# Serdica
## Mathematical Journal
# Сердика
## Математическо списание



For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **23** (1997), 113-124

**Serdica**
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

# ON THE KAM – THEORY CONDITIONS FOR THE KIRCHHOFF TOP

Ognyan Christov

*Communicated by V. Kanev*

Abstract. In this paper the classical Kirchhoff case of motion of a rigid body in an infinite ideal fluid is considered. Then for the corresponding Hamiltonian system on the zero integral level, the KAM theory conditions are checked. In contrast to the known similar results, there exists a curve in the bifurcation diagram along which the Kolmogorov's condition vanishes for certain values of the parameters.

**1. Introduction.** The question of integrability of Hamiltonian systems is one of the oldest problems of classical mechanics [1]. Classical results due to Poincare and Bruns show that most of the Hamiltonian systems are not integrable. This has lead Poincare [2] to define the main problem of dynamics to be the study Hamiltonian systems which are close to integrable ones. The most powerful approach to such systems is KAM – theory. Before giving a brief account of KAM – theory we remind the structure of the integrable Hamiltonian systems.

1991 *Mathematics Subject Classification:* 58F07, 70Exx
*Key words:* KAM-theory, Kirchhoff top, Abelian integrals
*Partially supported by Grant MM523/95 with Ministry of Science and Technologies.

The phase space of the generic integrable Hamiltonian systems with $n$–degrees of freedom is foliated into invariant manifolds the typical fibre being a $n$–dimensional torus, on which the motion is quasiperiodic. A natural question is whether small perturbations destroy these tori. KAM – theory gives conditions for the integrable systems which guarantee the survival of most of the invariant tori. The conditions are given in terms of so-called action – angle variables $J_1, J_2, \ldots, J_n$; $\varphi_1, \varphi_2, \ldots, \varphi_n$. Without going into details, we remind that the action – angle variables can be introduced for any integrable system locally near a fixed torus and have a property that $\mathbf{J} = (J_1, J_2, \ldots, J_n)$ maps a neighbourhood of a fixed torus on an open subset of $R^n$. The functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the co-ordinates on any of the nearby tori. Moreover the first integrals become functions of the action variables $J_1, J_2, \ldots, J_n$. At last to any fixed torus there corresponds an invariant torus on which the motion is quasiperiodic with frequencies $(\omega_1(\mathbf{J}), \ldots, \omega_n(\mathbf{J})) = (\partial H/\partial J_1, \ldots, \partial H/\partial J_n)$ (see [3] for details).

One condition, stated by Kolmogorov [3, app.8 and literature there] on the Hamiltonian of the integrable system that ensures the survival of most of the invariant tori under small perturbations is that the frequency map

$$\mathbf{J} \to (\omega_1(\mathbf{J}), \omega_2(\mathbf{J}), \ldots, \omega_n(\mathbf{J}))$$

should be non-degenerate. Analytically this means that the Hesseian

$$\det\left(\frac{\partial^2 H}{\partial J_j \partial J_k}\right) \; j,k = 1,\ldots,n \tag{1}$$

does not vanish. We should note that the measure of the surviving tori decreases with the increase of both perturbation and measure of the set where the above hesseian is too close to zero.

Another condition of this type, stated by Arnold and Moser [3, appl. 8] is that of isoenergetical non-degeneracy which we explain next. Fix the energy level $H_0 = h_0$. If we write the Hamiltonian $H_0$ in action variables, then define the following map $F_{h_0}$ from the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into the projective space $P^{n-1}$:

$$F_{h_0} : \mathbf{J} \to (\omega_1(\mathbf{J}) : \omega_2(\mathbf{J}) : \ldots : \omega_n(\mathbf{J}))$$

then the system is isoenergetically non-degenerate if the map $F_{h_0}$ is a homeomorphism. Analytically the isoenrgetical non-degeneracy is tantamount to nonvanishing of the determinant

$$\det \begin{pmatrix} \dfrac{\partial^2 H_0}{\partial \mathbf{J}^2} & \dfrac{\partial H_0}{\partial \mathbf{J}} \\ \dfrac{\partial H_0}{\partial \mathbf{J}} & 0 \end{pmatrix}. \tag{2}$$

Usually for the interesting systems studied by mathematical physics KAM – theory conditions can be expressed in terms of Abelian integrals. This makes them complicated for checking. (see References in [4] for existing results)

In this paper we study the KAM – theory conditions for the special case of the so called Kirchhoff top of the motion of a rigid body in an ideal infinite fluid [5, 6, 7]. The approach from one paper of Horozov [8] is adopted. One should note that in this case the algebraic curve is of 4th degree and the asymptotic behaviour of the corresponding Abelian integrals is different. In contrast to the similar works where the Hesseian is negative it turns out that in this case the Hesseian vanishes on a curve in bifurcation diagram for certain values of the parameters.

**2. The Kirchhoff top.** The free motion of a rigid body in an infinite ideal fluid can be described by the Kirchhoff equations [6]

$$\begin{aligned} \dot{\mathbf{y}} &= \mathbf{y}\times\omega + \mathbf{x}\times\mathbf{u}, \\ \dot{\mathbf{x}} &= \mathbf{x}\times\omega, \end{aligned} \tag{3}$$

where $\omega = \partial H/\partial \mathbf{y}$, $\mathbf{u} = \partial H/\partial \mathbf{x}$ and $H(\mathbf{x}, \mathbf{y})$ is a positive definite quadratic form

$$H = \frac{1}{2}\{(A\mathbf{y}, \mathbf{y}) + (B\mathbf{y}, \mathbf{x}) + (C\mathbf{x}, \mathbf{x})\} \tag{4}$$

The vectors $\omega$ and $\mathbf{y}$ are called the angular velocity and the kinetic momentum of the body, $\mathbf{x}$ and $\mathbf{u}$ are called the impulsive force and the impulsive momentum. The Kirchhoff equations (2) have three integrals $F_1 = H$, $F_2 = (\mathbf{y}, \mathbf{x})$, and $F_3 = (\mathbf{x}, \mathbf{x})$. In general, the equations (2) are not integrable. The classical Kirchhoff case [5, 1] (usually called the Kirchhoff top) is when the body is axially symmetric. In this case, $A = \operatorname{diag}(A_1, A_1, A_3)$, $B = \operatorname{diag}(B_1, B_1, B_3)$, $C = \operatorname{diag}(C_1, C_1, C_3)$, and there is an additional integral, namely the momentum with the respect to the symmetry axis

$$L(\mathbf{x}, \mathbf{y}) = y_3. \tag{5}$$

Then the system can be integrated via elliptic functions.

Define a Poisson bracket on $R^6$ by putting

$$\begin{aligned} \{y_i, y_j\} &= \epsilon_{ijk} y_k, \quad \{y_i, x_j\} = \epsilon_{ijk} x_k, \\ \{x_i, x_k\} &= 0, \qquad\quad i, j, k = 1, 2, 3. \end{aligned} \tag{6}$$

and extend $\{\cdot, \cdot\}$ to $C^\infty(R^6)$ in the natural way. Then, the Kirchhoff equations (2) can be expressed in the Hamiltonian form

$$\dot{y_i} = \{H, y_i\}, \qquad \dot{x_i} = \{H, x_i\}, \; i = 1, 2, 3. \tag{7}$$

However, the bracket (5) is degenerate – every smooth function commutes with the integrals $F_2$ and $F_3$. This permits us to restrict $\{\cdot, \cdot\}$ to the integral surface level $M_c \stackrel{\text{def}}{=} \{F_2 = c, F_3 = 1\}$ in the usual way. Thus, the Kirchhoff top is effectively a two degree of freedom system. It can be easily checked that the symplectic leaves $M_c$ are all diffeomorphic to $T^*S^2$ by the transformation $y = z + cx$ (we identify $TS^2$ with $T^*S^2$ via the standard Riemannian metric).

To simplify computation we shall consider the special case when the body has three mutually perpendicular planes of symmetry (like ellipsoid for example). In this case $B = 0$. Using $(\mathbf{x}, \mathbf{x}) = 1$ and a rescaling of time, the Hamiltonian can be written in the following way

$$H = \frac{1}{2}\{(y_1^2 + y_2^2) + Ay_3^2 + Cx_3^2\}. \tag{8}$$

Here $A > 1/2$, $C$ are some parameters which depend on the shape of the rigid body and the density of the fluid. There are two topologically different cases: $A > 1$, $C > 0$ – prolate (ovrary) ellipsoid case and $A > 1/2$, $C < 0$ – oblate (planetary) ellipsoid case.

On $T^*S^2$ the following coordinates are introduced [9]:

$$\begin{aligned} x_1 = \cos\theta\cos\phi, \quad z_1 &= p_\phi \tan\theta\cos\phi - p_\theta\sin\theta, \\ x_2 = \cos\theta\sin\phi, \quad z_2 &= p_\phi \tan\theta\sin\phi + p_\theta\cos\theta, \\ x_3 = -\sin\theta, \quad z_3 &= p_\phi. \end{aligned}$$

where $\theta \in (-\pi/2, \pi/2)$, $\phi \in (0, 2\pi)$, $p_\phi$, $p_\theta \in R$. The Poisson brackets (6) reduce to

$$\begin{aligned} \{\theta, \phi\} &= \{p_\theta, \phi\} = \{p_\phi, \theta\} = 0, \\ \{\theta, p_\theta\} &= \{\phi, p_\phi\} = 1, \{p_\theta, p_\phi\} = c\cos\theta. \end{aligned}$$

The corresponding 2-form is $d\sigma$, where

$$\sigma = p_\theta d\theta + p_\phi d\phi - -c\sin\theta d\phi. \tag{9}$$

Thus the symplectic structure on $M_c$ is the standard symplectic structure on $T^*S^2$ distorted by the magnetic term $F = -c\cos\theta d\theta \wedge d\phi$. As noted by Novikov in [9], the Kirchhoff top is mathematically equivalent to a classical charged particle moving on $S^2$ under the influence of monopole like magnetic field given by $F$, albeit with a nonstandard metric, gyroscopic terms and an unusual potential.

In this paper we consider the Kirchhoff top in a special case i.e. on the special integral level $M_0(c = 0)$ (see the final remarks on the general case).

**Remark.** We note that the particular case $A = 1$ is equivalent to the system governed the so called square potential spherical pendulum, studied by [10]. See also [4] for additional comments.

Then, in the coordinates $\phi$, $\theta$, $p_\phi$, $p_\theta$ the Hamiltonian (8) reads ($C = 1$)

$$H = \frac{1}{2}(p_\theta^2 + p_\phi^2(\tan^2\theta + A) + \sin^2\theta) = h \tag{10}$$

with a second integral $L = p_\phi = l$.

In order to introduce the action–angle variables, we need to find the set of regular values of energy-momentum map

$$EM : TS^2 \to R^2 : (\mathbf{x}, \mathbf{y}) \to (H, L).$$

**Lemma 2.1** [7]. *The regular values of the energy-momentum map are given by $U_r = \{(h, l) \in R^2, h > Al^2/2\} \setminus (1/2, 0)$ Moreover, for each $(h, l) \in U_r$ the level sets $EM^{-1}(h, l)$ is diffeomorphic to the two-torus $T_{h,l}$ .*

Choose a basis $\gamma_1, \gamma_2$ of the homology group $H_1(T_{h,l}, Z)$ with the following representatives. For $\gamma_1$ take the curve on $T_{h,l}$ defined by fixing $\theta, p_\theta, p_\phi$ and letting $\phi$ run through $[0, 2\pi]$. For $\gamma_2$ fix $\phi$ and let $\theta, p_\theta$ make one circle on the curve given by the equation

$$p_\theta^2 + l^2(\tan^2\theta + A) + \sin^2\theta = 2h.$$

Now, following [4] we can define the action co-ordinates $J_1, J_2$ by the formula

$$J_j = \int_{\gamma_j} \sigma, \quad j = 1, 2,$$

where $\sigma$ is canonical one - form (8) with $c = 0$. Trivial computations give

$$J_1 = 2\pi l, \tag{11}$$

(12) $$J_2 = \int_{\gamma_2} p_\theta d\theta = 2\int_{\theta_-}^{\theta_+} \sqrt{2h - \sin^2\theta - l^2(A + \tan^2\theta)}d\theta,$$

where $\theta_+ > \theta_-$ are the two roots of the equation

$$l^2(\tan^2\theta + A) + \sin^2\theta = 2h$$

in the interval $(-\pi/2, \pi/2)$. For later use we make a change of the variables in the integral defining $J_2$ . Put $z = \sin\theta$, $y^2 = 2h(1-z^2) - l^2(A+z^2(1-A)) - z^2(1-z^2)$. Denote the oval of the curve

$$\Gamma_{h,l} = \{(y, z) : y^2 = 2h(1 - z^2) - l^2(A + z^2(1 - A)) - z^2(1 - z^2)\}$$

by $\gamma$. Then we have

(13) $$\psi(h, l) \stackrel{\mathrm{def}}{=} J_2 = \int_\gamma \frac{ydz}{1 - z^2}.$$

**Remark.** It turns out that the action variables are not defined globally due to the existing of monodromy. However, one can construct them explicitly (see [10] for details) in the two simply connected domains where the global action variables exist. We continue our consideration in anyone of these domains.

**3. Main results.** Denote by $\tilde{H}(J_1, J_2)$ the Hamiltonian of the considered system in action co-ordinates. We state the theorems which are the aim of this paper.

**Theorem 1.**

**(i)** *When $1/2 < A \le 1$, for $(h, l) \in U_r$ the following determinant*

(14) $$\det\begin{pmatrix} \dfrac{\partial^2\tilde{H}}{\partial J_1^2} & \dfrac{\partial^2\tilde{H}}{\partial J_1\partial J_2} \\ \dfrac{\partial^2\tilde{H}}{\partial J_1\partial J_2} & \dfrac{\partial^2\tilde{H}}{\partial J_2^2}\end{pmatrix}.$$

*does not vanish.*

**(ii)** *When $A > 1$, there exists a curve in $U_r$, passing through the singular point $(1/2, 0)$ on which the above determinant vanishes.*

**Theorem 2.** *The isoenergetical non-degeneracy condition is fulfilled almost everywhere in $U_r$.*

$$\det\begin{pmatrix} \dfrac{\partial^2 \tilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \tilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \tilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \tilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \tilde{H}}{\partial J_2^2} & \dfrac{\partial \tilde{H}}{\partial J_2} \\ \dfrac{\partial \tilde{H}}{\partial J_1} & \dfrac{\partial \tilde{H}}{\partial J_2} & 0 \end{pmatrix} \not\equiv 0. \tag{15}$$

We shall give the conditions (14) and (15) the explicit form in terms of Abelian integrals of the second kind. Using the expressions (11) and (13) for $J_1$ and $J_2$ we can determine $\tilde{L}$ and $\tilde{H}$ implicitly from the equations

$$J_1 = 2\pi \tilde{L}, J_2 = \psi(\tilde{L}, \tilde{H}).$$

The following lemma gives the expression (14) in terms of Abelian integrals.

**Lemma 3.1** ([10])**.**

$$(2\pi)^2 \left(\frac{\partial \psi}{\partial h}\right)^4 \det\begin{pmatrix} \dfrac{\partial^2 \tilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \tilde{H}}{\partial J_1 \partial J_2} \\ \dfrac{\partial^2 \tilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \tilde{H}}{\partial J_2^2} \end{pmatrix} = \det\begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial l} \\ \dfrac{\partial^2 \psi}{\partial h \partial l} & \dfrac{\partial^2 \psi}{\partial l^2} \end{pmatrix}.$$

Obviously we have

$$\psi_h = \int_\gamma \frac{dz}{y} \neq 0$$

in $U_r$. So, instead of proving the theorem with the expression (14) it is enough to consider the last Hesseian which we denote by $D$. Similarly we have that the isoenrgetical non-degeneracy condition is reduced to $\psi_{ll}$ up to nonvanishing factor. We denote $D_1 = \psi_{ll}$ for simplicity.

Next we would like to show that the entries of $D$ and $D_1$ can be represented as elliptic integrals. Introduce functions

$$w_j(h,l) = \int_\gamma \frac{z^j dz}{y^3}, \ j = 0, 1, \ldots \tag{16}$$

It was proven in [11] that $w_0 \neq 0$ in $U_r$. Then obviously we have also that $w_2 \neq 0$ in $U_r$. The next lemma gives a representation of $D$ as a quadratic form in $w_0$, $w_2$.

**Lemma 3.2.** *The determinant $D$ has the following representation*

$$\begin{aligned} D &= w_2^2\{1-(1-A)[2h+1+l^2(1-A)]\}+ \\ & \quad w_0w_2\{2(1-A)(2h-Al^2)-A[2h+1+l^2(1-A)]\}+ \\ & \quad w_0^2\{(2h-Al^2)(2A-1)\}. \end{aligned} \tag{17}$$

Similarly we have for $D_1$.

$$D_1 = \psi_{ll} = [(2h-Al^2)(1-A)-2hA]w_0 - [2h(1-A)-A]w_2. \tag{18}$$

It is seen that $D$ and $D_1$ do not depend on the sign of $l$. That is why it is enough to prove theorems only for $l \geq 0$. We denote $U_r^+ = U_r \bigcap l \geq 0$.

**4. The Kolmogorov's condition.** The idea of the proof (see [8]) is first to evalute the sign of the determinant $D$ on line $l=0$ and then to spread it along curves which filled all $U_r^+$. The monotonicity of certain function plays essential role. The proof is done in several steps and the cases $A<1$, $A=1$, $A>1$ are considered separately. Here we only state the corresponding results. (see [4] for the proofs)

**Lemma 4.1.** *Let $l=0$. Then the functions $w_0, w_2$ satisfy the following system of Picard-Fuchs equations:*

$$\begin{aligned} 2h(1-2h)\frac{dw_0}{dh} &= 3w_2+2(3h-1)w_0, \\ (1-2h)\frac{dw_2}{dh} &= 3w_2+w_0. \end{aligned} \tag{19}$$

We also need the function

$$\sigma(h) = \frac{w_2(h,0)}{w_0(h,0)} \tag{20}$$

Easy calculations give that $\sigma$ satisfies the Riccati equation

$$2h(1-2h)\frac{d\sigma}{dh} = -3\sigma^2+2\sigma+2h \tag{21}$$

When $l = 0$ the expression for $D$ factors:

$$D = w_0^2 \sigma_1 \sigma_2,$$

where

$$\sigma_1(h) = \sigma(h) - 1,\ \sigma_2(h) = [A + (A-1)2h]\sigma(h) - 2h(2A-1).$$

From (21) we obtain easily the Riccati equations for $\sigma_1$, $\sigma_2$. They play decissive role in evaluting the signs of these functions.

We need also some other functions both for study of $\sigma$, $\sigma_1$, $\sigma_2$ and for the case $l > 0$. In order to introduce them we put the family of curves $\Gamma_{h,l}$ into the normal form:

$$\Gamma_p = \left\{ (u, v) \in C^2 : \frac{v^2}{2} = \frac{u^4}{4} - u^2 + p \right\}, \tag{22}$$

by rescaling $y = \alpha v$, $z = \beta u$ where

$$\beta^2 = [2h + 1 + (1-A)l^2]/4, \alpha^2 = 2\beta^4. \tag{23}$$

If we put

$$p(h,l) = \frac{4(2h - Al^2)}{[2h + 1 + (1-A)l^2]^2} \tag{24}$$

we get (22). In these variables the integrals $w_0(h,l), w_2(h,l)$ become

$$w_0 = \frac{\beta}{\alpha^3} \int_{\gamma(p)} \frac{du}{v^3},\ w_2 = \frac{\beta^3}{\alpha^3} \int_{\gamma(p)} \frac{u^2 du}{v^3}.$$

Introduce the new functions

$$\theta_0(p) = \int_{\gamma(p)} \frac{du}{v^3}, b \quad \theta_2(p) = \int_{\gamma(p)} \frac{u^2 du}{v^3}$$

and their ratio

$$\varrho(p) = \theta_2(p)/\theta_0(p).$$

In these notations we have

$$\sigma(h) = (2h+1)\varrho(p(h,0))/4. \tag{25}$$

Obviously the asymptotic behaviour of the functions $\sigma$, $\sigma_1$, $\sigma_2$ can be found easily from the function $\varrho(p)$. We state the corresponding result in the following lemma:

**Lemma 4.2.** *The function $\varrho(p)$ is strictly monotonic decreasing in the interval* $(0,1)$. $\varrho(0)=\frac{8}{3}, \varrho(1)=2$.

The proof is based on some elements of the Picard-Lefschetz theory and the Riccati equation for the function $\varrho$.

Now we begin with case $l=0$.

**Lemma 4.3.** *The function $\sigma_1(h)$ satisfies the following inequalities:*

**(i)** $\sigma_1(h)<0$, *when* $h\in(0,1/2)$;

**(ii)** $\sigma_1(h)>0$, *when* $h\in(1/2,\infty)$.

Similarly, we have

**Lemma 4.4.** *The function $\sigma_2(h)$ satisfies the following inequalities:*

**(i)** *Let* $1/2<A<1$, $\sigma_2(h)>0$ *in the region* $(0,1/2)$, $\sigma_2(h)<0$ *in the region* $(1/2,\infty)$;

**(ii)** *Let* $A>1, \sigma_2(h)>0$ *in the region* $(0,1/2)\bigcup(1/2,\infty)$.

**Corollary.** *For* $l=0$

**(i)** $D<0$, *when* $1/2<A<1$;

**(ii)** $D<0$ *in* $(0,1/2)$,
$D>0$ *in* $(1/2,\infty)$ *when* $A>1$.

Next we turn to the case $l>0$.

**Lemma 4.5.** *For $h>0$, $l>0$ we have the representation*

$$D=w_0^2\nu^2F(p,\nu) \tag{26}$$

*where* $\nu=\beta^2$ *and*

$$F(p,\nu)=\varrho^2(p)[1-4(1-A)\nu]+4[2(1-A)p\nu-A]\varrho(p)+4p(2A-1) \tag{27}$$

*The functions $\nu(h,l), p(h,l)$ map the set $U_r\bigcap\{h>0,l>0\}$ diffeomorphically on the set*

$$V_r=\left\{(p,\nu):\nu\in(1/4,\infty),p\in\left(0,\frac{4\nu-1}{4\nu^2}\right)\right\}.$$

Note that the line $l = 0$ maps on line $p = \dfrac{4\nu - 1}{4\nu^2}$.

When $A = 1$, $F$ factors. One of the terms is positive in $U_r$. Using the Riccati equation for the other function we reach that $F < 0$ and hence $D < 0$ for $A = 1$.

**Lemma 4.6.** *For any fixed $p \in (0, 1)$ the function $F(p, \ )$ is a*

**(i)** *strictly decreasing function of $\nu$ , when $1/2 < A < 1$;*

**(ii)** *strictly increasing function of $\nu$ , when $A > 1$.*

**Lemma 4.7.**

**(i)** *For all $(p, \nu) \in V_r$ $F(p, \nu) < 0$ when $A < 1$;*

**(ii)** *There exists a curve in $V_r$ along which $F(p, \nu) = 0$, when $A > 1$.*

The last lemma finishes the proof of the Theorem 1.

**5. The isoenergetical non-degeneracy condition.** In order to prove Theorem 2, we need only to check that $\psi_{ll}$ is nonzero in one point of $U_r$. We take for instance the point $h = A/2, l = 0$ for the case $A < 1$, the point $h = 1/3, l = 0$ for the case $A > 1$ and substitute them in (18). Then, using the results from the previous paragraph, it is seen that $\psi_{ll} \neq 0$ in these points. Hence, due to analyticity $\psi_{ll}$ is almost everywhere nonzero in $U_r$.

However, for the particular case $A = 1$ we can improve the above result, namely

**Theorem 3.** *Let $A = 1$.*

**(i)** *For $h \in (0, 1/2] \bigcup [1, \infty)$, $D_1 = \psi_{ll} \neq 0$.*

**(ii)** *For $h \in (1/2, 1)$, $\psi_{ll}$ has exactly two zeros.*

Note that in this case $\psi_{ll} = \dfrac{\beta\theta_0}{\alpha^3}\left[\varrho - \dfrac{4\nu - 1}{\nu}\right]$, where $\nu = \dfrac{2h + 1}{4}$. Then, the proof of theorem 3 follows the idea of [12].

**Remark.** It turns out that in the general case $c \neq 0$ the verification of the KAM – theory conditions is much more complicated and has not been done rigorously. However, the results from this paper afford us to claim that the KAM – theory conditions for the general Kirchhoff top are fulfilled almost everywhere.

## REFERENCES


[1] B. Dubrovin, I. Krichever, S. Novikov. Dynamical Systems IV, Encyclopedia of Mathematical Sciences. Springer, Berlin, 1987.

[2] H. Poincare. Les Methodes Nouvelles de la Mecanique Celeste, vol. I. Gautier-Villars, Paris, 1892.

[3] V. Arnold. Mathemathical Methods of Classical Mechanics. Springer, Berlin, 1980.

[4] O. Christov. On the Kolmogorov's condition for a special case of the Kirchhoff top. *J. Math. Phys.* **35**, *(7)* (1994), 3448-3463.

[5] G. Kirchhoff. Über die Bewegung eines Rotations in einer Flussigkeit. *J. Reine Angew. Math.* **71** (1870), 237-262.

[6] L. Milne-Thomson. Theoretical Hydrodynamics. McMillan, New York, 1968.

[7] L. Bates, M. Zou. Degeneration of Hamiltonian monodromy cycles. *Nonlinearity* **6**, *2* (1993), 313-335.

[8] E. Horozov. Perturbation of the spherical pendulum and Abelian integrals. *J. Reine Angew. Math.* **408** (1990), 114-135.

[9] S. Novikov. The Hamilton formalism and a many-valued analogue of Morse theory. *Russian Math. Surveys.* **37** (1982), 1-56.

[10] M. Zou. Kolmogorov's condition for one square potential spherical pendulum. *Phys. Lett. A* **166** (1992), 321-325.

[11] S. N. Chow, J. Sanders. On the number of critical points of the period. *J. Differential Equations* **64**, (1986), 51-66.

[12] E. Horozov. On the isoenergetical non-degeneracy of spherical pendulum. *Phys. Lett. A* **173** (1993), 279-283.


*Faculty of Mathematics and Informatics*
*Sofia University*
*5 J. Bouchier str.*
*1164 Sofia, Bulgaria*

*Received November 20, 1996*