is generically reduced, that has dimension* 125*, and that dominates* $\mathbf{P}^{125}$.

Proof. By the work of Clemens and Katz (see [8, Theorem 2.1, p. 153]), there is a pair $(C, F) \in I_d$ such that F is smooth along C and such that the normal sheaf $\mathcal{N}_{C/F}$ has a balanced decomposition, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Hence, Lemma 2.3 yields the assertion. □

Proposition 2.5. *Assume that I'_d does not dominate* $\mathbf{P}^{125}$*. Then* $d \leq 24$*, and the principal component $\overline{I}_{d,0}$ is the one and only component of I_d that dominates* $\mathbf{P}^{125}$.

Proof. By Lemma 2.4, there is at least one component of I_d that dominates $\mathbf{P}^{125}$. Given any such component, it cannot lie in I'_d by hypothesis; so it lies in the closure of the complement of I'_d, namely, $\overline{I}_{d,0}$. So $\overline{I}_{d,0}$ is nonempty. Hence Lemma 2.1 implies that $d \leq 24$ and that $\overline{I}_{d,0}$ is a component. The remaining assertions now follow. □

bthLemma 2.6. Let $\tilde{I}_d$ be a component of I_d, and assume that $\tilde{I}_d$ is generically reduced, has dimension 125, and dominates $\mathbf{P}^{125}$. Let $F \in \mathbf{P}^{125}$ be a general quintic, and let Φ be the set of C with $(C, F) \in \tilde{I}_d$. Then Φ is finite and nonempty. Moreover, at each C in Φ, the Hilbert scheme of F is reduced of dimension 0; in fact, each C is embedded in F with balanced normal sheaf, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

Proof. The set $\{(C, F) | C \in \Phi\}$ is simply the fiber of $\tilde{I}_d$ over F. So it is finite and nonempty, because $\tilde{I}_d$ has dimension 125 and dominates $\mathbf{P}^{125}$ and because F is general. By the same token, this fiber lies in the smooth locus of $\tilde{I}_d$, which is nonempty because $\tilde{I}_d$ is generically reduced. Hence, by Sard's lemma, the differential of the projection $I_d \to \mathbf{P}^{125}$ is surjective along Φ. Therefore, the remaining assertions follow from Lemma 2.3. □

Theorem 2.7. *Assume that I'_d does not dominate* $\mathbf{P}^{125}$*. Then* $d \leq 24$*. Let F be a general quintic threefold in* $\mathbf{P}^4$*, and in the Hilbert scheme of F, form the open subscheme of rational, smooth and irreducible curves C of degree d. Then this subscheme is finite, nonempty, and reduced; in fact, each C is embedded in F with balanced normal sheaf* $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$*, and possesses the properties* (1) *to* (4) *of Proposition* 2.2.

Proof. Proposition 2.2 applies, so its properties (1) to (4) hold (but, so far, possibly are vacuous). By Proposition 2.5 above, $d \leq 24$, and $\overline{I}_{d,0}$ is the one and only component of I_d that dominates $\mathbf{P}^{125}$. This component is generically reduced and has dimension 125 by Lemma 2.1, or alternatively by Lemma 2.4. Hence Lemma 2.6 yields the remaining assertions. □

3. Other subsets. In this section, we'll prove Proposition 3.2, which asserts that I_d is reducible for $d \geq 12$. We'll proceed by introducing and studying some basic subsets J_d^e and K_d of I_d. For $d \geq 12$, they provide one or more components of I_d, which do not dominate $\mathbf{P}^{125}$. After proving the proposition, we'll make two remarks; the first discusses a refinement of the condition that I'_d does not dominate $\mathbf{P}^{125}$, and the second discusses the location in I_d of the pair (C, F) of Clemens and Katz.

The subsets J_d^e and K_d of I_d are the following:

(a) J_d^e is the set of pairs $(C, F) \in I_d$ such that C spans a hyperplane H and lies on a smooth surface S of degree e in H;

(b) K_d is the set of pairs $(C, F) \in I_d$ such that C spans a hyperplane H and $H^0(\mathcal{I}_{C/H}(5)) = 0$ where $\mathcal{I}_{C/H}$ is the ideal of C in H.

Lemma 3.1. *The dimensions of the above sets are as follows:*

$$\dim J_d^2 = 2d + 101 \quad \textit{for} \quad d \geq 10; \qquad \dim J_d^3 = d + 101 \quad \textit{for} \quad d \geq 15;$$
$$\dim J_d^4 \leq 97 \quad \textit{for} \quad d \geq 20; \qquad \dim K_d = 4d + 73 \quad \textit{for} \quad d \geq 11.$$

None of these sets dominates $\mathbf{P}^{125}$. *Moreover,* K_d *is empty for* $d \leq 10$.

Proof. Fix $(C, F) \in J_d^e$. By definition, C spans a hyperplane H and lies on some smooth surface S of degree e in H. If $d \geq e^2$, then S is uniquely determined; otherwise, C would lie in the intersection of two different smooth surfaces of degree e in H, so C would be equal to this intersection, and so would have nonzero genus. Furthermore, if $d \geq 5e$, then S lies in F; otherwise, the intersection of S and F would be a curve containing C, so C would be equal to this intersection, and so would have nonzero genus.

Vary $(C, F) \in J_d^e$, and form the space $\tilde{J}_d^e$ of corresponding triples (C, S, F). If $e \leq 5$ and $d \geq 5e$, then, by the preceding argument, the projection $\tilde{J}_d^e \to J_d^e$ is bijective, so J_d^e and $\tilde{J}_d^e$ have the same dimension and the same image in $\mathbf{P}^{125}$. We'll now compute this dimension and image for $e = 2, 3, 4$. The fiber of $\tilde{J}_d^e$ over a pair (S, F) consists of all C in S. So this fiber has dimension $2d - 1$ if $e = 2$, dimension $d - 1$ if $e = 3$, and dimension at most 0 if $e = 4$. These dimensions are well known, and they are easy to check. (For $e = 3$, use [5, 4.8, p. 401] and [5, 4.8, p. 407]. For $e = 4$, note that there are at most finitely many C on a given S because the normal sheaf of C is equal to $\mathcal{O}_{\mathbf{P}^1}(-2)$; in fact, a general S can contain no C because all the curves on it are complete intersections by Noether's theorem).

The F containing a fixed S form a space of dimension $h^0(\mathcal{I}_S(5)) - 1$. To compute it, use the natural exact sequence of ideals,

$$0 \to \mathcal{I}_H \to \mathcal{I}_S \to \mathcal{I}_{S/H} \to 0.$$

The first term is equal to $\mathcal{O}_{\mathbf{P}^4}(-1)$ and the third to $\mathcal{O}_{\mathbf{P}^3}(-e)$. Hence

$$h^0(\mathcal{I}_S(5)) = h^0(\mathcal{O}_{\mathbf{P}^4}(4)) + h^0(\mathcal{O}_{\mathbf{P}^3}(5-e)) = 70 + \binom{8-e}{3}.$$

The various S in a fixed H form a space of dimension $\binom{3+e}{3} - 1$, and the various H form a $\mathbf{P}^4$. Hence the various pairs (S, F) form a space of dimension,

$$\begin{aligned} (70+20-1)+(10-1+4) &= 102 \quad \text{if} \quad e=2,\\ (70+10-1)+(20-1+4) &= 102 \quad \text{if} \quad e=3,\\ (70+4-1)+(35-1+4) &= 111 \quad \text{if} \quad e=4. \end{aligned}$$

These numbers are less than 125. Therefore, J_d^e doesn't dominate $\mathbf{P}^{125}$ for $e = 2, 3, 4$ and $d \geq 5e$. Furthermore,

$$\begin{aligned} &\dim J_d^2 = (2d-1) + 102 = 2d + 101 \quad \text{for} \quad d \geq 10,\\ &\dim J_d^3 = (d-1) + 102 = d + 101 \quad \text{for} \quad d \geq 15,\\ &\dim J_d^4 \leq 0 + 111 = 111 \quad \text{for} \quad d \geq 20. \end{aligned}$$

Thus the assertions about the J_d^e are proved.

To analyze K_d, observe that, in M_d, the C that lie in a fixed hyperplane H form a closed subset of dimension $4d$, and that, in this closed subset, those C with $h^0(\mathcal{I}_{C/H}(5)) = 0$ form an open subset by upper semi-continuity of dimension. This open set is nonempty, so of dimension $4d$, if and only if $d \geq 11$; indeed, the maximal rank theorem for rational curves in $\mathbf{P}^3$ [6, Theorem 0.1, p. 209] implies that, for a general C in H, the natural map,

$$H^0(H, \mathcal{O}_H(5)) \to H^0(C, \mathcal{O}_C(5)),$$

is injective if and only if $d \geq 11$, because the source and target have dimensions 56 and $5d + 1$. Hence K_d is empty for $d \leq 10$. On the other hand, since the various H form a $\mathbf{P}^4$, the image of K_d in M_d therefore has dimension $4d + 4$ for $d \geq 11$.

Whenever $C \subset H$ and $H^0(\mathcal{I}_{C/H}(5)) = 0$, the natural inclusion map,

$$H^0(\mathcal{I}_H(5)) \to H^0(\mathcal{I}_C(5)), \tag{3.1}$$

is bijective. Since the source has dimension 70, the fiber in K_d over C is a $\mathbf{P}^{69}$. Hence K_d has dimension $(4d + 4) + 69$, or $4d + 73$. Moreover, since (3.1) is bijective, the image of K_d in $\mathbf{P}^{125}$ is equal to the set of quintics F that contain a hyperplane. The latter set has dimension $69 + 4$, or 73. The proof is now complete. □

Proposition 3.2. *If $d \geq 12$, then I_d is reducible. In fact, if $d \geq 13$, then I_d has a component of dimension at least* 126*, as well as one of dimension* 125.

P r o o f. On the one hand, I_d always has at least one component that has dimension 125 and that dominates $\mathbf{P}^{125}$ by Lemma 2.4. On the other hand, if $d \geq 10$, then I_d has a subset, namely J_d^2, that has dimension $2d + 101$ and that doesn't dominate $\mathbf{P}^{125}$ by Lemma 3.1. Hence, if $d \geq 13$, then $\dim J_d^2 \geq 126$, and so I_d has a component of dimension at least 126, as well as one of dimension 125. Suppose $d = 12$. Then J_d^2 has dimension 125, but doesn't dominate $\mathbf{P}^{125}$. So J_d^2 cannot lie in the component of I_d that dominates $\mathbf{P}^{125}$. Hence I_d is still reducible. Thus the proposition is proved.

Remark 3.3. For $d \leq 24$, it is not unreasonable to hope that the complement I_d'' of $\overline{I}_{d,0}$ in I_d lies in the closure of the union of J_d^2, J_d^3, and K_d, and that this union doesn't dominate $\mathbf{P}^{125}$. If this hope is confirmed, then I_d' doesn't dominate $\mathbf{P}^{125}$ either, because $I_d' - I_d''$ is equal to $\overline{I}_{d,0} - I_{d,0}$ and so has dimension at most 124. Hence, then the conclusions of Proposition 2.2, Proposition 2.5, and Theorem 2.7 will hold.

Lemma 3.1 supports this hope. Indeed, the lemma implies that K_d for $d \geq 13$ yields another component of I_d that doesn't dominate $\mathbf{P}^{125}$, and that J_d^3 for $d \geq 24$ yields one too, but that J_d^4 for $d \geq 20$ does not. Of course, to confirm our hope, we must handle the J_d^e for the d and e not covered by Lemma 3.1, and we must handle the subset of I_d of pairs (C, F) such that C spans a hyperplane H and lies on a singular, reduced and irreducible, surface of degree e in H, but not on a smooth one. However, we may assume that $e \leq 5$, because C lies in the intersection of H and F, and the latter will be a surface of degree 5 for a suitable F if $(C, F) \notin K_d$. Of course, we may assume $d \geq 10$ because I_d is irreducible for $d \leq 9$ by [7, Proposition 2.2]. Moreover, we may assume that $e \geq 3$ because, if C lies in a plane, then d is 1 or 2, and if C lies on a 2-dimensional singular quadric cone, then $d \leq 3$ by [5, Exemple 2.9, p. 384].

To confirm our hope, we must also handle the subset of I_d of pairs (C, F) such that C spans $\mathbf{P}^4$ and lies on a hypersurface T of degree t with $2 \leq t \leq 5$. Now, for $t = 2, 3, 4$, this subset does not trivially yield a new component of I_d. Indeed, locally $dt + 1$ conditions must be satisfied for a C in M_d to lie on a given T, and each such C lies at least in the reducible quintics F of the form $T + T'$. Hence, the various such (C, F) form a space of dimension at least,

$$(5d + 1) - (dt + 1) + \binom{4 + t}{4} - 1 + \binom{9 - t}{4} - 1.$$

This number is equal to $3d + 48$ for $t = 2$, to $2d + 48$ for $t = 3$, and to $d + 73$ for $t = 4$. Its maximum value is achieved for $d = 24$ and $t = 2$, and this maximum is only 120, not enough to yield a new component.

It is less likely (as well as unnecessary) that I'_d lies in the closure of the union of J^2_d, J^3_d, and K_d. In other words, there may be pairs (C, F) outside this closure, yet in $\overline{I}_{d,0} - I_{d,0}$. For example, such a pair might arise from a curve C of degree 9 that spans $\mathbf{P}^4$ and has a 7-secant.

Remark 3.4. It is interesting to look at the pair (C, F) found by Katz [8, p. 153], and observe where it sits in I_d. Katz began with the curve $C \in M_d$ constructed by Mori [9]. It lies on a smooth quartic surface S in a hyperplane H in $\mathbf{P}^4$. So it lies in all the reducible quintic surfaces $S + L$ where L is a plane in H. Hence $h^0(\mathcal{I}_{C/H}(5)) \geq 4$. So $(C, F) \notin K_d$. Moreover, if $d \geq 12$, then C cannot lie on a cubic surface (otherwise it would lie on the intersection of this cubic with S), and so $(C, F) \notin J^3_d$. Similarly, if $d \geq 8$, then $(C, F) \notin J^2_d$.

If $d \geq 10$, then $(C, F) \notin I_{d,0}$. Indeed, $H^1(\mathcal{I}_C(5)) = H^1(\mathcal{I}_{C/H}(5))$ because of the exact sequence of twisted ideals,

$$0 \to \mathcal{I}_H(5) \to \mathcal{I}_C(5) \to \mathcal{I}_{C/H}(5) \to 0,$$

whose first term is equal to $\mathcal{O}_{\mathbf{P}^4}(4)$. Hence it's enough to check that $h^1(\mathcal{I}_{C/H}(5)) > 0$. Now, the usual long exact cohomology sequence yields

$$\begin{aligned} h^1(\mathcal{I}_{C/H}(5)) &= h^0(\mathcal{I}_{C/H}(5)) - h^0(\mathcal{O}_H(5)) + h^0(\mathcal{O}_C(5)) \\ &\geq (5d + 1) - 56 + 4 = 5d - 51. \end{aligned}$$

Hence $h^1(\mathcal{I}_{C/H}(5)) > 0$ if $d \geq 11$. A more sophisticated, but well-known, argument works for $d \geq 10$. Namely, C has a $(d-3)$-secant line; it's the curve D in [9, p.129]. By Bezout's theorem, D lies in every hypersurface of degree $d-4$ containing C. So C is not cut out by such hypersurfaces. Hence, C is $(d-4)$-irregular. Therefore, $H^1(\mathcal{I}_C(d-5))$ is nonvanishing since $H^q(\mathcal{I}_C(d-4-q))$ vanishes for

$q \geq 2$. It follows that $H^1(\mathcal{I}_C(5))$ is nonvanishing if $d \geq 10$. Thus there's some content to our conjecture that $(C, F) \in \overline{I}_{d,0}$ for $d \leq 24$.

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