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OSCILLATION THEOREMS FOR PERTURBED SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. Some oscillation criteria for solutions of a general perturbed second order ordinary differential equation with damping

$$(r(t)x'(t))' + h(t)f(x)x'(t) + \psi(t,x) = H(t,x(t),x'(t))$$

with alternating coefficients are given. The results obtained improve and extend some existing results in the literature.

1. Introduction. In this paper we are concerned with the problem of oscillation of nonlinear perturbed second order ordinary differential equation with damping

$$(r(t)x'(t))' + h(t)f(x(t))x'(t) + \psi(t,x(t)) = H(t,x(t),x'(t)) \tag{1}$$

where ($' = d/dt$).

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Throughout this paper, we restrict our attention only to the solutions of equation (1) which exist on some ray $[t_0, \infty)$. Such a solution is said to be oscillatory if it has an infinite number of zeros, and otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In the last two decades the problem of finding sufficient conditions for the oscillation of all solutions of ordinary differential equations has begun to receive more and more attention. An interesting case is that of establishing oscillation criteria for the perturbed equation (1) with damping and/or related equations which involve the average behavior of the integral of the alternating coefficient $q(t)$. As a contribution to this study we refer to the papers of Butler [1], Elabbasy [2], Grace et al. [3], [4] and [5], Graef et al [6], Nagabuchi et al [7], Philos [8], Wong [9], Yan [10] and Yeh [11] and the references cited in.

In the sequel we assume that $f : R \to R$, $h : [t_0, \infty) \to R$ and $r : [t_0, \infty) \to (0, \infty)$, $t_0 \geq 0$ are continuous functions. $\psi : [t_0, \infty) \times R \to R$, $H : [t_0, \infty) \times R \times R \to R$ are continuous functions such that $\dfrac{\psi(t,x)}{g(x)} \geq q(t)$ and $\dfrac{H(t,x,x')}{g(x)} \leq p(t)$ for $x \neq 0$ and $t \in [t_0, \infty)$, where $p, q : [t_0, \infty) \to R$ are continuous functions and $g : R \to R$ is differentiable function such that

$$(\mathrm{c}_1) \qquad xg(x) > 0 \quad \text{and} \quad g'(x) = \frac{d}{dx} g(x) \geq k > 0 \quad \text{for } x \neq 0.$$

We say that (1) is sublinear if $g(x)$ satisfies

$$(\mathrm{c}_2) \qquad 0 < \int_0^{\varepsilon} \frac{du}{g(u)} < \infty, \quad 0 < \int_0^{-\varepsilon} \frac{du}{g(u)} < \infty, \quad \varepsilon > 0;$$

(1) is superlinear if $g(x)$ satisfies

$$(\mathrm{c}_3) \qquad 0 < \int_{\varepsilon}^{\infty} \frac{du}{g(u)} < \infty, \quad 0 < \int_{-\varepsilon}^{-\infty} \frac{du}{g(u)} < \infty, \quad \varepsilon > 0;$$

(1) is a mixed type if $g(x)$ satisfies

$$(\mathrm{c}_4) \qquad 0 < \int_0^{\infty} \frac{du}{g(u)} < \infty, \quad 0 < \int_0^{-\infty} \frac{du}{g(u)} < \infty.$$

We see easily that, if $g(x) = g_1(x) + g_2(x)$ where g_1 is sublinear and g_2 is superlinear, then (c_4) is satisfied.

Yeh [11] considered equation (1) with $r(t) \equiv 1$, $f(x) \equiv 1$, $p(t) \equiv 0$, that is

$$(2) \qquad x''(t) + h(t)x'(t) + q(t)g(x) = 0$$

where $h, q \in c[t_0, \infty)$ and $xg(x) > 0$, $g'(x) \geq k > 0$ for $x \neq 0$. He proved that

(i)
$$\lim_{t\to\infty} \sup \int_{t_0}^{t} (t-s)^{n-1} sq(s)ds = \infty,$$

(ii)
$$\lim_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} s\left[(t-s)\left(h(s)-\frac{1}{n}\right)+n-1\right]^2 (t-s)^{n-3} ds < \infty$$

for some integer $n \geq 3$ are sufficient conditions for the oscillation of (2).

In [7] Nagabuchi et al. has extended and improved Yeh's result to the equation

(3)
$$(r(t)x'(t))' + h(t)x'(t) + q(t)g(x) = 0$$

where $g'(x) \geq k > 0$ for $x \neq 0$ and proved that (3) is oscillatory if there exists a continuously differentiable function $\rho(t)$ on $[t_0, \infty)$ and a constant $\alpha \in (1, \infty)$ such that

$$\lim_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha}\rho(s)q(s) -$$
$$-\frac{1}{4k}\left[(t-s)\frac{\rho(s)h(s)}{r(s)} + \alpha\rho(s) - (t-s)\rho'(s)\right]^2 (t-s)^{\alpha-2}\frac{r(s)}{\rho(s)} ds = \infty.$$

If $g(x) \equiv x$ and $k = 1$ then the result of Nagabuchi implies Yan's result in [10].

The purpose of this paper is to contribute further in this direction and to establish sufficient conditions for the oscillation of a broad class of second order nonlinear equations of the type (1). As a consequence, we are able to extend and improve a number of previously known oscillation results.

2. Main results.

Theorem 1. *Suppose* (c_1) *and* (c_3) *hold. Furthermore, assume that*

(c_5) $xf(x) > 0$ *for* $x \neq 0$ *and* $\displaystyle\int_{\pm\varepsilon}^{\pm\infty} \frac{f(u)}{g(u)} du < \infty$, $\varepsilon > 0$,

(c_6) $r(t)$ *is bounded for* $t \in [t_0, \infty)$ *i.e.,* $0 < r(t) \leq a$, $a > 0$,

(c_7) *there exists a continuously differentiable function* $\rho(t)$ *on* $[t_0, \infty)$ *such that* $\rho(t) > 0$, $\rho'(t) \geq 0$ *and* $\rho''(t) \leq 0$ *on* $[t_0, \infty)$, *and*

$$(r(t)\rho'(t))' \leq 0 \quad \text{and} \quad \rho(t)h(t) \leq 0, \quad (\rho(t)h(t))' \leq 0 \quad \text{for} \quad t \geq t_0,$$

(c$_8$) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s)(q(s)-p(s))ds > -\infty,$

(c$_9$) $\limsup_{t\to\infty} \left(\int_{t_0}^{t} \frac{1}{\rho(s)} ds\right)^{-1} \int_{t_0}^{t} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(u)(q(u)-p(u))duds = \infty.$

Then equation (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of the differential equation (1). Without loss of generality, we may assume that $x(t) > 0$ on $[T, \infty)$ for some $T \geq t_0$ (the case $x(t) < 0$ can be treated similarly). Define

$$w(t) = \frac{\rho(t)r(t)x'(t)}{g(x(t))} \quad \text{for all} \quad t \geq T.$$

This and (1) imply

$$w'(t) - \rho(t)(p(t)-q(t)) + \frac{r(t)\rho'(t)x'(t)}{g(x)} - \frac{\rho(t)h(t)f(x)x'(t)}{g(x)} - \frac{\rho(t)r(t)x'^2(t)g'(x)}{g^2(x)}.$$

Hence, for all $t \geq T$, we have

(4)

$$\int_{T}^{t} \rho(s)(q(s)-p(s))ds \leq -w(t)+w(T)+\int_{T}^{t} \frac{r(s)\rho'(s)x'(s)}{g(x(s))}ds - \int_{T}^{t} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds.$$

The first two integrals in R.H.S of the above inequality are bounded from above. This can be seen by applying the Bonnet theorem, for each $t \geq T$ there exist $\eta, \xi \in [T, t]$ such that

$$\int_{T}^{t} \frac{r(s)\rho'(s)x'(s)ds}{g(x(s))} = r(T)\rho'(T) \int_{x(T)}^{x(\xi)} \frac{du}{g(u)}.$$

Since

$$\int_{x(T)}^{x(\xi)} \frac{du}{g(u)} \leq \begin{cases} 0, & \text{if} \quad x(\xi) < x(T) \\ \displaystyle\int_{x(T)}^{\infty} \frac{du}{g(u)}, & \text{if} \quad x(\xi) \geq x(T) \end{cases}$$

and $r(T)\rho'(T) \geq 0$, it follows that

$$\int_{T}^{t} \frac{r(s)\rho'(s)x'(s)ds}{g(x(s))} \leq k_1 \quad \text{where} \quad k_1 = r(T)\rho'(T) \int_{x(T)}^{\infty} \frac{du}{g(u)},$$

and

$$\begin{aligned} -\int_{T}^{t} \frac{\rho(s)h(s)f(x)x'(s)}{g(x)}ds &= -\rho(T)h(T) \int_{x(T)}^{x(\eta)} \frac{f(u)}{g(u)}du \leq k_1' \\ &= -\rho(T)h(T) \int_{x(T)}^{\infty} \frac{f(u)}{g(u)}du. \end{aligned}$$

Hence, we have from (4)

$$\int_T^t \rho(s)(q(s)-p(s))ds \le -w(t)+k_2-\int_T^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds \tag{5}$$

where $k_2 = k_1 + k_1' + w(T)$, or, by virtue of condition (c_1),

$$\int_T^t \rho(s)(q(s)-p(s))ds \le -w(t)+k_2-k\int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds. \tag{6}$$

Now, we consider the behavior of x':

Case 1. x' is oscillatory. Then, there exists an infinite sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $x'(t_n) = 0$. Thus, (6) gives

$$\int_T^{t_n} \rho(s)(q(s)-p(s))ds \le k_2-k\int_T^{t_n} \frac{w^2(s)}{r(s)\rho(s)}ds.$$

By virtue of condition (c_8), we get $\dfrac{w^2(t)}{r(t)\rho(t)} \in L^1(T,\infty)$. Thus, there exists a positive constant N such that

$$\int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds \le N \quad \text{for every} \quad t \ge T.$$

For $t \ge T$, we may use the Schwarz inequality to obtain

$$\begin{aligned}\left|-\int_T^t \frac{w(s)}{\rho(s)}ds\right|^2 &= \left|\int_T^t \left(\frac{r(s)}{\rho(s)}\right)^{1/2}[w(s)(r(s)\rho(s))^{-1/2}]ds\right|^2 \\ &\le \int_T^t \frac{r(s)}{\rho(s)}ds\int_T^t \frac{w^2(s)}{r(s)\rho(s)}ds \le N\int_T^t \frac{r(s)}{\rho(s)}ds \le Na\int_T^t \frac{ds}{\rho(s)}.\end{aligned}$$

Hence, for every $t \ge T$ we have

$$-\int_T^t \frac{w(s)}{\rho(s)}ds \le \sqrt{Na}\left(\int_T^t \frac{ds}{\rho(s)}\right)^{1/2}.$$

Furthermore, (6) gives

$$\int_T^t \rho(s)(q(s)-p(s))ds \le -w(t)+k_2$$

hence,

$$\int_T^t \frac{1}{\rho(s)} \int_T^s \rho(u)(q(u)-p(u))duds \le -\int_T^t \frac{w(s)}{\rho(s)}ds + k_2 \int_T^t \frac{ds}{\rho(s)}. \tag{7}$$

Condition (c_7) implies that $\rho(t) \le \mu t$ for all large t, where μ is positive constant. This ensures that $\int_{t_0}^{\infty} \frac{dt}{\rho(t)} = \infty$.

Hence, we get from (7) for $t \ge T$

$$\begin{aligned}\int_T^t \frac{1}{\rho(s)} \int_T^s \rho(u)(q(u)-p(u))duds &\le \sqrt{Na}\left(\int_T^t \frac{ds}{\rho(s)}\right)^{1/2} + k_2 \int_T^t \frac{ds}{\rho(s)} \\ &\le (\sqrt{Na}+k_2)\int_T^t \frac{ds}{\rho(s)}.\end{aligned}$$

Dividing by $\int_T^t \frac{ds}{\rho(s)}$ and then taking the upper limit gives a contradiction to (c_9).

Case 2. $x'(t) > 0$ for $t \ge T_1 > t_0$. Then, it follows from (6) that

$$\int_{T_1}^t \rho(s)(q(s)-p(s))ds \le k_2$$

and consequently

$$\left(\int_T^t \frac{ds}{\rho(s)}\right)^{-1} \int_T^t \frac{1}{\rho(s)} \int_T^s \rho(u)(q(u)-p(u))duds \le k_2$$

contradicts (c_9).

Case 3. $x'(t) < 0$ for $t \ge T_2 > t_0$. If $\frac{w^2(t)}{r(t)\rho(t)} \in L^1(T_2,\infty)$, then we can follow the procedure of Case 1 to arrive at a contraction to (c_9). Suppose now $\frac{w^2(t)}{r(t)\rho(t)} \notin L^1(T_2,\infty)$. By virtue of ($c_8$), we get from (5) for some constant β

$$-w(t) \ge \beta + \int_{T_2}^t \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds \quad \text{for every} \quad t \ge T_2. \tag{8}$$

Since $\frac{w^2(t)}{r(t)\rho(t)} \notin L^1(T_2,\infty)$, there exists $T_3 \ge T_2$ such that

$$M = \beta + \int_{T_2}^{T_3} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds > 0.$$

Thus (8) ensures $w(t)$ is negative on $[T_3, \infty)$. By using (8), we have from (5) for every $t \geq T_3$

$$-w(t)\left\{\beta + \int_{T_2}^{t} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds\right\}^{-1} \geq 1$$

or

$$\frac{w^2(t)g'(x)}{r(t)\rho(t)}\left\{\beta + \int_{T_2}^{t} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds\right\}^{-1} \geq \frac{x'(s)g'(x)}{g(x)}.$$

An integration from T_3 to t yields

$$\log\left[\left\{\beta + \int_{T_2}^{t} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds\right\}\Big/M\right] \geq \log\left[\frac{g(x(T_3))}{g(x)}\right].$$

Then,

$$\beta + \int_{T_2}^{t} \frac{w^2(s)g'(x(s))}{r(s)\rho(s)}ds \geq M'g(x) \quad \text{for every} \quad t \geq T_3$$

where $M' = Mg(x(T_3)) > 0$.

Hence, from (8) we get

$$-w(t) \geq M'g(x)$$

i. e.

$$x'(t) \leq -\frac{M'}{r(t)\rho(t)}.$$

So,

$$x(t) \leq x(T_3) - M'\int_{T_3}^{t} \frac{ds}{r(s)\rho(s)} \leq x(T_3) - \frac{M'}{a}\int_{T_3}^{t} \frac{ds}{\rho(s)}$$

it follows, that $x(t) \to -\infty$ as $t \to \infty$, a contradiction. This completes the proof. □

The function ρ satisfying (c_7) may be taken to be $\rho(t) = t^\alpha$, $\alpha \in [0,1]$. Thus we have the following corollary of Theorem 1.

Corollary 1. *Equation* (1) *is oscillatory if* (c_1), (c_3) *hold and* $\gamma(t) = \alpha t^{\alpha-1}r(t) + ct^\alpha h(t) \geq 0$, $\gamma' \leq 0$ *for some* $\alpha \in [0,1]$,

$(c_8)'$ $\liminf_{t\to\infty} \int_{t_0}^{t} s^\alpha(q(s) - p(s))ds > -\infty,$

$$(c_9)' \quad \begin{cases} \lim_{t\to\infty} \frac{1}{\log t}\int_{t_0}^{t} \frac{1}{s}\int_{t_0}^{s} u(q(u) - p(u))duds = \infty, & \text{if } \alpha = 1 \\ \lim_{t\to\infty} \frac{1}{t^{1-\alpha}}\int_{t_0}^{t} \frac{1}{s^\alpha}\int_{t_0}^{s} u^\alpha(q(u) - p(u))duds = \infty, & \text{if } 0 \leq \alpha < 1. \end{cases}$$

In order to discuss the next two theorems, we need the following lemma, which is an extension of Wong's lemma [8].

Lemma. *Let*

(c_{10}) $\lim_{t\to\infty} inf \int_T^t q(s)ds \geq 0$ *for all large* T,

(c_{11}) $H(t,x,0) = 0$ *for all* $t \in [t_0,\infty)$, $x \neq 0$.

Then every nonoscillatory solution of (1) *which is not eventually a constant must satisfy* $x(t)x'(t) > 0$ *for all large* t.

The proof of this lemma is similar to that of Wong, and hence will be omitted.

The following is an extension of results of Nagabuchi [7], Yan [10] and Yeh [11].

Theorem 2. *Suppose that* (c_1), (c_{10}) *and* (c_{11}) *hold. Moreover, assume that*

(c_{12}) $h(t) \geq 0$ *for* $t \geq t_0$ *and* $f(x) \geq -c$; $c \geq 0$ *for* $x \in R$,

(c_{13}) *there exists a differentiable function* $\phi : [t_0,\infty) \to (0,\infty)$ *and continuous functions*

$$h, H : D \equiv \{(t,s) : t \geq s \geq t_0\} \to R$$

where H *has a continuous and nonpositive partial derivative on* D *with respect to the second variable such that*

$$H(t,t) = 0 \quad \text{for} \quad t \geq t_0, \quad H(t,s) > 0 \quad \text{for} \quad t > s \geq t_0$$

and

(c_{14}) $-\dfrac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)}$ *for all* $(t,s) \in D$.

Then equation (1) *is oscillatory if*

(c_{15}) $$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^t \Bigg\{ \phi(s)H(t,s)(q(s)-p(s)) - \frac{1}{4k}\left[r(s)\phi(s)\left(h(t,s) - \left(\frac{ch(s)}{r(s)} + \frac{\phi'(s)}{\phi(s)}\right)\sqrt{H(t,s)}\right)\right]^2 \Bigg\} ds = \infty.$$

P r o o f. Let $x(t)$ be a nonoscillatory solution of (1) and assume that $x(t) > 0$ for $t \geq t_0$. It follows from the above lemma that $x'(t) > 0$ on $[t_1,\infty)$ for some $t_1 \geq t_0$. Define $w(t) = \dfrac{\phi(t)r(t)x'(t)}{g(x(t))}$ for $t \geq t_1$. This and equation (1) imply

$$w'(t) \leq \phi(t)(p(t)-q(t)) + \left(\frac{ch(t)}{r(t)} + \frac{\phi'(t)}{\phi(t)}\right) w(t) - \frac{kw^2(t)}{r(t)\phi(t)}.$$

Hence, for all $t \geq t_1$, we have

$$\int_{t_1}^{t} \phi(s)H(t,s)(q(s)-p(s))ds \leq -\int_{t_1}^{t} H(t,s)w'(s)ds +$$

$$+\int_{t_1}^{t} H(t,s)\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)w(s)ds - k\int_{t_1}^{t}\frac{H(t,s)w^2(s)}{r(s)\phi(s)}ds =$$

$$= H(t,t_1)w(t_1) - \int_{t_1}^{t}\left(\frac{-\partial H}{\partial s}\right)w(s)ds +$$

$$+\int_{t_1}^{t} H(t,s)\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)w(s)ds - k\int_{t_1}^{t}\frac{H(t,s)w^2(s)}{r(s)\phi(s)}ds =$$

$$= H(t,t_1)w(t_1) -$$

$$-\int_{t_1}^{t}\left[\left(h(t,s)\sqrt{H(t,s)}-H(t,s)\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)w(s)+\frac{kH(t,s)w^2(s)}{r(s)\phi(s)}\right]ds=$$

$$= H(t,t_1)w(t_1) -$$

$$-\int_{t_1}^{t}\left[\sqrt{\frac{kH(t,s)}{r(s)\phi(s)}}w(s)-\frac{\sqrt{r(s)\phi(s)}}{2\sqrt{k}}\left(h(t,s)-\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)\right]^2 ds+$$

$$+\frac{1}{4k}\int_{t_1}^{t} r(s)\phi(s)\left(h(t,s)-\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2 ds \leq$$

$$\leq H(t,t_1)w(t_1) + \frac{1}{4k}\int_{t_1}^{t} r(s)\phi(s)\left(h(t,s)-\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2 ds.$$

Hence,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)\phi(s)(q(s)-p(s)) - \frac{r(s)\phi(s)}{4k}\times\right.$$

$$\left.\times\left(h(t,s)-\sqrt{H(t,s)}\left(\frac{ch(s)}{r(s)}+\frac{\phi'(s)}{\phi(s)}\right)\right)^2\right]ds \leq w(t_1),$$

a contradiction to (c_{15}). This completes the proof. □

Corollary 2. *Suppose that condition* (c_{15}) *in Theorem* 2 *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \phi(s)H(t,s)(q(s)-p(s))ds = \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\phi(s) \left(h(t,s) - \sqrt{H(t,s)} \left(\frac{ch(s)}{r(s)} + \frac{\phi'(s)}{\phi(s)}\right)\right)^2 < \infty.$$

Then the conclusion of Theorem 2 *holds.*

Remark. The functions $H(t,s) = (t-s)^{n-1}$, $t \ge s \ge t_0$, where n is an integer with $n > 2$, and $h(t,s) = (n-1)(t-s)^{(n-3)/2}$, $t \ge s \ge t_0$ are continuous and satisfy (c_{14}). Therefore, the results in [7], [10] and [11] can be obtained from Theorem 2 as special cases.

Theorem 3. *Suppose that* (c_1), (c_4), (c_{10}), (c_{11}) *and* (c_{12}) *hold. Furthermore, assume that*

(c_{16}) $r'(t) \le 0$ *for* $t \ge t_0$,

(c_{17}) *there exists a continuously differentiable function* $\rho : [t_0,\infty) \to (0,\infty)$, $\rho'(t) \ge 0$ *and* $\rho''(t) \le 0$; *and*

(c_{18}) $\sigma(t) = 2\rho'(t)r(t) + ch(t)\rho(t) \ge 0$ *and* $\sigma'(t) \le 0$ *for* $t \ge t_0$.

Then equation (1) *is oscillatory if*

(c_{19}) $\displaystyle\limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1}\rho(s)(q(s)-p(s))ds = \infty$, *for* $n \ge 2$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and assume that $x(t) > 0$ for $t \ge 0$. It follows from the lemma that $x'(t) > 0$ for $t \ge T \ge t_0$. Define $w(t) = \rho(t)\displaystyle\int_T^t \frac{r(s)x'(s)}{g(x(s))}ds$, $t \le T$. Therefore $w(t)$ is well defined for $t \ge T$. From this and equation (1) we have

$$w''(t) \le \rho(t)(p(t)-q(t)) + \frac{\sigma(t)x'(t)}{g(x(t))} + \rho''(t)\int_T^t \frac{r(s)x'(s)}{g(x(s))}ds.$$

But, by the Bonnet theorem, for a fixed $t \ge T$ and some $\xi \in [T,t]$

$$\int_T^t \frac{r(s)x'(s)}{g(x(s))}ds = r(T)\int_T^{\xi} \frac{x'(s)ds}{g(x(s))} = r(T)\int_{x(T)}^{x(\xi)} \frac{du}{g(u)}$$

and hence, since $r(T) > 0$ and

$$\int_{x(T)}^{x(\xi)} \frac{du}{g(u)} = \int_{x(T)}^{0} \frac{du}{g(u)} + \int_{0}^{x(\xi)} \frac{du}{g(u)} > \int_{x(T)}^{0} \frac{du}{g(u)} = -k_1$$

where $k_1 = \displaystyle\int_0^{x(T)} \frac{du}{g(u)}$, we have $\displaystyle\int_T^t \frac{r(s)x'(s)ds}{g(x(s))} \geq -k_1 r(T) = -k_2.$

Therefore,

$$w''(t) \leq \rho(t)(p(t) - q(t)) + \frac{\sigma(t)x'(t)}{g(x(t))} - k_2\rho''(t).$$

Hence, for all $t \geq T$, we have

$$\int_T^t (t-s)^{n-1}\rho(s)(q(s) - p(s))ds \leq -\int_T^t (t-s)^{n-1} w''(s)ds +$$

$$+\int_T^t \frac{(t-s)^{n-1}\sigma(s)x'(s)ds}{g(x(s))} - k_2\int_T^t (t-s)^{n-1}\rho''(s)ds = I_1 + I_2 + I_3.$$

The integrals I_i, $i = 1, 2, 3$ can be estimated as follows.
If n=2, then

$$\begin{aligned} I_1 &= -\int_T^t (t-s)w''(s)ds = (t-T)w'(T) - w(t) \leq \\ &\leq (t-T)w'(T) + |w(t)| \leq \\ &\leq (t-T)w'(T) + r(T)\rho(t)\int_0^\infty \frac{du}{g(u)}. \end{aligned}$$

Condition (c_{17}) implies that $\rho(t) \leq \mu t$, $\mu > 0$. Thus

$$I_1 \leq (t-T)w'(T) + \left(\mu r(T)\int_0^\infty \frac{du}{g(u)}\right)t.$$

If $n > 2$, then

$$\begin{aligned} I_1 &= -\int_T^t (t-s)^{n-1}w''(s)ds = \\ &= (t-T)^{n-1}w'(T) - (n-1)(n-2)\int_T^t (t-s)^{n-3}w(s)ds \leq \\ &\leq (t-T)^{n-1}w'(T) + (n-1)(n-2)\int_T^t (t-s)^{n-3}|w(s)|ds. \end{aligned}$$

Hence

$$\begin{aligned}
I_1 &\le (t-T)^{n-1}w'(T) + (n-1)(n-2)\mu r(T)\int_0^\infty \frac{du}{g(u)}\int_T^t s(t-s)^{n-3}ds \le \\
&\le (t-T)^{n-1}w'(T) + \mu r\int_0^\infty \frac{du}{g(u)}\cdot (t-T)^{n-1} = \\
&= \left(w'(T) + \mu r(T)\int_0^\infty \frac{du}{g(u)}\right)(t-T)^{n-1}, \quad \text{for} \quad n > 2.
\end{aligned}$$

For I_2, by using the Bonnet theorem, we obtain for a fixed $t \ge T$ and some $\eta \in [T,t]$

$$\begin{aligned}
I_2 = \int_T^t \sigma(s)(t-s)^{n-1}\frac{x'(s)}{g(x(s))}ds &= \sigma(T)(t-T)^{n-1}\int_{x(T)}^{x(\eta)}\frac{du}{g(u)} \le \\
&\le \sigma(T)(t-T)^{n-1}\int_0^\infty \frac{du}{g(u)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
I_3 &= -k_2\int_T^t (t-s)^{n-1}\rho''(s)ds = \\
&= k_2(t-T)^{n-1}\rho'(T) - k_2(n-1)\int_T^t (t-s)^{n-1}\rho'(s)ds \le \\
&\le k_2(t-T)^{n-1}\rho'(T).
\end{aligned}$$

Therefore, taking into account the above estimates for I_i; $i = 1,2,3$ we conclude

$$\frac{1}{t^{n-1}}\int_T^t (t-s)^{n-1}\rho(s)(q(s)-p(s))ds \le$$

$$\left(w'(T) + \mu r(T)\int_0^\infty \frac{du}{g(u)} + \sigma(t)\int_0^\infty \frac{du}{g(u)} + k_2\rho'(T)\right)\left(1-\frac{T}{t}\right)^{n-1}.$$

Taking the upper limit as $t \to \infty$, we get a contradiction to (c_{19}). This completes the proof. □

Theorem 3 can be used to some cases where some other known oscillation criteria cannot be used. For example, consider the differential equation

$$(t^\lambda x'(t))' + (t\sin t + t + t^2)(x^3(t) + x^{1/3}(t)) = \frac{t\cos t \sin x'(t)(x^5(t) + x^{7/3}(t))}{(x^2(t)+1)},$$

where $\lambda \leq 0$, $t \geq t_0 = \pi/2$.

Taking $g(x) = x^3(t) + x^{1/3}(t)$, we see that $\dfrac{\psi(t,x)}{g(x)} > t\sin t + t = q(t)$, and $\dfrac{H(t,x,x')}{g(x)} \leq t = p(t)$. Indeed, we have from

$$\liminf_{t\to\infty} \int_{\pi/2}^{t} (s.\sin s + s)ds) \geq 0,$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_{\pi/2}^{t}(t-s).s.s.\sin s ds = \limsup_{t\to\infty}\left\{\int_{\pi/2}^{t} s^2 \sin s ds - \frac{1}{t}\int_{\pi/2}^{t} s^3 \sin s ds\right\}$$

$$= \limsup_{t\to\infty}\left\{-t\sin t - 4\cos t - \frac{6}{t} + \frac{6\sin t}{t} - \pi + \frac{3\pi^2}{4}\right\} = \infty$$

and Theorem 3 implies that the above equation is oscillatory. That this equation is oscillatory does not appear to be deducible from other known oscillation criteria.

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