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**ON THE DIFFERENCE OF 4-GONAL LINEAR SYSTEMS  
ON SOME CURVES**

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ABSTRACT. Let  $C = (C, g_4^1)$  be a tetragonal curve. We consider the scrollar invariants  $e_1, e_2, e_3$  of  $g_4^1$ . We prove that if  $W_4^1(C)$  is a non-singular variety, then every  $g_4^1 \in W_4^1(C)$  has the same scrollar invariants.

**0. Introduction.** Let  $C$  be a complete non-singular curve defined over an algebraically closed field  $k$  with  $\text{char}(k) \neq 2$ . Let  $g = g(C)$  be the genus of  $C$  and let  $g_d^1$  be a base-point-free linear system on  $C$  of degree  $d$  and projective dimension 1. A pair  $C = (C, \mathcal{O}(g_d^1))$  is called a  $d$ -gonal curve if  $C$  does not admit a linear system of degree  $e < d$ . If a  $C = (C, \mathcal{O}(g_d^1))$  is a  $d$ -gonal curve, then  $g_d^1$  is a (base-point-free) complete linear system. Now we consider a pair  $C = (C, \mathcal{O}(g_d^1))$  such that  $g_d^1$  is a complete base-point-free of degree  $d$ . Let  $\omega_C$  be a canonical sheaf on  $C$ , let  $\mathcal{L} = \mathcal{O}(g_d^1)$  and let  $F_i = \Gamma(C, \omega_C \otimes \mathcal{L}^{\otimes -i})$ . If  $p: C \rightarrow \mathbf{P}^1$  is the map which corresponds to  $g_d^1$  then  $p_*\omega_C \cong \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{d-1}) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$  and  $F_i \cong \Gamma(\mathbf{P}^1, p_*\omega_C \otimes \mathcal{O}_{\mathbf{P}^1}(-i))$ . The modules  $F_i$  ( $i = 1, 2, \dots$ ) give a filtration,

$$F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$$

and by the definition of  $\{F_i\}_{i=0}^\infty$  we have injective maps

$$F_0/F_1 \hookrightarrow F_1/F_2 \hookrightarrow \cdots \hookrightarrow F_n/F_{n+1} \hookrightarrow \cdots$$

By Riemann-Roch's Theorem,  $\dim F_0/F_1 = d - 1$ . Now we define the scrollar invariants  $e_i = e_i(\mathcal{L})$  ( $i = 1, \dots, d - 1$ ) by

$$e_i = e_i(\mathcal{L}) = \#\{j \in \mathbb{N}; \dim (F_{j-1}/F_j) \geq i\} - 1 \quad (i = 1, 2, \dots, d - 1)$$

and we put  $e_0 = e_0(\mathcal{L}) = 0$ . Let  $W_d^r(C)$  be a subscheme of a Picard variety  $\text{Pic}^d(C)$  roughly defined as follows:

$$W_d^r(C) \stackrel{\text{as a set}}{=} \{\mathcal{L} \in \text{Pic}^d(C) \mid \dim \Gamma(C, \mathcal{L}) \geq r + 1\}.$$

The precise definition is found in Arbarello, Cornalba, Griffiths, Harris [1] p. 176. If  $C = (C, g_d^1)$  is a hyperelliptic curve or a trigonal curve, then the scrollar invariants of any  $g_d^1 \in W_d^1(C)$  depend only on the curve  $C$ . We now assume that  $C = (C, g_d^1)$  is a tetragonal curve. If  $C = (C, g_d^1)$  is an elliptic-hyperelliptic curve, then there is a  $\pi : C \rightarrow E$  where  $E$  is an elliptic curve and  $\deg \pi = 2$ . Then  $W_4^1(C) = \pi^*W_2^1(E)$ . Hence the scrollar invariants of any  $g_4^1 \in W_4^1(C)$  depend only on the curve  $C$ . If  $g \leq 4$ , then  $C$  is a trigonal curve. So we assume  $5 \leq g$  and  $C$  is not an elliptic-hyperelliptic curve.

**Definition A.** Let  $C_1 \subset \mathbb{P}^2$  be a plane curve and let  $P \in C_1$  be a double point. We call that  $P$  is an  $r$ -fold node if  $P$  is analytically isomorphic to the singularity at  $(0,0)$  of the curve  $y^2 = x^{2r}$  in  $\mathbb{A}^2$  and we call that  $P$  is an  $r$ -fold cusp if  $P$  is analytically isomorphic to the singularity at  $(0,0)$  of the curve  $y^2 = x^{2r+1}$  in  $\mathbb{A}^2$  (see Hartshorne [9] p. 38 Exercise 5.14(d)).

**Theorem A (Main Theorem).** Let  $C$  be a tetragonal curve of genus  $g$ , where  $6 \leq g \leq 8$ . Assume that  $C$  is not elliptic-hyperelliptic and  $\#(W_4^1(C)) \geq 2$ . For any  $g_4^1 \in W_4^1(C)$ , there is a divisor  $D = D_{g_4^1}$  such that  $|K_C - g_4^1 - D|$  gives a birational morphism  $\rho = \rho_{g_4^1} : C \rightarrow C_1 \subset \mathbb{P}^2$ ,  $\deg(C_1) = 6$  and every singular point of  $C_1$  has multiplicity 2. Let  $k_4^1 \in W_4^1(C)$ . Then there is a  $g_4^1 \in W_4^1(C)$  and a  $P \in \text{Sing}(C_1) = \text{Sing}(\rho_{g_4^1}(C))$  such that  $k_4^1$  is given by a cut out of lines which pass through  $P$ . And we have the following:

I) The following statements are equivalent:

- 1)  $k_4^1 \in W_4^1(C)$  is a reduced point
- 2)  $P$  is an ordinary node or an ordinary cusp
- 3)  $k_4^1$  is of type  $(1,1,1)$  if  $g = 6$ ,  $(2,1,1)$  if  $g = 7$  and  $(2,2,1)$  if  $g = 8$ .

II) The following statements are equivalent:

- 1)  $P$  is a 2-fold node or a 2-fold cusp
- 2)  $k_4^1$  is of type  $(2,1,0)$  if  $g = 6$ ,  $(2,2,0)$  if  $g = 7$  and  $(3,1,1)$  if  $g = 8$ .

III) The following statements are equivalent:

- 1)  $P$  is a 3-fold node or a 3-fold cusp
- 2)  $k_4^1$  is of type  $(3,1,0)$  if  $g = 7$ .

As a corollary of Theorem A, we have the following:

**Corollary A.** *Assume that  $C$  is a tetragonal curve and  $C$  is not an elliptic-hyperelliptic curve. If  $g \geq 10$ , then  $C$  has only one  $g_4^1$ . If  $5 \leq g \leq 9$  and  $W_4^1(C)$  is reduced, then there exist integers  $e_1 \geq e_2 \geq e_3 \geq 0$  such that any  $g_4^1 \in W_4^1(C)$  has  $e_1, e_2, e_3$  for its scrollar invariants.*

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#### NOTATIONS

$\text{char}(k)$ : The characteristic of a field  $k$

$\mathcal{O}_A$ : The structure sheaf of a variety  $A$

$f^*$ : The pull back defined by a morphism  $f$

$f_*$ : The direct image defined by a morphism  $f$

$\deg(f)$ : The degree of a finite morphism  $f$

$|\mathcal{L}|$ : The complete linear system defined by an invertible sheaf  $\mathcal{L}$

$\phi_V$ : The rational map defined by a linear system  $V$

$\mathcal{O}_A(D)$ : The invertible sheaf associated with a divisor  $D$

$\Gamma(A, \mathcal{F})$ : The global sections of a sheaf  $\mathcal{F}$

$K_A$ : A canonical divisor on a non-singular variety  $A$

$\omega_A$ : The canonical invertible sheaf on a non-singular variety  $A$

$\mathbb{P}(\mathcal{E})$ : The projective bundle  $\text{Proj}(\bigoplus_{n=0}^{\infty} \text{Sym}^n \mathcal{E})$  defined by a locally free sheaf  $\mathcal{E}$  on a variety  $Y$

**1. Preliminary and Known Facts.** Let  $C$  be a non-singular curve of genus  $g$  defined over an algebraically closed field  $k$ . Let  $g_d^1$  be one of a base-point-free linear system on  $C$  of degree  $d$  and projective dimension 1. We assume that  $C$  is a tetragonal curve, i.e.  $C$  admits a  $g_4^1$  but does not admit a  $g_e^1$  for every  $e < 4$ . We know that if  $g \leq 4$ , then  $C$  is a hyperelliptic curve or a trigonal curve. So we assume that  $g \geq 5$ . The following results are proved in [10].

**Theorem 1.** *If  $|(e_{d-1} + 2)g_d^1|$  is birationally very ample, then*

$$e_{i-1} \leq e_i + e_{d-1} + 2$$

for any  $i \in \mathbb{Z}/d\mathbb{Z}$ .

**Theorem 2.** *Let  $e_1, e_2, e_3$  and  $g \geq 5$  be integers such that*

$$e_1 \leq e_2 + e_3 + 2, \quad e_2 \leq 2e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3,$$

then there is a tetragonal curve  $C = (C, g_4^1)$  of genus  $g$  such that  $\mathcal{O}(g_4^1)^{\otimes e_3+2}$  is birationally very ample and  $e_1 = e_1(g_4^1), e_2 = e_2(g_4^1), e_3 = e_3(g_4^1)$ .

**Theorem 3.** *Let  $C = (C, g_4^1)$  be a tetragonal curve of genus  $g$  with scrollar invariants  $e_1, e_2, e_3$ . If  $\mathcal{O}(g_4^1)^{\otimes e_3+2}$  is not birationally very ample, then there exists a curve  $C = (C_1, h_2^1)$  of genus  $e_3 + 1$  with a pencil of degree 2 and a map  $\pi : C \rightarrow C_1$  of degree 2 such that  $g_4^1 = \pi^*(h_2^1)$ .*

Hence we have the following result.

**Corollary 1.** *Let  $e_1, e_2, e_3$  and  $g \geq 5$  be integers. Then there exists a tetragonal curve  $C = (C, g_4^1)$  of genus  $g$  such that  $e_1 = e_1(g_4^1), e_2 = e_2(g_4^1), e_3 = e_3(g_4^1)$  if and only if*

$$e_1 \leq e_2 + e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3.$$

We now assume that  $C$  is not elliptic-hyperelliptic. For  $g=5$ , we have the following result. Let  $C \hookrightarrow \mathbb{P}^4$  be the canonical embedding. Let  $\delta \cong \mathbb{P}^2$  be the linear system of quadrics in  $\mathbb{P}^4$  containing  $C$ ,  $\Gamma$  is the locus of quadrics of rank  $\leq 4$  and  $\Gamma'$  is the locus of quadrics of rank  $\leq 3$ . We know the following:

**Proposition 1.** *If  $C$  is a tetragonal curve, then a general  $Q \in \delta$  is non-singular.*

By Proposition 1, we have that  $\Gamma \subset \mathbb{P}^2$  is a plane curve of degree 5. Let  $\mathcal{L} \in W_4^1(C)$  and let  $Q_{\mathcal{L}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2) \oplus \mathcal{O}_{\mathbb{P}^1}(e_3)) \subset \mathbb{P}^4$ , where  $e_i = e_i(\mathcal{L})$ . As  $Q_{\mathcal{L}}$  is contained in  $\Gamma$  (see E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] p. 240 Theorem 2.1), we have the morphism

$$\phi : W_4^1(C) \rightarrow \Gamma$$

given by

$$\phi(\mathcal{L}) = Q_{\mathcal{L}}.$$

Then we know the following Theorem:

**Theorem 4.**  $(W_4^1(C))_{\text{sing}} = \phi^{-1}(\Gamma') = \{\mathcal{L} \mid \mathcal{L} \in W_4^1(C), \mathcal{L}^{\otimes 2} \cong \omega_C\}$ .

As a corollary of Theorem 4, we have the following:

**Corollary 2.** *Let  $C$  be a tetragonal curve of genus 5. If  $W_4^1(C)$  is non-singular, then any  $g_4^1 \in W_4^1(C)$  has same  $e_1, e_2, e_3$ .*

We now assume that  $C$  is a tetragonal curve  $C$  of genus 6 which is not elliptic-hyperelliptic. Then we know the following results. Let  $\mathcal{L}$  be a tetragonal linear system (therefore  $\mathcal{L}$  is a base-point-free linear system) on  $C$ . Then the line bundle  $\omega_C \otimes \mathcal{L}^{\otimes -1}$  defines a base-point-free linear system of degree 6 and of projective dimension 2 on  $C$  because if  $\omega_C \otimes \mathcal{L}^{\otimes -1}$  has a base point, then  $\omega_C \otimes \mathcal{L}^{\otimes -1}$  defines a map  $\phi : C \rightarrow C_0 \subset \mathbb{P}^2$  such that  $\deg(\phi)\deg(C_0) \leq 5$ . If  $\deg(\phi) \geq 2$ , then  $C$  is a trigonal curve or a hyperelliptic curve. Therefore  $C$  has a singular plane curve model of degree  $\leq 5$ . Hence  $C$  has a trigonal linear system or a hyperelliptic linear system because  $C_0$  must have a singular point and the lines in  $\mathbb{P}^2$  which pass through one of the singular points of  $C_0$  induces a trigonal linear system or a hyperelliptic linear system. This is a contradiction. So  $\omega_C \otimes \mathcal{L}^{\otimes -1}$  defines a base-point-free linear system. As we assume that  $C$  is not an elliptic-hyperelliptic curve, therefore  $\deg(\phi) = 1$ . So  $C$  has a singular plane curve model  $C_0$  of degree 6, therefore the arithmetic genus  $p_a(C_0)=10$ . As  $C$  is not trigonal and not hyperelliptic, every singular point of  $C_0$  is multiplicity 2. So  $C_0$  has just 4 singular points (including infinitely near singular points).

**Proposition 2.** *Every member of  $W_4^1(C)$  is given by a cut out of the lines in  $\mathbb{P}^2$  which pass through one of the singular points of  $C_0$  or is given by a cut out of conics in  $\mathbb{P}^2$  which pass through 4 singular points (singular points of  $C_0$  and its infinitely near singular points of  $C_0$ ).*

*Proof.* See Griffiths, Harris [7] p. 210.  $\square$

Let  $g_4^1, k_4^1 \in W_4^1(C)$  and let  $C_1$  be the singular plane curve model defined by  $|K_C - g_4^1|$ . Assume that  $k_4^1$  is given by a cut out of conics which pass through 4 singular points. Then we consider a singular plane curve model defined by  $|K_C - k_4^1|$  and let  $h_4^1$  be a tetragonal linear system given by a cut out of the lines. We consider a singular plain curve model defined by  $|K_C - h_4^1|$ . Then  $k_4^1$  is given by a cut out of the lines. So we may assume that  $k_4^1 \in W_4^1(C)$  is given by a cut out of the lines. Let  $P \in \text{Sing}(C_1)$  be a singular point corresponds to  $k_4^1$ . By the definition of  $(e_1, e_2, e_3)$ , we have the following proposition.

**Proposition 3.** *The following conditions are equivalent:*

- 1)  $g_4^1 \in W_4^1(C)$  is a non-reduced point
- 2)  $g_4^1$  is type  $(2,1,0)$ .
- 3)  $P$  is a 2-fold node or a 2-fold cusp.

So we have the following theorem:

**Theorem 5.** *Let  $C$  be of genus 6. If  $W_4^1(C)$  is reduced of dimension 0, then every  $g_4^1$  is of type  $(1,1,1)$ .*

The following result is given by the adjunction formula on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

**Lemma 1 (Riemann).** *Let  $C$  be a Riemann surface of genus  $g$ , let  $f_1, f_2$  be a meromorphic function on  $C$  of  $\deg(f_1) = d_1, \deg(f_2) = d_2$  and  $k(C) = k(f_1, f_2)$ . Then  $g \leq (d_1 - 1)(d_2 - 1)$ .*

By Lemma 1, we have the following result.

**Theorem 6.** *Assume that  $C$  is a tetragonal curve and it is not an elliptic-hyperelliptic curve of genus  $g$ . If  $g \geq 10$ , then  $C$  has only one  $g_4^1$ .*

*Proof.* We assume that  $\#(W_4^1(C)) \geq 2$ . Then we can take distinct tetragonal (therefore base-point-free) linear systems  $g_4^1, h_4^1$  on  $C$ . Let  $\phi : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  be defined by  $\phi = (g_4^1, h_4^1)$ . Then  $g_4^1 = p_1 \cdot \phi$  and  $h_4^1 = p_2 \cdot \phi$  where  $p_1$  and  $p_2$  are projections. Therefore  $\deg \phi = 1, 2$  or  $4$ . Let  $C_0 = \phi(C)$ . If  $\deg \phi = 4$ , then  $C_0$  is a rational curve so  $g_4^1 = h_4^1$ . If  $\deg \phi = 2$ , then  $C_0$  is linearly equivalent to  $2l + 2m$  where  $l = \text{pt} \times \mathbb{P}^1$  and  $m = \mathbb{P}^1 \times \text{pt}$ . Therefore  $C_0$  is a rational curve or an elliptic curve by the adjunction formula. Therefore we have that  $\deg(\phi) = 1$ . Therefore we prove Theorem 6 by Lemma 1.  $\square$

The following theorem is found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 189 (4.2) Proposition).

**Theorem 7.** *Let  $\mathcal{L} \in W_d^r(C) \setminus W_d^{r+1}(C)$ . Then the tangent space  $T_{\mathcal{L}}(W_d^r(C))$  is isomorphic to  $(\text{im} \mu_0)^\perp \subset H^1(C, \mathcal{O}_C)$  where  $\mu : \Gamma(C, \mathcal{L}) \otimes \Gamma(C, \omega_C \otimes \mathcal{L}^{-1}) \rightarrow \Gamma(C, \omega_C)$  is the cup product map and  $(\text{im} \mu_0)^\perp$  denotes the complement space of  $\text{im} \mu_0 \subset \Gamma(C, \omega_C)$ .*

The following is also found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 191 (5.1) Theorem and p. 193 (5.2) Theorem).

**Theorem 8 (Martens-Mumford).** *If  $g \geq 3, 2 \leq d \leq g - 1$  and  $0 < 2r \leq d$ , then  $\dim W_d^r(C) \leq d - 2r$ . If  $C$  is a non-hyperelliptic curve and  $d \leq g - 2$ , then  $\dim W_d^r(C) \leq d - 2r - 1$  and if there is a component  $X \subset W_d^r(C)$  such that  $\dim X = d - 2r - 1$ , then  $C$  is either trigonal, elliptic-hyperelliptic or smooth plane quintic.*

**2. The proof of Main Theorem.** We now prove Theorem A. By Proposition 3, we may assume that  $C$  is a non-singular curve of genus  $g=7$  or  $8$ . We have already assumed that  $C$  is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system  $g_4^1$ . Moreover we may assume that  $C$  is not an elliptic-hyperelliptic curve. Let  $g_4^1, h_4^1 \in W_4^1(C)$

be such that  $g_4^1 \neq h_4^1$ . Let  $\rho = (g_4^1, h_4^1) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . As  $C$  is not an elliptic-hyperelliptic curve, so  $\rho$  is a birational morphism to its image. And every singular point of  $\rho(C)$  has multiplicity 2 because  $\rho(C) \subset \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$  is of degree 8 and  $C$  is neither hyperelliptic, nor trigonal and nor elliptic-hyperelliptic, i.e. if  $\rho(C)$  has a singular point  $P$  such that  $\text{mult}_P(\rho(C)) \geq 3$ , then we have a singular plane curve model of degree  $\leq 5$  and such curve is hyperelliptic or trigonal or elliptic-hyperelliptic. Let  $l = \text{pt} \times \mathbb{P}^1$  and  $m = \mathbb{P}^1 \times \text{pt}$  and let  $C_0 = \rho(C)$ . Take one singular point  $P \in C_0$  and take  $l_1 \ni P$  and  $m_1 \ni P$  such that  $l \sim l_1$  and  $m \sim m_1$  where  $\sim$  means a linear equivalence. We consider a blowing-up  $\pi_1 : T_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  at  $P$ . Let  $l_0$  be a proper transform of  $l_1$ , let  $m_0$  be a proper transform of  $m_1$ , let  $F_0$  be the exceptional divisor and let  $\tilde{C}$  be a proper transform of  $C_0$ . We first assume that  $g=7$ . Then  $\tilde{C}$  has one singular point  $Q$ . So we consider a blowing-up  $\pi_2 : T_2 \rightarrow T_1$  at  $Q$ , let  $S = T_2$ ,  $\pi = \pi_1 \cdot \pi_2$ , let  $E_1 =$  the total transform of  $l_0$ ,  $E_2 =$  the total transform of  $m_0$  and  $E_3 =$  the exceptional divisor of  $\pi_2$  and let  $L = \pi_2^*(F_0 + l_0 + m_0)$ . Let  $\phi$  be the morphism defined by the complete linear system  $|L|$ . By the definition of  $S$ , the proper transform of  $C_0$  in  $S$  is  $C$  and  $C \sim 6L - 2E_1 - 2E_2 - 2E_3$ . Moreover  $C_1 = \phi(C)$  is a (singular) plane curve model of  $C$  such that  $\text{deg}(C_1) = 6$  and  $\phi : C \rightarrow C_1 \subset \mathbb{P}^2$  is a normalization map. By elementary arguments, we have that  $g_4^1$  is given by a cut out of the lines which pass through one of the singular point of  $C_1 \subset \mathbb{P}^2$  because this linear system corresponds to the linear system  $|L - E_1|$ . As  $K_C - g_4^1 \sim K_S + C - (L - E_1)|C$ ,  $K_C - g_4^1 \sim K_S + C - (L - E_1)|C \sim 2L - E_2 - E_3|C$  and  $\dim|K_C - g_4^1| = \dim|2L - E_2 - E_3| = 3$ . Hence  $|K_C - g_4^1|$  is birationally very ample but not very ample because  $L - E_1 - E_2$  contracts to one point, so  $|K_C - g_4^1 - (L - E_1 - E_2)|C|$  gives  $\phi : C \rightarrow \mathbb{P}^2$  and  $\phi(C)$  is a (singular) plane curve of  $C$  such that  $\text{deg}(\phi(C)) = 6$ . We put  $D = L - E_1 - E_2|C$ . Let  $k_4^1 \in W_4^1(C)$  be such that  $g_4^1 \neq k_4^1$ . Then  $\dim|g_4^1 + k_4^1| = 3$ . Hence  $\dim|K_C - g_4^1 - k_4^1| = 1$ . Therefore  $\dim|K_C - g_4^1 - k_4^1 - D| = 0$  by the above. This implies that every  $k_4^1 \in W_4^1(C)$  such that  $k_4^1 \neq g_4^1$  is given by a cut out of the lines which pass through one of a singular points of  $\phi(C)$ . Now we assume that  $g = 8$ . We put  $S = T_1$  and  $\pi = \pi_1$ , let  $E_1 = m_0$ ,  $E_2 = l_0$  and let  $L = F_0 + l_0 + m_0$ . Let  $\phi$  be the morphism defined by the complete linear system  $|L|$ . By the definition of  $S$ , the proper transform of  $C_0$  in  $S$  is  $C$  and  $C \sim 6L - 2E_1 - 2E_2$ . Moreover  $C_1 = \phi(C)$  is also a (singular) degree 6 plane curve model of  $C$  and  $\phi : C \rightarrow C_1 \subset \mathbb{P}^2$  is a normalization map and we have that  $g_4^1$  is given by a cut out of the lines which pass through one of the singular point of  $C_1 \subset \mathbb{P}^2$  because this linear system corresponds to the linear system  $|L - E_1|$ . As  $K_C - g_4^1 \sim K_S + C - (L - E_1)|C$ ,  $K_C - g_4^1 \sim K_S + C - (L - E_1)|C \sim 2L - E_2|C$  and  $\dim|K_C - g_4^1| = \dim|2L - E_2| = 4$ . Hence  $|K_C - g_4^1|$  is birationally very ample but not very ample because  $E_1$  contracts to one point, so  $|K_C - g_4^1 - (E_1)|C|$  gives a birational morphism  $C \rightarrow \mathbb{P}^3$  because  $C$  is not hyperelliptic, not trigonal and not elliptic-hyperelliptic. As



$K_C - g_4^1 - (E_1|C) \sim 2L - E_1 - E_2|C$ ,  $\dim|K_C - g_4^1 - (E_1|C)| = \dim|2L - E_1 - E_2| = 3$  and  $L - E_1 - E_2$  is contracted to one point by the linear system  $|2L - E_1 - E_2|$ ,  $C \rightarrow \mathbb{P}^3$  is not very ample. Therefore we have a morphism  $\phi : C \rightarrow \mathbb{P}^2$  and  $\phi(C)$  is a (singular) plane curve of  $C$  such that  $\deg(\phi(C)) = 6$ . We put  $D = L - E_1 - E_2|C$ . Let  $k_4^1 \in W_4^1(C)$  such that  $g_4^1 \neq k_4^1$ . Then  $\dim|g_4^1 + k_4^1| = 3$ . Hence  $\dim|K_C - g_4^1 - k_4^1| = 2$ . Therefore  $\dim|K_C - g_4^1 - k_4^1 - D| = 0$  by the above. This implies that every  $k_4^1 \in W_4^1(C)$  such that  $k_4^1 \neq g_4^1$  is given by a cut out of the lines which pass through one of a singular points of  $\phi(C)$ .  $\square$

We now prove the following lemma:

**Lemma 2.** *Let  $C \rightarrow \phi(C) \subset \mathbb{P}^2$  be a singular plane curve model of  $C$  constructed as above. Let  $P \in \phi(C) \subset \mathbb{P}^2$  be a singular point and let  $k_4^1$  be a tetragonal linear system given by a cut out of the lines which pass through  $P$ . Then  $(e_1, e_2, e_3) = (2, 1, 1)$ ,  $(2, 2, 0)$  or  $(3, 1, 0)$  if  $g = 7$  and  $(e_1, e_2, e_3) = (3, 1, 1)$  or  $(2, 2, 1)$  if  $g = 8$ . And  $k_4^1 \in W_4^1(C)$  is reduced point if and only if  $P$  is ordinary node or ordinary cusp. Moreover  $P$  is ordinary node or ordinary cusp if and only if  $(e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2, 1, 1)$  if  $g=7$  and  $(e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2, 2, 1)$  if  $g=8$ .  $P$  is 2-fold node or 2-fold cusp if and only if  $(e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2, 2, 0)$  if  $g=7$ ,  $(e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (3, 1, 1)$  if  $g=8$ .  $P$  is 3-fold node or 3-fold cusp if and only if  $(e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (3, 1, 0)$  if  $g=7$ .*

**Proof.** Let  $e_i = e_i(k_4^1)$  ( $i=1, 2, 3$ ). We first assume that  $g=7$ . By Corollary 1, the possibilities are  $(e_1, e_2, e_3) = (2, 2, 0)$ ,  $(3, 1, 0)$ ,  $(2, 1, 1)$ . By Theorem 8, we have that  $\dim W_4^1(C) = 0$ . Therefore  $k_4^1 \in W_4^1(C)$  is reduced point if and only if  $k_4^1$  is of type  $(2, 1, 1)$  by Theorem 7. Hence we only have to prove that  $\dim \Gamma(C, \mathcal{O}(2k_4^1)) = 3$  if and only if  $P$  is ordinary node or ordinary cusp. Let  $\psi = \psi_1 \psi_2 \psi_3$  where  $\psi_1 : S_1 \rightarrow \mathbb{P}^2$  is a blowing-up at  $P_0 = P \in \mathbb{P}^2$ ,  $\psi_2 : S_2 \rightarrow S_1$  is a blowing-up at  $P_1 \in S_1$  and  $\psi_3 : S \rightarrow S_2$  is a blowing-up at  $P_2 \in S_2$  such that  $F_1 = (\psi_2 \psi_3)^*(F'_1)$ ,  $F_2 = \psi_3^*(F'_2)$  where  $F'_1$  is the exceptional divisor of  $\psi_1$ ,  $F'_2$  is the exceptional divisor of  $\psi_1$  and  $F_3$  is the exceptional divisor of  $\psi_3$ . As  $k_4^1 = L - F_1|C$ , so we have

$$0 \rightarrow \mathcal{O}(2L - 2F_1 - C) \rightarrow \mathcal{O}(2L - 2F_1) \rightarrow \mathcal{O}_C(2k_4^1) \rightarrow 0.$$

and  $2L - 2F_1 - C \sim -4L + 2F_2 + 2F_3$  because  $C \sim 6L - 2F_1 - 2F_2 - 2F_3$ . As  $4L - 2F_2 - 2F_3$  is linearly equivalent to some effective divisor, we have  $h^0(S, \mathcal{O}(2L - 2F_1 - C)) = 0$ . By Serre's duality,  $h^2(S, \mathcal{O}(2L - 2F_1 - C)) = h^0(S, \mathcal{O}(L + F_1 - F_2 - F_3))$ . By Definition A,  $P$  is an ordinary node or an ordinary cusp if and only if  $P_1 \notin F'_1$ ,  $P$  is a 2-fold node or a 2-fold cusp if and only if  $P_1 \in F'_1$  but  $P_2 \notin \psi_2^* F'_1$  and  $P$  is a 3-fold node or a 3-fold cusp if and only if  $P_1 \in F'_1$  and  $P_2 \in \psi_2^* F'_1$ . Hence we have that  $P$  is an ordinary node or ordinary cusp if and only if  $h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 1$  and  $P$  is an  $r$ -fold node or an  $r$ -fold cusp ( $r = 2, 3$ ) if and only if  $h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 2$ . As  $h^0(S, \mathcal{O}(2L - 2F_1)) = 3$

and  $h^2(S, \mathcal{O}(2L - 2F_1)) = h^0(S, \mathcal{O}(-5L + 3F_1 + F_2 + F_3)) = 0$  (by Serre's duality and  $5L - 3F_1 - F_2 - F_3$  is linearly equivalent to an effective divisor), we have  $h^1(S, \mathcal{O}(2L - 2F_1)) = 0$  by Riemann-Roch's Theorem (see Hartshorne [9] p. 362 Theorem 1.6). Therefore  $P$  is an ordinary node or an ordinary cusp if and only if  $(e_1, e_2, e_3) = (2, 1, 1)$  and  $P$  is an  $r$ -fold node or an  $r$ -fold cusp ( $r = 2, 3$ ) if and only if  $(e_1, e_2, e_3) = (3, 1, 0), (2, 2, 0)$ . By the same calculation, we have that  $(e_1, e_2, e_3) = (3, 1, 0)$  if and only if  $h^2(S, \mathcal{O}(3L - 3F_1 - C)) = h^0(S, \mathcal{O}(2F_1 - F_2 - F_3)) = 1$ . Hence  $(e_1, e_2, e_3) = (3, 1, 0)$  if and only if  $P$  is a 3-fold node or a 3-fold cusp. We now assume that  $g=8$ . By Corollary 1, the possibilities are  $(e_1, e_2, e_3) = (3, 2, 0), (3, 1, 1), (2, 2, 1)$ . As  $\dim W_4^1(C) = 0$ . Therefore  $k_4^1 \in W_4^1(C)$  is reduced point if and only if  $k_4^1$  is of type  $(2, 2, 1)$  or  $(3, 1, 1)$  by Theorem 7. And  $\dim \Gamma(C, \mathcal{O}(2k_4^1)) = 3$  if and only if  $P$  is an ordinary node or an ordinary cusp. So we first prove that  $\dim \Gamma(C, \mathcal{O}(2k_4^1)) = 3$  if and only if  $k_4^1$  is of type  $(2, 2, 1)$ . Let  $\psi = \psi_1 \psi_2$  where  $\psi_1 : S_1 \rightarrow \mathbb{P}^2$  is a blowing-up at  $P_0 = P \in \mathbb{P}^2$ ,  $\psi_2 : S_2 \rightarrow S_1$  is a blowing-up at  $P_1 \in S_1$  such that  $F_1 = \psi_2^*(F'_1)$  where  $F'_1$  is the exceptional divisor of  $\psi_1$ ,  $F_2$  is the exceptional divisor of  $\psi_1$ . As  $k_4^1 = L - F_1|_C$ , so we have

$$0 \rightarrow \mathcal{O}(3L - 3F_1 - C) \rightarrow \mathcal{O}(3L - 3F_1) \rightarrow \mathcal{O}_C(3k_4^1) \rightarrow 0.$$

and  $3L - 3F_1 - C \sim -3L - F_1 + 2F_2$  because  $C \sim 6L - 2F_1 - 2F_2$ . We have  $h^0(S, \mathcal{O}(3L - 3F_1 - C)) = 0$  and by Serre's duality,  $h^2(S, \mathcal{O}(3L - 3F_1 - C)) = h^0(S, \mathcal{O}(2F_1 - F_2))$ . As  $P$  is an ordinary node or an ordinary cusp,  $h^0(S, \mathcal{O}(2F_1 - F_2)) = 0$ . Therefore we have that  $h^1(S, \mathcal{O}(3L - 3F_1 - C)) = 1$ . As  $h^0(S, \mathcal{O}(3L - 3F_1)) = 4$  and  $h^2(S, \mathcal{O}(3L - 3F_1)) = h^0(S, \mathcal{O}(-6L + 4F_1 + F_2)) = 0$ , we have  $h^1(S, \mathcal{O}(3L - 3F_1)) = 0$  by Riemann-Roch's Theorem. Therefore we have that  $\dim \Gamma(C, \mathcal{O}(3k_4^1)) = 5$ . Hence  $k_4^1$  is of type  $(2, 2, 1)$ . And  $(e_1, e_2, e_3) = (3, 2, 0)$  if and only if  $P$  is a 2-fold node or a 2-fold cusp by the same calculation.  $\square$

**Proof of Corollary A.** We now prove Corollary A. By Theorem 6, we may assume that  $C$  is a non-singular curve of genus  $g = 7, 8$  or  $9$ . We have already assumed that  $C$  is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system  $g_4^1$ . Moreover we assume that  $C$  is not an elliptic-hyperelliptic curve. If there is only one  $g_4^1$  on  $C$ , there is nothing to prove. So we may assume that there are  $g_4^1, h_4^1 \in W_4^1(C)$  such that  $g_4^1 \neq h_4^1$ . If  $g = 7$  or  $8$ , then Corollary A holds by Theorem A. Therefore the remaining case is  $g=9$  case. But in this case, if  $\#W_4^1(C) \geq 2$ , then we have an embedding  $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . In this case one easily proves that  $h^0(C, \mathcal{O}(K_C - 3g_4^1)) = h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-l + 2m)) = 0$ , thus the invariants of  $g_4^1$  are  $(2, 2, 2)$ . Therefore the  $e$ -numbers of any  $k_4^1 \in W_4^1(C)$  are the same and equal to  $(2, 2, 2)$ . This proves Corollary A.  $\square$

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