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INTEGRAL MANIFOLDS AND BOUNDED SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. Sufficient conditions for the existence of bounded solutions of singularly perturbed impulsive differential equations are obtained. For this purpose integral manifolds are used.

1. Introduction. The impulsive systems of differential equations are in adequate apparatus for mathematical simulation of numerous real processes and phenomena studied in physics, biology, population dynamics, biotechnologies, control, economics, etc. Such processes and phenomena are characterized by the fact that at certain moments of their evolution they undergo rapid changes. That is why in their mathematical simulation it is convenient to neglect the duration of these changes and assume that such processes and phenomena change their state momentarily, by jumps.

In the recent years these equations have been the object of numerous investigations [1]–[8].

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In the present paper the questions of the existence of bounded solutions of singularly perturbed impulsive differential equations are considered.

2. Preliminary notes and definitions. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$. Let $M = (0, \mu)$ $\mu = \text{const} > 0$; E_n is the unit matrix of type $n \times n$; $V_\rho = \{x \in \mathbb{R}^n : \|x\| \leq \rho\}$, $\rho = \text{const} > 0$; $V \subset V_\rho$.

Let $t_i \in \mathbb{R}$, $t_i < t_{i+1}, i = \pm 1, \pm 2, \ldots$ and $\lim\limits_{i \to \pm\infty} t_i = \pm\infty$.

We consider the following system of impulsive differential equations

$$\begin{cases} \dot{z} = A(t)z, t \neq t_i, \\ \Delta z(t_i) = A_i(x(t_i)), i = \pm 1, \pm 2, \ldots, \end{cases} \tag{1}$$

where $t \in \mathbb{R}$, $z \in \mathbb{R}^n$; $\Delta z(t_i) = z(t_i + 0) - z(t_i - 0)$.

Such systems are characterized by the fact that under the action of a force of negligible duration the mapping point of the extended phase space at the moments $t = t_i, i = \pm 1, \pm 2, \ldots$ jumps from the position $(t_i, z(t_i))$ to the position $(t_i, z(t_i) + A_i z(t_i)$.

In the paper consider the system of singularly perturbed impulsive differential equations

$$\begin{cases} \dot{x}(t) = B(t)x + f(t, x, y, \mu), t \neq t_i, \\ \Delta x(t_i) = B_i x(t_i) + I_i(x(t_i), y(t_i), \mu), \\ \mu \dot{y}(t) = D(t)y + h(t, x, y, \mu), t \neq t_i, \\ \Delta y(t_i) = D_k y(t_i) + J_i(x(t_i), y(t_i), \mu), i = \pm 1, \pm 2, \ldots, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^m$; $y \in \mathbb{R}^n$; $t \in \mathbb{R}$; $\mu \in M$ is a small parameter; $B(t)$ and $D(t)$ are matrix-functions of type $m \times m$ and $n \times n$ respectively; B_i and D_i are constant matrices of type $m \times m$ and $n \times n$ respectively; $f : \mathbb{R} \times \mathbb{R}^m \times V \times M \to \mathbb{R}^m$; $h : \mathbb{R} \times \mathbb{R}^m \times V \times M \to \mathbb{R}^n$; $I_k^1 : \mathbb{R}^m \times V \times M \to \mathbb{R}^m$; $I_k^2 : \mathbb{R}^m \times V \times M \to \mathbb{R}^n$; $\Delta x(t_i) = x(t_i + 0) - x(t_i - 0)$; $\Delta y(t_i) = y(t_i + 0) - y(t_i - 0)$, $i = \pm 1, \ldots$

Definition 1. *We call an arbitrary manifold G in the extended phase space of the system* (2) integral manifold, *if* $(t_0, x(t_0), y(t_0)) \in G$ *implies* $(t, x(t), y(t)) \in G$, $t_0 \in \mathbb{R}$, $t \geq t_0$.

Introduce the following notations:

$E = \{\varphi : \mathbb{R} \times \mathbb{R}^m \times M \to \mathbb{R}^n$, $\varphi = \varphi(t, x, \mu)$ is continuous with respect to its arguments x and μ, and it is piecewise continuous on $t \in \mathbb{R}$ with points of discontinuity of the first kind $t = t_i$, $i = \pm 1, \pm 2, \ldots$ at which it is continuous from the left $|\varphi(t,x,\mu)| = \sup\{\|\varphi(t,x,\mu)\| : (t,x,\mu) \in \mathbb{R}\times\mathbb{R}^m\times M\}$ is the norm of the function $\varphi \in E$.

Let $\rho = \text{const} > 0$, $\eta = \text{const} > 0$.

$L(\rho,\eta) = \{\varphi \in E : |\varphi(t,x,\mu)| \le \rho, |\varphi(t,\tilde{x},\mu) - \varphi(t,x,\mu)| \le \eta\|\tilde{x} - x\|, t \in \mathbb{R}, \tilde{x}, x \in \mathbb{R}^m, \mu \in M\}$.

Definition 2. *The set*

$$J = \{(t,x,y,\mu) \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n\times M : y = \varphi(t,x,\mu), \varphi \in L(\rho,\eta)\} \tag{3}$$

is called integral manifold of class $L(\rho,\eta)$ or (ρ,η)-integral manifold.

Definition 3. *The function $\varphi(t,x,\mu)$ from* (2) *is called a* parameter function *with respect to the integral manifold J.*

In the present paper for arbitrary ρ and η sufficient conditions for existence of bounded solutions with the method of integral manifolds for the system (2) are found.

Together with system (2) consider the linear systems of impulsive differential equations

$$\begin{cases} \dot{x}(t) = B(t)x, t \ne t_i, \\ \Delta x(t_i) = B_i(x(t_i), i = \pm1, \pm2, \dots, \end{cases} \tag{4}$$

and

$$\begin{cases} \mu\dot{y}(t) = D(t)y, t \ne t_i, \\ \Delta y(t_i) = D_i y(t_i), i = \pm1, \pm2, \dots. \end{cases} \tag{5}$$

Introduce the following conditions:

H1. The matrix function $B(t)$ is continuous for $t \in \mathbb{R}$.

H2. The matrix function $D(t)$ is continuous for $t \in \mathbb{R}$.

H3. The function $f : \mathbb{R}\times\mathbb{R}^m\times V\times M \to \mathbb{R}^m$ is continuous every where except $(t,x,y,\mu) \in \mathbb{R}\times\mathbb{R}^m\times V\times M$, $f(t_i,x,y,\mu) = f(t_i - 0,x,y,\mu)$ and $f(t_i + 0,x,y,\mu)$ exists, $i = \pm1, \pm2, \dots$.

H4. The function $h : \mathbb{R}\times\mathbb{R}^m\times V\times M \to \mathbb{R}^n$ is continuous every where except $(t,x,y,\mu) \in \mathbb{R}\times\mathbb{R}^m\times V\times M$, $h(t_i,x,y,\mu) = h(t_i - 0,x,y,\mu)$ and $h(t_i + 0,x,y,\mu)$ exists, $i = \pm1, \pm2, \dots$.

H5. The functions I_i are continuous in $\mathbb{R}^m \times V \times M$.

H6. The functions J_i are continuous in $\mathbb{R}^m \times V \times M$.

H7. The Cauchy matrix $X(t,s)$ of system (4) satisfies the inequality

$$\|X(t,s)\| \leq Ke^{\alpha|t-s|},$$

where $t,s \in \mathbb{R}$; $\alpha = \text{const} > 0$; $K = \text{const} > 0$.

H8. The eigenvalues $\lambda_k = \lambda_k(t), k = 1,\ldots,n$ of the matrix $D(t)$ satisfy the inequalities

$$Re\lambda_k(t) \leq -\Delta < 0, k = 1,\ldots,n.$$

H9. $\|E + D_i\| < l, l = \text{const} > 0$, $i = \pm 1, \pm 2, \ldots$.

H10. There exists $\kappa = \text{const} > 0$ such that

$$i(s,t) \leq \kappa(t-s),$$

where $i(s,t)$ is the number of the points t_i in the interval (s,t).

Remark 1. We shall note that sufficient conditions under which the inequality from **H7** is valid, are given in [5] and [6].

Theorem 1 [8]**.** *Let the following conditions hold:*

1. *Conditions* **H1** – **H10** *are met.*

2. *There exists a constant* $L > 0$ *such that*

$$\|f(t,\overline{x},\overline{y},\mu)-f(t,x,y,\mu)\|+\|I_i(\overline{x},\overline{y},\mu)-I_i(x,y,\mu)\| \leq L(\|\overline{x}-x\|+\|\overline{y}-y\|),$$

$$\|h(t,\overline{x},\overline{y},\mu)-h(t,x,y,\mu)\|+\|I_i(\overline{x},\overline{y},\mu)-I_i(x,y,\mu)\| \leq L(\|\overline{x}-x\|+\|\overline{y}-y\|),$$

where $\overline{x}, x \in \mathbb{R}^m; \overline{y}, y \in V; t \in \mathbb{R}, \mu \in M, k = \pm 1, \pm 2, \ldots$.

3. *There exists a constant* $Q > 0$ *such that*

$$\|h(t,x,\mu)\| \leq Q, \|J_i(x,y,\mu)\| \leq Q,$$

where $(t,x,y,\mu) \in \mathbb{R} \times \mathbb{R}^m \times V \times M,\ k = \pm 1, \pm 2, \ldots$.

Then for all numbers $\rho > 0$ and $\eta > 0$ there exist a constants $\mu^* > 0$, $Q^* > 0$, $L^* > 0$ such that if $\mu \in (0, \mu^*]$, $Q \in (0, Q^*]$ and $L \in (0, L^*]$ then for the system (1) there exists an (ρ, η)–integral manifold.

Corollary 1. *If ρ, η, L and Q are functions of the variable μ such that $\rho(\mu) \to 0$, $\eta(\mu) \to 0$, $L(\mu) \to 0$ and $Q(\mu) \to 0$ as $\mu \to 0$ then there exists a constant μ^* such that for each $\mu \in (0, \mu^*]$ for the system (2) there exists an (ρ, η)-integral manifold.*

(6)
$$\begin{cases} \dot{x}(t) = B(t)x + f(t, x, \varphi(t, x, \mu), \mu), t \neq t_i, \\ \Delta x(t_i) = B_i x(t_i) + I_i(x(t_i), \varphi(t_i, x(t_i), \mu), \mu), i = \pm 1, \pm 2, \dots . \end{cases}$$

Introduce the following conditions:

H11. For the system

(7)
$$\begin{cases} \dot{x}(t) = B(t)x + f(t, x, 0, 0), t \neq t_i, \\ \Delta x(t_i) = B_i x(t_i) + I_i(x(t_i), 0, 0), \mu), \mu), i = \pm 1, \pm 2, \dots \end{cases}$$

there exists a bounded solution $x = p^0(t)$, $t \in \mathbb{R}$.

H12. The derivate $\dfrac{\partial g}{\partial x}$ of the function $g(t, x, y, \mu) = B(t)x + f(t, x, y, \mu)$ is piecewise continuous function with points of discontinuity of the first kind at the moments $t = t_i, i = \pm 1, \pm 2, \dots$.

H13. For the system

(8)
$$\begin{cases} \dot{X}(t) = C(t)X, t \neq t_i, \\ \Delta X(t_i) = C_i X(t_i), i = \pm 1, \pm 2, \dots , \end{cases}$$

where

$$C(t) = B(t) + \frac{\partial}{\partial x} f(t, p^0(t), 0, 0),$$

$$C_i = B_i + \frac{\partial}{\partial x} I_i(t, p^0(t_i), 0, 0), i = \pm 1, \pm 2, \dots ,$$

there exists a fundamental matrix $\phi(t)$ such that for

$$G(t, s) = \begin{cases} \phi(t) P_k \phi(s)^{-1}, t \leq s, \\ \phi(t)(P_k - E_m)\phi(t)^{-1}, s > t \end{cases}$$

the following inequality hold

(9) $$\|G(t,s)\| \le N_1 e^{-\gamma_1|t-s|},$$

where $P_k = \mathrm{diag}[E_k, 0], N_1, \gamma_1 > 0, t \in \mathbb{R}, s \in \mathbb{R}$.

3. Main results. We set

(10) $$x = p^0(t) + v$$

and from H11, (6) it follows that

(11) $$\begin{cases} \dot{v} = C(t)v + r(t,v,\mu), t \ne t_i, \\ \Delta v(t_i) = C_i v(t_i,\mu) + \tilde{I}_i(v(t_i,\mu),\mu), i = \pm 1, \pm 2, \dots, \end{cases}$$

where

$$r(t,0,\mu) = f(t,p^0+v,\varphi(t,p^0+v,\mu),\mu) - f^0(t,p^0(t),0,0) - \\ -\frac{\partial}{\partial x} f(t,p^0(t),0,0)v,$$

$$\tilde{I}_i(v,\mu) = I_i(p^0+v,\varphi(t,p^0+v,\mu)) - I_i(p^0,0,0) - \frac{\partial}{\partial x} I_i(p^0,0,0)v.$$

Lemma 1. *Let the following conditions be fulfilled:*

1. *The conditions of Theorem* 1 *are met.*
2. *There exists* $\omega = \omega(\mu)$, $\omega(\mu) \to 0, \mu \to 0$ *such that*

   $$\|f(t,x,0,\mu)\| + \|I_i(x,0,\mu)\| \le \omega(\mu),$$

   where $(t,x,\mu) \in \mathbb{R} \times \mathbb{R}^m \times M$.
3. *The conditions* **H11** *and* **H12** *are met.*
4. *The functions* $\rho = \rho(\mu)$ *and* $\eta = \eta(\mu)$ *are such that* $\rho(\mu) \to 0$, $\eta(\mu) \to 0$ *for* $\mu \to 0$.

Then for $v, \overline{v} \in V_\sigma$ $(0 < \sigma < \rho)$ *it follows*

(12) $$\|r(t,0,\mu)\| + \|\tilde{I}_i(0,\mu)\| \le v(\mu),$$

(13) $$\|r(t,\overline{v},\mu) - r(t,v,\mu)\| + \|\tilde{I}_i(\overline{v},\mu) - \tilde{I}_i(v,mu)\| \le L_1(\mu)\|\tilde{v} - v\|,$$

where $v(\mu) \to 0$ $L_1(\mu) \to 0$ *for* $\mu \to 0, t \in \mathbb{R}, i = \pm 1, \pm 2, \ldots$.

P r o o f. From (10) we obtain (12). Set

$$\overline{\varphi}(t) = \overline{\varphi}(t, p^0 + \overline{v}, \mu),$$

$$v_\xi = v + \xi(\overline{v} - v),$$

where $\xi \in (0, 1]$.

From the theorem of everage values and from Lemma 1 it follows

$$\|r(t, \overline{v}, \mu) - r(t, v, \mu)\| + \|\tilde{I}_i(\overline{v}, \mu) - \tilde{I}_i(v, \mu)\| \leq \|f(t, p^0 + \overline{v}, 0, \mu)$$
$$-f(t, p^0 + \overline{v}, \varphi, \mu)\| + \|f(t, p^0 + \overline{v}, \varphi, \mu) - f(t, p^0 + \overline{v}, 0, \mu)$$
$$-[f(t, p^0 + v, \varphi, \mu) - f(t, p^0 + v, 0, \mu)]\| + \|f(t, p^0 + \overline{v}, 0, \mu)$$
$$-f(t, p^0, 0, \mu) - \frac{\partial}{\partial x} f(t, p^0, 0, 0)\overline{v} - [f(t, p^0 + v, 0, \mu) - f(t, p^0, 0, \mu)]$$
$$-\frac{\partial}{\partial x} f(t, p^0, 0, \mu)v\| + \|I_i(p^0(t_i) + \overline{v}(t_i), \overline{\varphi}(t_i), \mu) - I_i(p^0(t_i) + \overline{v}(t_i), \varphi(t_i), \mu)\|$$
$$+\|I_i(p^0(t_i) + \overline{v}(t_i), \varphi(t_i), \mu) - I_i(p^0(t_i) + \overline{v}(t_i), 0, \mu)$$
$$-[I_i(p^0(t_i) + v(t_i), \varphi(t_i), \mu)[I_i(p^0(t_i) + v(t_i), 0, \mu) - I_i(p^0(t_i), 0, \mu)]$$
$$-I_i(p^0(t_i) + v(t_i), 0, \mu)]\| + \|I_i(p^0(t_i) + v(t_i), 0, \mu)$$
$$-I_i(p^0(t_i) + v(t_i), 0, \mu) - \frac{\partial}{\partial x} I_i(p^0, 0, 0)\overline{v} - [I_i(p^0(t_i) + v(t_i), 0, \mu) - I_i(p^0(t_i), 0, \mu)]$$
$$-\frac{\partial}{\partial x} I_i(p^0, 0, \mu)v\| \leq L_1(\mu)\|\overline{v} - v\|,$$

where

$$L_1 = L_\eta + \max_{0 \leq \xi \leq 1} \|\frac{\partial}{\partial x} f(t, p^0 v_\xi, \varphi, \mu) - \frac{\partial}{\partial x} f(t, p^0 + v_\xi, 0, \mu)\| +$$
$$+ \max_{0 \leq \xi \leq 1} \|\frac{\partial}{\partial x} f(t, p^0 + v_\xi, 0, \mu) - \frac{\partial}{\partial x} f(t, p^0 + v_\xi, 0, 0)\|$$
$$+ \max_{0 \leq \xi \leq 1} \|\frac{\partial}{\partial x} I_i(p^0 + v_\xi, \varphi, \mu) - \frac{\partial}{\partial x} I_i(p^0 + v_\xi, 0, \mu)\|$$
$$+ \max_{0 \leq \xi \leq 1} |\frac{\partial}{\partial x} I_i(p^0 + v_\xi, 0, \mu) - \frac{\partial}{\partial x} I_i(p^0, 0, 0)\|. \qquad \square$$

Theorem 2. *Let the following conditions hold:*

1. *The conditions of Theorem* 1 *are met.*
2. *Conditions* **H10** – **H12** *are met.*
3. *For the function* f *and* I_i *there exists* $\omega(\mu) \geq 0$ *such that* $\omega(\mu) \to 0$ *for* $\mu \to 0$, *and*
$$\|f(t,x,0,\mu)\| + \|I_i(x,0,\mu)\| \leq \omega(\mu),$$
for $(t,x,\mu) \in \mathbb{R} \times \mathbb{R}^m \times M$, $i = \pm 1, \pm 2, \ldots$.
4. *The functions* $\rho = \rho(\mu)$ *and* $\eta = \eta(\mu)$ *are such that* $\rho(\mu) \to 0$, $\eta(\mu) \to 0$ *for* $\mu \to 0$.

Then there exists $\overline{\mu} > 0$, $\overline{\mu} \leq \mu^*$ *such that for* $\mu \in (0, \overline{\mu}]$ *for the system* (2) *there exists a bounded solution.*

Proof. With PC_σ, $0 < \sigma < \rho$ we denote the space of all functions $v(t,\mu)$ map the set $\mathbb{R} \times M$ in the set $\mathbb{R}^m$ which are piecewise continuous with discontinuity of the first kind in the points $t = t_i$, $i = \pm 1, \pm 2, \ldots$, and they are continuos with respect to μ and the inequality

$$\|v(t,\mu)\| \leq \sigma$$

for $t \in \mathbb{R}, \mu \in M$ holds.
In this space we shall investigate the operator S, where

$$Sv = \int_{-\infty}^{\infty} G(t,s) r(s, v(s,\mu), \mu) ds + \sum_{i=-\infty}^{\infty} G(t,t_i) \tilde{I}_i(v(t_i,\mu),\mu). \tag{14}$$

From (9) and Lemma 1 we obtain

$$\begin{aligned} \|Sv\| \leq & \int_{-\infty}^{t} \|G(t,s)\| (\|r(s,v(s),\mu) - r(s,0,\mu)\|) ds \\ & + \int_{t}^{\infty} \|G(t,s)\| (\|r(s,v(s),\mu) - r(s,0,\mu)\|) ds \\ & + \sum_{t_i < t} \|G(t,t_i)\| (\|\tilde{I}_i(t_i,v(t_i),\mu) - \tilde{I}_i(t_i,0,\mu))\| + \|\tilde{I}_i(t_i,0,\mu)\|) \\ & + \sum_{t < t_i} \|G(t,t_i)\| (\|\tilde{I}_i(t_i,v(t_i),\mu) - \tilde{I}_i(t_i,0,\mu))\| + \|\tilde{I}_i(t_i,0,\mu)\|) \\ \leq & \; 2N_1 \Big(\frac{1}{\gamma_1} + \frac{1}{e^{\gamma_1} - 1}\Big)(L_1(\mu)\sigma + v(\mu)). \end{aligned} \tag{15}$$

On the other hand

$$\begin{aligned}\|S\tilde{v}-Sv\| \leq\ & \int_{-\infty}^{t}\|G(t,s)\|\|r(s,\tilde{v},\mu)-r(s,v,\mu)\|ds \\ & +\int_{t}^{\infty}\|G(t,s)\|\|r(s,\tilde{v},\mu)-r(s,v,\mu)\|ds \\ & +\sum_{t_i<t}\|G(t,t_i)\|(\|\tilde{I}_i(t_i,\tilde{v}(t_i),\mu)-\tilde{I}_i(t_i,v(t_i),\mu))\|) \\ & +\sum_{t<t_i}\|G(t,t_i)\|(\|\tilde{I}_i(t_i,\tilde{v}(t_i),\mu)-\tilde{I}_i(t_i,v(t_i),\mu))\|) \\ \leq\ & N_1L_1(\mu)(\frac{1}{\gamma_1}+\frac{1}{e^{\gamma_1}-1})|\tilde{v}-v|,\end{aligned} \tag{16}$$

where $|v| = sup\{\|v(t,\mu)\|, t\in\mathbb{R}, \mu\in M\}$. From (15) and (16) it follows that there exists $\overline{\mu}>0$, $\overline{\mu}\leq\mu^*$ such that for $\mu\in(0,\overline{\mu}]$ we obtain

$$2N_1(\frac{1}{\gamma_1}-\frac{1}{e^{\gamma_1}-1})(L_1(\mu)\sigma+v(\mu))\leq\sigma, \tag{17}$$

$$2N_1(\frac{1}{\gamma_1}-\frac{1}{e^{\gamma_1}-1})(L_1(\mu))<1. \tag{18}$$

From (17) it follows that $Sv\in PC_\sigma$ and from (20) it follows that S is contracting operator.

Then for the equality $v=Sv$ for $\mu\in(0,\overline{\mu}]$ there exists only one solution $v\in PC_\sigma$.

From the relations

$$\frac{\partial}{\partial t}G(t,s)=C(t)G(t,s), t\neq t_i,$$

$$G(t_i+0,t)-G(t_i,t)=C_iG(t_i,t),$$

$$G(t,t-0)-G(t,t+0)=E_m, t\neq t_i$$

it follows that

$$v(t)=\int_{-\infty}^{\infty}G(t,s)r(s,v(s),\mu)ds+\sum_{i=-\infty}^{\infty}G(t,t_i)\tilde{I}_i(v(t_i),\mu)$$

is solution of (11).

We set $p(t)=p^0(t)+v(t,\mu)$, and from (10) it follows that $p(t)$ is solution of (6). Then (p,q) is bounded solution of the system (2). □

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