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# HIGHEST WEIGHT MODULES OF $W_{1+\infty}$, DARBOUX TRANSFORMATIONS AND THE BISPECTRAL PROBLEM

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Abstract. This paper is a survey of our recent results on the bispectral problem. We describe a new method for constructing bispectral algebras of any rank and illustrate the method by a series of new examples as well as by all previously known ones. Next we exhibit a close connection of the bispectral problem to the representation theory of $W_{1+\infty}$–algerba. This connection allows us to explain and generalise to any rank the result of Magri and Zubelli on the symmetries of the manifold of the bispectral operators of rank and order two.

**Introduction.** In this paper we announce the existence of a large class of *bispectral* ordinary differential operators and a series of their properties. Following [10] we call an operator $L(x, \partial_x)$ bispectral if it possesses a family of eigenfunctions

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$\Psi(x,z)$, which are also eigenfunctions for another differential operator $\Lambda(z,\partial_z)$, but this time in the "spectral parameter" $z$, to wit

$$L(x,\partial_x)\Psi(x,z) = f(z)\Psi(x,z) \tag{1}$$

$$\Lambda(z,\partial_z)\Psi(x,z) = \Theta(x)\Psi(x,z) \tag{2}$$

for some functions $f(z), \Theta(x)$.

The problem of describing bispectral operators has its roots in several mathematical and physical issues, e.g. – computer tomography. For more motivation and background we recommend [10, 12, 13].

The first general result in the direction of classifying bispectral operators belongs to J. J. Duistermaat and F. A. Grünbaum [10] who determined all second order bispectral operators $L$. Their answer is as follows. If we write the operator $L$ as

$$L = \partial_x^2 + u(x),$$

the bispectral potentials $u(x)$ are given (up to translations and rescalings of $x$ and $z$) apart from the obvious Airy ($u(x) = x$) and Bessel ($u(x) = cx^{-2}$) potentials by potentials obtained by finitely many "rational" Darboux transformations from $u(x) = 0$ and $u(x) = (-1/4)x^{-2}$.

Every operator $L(x,\partial_x)$ can be considered as an element of a maximal algebra $\mathcal{A}$ of commuting ordinary differential operators [7]. Led by this observation G. Wilson [23] introduced the following terminology. He called such an algebra *bispectral* if there exists a joint eigenfunction $\Psi(x,z)$ for the operators $L$ in $\mathcal{A}$ that satisfy also the eq. (2). The dimension of the space of eigenfunctions $\Psi(x,z)$ is called a *rank* of the commutative algebra $\mathcal{A}$ (see e.g. [19]). This number coincides with the greatest common divisor of the orders of the operators in $\mathcal{A}$. Wilson classified all rank 1 bispectral algebras [23].

Our construction puts in a general context all previously known results and explains them (to some extent) from a representation-theoretic point of view. The main results of the present paper are as follows.

First, starting with certain generalizations of Bessel functions we construct highest weight modules $\mathcal{M}_\beta$ of the Lie algebra $W_{1+\infty}$ with highest weight vectors – tau-functions.

Second, performing a large class of Darboux transformations on polynomials of the corresponding to these tau-functions differential operators (Bessel operators) we construct families of bispectral algebras of any rank $N$. There is a similar construction for the higher order generalizations of Airy operators.

Third, we show that the manifolds of bispectral operators obtained by Darboux transformations on powers of Bessel operators are in one to one correspondence with the manifolds of tau-functions lying in the modules $\mathcal{M}_\beta$. An immediate corollary is that they are preserved by hierarchies of symmetries generated by subalgebras of $W_{1+\infty}$.

At the end we point out that the suggested method allows to obtain the bispectral algebras algorithmically despite that we use highly transcendental functions like Bessel and Airy ones. We covered all classes and examples of bispectral operators which we know from the literature [10, 23, 24, 13], etc. We conjecture that all bispectral scalar differential operators can be obtained in this way.

The proofs of the results presented here are performed in [2]–[5].

**1. Darboux transformations and bispectral algebras.** The framework of our construction is Sato's theory of KP–hierarchy [21, 22, 18]. In particular our eigenfunctions are Baker (or wave) functions $\Psi_V(x, z)$ corresponding to points (or planes) $V$ in Sato's Grassmannian $Gr$. We obtain our bispectral algebras by applying a version of *Darboux transformations* on specific wave functions which we call *Bessel* (and *Airy*) *wave functions.*

For $\beta \in \mathbb{C}^N$ such that $\sum_{i=1}^N \beta_i = N(N-1)/2$ we introduce the ordinary differential operator

$$P_\beta(D_z) = (D_z - \beta_1)(D_z - \beta_2)\cdots(D_z - \beta_N) \tag{3}$$

where $D_z = z\partial_z$ and consider the differential equation

$$P_\beta(D_z)\Phi_\beta(z) = z^N \Phi_\beta(z). \tag{4}$$

For every sector $S$ with a center at the irregular singular point $z = \infty$ and an angle less than $2\pi$ the equation (4) has a solution $\Phi_\beta$ with an asymptotics

$$\Phi_\beta(z) \sim \Psi_\beta(z) = e^z\Big(1 + \sum_{k=1}^{\infty} a_k(\beta) z^{-k}\Big) \tag{5}$$

for $|z| \to \infty$, $z \in S$. Here $a_k(\beta)$ are symmetric polynomials in $\beta_i$. The function $\Phi_\beta(z)$ can be taken to be (up to a rescaling) the Meijer's $G$-function $G_{0N}^{N0}((-z/N)^N|(1/N)\beta)$ (see [6], §5.3). The next definition is fundamental for the present paper.

**Definition 1.** *Bessel wave function is called the function* $\Psi_\beta(x,z) = \Psi_\beta(xz)$ (cf. [11, 14, 24]). *The Bessel operator* $L_\beta$ *is defined as*

$$L_\beta(x,\partial_x) = x^{-N} P_\beta(D_x). \tag{6}$$

*A Bessel wave function* $\Psi_\beta$ *defines a plane* $V_\beta \in Gr$ *(called* Bessel plane*) by the standard procedure:*

$$V_\beta = \mathrm{span}\{\partial_x^k \Psi_\beta(x,z)|_{x=1}\}.$$

*We denote by* $\tau_\beta(t)$ *the corresponding tau-function and call it* Bessel tau–function.

Classically, a Darboux transformation [8] of a differential operator $L$ presented as a product $L = QP$ is defined by exchanging the places of the factors, i.e. $\overline{L} = PQ$. Obviously, if $\Psi(x,\lambda)$ is an eigenfunction of $L$, i.e. $L(x,\partial_x)\Psi(x,\lambda) = \lambda\Psi(x,\lambda)$ then $P\Psi(x,\lambda)$ is an eigenfunction of $\overline{L}$. Our definition of a Darboux transformation puts the emphasis rather on the eigenfunctions $\Psi(x,\lambda)$ and $P\Psi(x,\lambda)$ than on the operators $L$ and $\overline{L}$.

**Definition 2.** *We say that a plane* $W$ *(or the corresponding wave function* $\Psi_W(x,z)$*) is a* Darboux transformation *of the plane* $V$ *(respectively wave function* $\Psi_V(x,z)$*) iff there exist monic polynomials* $f(z)$, $g(z)$ *and differential operators* $P(x,\partial_x)$, $Q(x,\partial_x)$ *such that*

$$\Psi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Psi_V(x,z), \tag{7}$$

$$\Psi_V(x,z) = \frac{1}{f(z)} Q(x,\partial_x)\Psi_W(x,z). \tag{8}$$

Simple consequences of Definition 2 are the identities

$$PQ\Psi_W(x,z) = f(z)g(z)\Psi_W(x,z), \tag{9}$$

$$QP\Psi_V(x,z) = f(z)g(z)\Psi_V(x,z). \tag{10}$$

The operator $\overline{L} = PQ$ is a Darboux transformation of $L = QP$. Obviously (7) implies the inclusion

$$gW \subset V. \tag{11}$$

Conversely, if (11) holds there exists $P$ satisfying (7). Therefore $W$ is a Darboux transformation of $V$ iff

$$fV \subset W \subset \frac{1}{g}V \tag{12}$$

for some polynomials $f(z)$, $g(z)$.

Obviously some of the Bessel planes $V_\beta$, $\beta \in \mathbb{C}^N$ can be obtained by a Darboux transformation from other Bessel planes $V_{\beta'}$, $\beta' \in \mathbb{C}^{N'}$, $N' < N$. Of course, we are interested in $\beta$ which do not have this property. We call them *generic*.

To each plane $W$ one can associate the *spectral algebra* $A_W$ of polynomials $f(z)$ that leave $W$ invariant. For each $f(z) \in A_W$ one can show that there exists a unique differential operator $L_f(x, \partial_x)$, the order of $L_f$ being equal to the degree of $f$, such that

$$L_f(x, \partial_x)\Psi_W(x, z) = f(x)\Psi_W(x, z). \tag{13}$$

We denote the commutative algebra of these operators by $\mathcal{A}_W$.

**Proposition 1.** *For a generic* $\beta \in \mathbb{C}^N$ *we have*

$$A_{V_\beta} = \mathbb{C}[z^N], \quad \mathcal{A}_{V_\beta} = \mathbb{C}[L_\beta]. \tag{14}$$

For a Darboux transformation of a Bessel plane $V = V_\beta$ with generic $\beta \in \mathbb{C}^N$ Proposition 1 and (10) imply

$$f(z)g(z) = h(z^N), \tag{15}$$

$$QP = h(L_\beta) \tag{16}$$

for some polynomial $h$. The operator $P$ is determined by its kernel which is a subspace of $\mathrm{Ker} h(L_\beta)$. The latter is described in the following lemma (see e.g. [15]).

**Lemma 1.** *Let* $h(z)$ *be a polynomial*

$$h(z) = z^{d_0}\left(z - \lambda_1^N\right)^{d_1} \cdots \left(z - \lambda_r^N\right)^{d_r}, \quad \lambda_i^N \neq \lambda_j^N, \ \lambda_0 = 0, \ d_i \geq 0. \tag{17}$$

*Then we have*

(i) $\mathrm{Ker}h(L_\beta) = \bigoplus_{i=0}^{r} \mathrm{Ker}\left(L_\beta - \lambda_i^N\right)^{d_i}$.

(ii) $(L_\beta)^d = L_{\beta^d}$, *where*

$$\beta^d = (\beta_1, \beta_1 + N, \ldots, \beta_1 + (d-1)N, \ldots, \beta_N, \ldots, \beta_N + (d-1)N). \tag{18}$$

(iii) *If* $\{\beta_1, \ldots, \beta_N\} = \{\underbrace{\alpha_1, \ldots, \alpha_1}_{k_1}, \ldots, \underbrace{\alpha_s, \ldots, \alpha_s}_{k_s}\}$ *with distinct* $\alpha_1, \ldots, \alpha_s$, *then*

$$\mathrm{Ker}L_\beta = \mathrm{span}\left\{x^{\alpha_i}(\ln x)^k\right\}_{1\le i\le s,\ 0\le k\le k_i-1}.$$

(iv) *For* $\lambda \neq 0$

$$\mathrm{Ker}\left(L_\beta - \lambda^N\right)^d = \mathrm{span}\left\{\partial_\lambda^k \Psi_\beta(x, \lambda\varepsilon^j)\right\}_{0\le k\le d-1,\ 0\le j\le N-1},$$

*where* $\varepsilon = e^{2\pi i/N}$ *is an* $N$*-th root of unity.*

We call $\mathrm{Ker}P$ *homogeneous* and $\mathbb{Z}_N$*-invariant* iff it has a basis which is a union of:

(i) Several groups of elements supported at 0 of the form:

$$\partial_y^l\Big(\sum_{k=0}^{k_0}\sum_{j=0}^{\mathrm{mult}(\beta_i+kN)-1} b_{kj}x^{\beta_i+kN}y^j\Big)\Big|_{y=\ln x}, \quad 0\le l\le j_0, \tag{19}$$

where $\mathrm{mult}(\beta_i + kN) :=$ multiplicity of $\beta_i + kN$ in $\bigcup_{j=1}^{N}\{\beta_j + N\mathbb{Z}_{\ge 0}\}$ and $j_0 = \max\{j | b_{kj} \neq 0 \text{ for some } k\}$.

(ii) Several groups of elements supported at the points $\varepsilon^i\lambda$ ($0 \le i \le N-1$, $\lambda \neq 0$) of the form:

$$\sum_{k=0}^{k_0} a_k\varepsilon^{ki}\partial_z^k\Psi_\beta(x,z)|_{z=\varepsilon^i\lambda}, \quad 0\le i\le N-1. \tag{20}$$

Instead of (20) we can also take

$$\sum_{k=0}^{k_0} a_k D_z^k\Psi_\beta(x,z)|_{z=\varepsilon^i\lambda}, \quad 0\le i\le N-1. \tag{21}$$

Denote by $n_0$ the number of elements of the form (19) in the above basis of $\mathrm{Ker}P$ and by $n_j$ for $1 \le j \le r$ the number of groups of elements of the form (20) with $\lambda = \lambda_j$.

Now we give our fundamental definition.

**Definition 3.** *We say that the wave function $\Psi_W(x,z)$ is a* polynomial Darboux transformation *of the Bessel wave function $\Psi_\beta(x,z)$, $\beta \in \mathbb{C}^N$, iff* (7) *holds (for $V = V_\beta$) with $P(x,\partial_x)$ and $g(z)$ satisfying:*

$(i)$ *The kernel of the operator $P$ is homogeneous and $\mathbb{Z}_N$-invariant, i.e. it has a basis of the form* (19, 20)*.*

$(ii)$ *The polynomial $g(z)$ is given by*

$$g(z) = z^{n_0}\left(z^N - \lambda_1^N\right)^{n_1} \cdots \left(z^N - \lambda_r^N\right)^{n_r} \tag{22}$$

*where $n_j$ are the numbers defined above.*

*We denote the set of all such planes $W$ by $Gr_B(\beta)$ and put $Gr_B^{(N)} = \bigcup\limits_\beta Gr_B(\beta)$, $\beta \in \mathbb{C}^N$-generic.*

*We say that the polynomial Darboux transformation $\Psi_W(x,z)$ of $\Psi_\beta(x,z)$ is* monomial *iff*

$$g(z) = z^{n_0}.$$

*Denote the set of the corresponding planes $W$ by $Gr_{MB}(\beta)$ and put $Gr_{MB}^{(N)} = \bigcup_\beta Gr_{MB}(\beta)$, $\beta \in \mathbb{C}^N$-generic.*

The next theorem provides another equivalent definition of $Gr_B(\beta)$ and is used essentially in the proof of the bispectrality.

**Theorem 1.** *The wave function $\Psi_W(x,z)$ is a* polynomial Darboux transformation *of the Bessel wave function $\Psi_\beta(x,z)$, for generic $\beta \in \mathbb{C}^N$, iff* (7, 8, 15, 16) *hold* (*for $V = V_\beta$*) *and*

(i) *The operator $P$ from* (7) *has the form*

$$P(x,\partial_x) = x^{-n}\sum_{k=0}^{n} p_k(x^N)(x\partial_x)^k, \tag{23}$$

*where $p_k$ are rational functions, $p_n \equiv 1$.*

(ii) *There exists the formal limit*

$$\lim_{x\to\infty} e^{-xz}\Psi_W(x,z) = 1. \tag{24}$$

The limit in (24) is formal in the sense that it is taken in the coefficients at any power of $z$.

Following G. Wilson [23] we define the *bispectral involution* $b$ on the wave function $\Psi_W(x,z)$ by exchanging the places of $x$ and $z$:

$$\Psi_{bW}(x,z) = \Psi_W(z,x)$$

(provided the LHS is again a wave function).

**Theorem 2.** *If $W \in Gr_B(\beta)$ then $bW$ exists and $bW \in Gr_B(\beta)$.*

This means that $\Psi_{bW}(x,z)$ is a wave function and

$$\Psi_{bW}(x,z) = \frac{1}{g_{\mathrm{b}}(z)} P_{\mathrm{b}}(x,\partial_x)\Psi_\beta(x,z), \tag{25}$$

$$\Psi_\beta(x,z) = \frac{1}{f_{\mathrm{b}}(z)} Q_{\mathrm{b}}(x,\partial_x)\Psi_{bW}(x,z) \tag{26}$$

for some polynomials $g_{\mathrm{b}}, f_{\mathrm{b}}$ and operators $P_{\mathrm{b}}, Q_{\mathrm{b}}$ satisfying (23). We can derive explicit expressions for (25, 26) in terms of (7, 8) as follows. If the operators $P(x,\partial_x)$ and $Q(x,\partial_x)$ are written in the form

$$P(x,\partial_x) = \frac{1}{x^n p_n(x^N)} \sum_{k=0}^{n} p_k(x^N)(x\partial_x)^k, \tag{27}$$

$$Q(x,\partial_x) = \sum_{s=0}^{m} (x\partial_x)^s q_s(x^N) \frac{1}{x^m q_m(x^N)} \tag{28}$$

with polynomials $p_k, q_s$ then

$$P_{\mathrm{b}}(x,\partial_x) = \frac{1}{g(x)} \sum_{k=0}^{n} (x\partial_x)^k p_k(L_\beta(x,\partial_x)), \tag{29}$$

$$g_{\mathrm{b}}(z) = z^n p_n(z^N) \tag{30}$$

and

$$Q_{\mathrm{b}}(x,\partial_x) = \sum_{s=0}^{m} q_s\left(L_\beta(x,\partial_x)\right)(x\partial_x)^s \frac{1}{f(x)}, \tag{31}$$

$$f_{\mathrm{b}}(z) = z^m q_m(z^N). \tag{32}$$

An immediate corollary is the following result, which we state as a theorem because of its fundamental character.

**Theorem 3.** *If* $W \in Gr_B^{(N)}$ *then the wave function* $\Psi_W(x,z)$ *solves the bispectral problem, i.e. there exist operators* $L(x,\partial_x)$ *and* $\Lambda(z,\partial_z)$ *such that*

$$L(x,\partial_x)\Psi_W(x,z) = h(z^N)\Psi_W(x,z), \tag{33}$$

$$\Lambda(z,\partial_z)\Psi_W(x,z) = \Theta(x^N)\Psi_W(x,z). \tag{34}$$

*Moreover,*

$$\mathrm{rank} A_W = \mathrm{rank} A_{bW} = N. \tag{35}$$

The operators and the polynomials from (33) and (34) are given by:

$$L(x,\partial_x) = P(x,\partial_x)Q(x,\partial_x), \quad h(z^N) = f(z)g(z); \tag{36}$$

$$\Lambda(z,\partial_z) = P_{\mathrm{b}}(z,\partial_z)Q_{\mathrm{b}}(z,\partial_z), \quad \Theta(x^N) = f_{\mathrm{b}}(x)g_{\mathrm{b}}(x). \tag{37}$$

The whole bispectral algebra is given in the following proposition.

**Proposition 2.** *For* $W \in Gr_B(\beta)$ *with generic* $\beta \in \mathbb{C}^N$ *we have*

$$A_W = \left\{ u \in \mathbb{C}[z^N] \mid u(L_\beta)\mathrm{Ker}P \subset \mathrm{Ker}P \right\} \tag{38}$$

*and*

$$\mathcal{A}_W = \left\{ Pu(L_\beta)P^{-1} \mid u \in A_W \right\}. \tag{39}$$

**Remark 1.** $Gr_B^{(1)}$ coincides with the "adelic" Grassmannian $Gr^{ad}$ introduced by Wilson [23]. The rank 2 bispectral algebras containing an operator of order 2 (the "even case" of Duistermaat and Grünbaum [10]) are obtained from $Gr_{MB}^{(2)} \cap Gr^{(2)}$.

**Remark 2.** The (generalized higher) *Airy wave function* is defined by the following equations (see e.g. [9])

$$L_\alpha(x,\partial_x)\Psi_\alpha(x,z) = z^N\Psi_\alpha(x,z),$$
$$L_\alpha(x,\partial_x) = \partial_x^N - \alpha_0 x + \sum_{i=2}^{N-1} \alpha_i \partial_x^{N-i},$$

where $\alpha = (\alpha_0, \alpha_2, \alpha_3, \ldots, \alpha_{N-1}) \in \mathbb{C}^{N-1}$. The definition of *polynomial Darboux transformations* is similar to that in the Bessel case (see Theorem 1) with only minor modifications: $P$ is not necessarily $\mathbb{Z}_N$-invariant and $g(z)$ has to belong to $\mathbb{C}[z^N]$. After a suitable definition of the bispectral involution $b$ we proved analogs of theorems 2 and 3 [4]).

**Remark 3.** The above eigenfunction $\Psi_W(x,z)$ from (33, 34) is a formal series. However, if we substitute $\Phi_\beta(x,z) = \Phi_\beta(xz)$ for $\Psi_\beta(x,z)$ then

$$\Phi_W(x,z) = \frac{1}{g(z)} P(x,\partial_x)\Phi_\beta(x,z)$$

gives a convergent solution to the bispectral problem with the same operators $L(x,\partial_x)$ and $\Lambda(z,\partial_z)$ (see [4]).

**2. $W_{1+\infty}$ and monomial Darboux transformations.** The algebra $w_\infty$ of the additional symmetries of the KP–hierarchy is isomorphic to the Lie algebra of regular polynomial differential operators on the circle:

$$\mathcal{D} = \operatorname{span}\{z^\alpha \partial_z^\beta \mid \alpha,\beta \in \mathbb{Z},\ \beta \geq 0\}.$$

Its unique central extension [16, 17] will be denoted by $W_{1+\infty}$. This algebra gives the action of the additional symmetries on the tau-functions [1].

Denote by $c$ the central element of $W_{1+\infty}$ and by $W(A)$ the image of $A \in \mathcal{D}$ under the natural embedding $\mathcal{D} \hookrightarrow W_{1+\infty}$ (as vector spaces). The algebra $W_{1+\infty}$ has a basis

$$c,\ L_k^l = W(-z^k D^l), \qquad l,k \in \mathbb{Z},\ l \geq 0$$

where $D \equiv D_z = z\partial_z$. The commutation relations of $W_{1+\infty}$ can be written most conveniently in terms of generating series [17]

$$\Big[W(z^k e^{xD}), W(z^m e^{yD})\Big] = (e^{xm} - e^{yk}) W(z^{k+m} e^{(x+y)D}) + \delta_{k,-m} \frac{e^{xm} - e^{yk}}{1 - e^{x+y}} c.$$

We introduce the subalgebra $W_{1+\infty}(N)$ of $W_{1+\infty}$ spanned by $c$ and $L_{kN}^l$, $l,k \in \mathbb{Z}$, $l \geq 0$. It is a simple fact that $W_{1+\infty}(N)$ is isomorphic to $W_{1+\infty}$.

In the next theorem we sum up some of the results from [2] which will be needed in Theorems 5 and 6.

**Theorem 4.** *The functions $\tau_\beta(t)$ satisfy the constraints*

$$\begin{aligned}
&L_0^l \tau_\beta = \lambda_\beta(L_0^l)\tau_\beta, \quad l \geq 0,\\
&L_{kN}^l \tau_\beta = 0, \quad k > 0,\ l \geq 0,\\
&W\Big(z^{-kN} P_{\beta,k}(D) D^l\Big)\tau_\beta = 0, \quad k > 0,\ l \geq 0,
\end{aligned}$$

*where* $P_{\beta,k}(D) = P_\beta(D)P_\beta(D-N)\cdots P_\beta(D-N(k-1))$.

The first two constraints mean that $\tau_\beta$ is a highest weight vector with highest weight $\lambda_\beta$ of a representation of $W_{1+\infty}(N)$ in the module

(40) $$\mathcal{M}_\beta = \operatorname{span}\left\{ L^{l_1}_{k_1 N} \cdots L^{l_p}_{k_p N} \tau_\beta \middle| k_1 \leq \ldots \leq k_p < 0 \right\}.$$

In [2] we studied $\mathcal{M}_\beta$ as modules of $W_{1+\infty}$. We proved that they are *quasifinite* (see [17]) and we derived formulae for the highest weights and for the singular vectors.

The next theorem establishes the connection between the highest weight modules $\mathcal{M}_\beta$ and the monomial Darboux transformations.

**Theorem 5.** *If* $\tau_W$ *is a tau-function lying in the* $W_{1+\infty}(N)$*-module* $\mathcal{M}_\beta$ $(\beta \in \mathbb{C}^N)$ *then the corresponding plane* $W \in Gr_{MB}(\beta)$*. Conversely, if* $W \in Gr_{MB}(\beta)$ *then* $\tau_W \in \mathcal{M}_{\beta'}$ *for some* $\beta' \in \mathbb{C}^N$ *such that* $V_{\beta'} \in Gr_{MB}(\beta)$*.*

In general $\beta' \neq \beta$. A more precise version of the second part of Theorem 5 is given in [5]. Here we shall restrict ourselves only to the case when there are no logarithms in the basis (19) of $\operatorname{Ker} P$, i.e. when it is of the form

(41) $$f_k(x) = \sum_{i=1}^{dN} a_{ki} x^{\gamma_i}, \quad 0 \leq k \leq n-1,$$

where $\gamma = \beta^d$ (see (18)).

Let $W \in Gr_{MB}(\beta)$ be a monomial Darboux transformation of a Bessel plane $V_\beta$, $\beta \in \mathbb{C}^N$ with $g(z) = z^n$ and $\operatorname{Ker} P$ of the above type. Definition 3 in this case is equivalent to

(42) $$\gamma_i - \gamma_j \in N\mathbb{Z} \setminus 0 \quad \text{if } a_{ki}a_{kj} \neq 0, i \neq j.$$

We say that the element $f_k(x)$ of the above basis of $\operatorname{Ker} P$ is *associated* to $\beta_s$ $(1 \leq s \leq N)$ iff

(43) $$\gamma_i - \beta_s \in N\mathbb{Z}_{\geq 0} \quad \text{if } a_{ki} \neq 0.$$

Then up to a relabeling we can take a subset $\{\beta_s\}_{1 \leq s \leq M}$ such that

(44) $$\beta_s - \beta_t \notin N\mathbb{Z} \quad \text{for } 1 \leq s \neq t \leq M$$

and each element of the basis (41) of Ker$P$ is associated to some $\beta_s$ from this set. Denote by $n_s$ the number of elements associated to $\beta_s$ and set $n_s = 0$ for $s > M$. Then $n_1 + \cdots + n_N = n$. We put

$$\beta' = (\beta_1 + n_1 N - n, \beta_2 + n_2 N - n, \ldots, \beta_N + n_N N - n). \tag{45}$$

**Theorem 6.** *Let $W$ be a monomial Darboux transformation of the Bessel plane $V_\beta$ with* Ker$P$ *satisfying* (41, 42) *and $\beta'$ be as above. Then the tau-function $\tau_W$ of $W$ lies in the $W_{1+\infty}(N)$-module $\mathcal{M}_{\beta'}$.*

**Example 1.** Let $A = (0\ 1\ 0\ \ldots\ 0)$ and $\beta'_2 - \beta'_1 = N\alpha$, $\alpha \in \mathbb{Z}_{\geq 0}$. Set

$$\beta'' = (\beta'_1 - N, \beta'_2 + N, \beta'_3, \ldots, \beta'_N).$$

Then the module $\mathcal{M}_{\beta''}$ embeds in $\mathcal{M}_{\beta'}$. The singular vector $\tau_{\beta''}$ is given by

$$\tau_{\beta''} = W(P_1)\tau_{\beta'} + \mathrm{const} \cdot \tau_{\beta'}, \tag{46}$$

where

$$P_1 = -N(\alpha+1) z^{-N} \frac{P_{\beta'}(D_z)}{D_z - \beta'_1} \left( z^{-N} P_{\beta'}(D_z) \right)^{\alpha+1}. \tag{47}$$

Another important question posed by Duistermaat and Grünbaum [10] is about the existence of an hierarchy of symmetries leaving the manifold of bispectral operators invariant. The following two theorems answer this question for the manifold of monomial Darboux transformations.

**Theorem 7.** *Vector fields corresponding to $W^+_{1+\infty}(N)$ are tangent to the manifold $Gr^{(N)}_{MB}$ of monomial Darboux transformations. More precisely, if $W \in Gr^{(N)}_{MB}$ then*

$$\exp\left( \sum_{i=1}^{p} \lambda_i L^{l_i}_{Nk_i} \right) \tau_W$$

*is a tau-function associated to a plane from $Gr^{(N)}_{MB}$ for arbitrary $p \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $l_i, k_i \in \mathbb{Z}_{\geq 0}$.*

Let us define

$$\overline{L}_m = \frac{1}{N} \sum_{k \in \mathbb{Z} \setminus N\mathbb{Z}} {:}J_{mN-k} J_k{:}$$

where $J_k = L^0_k = W(-z^k)$. The operators $\overline{L}_m$, $m \in \mathbb{Z}$ form a Virasoro algebra with central charge $N - 1$ which we denote by $Vir_N$. Denote by $Vir^+_N$ the subalgebra spanned by $\overline{L}_m$, $m \geq 0$. Then we can formulate the following theorem which for $N = 2$ contains Magri–Zubelli's result [20].

**Theorem 8.** *The manifold* $Gr_B^{(N)} \cap Gr^{(N)}$ *is preserved by the vector fields corresponding to* $Vir_N^+$. *More precisely, if* $W \in Gr_B^{(N)} \cap Gr^{(N)}$ *then*

$$\exp\left(\sum_{i=1}^{p} \lambda_i \overline{L}_{k_i}\right) \tau_W$$

*is a tau-function associated to a plane from* $Gr_B^{(N)} \cap Gr^{(N)}$ *for arbitrary* $p \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $k_i \geq 0$.

**Remark 4.** $Gr_B^{(N)} \cap Gr^{(N)} = Gr_{MB}^{(N)} \cap Gr^{(N)}$.

**3. Explicit formulae and examples.** We shall begin with stating explicit formulae for the bispectral operators from (33, 34) in the case of monomial Darboux transformations when there are no logarithms in the basis (19) of Ker$P$. The general case of monomial Darboux transformation can be reduced to this one by taking a limit in all formulae (see [5]).

Let $\beta \in \mathbb{C}^N$ and $W \in Gr_{MB}(\beta)$. We use the notation from (7, 8, 25, 26). Then $h(z) = z^d$, $g(z) = z^n$, $f(z) = z^{dN-n}$ for some $n$, $d$. The kernel of $P$ has a basis of the form (41, 42) where $\gamma = \beta^d$ is from (18). We shall use multi-index notation for subsets $I = \{i_1 < \ldots < i_n\}$ of $\{1, \ldots, dN\}$. Let $A$ be the matrix $(a_{ki})$ and $A^I = (a_{k,i_l})_{0 \leq k,\, l \leq n-1}$ be the corresponding minor of $A$. Denote by $I^0$ the complement of $I$. We put $\gamma_I = \{\gamma_i\}_{i \in I}$; $(\delta_I)_i = 1$ for $i \in I$ and $= 0$ for $i \in I^0$; $\Delta_I = \prod\limits_{r<s} (\gamma_{i_r} - \gamma_{i_s})$. Let $I_{\min}$ be the subset of $\{1, \ldots, dN\}$ with $n$ elements such that $\det A^{I_{\min}} \neq 0$ and $\sum\limits_{i \in I_{\min}} \gamma_i$ be the minimum of all such sums, and set $p_I = \sum\limits_{i \in I} \gamma_i - \sum\limits_{i \in I_{\min}} \gamma_i$. Eq. (42) implies that these numbers are divisible by $N$.

**Proposition 3.** *In the above notation the operators and the polynomials from* (36, 37) *are given by the following formulae*:

(a) $g(z) = z^n$,

$$P = \left(\sum \det A^I \Delta_I x^{p_I}\right)^{-1} \left(\sum \det A^I \Delta_I x^{p_I} L_{\gamma_I}\right).$$

(b) $f(z) = z^{dN-n}$,

$$Q = \left(\sum \det A^I \Delta_I L_{\gamma_{I^0} - n\delta_{I^0}} x^{p_I}\right)\left(\sum \det A^I \Delta_I x^{p_I}\right)^{-1}.$$

(c) $g_{\mathrm{b}}(z) = z^n \sum \det A^I \Delta_I z^{p_I}$,
$P_{\mathrm{b}} = \sum \det A^I \Delta_I L_{\gamma_I} (L_\beta)^{p_I/N}$.

(d) $f_{\mathrm{b}}(z) = z^{dN-n} \sum \det A^I \Delta_I z^{p_I}$,
$Q_{\mathrm{b}} = \sum \det A^I \Delta_I (L_\beta)^{p_I/N} L_{\gamma_{I^0} - n\delta_{I^0}}$.

**Example 2.** Consider the case when $\beta^d = \gamma$ has different coordinates. Then choose the following basis of $\mathrm{Ker} L_\beta^d$:

$$\Phi_{(k-1)d+j}(x) = \mu_{kj} x^{\beta_k + (j-1)N}, \quad 1 \le k \le N,\ 1 \le j \le d,$$

where

$$\mu_{k,1} := 1, \quad \mu_{kj} := \mu_{k,j-1} \cdot \prod_{i=1}^{N} (\beta_i - \beta_k - (j-1)N)^{-1}.$$

In this basis the action of $L_\beta$ is quite simple: $L_\beta \Phi_{(k-1)d+j} = \Phi_{(k-1)d+j-1}$ for $2 \le j \le d$ and $= 0$ for $j = 1$. Let a basis of $\mathrm{Ker} P$ be

$$f_k(x) = \sum_{i=1}^{dN} a_{ki} \Phi_i(x), \quad k = 0, \dots, d-1.$$

Let $\beta_i - \beta_j \in N\mathbb{Z}$ for all $i$, $j$ and the matrix $A = (a_{ki})$ has the form:

$$A = \begin{pmatrix} t_0^{(1)} & & & & \dots & t_0^{(N)} & & & \\ t_1^{(1)} & t_0^{(1)} & & & \dots & t_1^{(N)} & t_0^{(N)} & & \\ t_2^{(1)} & t_1^{(1)} & t_0^{(1)} & & \dots & t_2^{(N)} & t_1^{(N)} & t_0^{(N)} & \\ \vdots & \vdots & & \ddots & & \vdots & \vdots & \ddots & \\ t_{n-1}^{(1)} & t_{n-2}^{(1)} & \dots & t_0^{(1)} & \dots & t_{n-1}^{(N)} & t_{n-2}^{(N)} & \dots & t_0^{(N)} \end{pmatrix}$$

Then the operator $L = P L_\beta P^{-1}$ is differential of order $N$ and solves the bispectral problem. For a generic $\beta \in \mathbb{C}^N$ the spectral algebra has rank $N$ (i.e. it is $\mathbb{C}[L]$). For $N = 2$ this is the *"even case"* of J. J. Duistermaat and F. A. Grünbaum [10] (see also [20]).

**Example 3.** All *bispectral algebras of rank 1* are polynomial Darboux transformations of the plane $H_+ = \mathrm{span}\{z^k\}_{k\geq 0}$ (see [23]). This corresponds to the $N = 1$ Bessel with

$$\beta = (0), \quad L_{(0)} = \partial_x, \quad V_{(0)} = H_+, \quad \tau_{(0)}(t) = 1, \quad \Psi_{(0)}(x,z) = e^{xz}.$$

The operator $L$ which solves the bispectral problem is a Darboux transformation of the operator $h\left(L_{(0)}\right) = h(\partial_x)$ with constant coefficients. The "adelic Grassmannian" $Gr^{ad}$, introduced by Wilson [23], coincides with $Gr_B((0))$ $(= Gr_B^{(1)})$.

In the last example we study the simplest polynomial Darboux transformation of a Bessel operator of order 2.

**Example 4.** For $N = 2$, $\beta = (1-\nu, \nu)$ the corresponding Bessel operator is

$$L_\beta = x^{-2}(D_x - (1-\nu))(D_x - \nu) = \partial_x^2 + \frac{\nu(1-\nu)}{x^2}, \quad D_x = x\partial_x.$$

Fix $\lambda \in \mathbb{C} \setminus \{0\}$ and set $h(z) = (z - \lambda^2)^2$. For fixed $\lambda, a \in \mathbb{C} \setminus \{0\}$ we take $\mathrm{Ker}P$ to have a basis

$$f_k(x) = \Psi_\beta(x, (-1)^k\lambda) + aD_x\Psi_\beta(x, (-1)^k\lambda), \qquad k = 0, 1.$$

After introducing the operator

$$P(a,\lambda,\mu) = \frac{1}{x^2 p_2(x^2)}\Big\{p_2(x^2)D_x^2 + p_1(x^2)D_x + p_0(x^2)\Big\},$$

where $\mu^2 = (a + 1 - a^2\nu(\nu-1))/a^2\lambda^2$, $p_2(x^2) = x^2 - \mu^2$, $p_1(x^2) = \mu^2 - 3x^2$, $p_0(x^2) = -\lambda^2x^4 + (2\lambda^2\mu^2 + (a+1)(2a-1)a^{-2})x^2 + ((a+1)a^{-2} - \lambda^2\mu^2)\mu^2$, the operators and polynomials from formulae (36, 37) are given by:

$$P = P(a,\lambda,\mu), \qquad g(z) = z^2 - \lambda^2; \tag{48}$$

$$Q = P^*(b,\lambda,\mu), \qquad f(z) = z^2 - \lambda^2; \tag{49}$$

$$P_\mathrm{b} = P(a,\mu,\lambda), \qquad g_\mathrm{b}(z) = z^2 - \mu^2; \tag{50}$$

$$Q_\mathrm{b} = P^*(b,\mu,\lambda), \qquad f_\mathrm{b}(z) = z^2 - \mu^2; \tag{51}$$

where $b = -a/(a+1)$ and "$*$" is the formal adjoint of differential operators (i.e. the unique antiautomorphism such that $\partial_x^* = -\partial_x$, $x^* = x$). The spectral algebras

$$\mathcal{A}_W = P\left(L_\beta - \lambda^2\right)^2 \mathbb{C}[L_\beta]P^{-1}$$

and

$$\mathcal{A}_{bW} = P_{\mathrm{b}} \left( L_\beta - \mu^2 \right)^2 \mathbb{C}[L_\beta] P_{\mathrm{b}}^{-1}$$

consist of operators of orders $4, 6, 8, 10, \ldots$

The above formulae (48–51) raise some interesting questions about the properties of the bispectral involution and of the so-called adjoint involution (see [4]).

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