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PSEUDORADIAL SPACES: FINITE PRODUCTS AND AN EXAMPLE FROM CH*

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Abstract. Aiming to solve some open problems concerning pseudoradial spaces, we shall present the following: Assuming CH, there are two semi-radial spaces without semi-radial product. A new property of pseudoradial spaces insuring the pseudoradiality of a product is presented.

Let X be a topological space. Call *a chain* in X any mapping from an infinite cardinal to X. If κ is a cardinal, then κ-*chain* is a mapping from κ to X. As usual, ω-chain is called *a sequence.* A chain $\langle x_\alpha : \alpha < \kappa\rangle$ *converges* to a point x (equivalently, x is *a limit point* of a chain $\langle x_\alpha : \alpha < \kappa\rangle$) if for every neighborhood U of x there is some $\gamma < \kappa$ such that $U \supseteq \{x_\alpha : \gamma < \alpha < \kappa\}$. A

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convergent chain $\langle x_\alpha : \alpha < \kappa \rangle$ is called *strict* or *strictly convergent*, if its limit does not belong to any $\overline{\{x_\alpha : \alpha < \beta\}}$, $\beta < \kappa$.

A set $T \subseteq X$ is called *chain-closed* or *radially closed* (*strictly chain-closed*, *sequentially closed*, κ*-chain-closed*, resp.), if every convergent chain (strict chain, convergent sequence, convergent λ-chain with $\lambda \leq \kappa$, resp.) with values in T has a limit in T, too. A set T is called κ*-closed*, if for each $M \in [T]^{\leq \kappa}$, $\overline{M} \subseteq T$.

A space X is called

— *sequential*, if every sequentially closed subset of X is closed;

— *radial* or *Fréchet chain-net*, if for every nonclosed $T \subseteq X$ and every point $x \in \overline{T}$ there is some chain in T which converges to x;

— *semi-radial*, if for every infinite cardinal κ and every $T \subseteq X$, whenever T is κ-chain-closed, then T is κ-closed;

— *almost radial*, if every strictly chain-closed subset of X is closed;

— *pseudoradial*, if every chain-closed subset of X is closed.

Radial and pseudoradial spaces were introduced by H. Herrlich in 1967 [8] and the intensive study of these classes was initiated by A. V. Arhangel'skii in late seventies [1]. The main open problem in this field is whether the class of compact Hausdorff pseudoradial spaces is finitely productive. Assuming CH or even $\mathfrak{c} \leq \omega_2$, the answer is affirmative, since under these assumptions each compact sequentially compact Hausdorff space is pseudoradial [12, 9]. Attempts for a ZFC result revealed the importance of the class of semi-radial spaces: The product of two compact Hausdorff pseudoradial spaces is pseudoradial, if one factor is semi-radial [3]. The product of two compact Hausdorff almost radial spaces is almost radial, if one factor is semi-radial [11]. Since radial $\Longrightarrow$ semi-radial $\Longrightarrow$ almost radial $\Longrightarrow$ pseudoradial, these theorems improve the previous results by J. Gerlits, Z. Nagy, Z. Frolík and G. Tironi [7, 6].

The aim of the present paper is to show that the class of semi-radial spaces is not finitely productive, answering thus a question from [2], and to introduce another condition, which suffices to guarantee the pseudoradiality of a product of pseudoradial compact Hausdorff spaces.

Example. *Assume CH. Then there are two compact Hausdorff radial spaces, whose product is not semi-radial.*

P r o o f. In order to get the desired spaces, we shall construct an auxiliary Hausdorff gap on the set of natural numbers $\mathbb{N}$ first. Assuming the Continuum Hypothesis, it is easy to find a transfinite sequence $\{A_\alpha, B_\alpha : \alpha \in \omega_1\}$ satisfying the following:

(i) Every A_α as well as B_α is an infinite subset of $\mathbb{N}$;

(ii) for every $\alpha < \omega_1$, $A_\alpha \cap B_\alpha$ is finite;

(iii) whenever $\alpha < \beta < \omega_1$, then $|A_\alpha \setminus A_\beta| < \omega$ and $|A_\beta \setminus A_\alpha| = \omega$; similarly, $|B_\alpha \setminus B_\beta| < \omega$ and $|B_\beta \setminus B_\alpha| = \omega$;

(iv) there is no set $C \subseteq \mathbb{N}$ satisfying $|C \setminus A_\alpha| = \omega$ and $|B_\alpha \cap C| < \omega$ for all $\alpha < \omega_1$;

(v) symmetrically, there is no set $C \subseteq \mathbb{N}$ satisfying $|C \setminus B_\alpha| = \omega$ and $|A_\alpha \cap C| < \omega$ for all $\alpha < \omega_1$;

Proceeding by a transfinite induction, enumerate all infinite subsets of $\mathbb{N}$ as $\{Z_\alpha : \alpha < \omega_1\}$ and choose arbitrarily two disjoint infinite subsets A_0, B_0 of the set Z_0 such that the difference $Z_0 \setminus (A_0 \cup B_0)$ is infinite. If $\alpha < \omega_1$ and all sets A_β, B_β are known for $\beta < \alpha$, then, assuming that the respective form of (i)–(iii) holds for them and that $\mathbb{N} \setminus (A_\beta \cup B_\beta)$ is infinite for each $\beta < \alpha$, it is possible to find two disjoint infinite sets A, B such that $A_\beta \setminus A$ is finite for all $\beta < \alpha$ as well as $B_\beta \setminus B$, and $\mathbb{N} \setminus (A \cup B)$ is infinite. (For the proof of this fact, see e.g., [5, Lemma 14.16].) If $|Z_\alpha \setminus (A \cup B)| < \omega$, then it is enough to set $A_\alpha = A$, $B_\alpha = B$, otherwise choose two disjoint infinite subsets A', B' of the set $Z_\alpha \setminus (A \cup B)$ such that the set $Z_\alpha \setminus (A \cup B \cup A' \cup B')$ is infinite and define $A_\alpha = A \cup A'$, $B_\alpha = B \cup B'$. This completes the inductive definition.

Since every infinite subset of $\mathbb{N}$ was taken into account, it should be clear that the resulting family $\{A_\alpha, B_\alpha : \alpha < \omega_1\}$ is as required.

Both spaces X and Y will be homeomorphic to the space $\delta(N)$ from [10]. The space X is a disjoint union of $\mathbb{N}$, $\{x_\alpha : \alpha < \omega_1\}$ and a one-point set $\{\infty_X\}$, similarly, the space Y is a disjoint union $\mathbb{N}\cup\{y_\alpha : \alpha < \omega_1\}\cup\{\infty_Y\}$. The topology of X is described by neighborhood systems as follows: Every $n \in \mathbb{N}$ is isolated, the neighborhood base of a point x_α consists of all sets $\{x_\gamma : \beta < \gamma \le \alpha\}\cup A_\alpha\setminus(A_\beta\cup F)$ for $\beta < \alpha$ and finite $F \subset \mathbb{N}$. Finally, the basic neighborhood of the point ∞_X has a form $\{\infty_X\} \cup \{x_\gamma : \beta < \gamma < \omega_1\} \cup \mathbb{N} \setminus (A_\beta \cup F)$ where $\beta < \omega_1$, $F \in [\mathbb{N}]^{<\omega}$.

The topology of the space Y is described quite analogously — replace in the previous definition all x's and X's by y's and Y's, and all A's by B's.

Let us verify that the space X is radial. Since X is 1st countable at all its points except ∞_X, it suffices to consider only two cases: If $M \subseteq \mathbb{N}$ is infinite and $\infty_X \in \overline{M}$, then, according to the definition of the topology of X, $|M \setminus A_\alpha| = \omega$ for each $\alpha < \omega_1$. By (iv), there must exist some $\delta < \omega_1$ with $B_\delta \cap M$ infinite. Whenever $\beta < \omega_1$, then by (ii) and (iii), $A_\beta \cap B_\delta$ is finite. Consequently each neighborhood of ∞_X contains all but finitely many points of the set $B_\delta \cap M$, so $B_\delta \cap M$ is a sequence contained in M and converging to ∞_X. If $M \subseteq \{x_\alpha : \alpha < \omega_1\}$ satisfies $\infty_X \in \overline{M}$, then M itself is a chain of length ω_1 converging to ∞_X, which follows immediately from the definition of the topology of X.

It is obvious that the radiality of Y can be proved quite analogously.

The verification of compactness of both spaces is yet easier and we shall leave it to the reader.

It remains to show that the space $X \times Y$ is not semi-radial. To this end, consider the following subset $D = \{(n,n) : n \in \mathbb{N}\} \cup \{(x_\alpha, \infty_Y) : \alpha < \omega_1\} \cup \{(\infty_X, y_\alpha) : \alpha < \omega_1\}$ of the product. We shall show that D is sequentially closed. Whenever the values of a convergent sequence belong to the set $\{(x_\alpha, \infty_Y) : \alpha < \omega_1\}$ or to the set $\{(\infty_X, y_\alpha) : \alpha < \omega_1\}$, then so does its limit, since both subsets are homeomorphic to the space ω_1 of all countable ordinals. If a sequence $\langle (n,n) : n \in C\rangle$ converges, then either the sequence $\langle n : n \in C\rangle$ converges in X to some x_α and so the same sequence converges to ∞_Y in Y, or it converges in X to ∞_X. In the latter case it cannot converge in Y to the point ∞_Y, because of (iv) and (v). Being convergent also in Y, it has to converge to some y_β. Therefore its limit again belongs to D.

However, if U is a neighborhood of the point ∞_X, then there is some

$\alpha < \omega_1$ and a finite set $F \subseteq \mathbb{N}$ such that $U \supseteq \mathbb{N} \setminus (A_\alpha \cup F)$, and if V is a neighborhood of ∞_Y, then $V \supseteq \mathbb{N} \setminus (B_\beta \cup G)$ for some $\beta < \omega_1$ and some finite $G \subseteq \mathbb{N}$. Consequently, $U \times V \cap D$ is infinite, since by (ii) and (iii), the set $\mathbb{N} \setminus (A_\alpha \cup B_\beta \cup F \cup G)$ is.

We have shown that D is sequentially closed but not ω-closed, thus the product space $X \times Y$ is not semi-radial. $\square$

We present here a compactness–like property of pseudoradial spaces that appears to be true for one of the factors in many cases in which the product of two pseudoradial spaces turns out to be pseudoradial. It is the following

Property. *If X is a compact Hausdorff pseudoradial space, γ a limit ordinal and $\langle A_\iota : \iota < \gamma \rangle$ a decreasing sequence of subsets of X, then $\bigcap_{\iota<\gamma} cl_1 A_\iota \neq \emptyset$.*

For a time we did not have any example of a pseudoradial space not satisfying the property. However recently A. Bella and I. Yaschenko [4] provided two such examples. One such example is easily described; let $X = (\omega_1 + 1) \times [0, 1]$, where $\omega_1 + 1$ is endowed with the usual order topology. Then X is compact and pseudoradial, but does not satisfy the property. They also give an example of a compact sequential space that does not satisfy the above property.

Theorem. *Suppose the property holds for one of two factors. Then the product of two compact Hausdorff pseudoradial spaces is pseudoradial.*

P r o o f. Suppose the contrary. For a triple (X, Y, C), where X and Y are compact Hausdorff pseudoradial spaces and C is a radially closed subset of the product, define $\lambda(X, Y, C)$ to be the minimal length of a chain ranging in $\pi_X[C]$ and converging to a point in $X \setminus \pi_X[C]$, if there is some, $\lambda(X, Y, C)$ is undefined, if there is no such chain.

Let us observe that our assumption implies that at least one $\lambda(X, Y, C)$ is defined. Choose a pair (X, Y) of compact Hausdorff pseudoradial spaces the product of which is not pseudoradial. Then there is a chain-closed set $C \subseteq X \times Y$ which is not closed. Select a point $(x, y) \in \overline{C} \setminus C$. Since the closed subspace $\{x\} \times Y$ is homeomorphic to Y, it is pseudoradial and thus $C \cap (\{x\} \times Y)$ is a closed set. Choose a closed neighborhood V of the point y such that $X \times V$ is disjoint with $C \cap (\{x\} \times Y)$. The set $C_1 = C \cap (X \times V)$ is an intersection of chain-closed and a closed set, so it is chain-closed and $(x, y) \in \overline{C_1}$. Since the space Y is compact, the projection π_X is a closed mapping, therefore the set $\pi_X[C_1]$ is nonvoid and not closed in X. Consequently, for a triple (X, V, C_1), the cardinal $\lambda(X, V, C_1)$ is defined.

Let λ be the minimum of all $\lambda(X, Y, C)$, where (X, Y) runs through all pairs of compact Hausdorff pseudoradial spaces without pseudoradial product and C through all radially closed subsets of $X \times Y$ such that $\lambda(X, Y, C)$ is defined. According to the previous observation, the cardinal λ is defined correctly.

Let us fix two spaces X, Y and a set $C \subseteq X \times Y$ such that $\lambda(X, Y, C) = \lambda$.

Let $\langle x_\alpha : \alpha < \lambda \rangle$ be a chain converging to a point $x \in X \setminus \pi_X[C]$ with all $x_\alpha \in \pi_X[C]$ and denote $F_\alpha = \pi_Y[\pi_X^{-1}\{x_\alpha\} \cap C]$. Each set F_α is a compact subset of Y.

Denote $A_\alpha = \bigcup_{\alpha < \beta < \lambda} F_\beta$.

If $\bigcap_{\alpha < \lambda} A_\alpha \neq \emptyset$, then we easily get a contradiction with the assumption $x \in X \setminus \pi_X[C]$: Choose a point $y \in \bigcap_{\alpha < \lambda} A_\alpha$ and denote by I the set $\{\alpha < \lambda : y \in F_\alpha\}$. Obviously, $|I| = \lambda$. The chain $\langle (x_\alpha, y) : \alpha \in I \rangle$ has all values in C, thus $(x, y) \in C$, since C is radially closed.

So, for the rest of the proof, let us assume that $\bigcap_{\alpha < \lambda} A_\alpha = \emptyset$. By the property, there is some $y \in \bigcap_{\alpha < \lambda} cl_1 A_\alpha$ and we may choose cardinals μ_α and chains $\langle y_\xi(\alpha) : \xi < \mu_\alpha \rangle$ for every $\alpha < \lambda$ so that each $y_\xi(\alpha)$ belongs to A_α and each chain $\langle y_\xi(\alpha) : \xi < \mu_\alpha \rangle$ converges to y.

Three cases are possible:

(i) The set $I = \{\alpha < \lambda : \mu_\alpha > \lambda\}$ is cofinal in λ. For each $\alpha \in I$ let $\gamma(\alpha)$ be the first β satisfying $\alpha < \beta < \lambda$ and $|\{\xi < \mu_\alpha : y_\xi(\alpha) \in F_\beta\}| = \mu_\alpha$. Since the set $F_{\gamma(\alpha)}$ is compact, $y \in F_{\gamma(\alpha)}$. However, the set $\{\gamma(\alpha) : \alpha \in I\}$ is cofinal in λ, so the point y belongs to all A_α, which contradicts the assumption $\bigcap_{\alpha < \lambda} A_\alpha = \emptyset$.

(ii) The set $J = \{\alpha < \lambda : \mu_\alpha = \lambda\}$ is cofinal in λ. If for each $\alpha \in J$ one can find some β satisfying $\alpha < \beta < \lambda$ and $|\{\xi < \lambda : y_\xi(\alpha) \in F_\beta\}| = \lambda$, then we reach the contradiction exactly as in the case (i). However, if there is some $\gamma \in J$ such that for every β, $\gamma < \beta < \lambda$, we have that $|\{\xi < \lambda : y_\xi(\gamma) \in F_\beta\}| < \lambda$, then there is an obvious possibility to diagonalize: Let M be the set of all $\alpha < \lambda$

such that there is some $y_\xi(\gamma) \in F_\alpha$; denote one such $y_\xi(\gamma)$ as y_α. Then the chain $\langle (x_\alpha, y_\alpha) : \alpha \in M \rangle$ converges to (x, y), a contradiction.

(iii) The remaining. It means that there is some $\gamma < \lambda$ such that $\mu(\alpha) < \lambda$ for all α, $\gamma < \alpha < \lambda$. Since $x \notin \pi_X[C]$, we have $(x, y) \notin C$. Similarly as in the observation at the beginning of this proof, choose a closed neighborhood U of a point x such that $(U \times Y) \cap (X \times \{y\}) \cap C = \emptyset$. There is some $\gamma < \lambda$ such that for all $\alpha > \gamma$, $x_\alpha \in \operatorname{int} U$, therefore for every $\alpha > \gamma$, $A_\alpha \subseteq \pi_Y[C \cap (U \times Y)]$. Now it is enough to denote $D = \{(y, x) : (x, y) \in C \;\&\; x \in U\}$: we obtain that $\lambda(Y, U, D) < \lambda$, contrary to the minimality of λ. $\square$

Since any compact radial space satisfies the property, the above theorem generalizes the theorem in [6] concerning the product of a compact pseudoradial and a compact radial space.

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