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REGULAR AND OTHER KINDS OF EXTENSIONS OF TOPOLOGICAL SPACES*

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Communicated by S. Gul'ko

Dedicated to the memory of Professor D. Doitchinov

Abstract. In this paper the notion of *SR-proximity* is introduced and in virtue of it some new proximity-type descriptions of the ordered sets of all (up to equivalence) regular, resp. completely regular, resp. locally compact extensions of a topological space are obtained. New proofs of the Smirnov Compactification Theorem [31] and of the Harris Theorem on regular-closed extensions [17, Thm. H] are given. It is shown that the notion of SR-proximity is a generalization of the notions of *RC-proximity* [17] and *Efremovič proximity* [15]. Moreover, there is a natural way for coming to both these notions starting from the SR-proximities. A characterization (in the spirit of M. Lodato [23, 24]) of the proximity relations induced by the regular extensions is given. It is proved that the injectively ordered set of all (up to equivalence) regular extensions of X in which X is 2-combinatorially embedded has a largest element $(\kappa X, \kappa)$. A construction of κX is proposed. A new class of regular spaces, called *CE-regular spaces*, is introduced; the class of all *OCE-regular spaces* of J. Porter and C. Votaw [29] (and, hence, the class of all regular-closed spaces) is its proper subclass. The CE-regular extensions of the regular spaces are studied. It is shown that SR-proximities can be interpreted as bases (or generators) of the subtopological regular nearness spaces of H. Bentley and H. Herrlich [4].

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1. Introduction. In this paper by a "space" we will always mean a "topological T_1-space" and by a "proximity" – some kind of generalized proximity. The *Efremovič proximity* appears here as *EF-proximity.*

Our purpose in this note is to obtain *proximity-type descriptions* of the regular and some other kinds of extensions of a (completely) regular space. The theory of the regular extensions is well developed in the papers of K. Morita [26] and A. K. Steiner & E. F. Steiner [32] (on the language of *generalized uniformities*), H. Bentley & H. Herrlich [4] (on the basis of the notion of *nearness*), D. Doitchinov [14] and G. Dimov & D. Doitchinov [13] (by means of special families of open filters, called T_3*-systems*), but, to the author's knowledge, there exists no such theory based on proximities. Moreover, it seems that the celebrated Smirnov Compactification Theorem [31] is the unique result where the *ordered set* of all equivalence classes of certain kinds of extensions is described on a *purely* proximity language. Indeed, a proximity-type description of the locally compact extensions was given by S. Leader in [22], but there, together with the proximity, the *boundedness* was used as a primitive term; the well-known D. Harris' description of the regular-closed extensions by means of the *RC-proximities* [17] has the lack that comparable RC-proximities need not give rise to comparable regular-closed extensions (the latter was shown by P. Sharma and S. Naimpally [30]), i.e. the Harris Theorem gives a bijection, but not an isomorphism; the same is true for the J. Porter and C. Votaw's description of the *OCE-regular* extensions of a regular space by means of the *OCER-proximities* [29]. Actually, at the present time the *nearness structures* are much more prefered to proximities as a tool for the study of various kinds of extensions (see, e.g., [21, 20, 4, 5]). The reason is that the proximity structure is less informative than the nearness structure. On the other hand, the utilization of proximities has the advantage that a proximity on a set X is a special kind of *binary relation* on the power set $Exp(X)$, while the nearness is a special kind of ∞*-ary relation* on $Exp(X)$. In this paper we will try to unite the advantages of the proximity and nearness structures by describing, using only a proximity-type language, some *bases* (or *generators*) of those nearness structures which are induced by regular extensions (i.e. of the *subtopological regular nearness structures* (see [4])). These bases (or generators) are called *SR-proximities* and are choosen in such a way that they contain the whole information, useful for the extensions, which could be extracted from the generated by them nearness structures. Our SR-proximities describe the filter traces of the regular extensions. A description of the filter traces of the regular extensions, but on the language of T_3-systems, was given earlier by D. Doitchinov in [14]. In reality, the notion of SR-proximity arose just as a possible solution of such a problem: find a reasonable method for producing T_3-systems.

In this sense, the present note was inspired by Doitchinov's paper [14], but the influence of the ideas from D. Harris' paper [17] has to be emphasized as well. Our solution relies on the Harris' notion of *R-proximity* [17] (and the term "SR-proximity" comes from "strong R-proximity"). It seemed hopeless to make use of the R-proximities for describing the regular extensions, because it was well-known (see D. Harris [17, 18]) that two non-equivalent regular extensions of a regular space could induce equal R-proximities on it and that, on the other hand, there exist R-proximities which cannot be induced by any regular extension. However, as it is proved here, every T_3-system in a regular space X can be obtained as such a family of (maximal) δ-round filters in X (where δ is some R-proximity on X) which, in turn, determines completely the R-proximity δ. In such a way, a technique for producing T_3-systems, which is habitual for those working with proximities, is developed. We hope that the present paper will demonstrate the usefulness of this technique.

The Section 2 contains the preliminaries. In Section 3 the ordered set of all (up to equivalence) regular extensions of a regular space is described on the basis of our notion of SR-proximity (see Theorem 3.8). This result is applied for obtaining a description of the *ordered set* of all (up to equivalence) regular-closed extensions of a regular space (see Theorem 3.11) (as it is well-known (see H. Herrlich [19]), this set could be also empty). Then it becomes obvious that there is a bijection between the set of the SR-proximities on a space X corresponding to the regular-closed extensions of X and the set of the Harris' RC-proximities on X (see Proposition 3.12). In such a natural way we come to the *notion* of RC-proximity and obtain a new proof of the Harris Theorem [17, Theorem H]. We even improve it, because our theorem gives an isomorphism, while the Harris Theorem gives a bijection. The fact that the "right" order in the set of all RC-proximities on a space X (i.e. that order which reflects the order in the family of all corresponding extensions of X) is not the usual one (see [30] for an example), but that which comes through the interpretation of RC-proximities as special kinds of SR-proximities, witnesses the rightness and naturality of our approach. In the same Section 3 we characterize those binary relations δ on the power set of a regular space X which are induced by a regular extension (Y, e) of X (i.e., for $A, B \subseteq X$, $A\delta B$ iff $cl_Y(e(A)) \cap cl_Y(e(B)) \neq \emptyset$) (see Theorem 3.16). Such relations, induced, however, by Hausdorff or T_1-extensions, were characterized by M. Lodato in [23, 24]. In Section 3 we show also that the notion of *SR-proximally continuous function* works well in the problems concerning the extensions of continuous functions (see Theorem 3.18 and Corollaries 3.19, 3.20). With Corollary 3.19 we rediscover, in fact, the solution to Problem II of D. Harris

[17], given by W. Hunsaker and P. Sharma (see [27]). In Section 4 we show that if X is a regular space then the set of all (up to equivalence) regular extensions of X in which X is *2-combinatorially embedded* (in the sense of E. Čech and J. Novák [9]) has an injectively largest element, which we denote by κX (see Theorem 4.2). We give a construction of the extension κX and study its properties and its relationships with the Alexandroff extension αX (see [1]) and the Stone-Čech compactification (see Propositions 4.9, 4.10 and Example 4.11). In the same section we introduce also the class of *CE-regular spaces* as a generalization of the J. Porter and C. Votaw's class of *OCE-regular spaces* [29] and, hence, of the class of regular-closed spaces. We show that the CE-regular spaces are precisely those regular spaces X for which $\kappa X = X$. We study the CE-regular extensions of regular spaces and we prove that if an R-proximity δ on a space X is induced by a regular extension of X then the set of all (up to equivalence) regular extensions of X inducing δ has an injectively largest element $(\kappa_\delta X, \kappa_\delta)$ which is a CE-regular extension of X (see Theorem 4.7). We show that there are normal spaces having CE-regular non-OCE-regular extensions (see Example 4.12). In Section 5 we describe, using the language of SR-proximities, the ordered set of all (up to equivalence) completely regular extensions of a completely regular space (see Theorem 5.2). Some non-proximity-type descriptions of these extensions were given by G. Dimov [10] (by means of special families of open filters, called *CR-systems* (i.e. in the spirit of Doitchinov's description of the regular extensions)) and by H. Bentley, H. Herrlich and R. Ori [5] (on the basis of nearness structures). Further, we obtain a new proof of the Smirnov Compactification Theorem (see Theorem 5.6) and show that the *notion* of Efremovič proximity arises naturally from the notion of SR-proximity (see Proposition 5.5). So, both the Efremovič proximity and RC-proximity of D. Harris are special kinds of SR-proximities. We end this section with a description (on the language of our SR-proximities) of the ordered set of all (up to equivalence) locally compact extensions of a completely regular space (see Theorem 5.9). Some other descriptions of the locally compact extensions were given by S. Leader [22] (by using the notion of *local proximity* in which the *boundedness* and proximity are *both* primitive terms), by V. Zaharov [36] (by means of some special vector lattices of functions) and by G. Dimov & D. Doitchinov [12] (on the basis of the notion of *supertopological space*). In the last Section 6 we show that SR-proximities can be interpreted as bases (or generators) of the subtopological regular nearness spaces of H. Bentley and H. Herrlich [4].

A great part of the results presented in this paper were announced without proofs in [11].

2. Preliminaries. We first fix some notations. If (X, τ) is a topological space and $A \subseteq X$ then by $cl_X(A)$ (or simply by $cl(A)$) we denote the closure of A in X. If x is a point of X then by $\mathcal{N}_X(x)$ (or simply by $\mathcal{N}(x)$) we denote the neighbourhood filter of x in (X, τ). Further, by $\boldsymbol{N}$ (or ω) we denote the set of all natural numbers, by $\boldsymbol{R}$ – the real line with its natural topology and by $\boldsymbol{I}$ – its subspace $[0, 1]$. If X is a set then by $Exp(X)$ we denote the power set of X. For $\mathcal{G} \subseteq Exp(X)$ and $A \subseteq X$, we write $\bigcap \mathcal{G}$ instead of $\bigcap\{G : G \in \mathcal{G}\}$ and $\mathcal{G} \cap A$ instead of $\{G \cap A : G \in \mathcal{G}\}$.

2.1. An *extension* of a space X is a pair (Y, e), where Y is a space and $e : X \longrightarrow Y$ is a dense embedding of X into Y. Two extensions (Y_i, e_i), $i = 1, 2$, of X are called *isomorphic* (or *equivalent*) if there exists a homeomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ e_1 = e_2$. Clearly, the relation of isomorphism is an equivalence in the class of all extensions of X. We write $(Y_1, e_1) \geq_0 (Y_2, e_2)$ (resp., $(Y_1, e_1) \geq (Y_2, e_2)$) if there exists a continuous mapping (resp., a continuous surjection) $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ e_1 = e_2$. These relations are orders (i.e. they are reflexive and transitive). We refer to them as to the *projective orders*. We write $(Y_1, e_1) \geq_i (Y_2, e_2)$ and say that the extension (Y_1, e_1) is *injectively larger* than the extension (Y_2, e_2) if there exists a continuous mapping $\varphi : Y_2 \longrightarrow Y_1$ such that $\varphi \circ e_2 = e_1$ and φ is a homeomorphism from Y_2 to the subspace $\varphi(Y_2)$ of Y_1. This relation is also an order. The equivalence relations associated with these three orders (i.e. (Y_1, e_1) projectively (injectively) larger than (Y_2, e_2) and conversely) coincide with the relation of isomorphism (defined above) on the class of all Hausdorff extensions of X (see [2]).

Notation 2.2. (a) The set of all (up to equivalence) regular (resp., completely regular; Hausdorff locally compact; compact Hausdorff) extensions of a space X will be denoted by $\boldsymbol{R}(X)$ (resp., $\boldsymbol{CR}(X)$; $\boldsymbol{LC}(X)$; $\boldsymbol{C}(X)$).

(b) Let X be a space and (Y, e) be an extension of X. If $A \subseteq X$ then by $Ex(A)$ (or $Ex_Y(A)$) we denote the set $Y \setminus cl_Y(e(X \setminus A))$.

2.3. Let (Y, e) be an extension of a space (X, τ). If $M \subseteq Y$ then the set $e^{-1}(M)$ is called the *trace of* M *on* X. Analogously, if $\mathcal{G}$ is a subset of $Exp(Y)$ then the family $T(\mathcal{G}) = e^{-1}(\mathcal{G})(= \{e^{-1}(U) : U \in \mathcal{G}\})$ is called the *trace of* $\mathcal{G}$ *on* X. For every point y of Y, let $T(y) = T(\mathcal{N}_Y(y))$, i.e. $T(y)$ is the trace of $\mathcal{N}_Y(y)$ on X. Then the family $\{T(y) : y \in Y\}$ is called the *filter trace* of (Y, e) on X. (Y, e) is called *strict extension* of X if $\{cl_Y(e(A)) : A \subseteq X\}$ is a base for the closed sets in Y. Every regular (and even every semi-regular) extension of a space is strict (see [33, 2]).

A filter $\mathcal{F}$ in a space (X, τ) is called *open* (resp. *regular*) if it has a filter base of open sets (resp., if it is an open filter and has a filter base of closed sets).

Let Σ be a family of open filters in a space (X, τ) which extends the family of neighbourhood filters of the space X. Defining a topology on the set Σ by taking as an open base the family $\{U^* : U \in \tau\}$, where $U^* = \{\mathcal{F} \in \Sigma : U \in \mathcal{F}\}$, and setting $\sigma(x) = \mathcal{N}_X(x)$ for every $x \in X$, we obtain that (Σ, σ) is a strict extension of X. Its filter trace on X is just the given family Σ. (Σ, σ) is called *the strict extension of* X *with filter trace* Σ (see [2]). Note that $U^* = Ex_\Sigma(U)$, for every $U \in \tau$.

Let X be a regular space. Then the strict extension of X with filter trace the family of all maximal regular filters in X will be denoted by αX. We will refer to it as to the *Alexandroff extension* of X. It was constructed by P. S. Alexandroff in his fundamental paper [1].

2.4. Let X be a set. A *basic proximity* on X (see [8]) is a symmetric binary relation δ on $Exp(X)$ satisfying the following four conditions:

(P1) $\emptyset \overline{\delta} A$ for every $A \subseteq X$ ($\overline{\delta}$ means "not-δ");

(P2) $A\delta A$ for every $A \neq \emptyset$;

(P3) $A\delta(B \cup C)$ iff $A\delta B$ or $A\delta C$;

(P4) If x and y are distinct points of X then $\{x\}\overline{\delta}\{y\}$.

The pair (X, δ) is called *basic proximity space.* If M is a subset of X then the *restriction* δ_M of δ to M is defined as follows: for $A, B \subseteq M$, $A\delta_M B$ iff $A\delta B$. It is easy to see that (M, δ_M) is a basic proximity space.

We write $A < B$ if $A\overline{\delta}(X \setminus B)$. When x is a point of X, we write $x\delta A$ and $x < A$ respectively in place of $\{x\}\delta A$ and $\{x\} < A$. A basic proximity δ on a set X is called an *R-proximity* (D. Harris [17]) if it satisfies the following axiom:

(P5) If $x \in X$ and $x < A$, then there is $B \subseteq X$ with $x < B < A$.

Let (X, δ) be a basic proximity space. Then the operator cl_δ on $Exp(X)$ defined by $cl_\delta(A) = \{x \in X : x\delta A\}$ is a Čech closure operator (see [8]). Hence $\tau_\delta = \{X \setminus A : A = cl_\delta(A)\}$ is a topology on X. If δ is an R-proximity then cl_δ is a topological (i.e. Kuratowski) closure operator and the topology τ_δ on X defined via cl_δ is regular (see [17]). If (X, τ) is a space, δ is a basic proximity on the set X and $\tau = \tau_\delta$ then we say that δ is a basic proximity *on the space* (X, τ).

A function $f : (X_1, \delta_1) \longrightarrow (X_2, \delta_2)$ between two basic proximity spaces (X_i, δ_i), $i = 1, 2$, is called *proximally continuous* if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$ $(A, B \subseteq X_1)$. If δ_i, $i = 1, 2$, are two basic proximities on a set X then we write $\delta_1 \geq \delta_2$ if the identity function $id : (X, \delta_1) \longrightarrow (X, \delta_2)$ is proximally continuous (i.e. if, for $A, B \subseteq X$, $A\delta_1 B$ implies $A\delta_2 B$).

A basic proximity δ on a set X is called *LO-proximity* (or *Lodato proximity*) if $cl_\delta(A)\ \delta\ cl_\delta(B)$ implies $A\delta B$. If δ is a LO-proximity on X then cl_δ is a

Kuratowski closure operator [24]. A LO-proximity which is also an R-proximity is called *LR-proximity* [18].

A basic proximity δ on a set X is called *EF-proximity* (or *Efremovič proximity*) [15] if it satisfies the following axiom:

(EF) If $A, B \subseteq X$ and $A < B$ then there exists $C \subseteq X$ such that $A < C < B$.

Let (X, δ) be a basic proximity space. A filter $\mathcal{F}$ in X is called *round* (or δ-*round*) if $\forall\, V \in \mathcal{F}\ \exists\, W \in \mathcal{F}$ such that $W < V$. The set of all maximal round filters in (X, δ) will be denoted by $\Sigma(X, \delta)$ or simply by $\Sigma(\delta)$. A round filter $\mathcal{F}$ is called *end* in (X, δ) (or δ-*end*) if it satisfies the following condition:

(E) $A < B$ implies that $(X \setminus A) \in \mathcal{F}$ or $B \in \mathcal{F}$.

The set of all ends in (X, δ) will be denoted by $\Sigma_{end}(X, \delta)$ or simply by $\Sigma_{end}(\delta)$. It is well known (see, e.g., [28, Theorem 6.9]) that $\Sigma_{end}(\delta) = \Sigma(\delta)$ if δ is an EF-proximity on X. If δ is a basic proximity on X then

$(*)$ $\Sigma_{end}(\delta) \subseteq \Sigma(\delta)$

(see, e.g, the proof of Theorem 6.7 in [28]), but, in general, the converse doesn't hold even for the R-proximities (see Corollary 3.17 below).

If X is a set, $A \subseteq X$ and $\mathcal{F}$ is a filter in X, then we say that $\mathcal{F}$ *meets* A if $A \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.

Let (X, δ) be a basic proximity space and $A, B \subseteq X$. We say that B *surrounds* (or δ-*surrounds*) A if every $\mathcal{F} \in \Sigma(\delta)$ which meets A contains B [17]. An R-proximity δ on a set X which satisfies the following axiom:

(RC) $A < B$ iff the subset B surrounds the subset A,

is called an *RC-proximity* [17].

If (Y, e) is an extension of a space X then the binary relation on $Exp(X)$ defined by "$A\delta B$ iff $cl_Y(e(A)) \cap cl_Y(e(B)) \neq \emptyset$" is a basic proximity on the space X; we refer to it as to the *proximity induced by the extension* (Y, e). It is proved in [17] that the proximity induced by a regular extension is an R-proximity.

Definition 2.5 (D. Doitchinov [14]). *Let X be a space and Σ be a family of open filters in X. If U and V are two subsets of X then we say that U* Σ-surrounds *V (or that V is* Σ-surrounded *by U) if every filter $\mathcal{F} \in \Sigma$ which meets V contains U. The family Σ is called a* T_3-system *if the following two conditions are fulfilled:*
(i) $\{\mathcal{N}(x) : x \in X\} \subseteq \Sigma$, *and*
(ii) $\forall\, \mathcal{F} \in \Sigma$ *and* $\forall\, U \in \mathcal{F}$ *there exists a* $V \in \mathcal{F}$ *which is Σ-surrounded by* U.

The set of all T_3-systems in X will be denoted by $\boldsymbol{T}_3(X)$.

Note that if $\mathcal{F} \in \Sigma$ and $\Sigma \in \boldsymbol{T}_3(X)$ then $\bigcap \mathcal{F} \neq \emptyset$ iff $\mathcal{F} = \mathcal{N}(x)$ for some $x \in X$ (see [13, 1.6(a)]).

Definition 2.6 ([13]). *Let X be a space and Σ_i, $i = 1, 2$, be two T_3-systems in X. We put $\Sigma_1 \leq_0 \Sigma_2$ if every element of Σ_2 contains some element of Σ_1. We put $\Sigma_1 \leq \Sigma_2$ if $\Sigma_1 \leq_0 \Sigma_2$ and, in addition, every element of Σ_1 is contained in some element of Σ_2. It is easy to see that the relations $\leq_0$ and $\leq$ are orders in the set $\boldsymbol{T}_3(X)$.*

Theorem 2.7 ([13]). *Let (X, τ) be a regular space. Then the ordered sets $(\boldsymbol{R}(X), \leq_0)$ (resp. $(\boldsymbol{R}(X), \leq)$) and $(\boldsymbol{T}_3(X), \leq_0)$ (resp. $(\boldsymbol{T}_3(X), \leq)$) are isomorphic. The isomorphism between these ordered sets is constructed as follows: to every T_3-system Σ in X corresponds the strict extension of X with filter trace Σ. (This extension will be denoted by $(r_\Sigma X, r_\Sigma)$.)*

In the following theorem the fundamental result of N. Bourbaki [7], concerning the extension of a continuous function from a dense subspace of a space into a regular space to a continuous mapping of the whole space, is presented in an equivalent form, appropriate for the problems regarded in this paper.

Theorem 2.8. *Let X be a topological space, (eX, e) be an extension of X, Y be a regular space and (rY, r) be a regular extension of Y. Let, further, Σ_e and Σ_r be the filter traces of (eX, e) on X and of (rX, r) on Y respectively. Let, finally, $f : X \longrightarrow Y$ be a continuous function. Then the following conditions are equivalent:*
(i) There exists a continuous function $F : eX \longrightarrow rY$ such that $F \circ e = r \circ f$;
(ii) For every $\mathcal{F} \in \Sigma_e$ there exists a $\mathcal{G} \in \Sigma_r$ such that $\mathcal{G}$ is contained in the filter in Y generated by the filter-base $f(\mathcal{F})$.

For all undefined here notions and notations see [16, 28].

3. Regular extensions.

Definition 3.1. *Let (X, δ) be an R-proximity space and Σ be a set of round filters in (X, δ) such that:*

(SR1) *All neighbourhood filters of the points of (X, τ_δ) are in Σ, and*

(SR2) *For $A, B \subseteq X$, $A\delta B$ is equivalent to the existence of an element $\mathcal{F}$ of Σ which does not contain the sets $X \setminus A$ and $X \setminus B$.*

Then the pair $\alpha = (\delta, \Sigma)$ is called an SR-proximity *and the pair (X, α) – an* SR-proximity space. *If (X, τ) is a topological space and (X, α), where $\alpha = (\delta, \Sigma)$, is an SR-proximity space such that $\tau = \tau_\delta$, then we say that α is an* SR-proximity on the space X. *The set of all SR-proximities on a space X will be denoted by* $\boldsymbol{SRProx}(X)$.

A function $f : (X,\alpha_1) \longrightarrow (Y,\alpha_2)$, where $\alpha_i = (\delta_i,\Sigma_i)$, $i = 1,2$, are SR-proximities, is called SR-proximally continuous *if for every $\mathcal{F} \in \Sigma_1$ there exists a $\mathcal{G} \in \Sigma_2$ such that $\mathcal{G}$ is contained in the filter in Y generated by the filter-base $f(\mathcal{F})$.*

Example 3.2. Let X be a regular space. Then, as it is shown in [17], defining a binary relation δ_w on $Exp(X)$ by setting $A\delta_w B$ iff $cl_X(A)\cap cl_X(B) \neq \emptyset$, one obtains an R-proximity on the space X (it is called *Wallman-proximity* on X and, when it is necessary, the complete notation $\delta_w(X)$ will be used). Put $\Sigma_X = \{\mathcal{N}(x) : x \in X\}$. Then, obviously, $\alpha_w = (\delta_w, \Sigma_X)$ is an SR-proximity on the space X. It will be called *Wallman SR-proximity* on X.

Since every R-proximity induces a regular topology (see [17]), we obtain the following fact: if Y is a topological space then the set $\boldsymbol{SRProx}(Y)$ is not empty if and only if the space Y is regular.

Proposition 3.3. (a) *The composition of two SR-proximally continuous functions is an SR-proximally continuous function.*

(b) *Condition (SR2) in 3.1 is equivalent to the following one:*

(SR2$'$) *For $A, B \subseteq X$, $A\delta B$ is equivalent to the existence of an element $\mathcal{F}$ of Σ which meets both A and B.*

(c) *If $f : (X,\alpha_1) \longrightarrow (Y,\alpha_2)$, where $\alpha_i = (\delta_i,\Sigma_i)$, $i = 1,2$, are SR-proximities, is an SR-proximally continuous function then $f : (X,\delta_1) \longrightarrow (Y,\delta_2)$ is a proximally continuous mapping.*

(d) *If $\alpha = (\delta,\Sigma)$ is an SR-proximity on a set X then $\Sigma \subseteq \Sigma_{end}(X,\delta)$. Hence $\Sigma \subseteq \Sigma(X,\delta)$.*

P r o o f. (a) and (b). The proofs are straightforward.

(c). Let $A, B \subseteq X$ and $A\delta_1 B$. Then, by (b), there exists an $\mathcal{F}$ of Σ_1 which meets both A and B. Thus the filter-base $f(\mathcal{F})$ meets both $f(A)$ and $f(B)$. There exists a $\mathcal{G} \in \Sigma_2$ such that $\mathcal{G}$ is contained in the filter in Y generated by the filter-base $f(\mathcal{F})$. Then $\mathcal{G}$ meets both $f(A)$ and $f(B)$. Hence $f(A)\delta_2 f(B)$.

(d). Let $\mathcal{F} \in \Sigma$. Then $\mathcal{F}$ is a round filter in (X,δ). Let's check the condition (E) from 2.4. If $A, B \subseteq X$ and $A < B$ then $A\overline{\delta}(X \setminus B)$. Hence, by (SR2) (see Definition 3.1), we have that $\mathcal{F}$ contains at least one of the sets $X \setminus A$ and B. So, $\mathcal{F}$ is an end in (X,δ). Therefore, $\Sigma \subseteq \Sigma_{end}(X,\delta)$. Hence, by $(*)$ (see 2.4.), $\Sigma \subseteq \Sigma(X,\delta)$. $\square$

Definition 3.4. *Let X be a regular space and $\alpha_i = (\delta_i,\Sigma_i)$, $i = 1,2$, be two SR-proximities on the space X. We put $(\delta_1,\Sigma_1) \leq_0 (\delta_2,\Sigma_2)$ if every* *element of* Σ_2 *contains some element of* Σ_1*. We put* $(\delta_1, \Sigma_1) \leq (\delta_2, \Sigma_2)$ *if* $(\delta_1, \Sigma_1) \leq_0 (\delta_2, \Sigma_2)$ *and, in addition, every element of* Σ_1 *is contained in some element of* Σ_2*. It is easy to see that the relations* $\leq_0$ *and* $\leq$ *are orders in the set* $\boldsymbol{SRProx}(X)$*.*

Proposition 3.5. *Let* X *be a regular space and* $\alpha_i = (\delta_i, \Sigma_i)$*,* $i = 1, 2$*, be two SR-proximities on the space* X*. Then:*
(i) $\alpha_1 \leq_0 \alpha_2$ *iff the identity* $id : (X, \alpha_2) \longrightarrow (X, \alpha_1)$ *is an SR-proximally continuous mapping, and*
(ii) $\alpha_1 \leq_0 \alpha_2$ *implies* $\delta_1 \leq \delta_2$*.*

Proof. The assertion (i) is obvious and (ii) follows from (i) and Proposition 3.3(c). □

Lemma 3.6. *Let* (X, α)*, where* $\alpha = (\delta, \Sigma)$*, be an SR-proximity space. Then* Σ *is a* T_3*-system.*

Proof. Let $\mathcal{F} \in \Sigma$ and $F \in \mathcal{F}$. Since $\mathcal{F}$ is a round filter, there exists a $G \in \mathcal{F}$ such that $G < F$, i.e. $G\overline{\delta}(X \setminus F)$. Then (SR2′) implies that every filter $\mathcal{P} \in \Sigma$ which meets G contains F. Hence F Σ-surrounds G. Thus, Σ is a T_3-system. □

Lemma 3.7. *Let* Σ *be a* T_3*-system in a space* (X, τ)*. Define a binary relation* δ *in* $Exp(X)$ *by letting* $A\delta B$ *iff there exists an element* $\mathcal{F}$ *of* Σ *which meets both* A *and* B*. Then the pair* (δ, Σ) *is an SR-proximity on the space* X*.*

Proof. We first show that δ is an R-proximity. It is clear that the conditions (P1), (P2) and (P4) from 2.4 are fulfilled. Since from $A\delta B$ or $A\delta C$ follows immediately that $A\delta(B \cup C)$, for checking (P3) it remains to show that the converse implication holds as well. We have that if $A\delta(B \cup C)$ then there exists a filter $\mathcal{F} \in \Sigma$ which meets both A and $B \cup C$. Suppose that $A\overline{\delta}B$ and $A\overline{\delta}C$. Then the filter $\mathcal{F}$ doesn't meet both B and C. Hence $X \setminus B \in \mathcal{F}$ and $X \setminus C \in \mathcal{F}$. Then $(X \setminus B) \cap (X \setminus C) \in \mathcal{F}$, i.e. $X \setminus (B \cup C) \in \mathcal{F}$. This is a contradiction, since $\mathcal{F}$ meets $B \cup C$. Hence the condition (P3) is fulfilled as well. So, δ is a basic proximity. For showing that it is an R-proximity, let $x \in X$, $A \subseteq X$ and $x < A$, i.e. $x\overline{\delta}(X \setminus A)$. This implies that $\mathcal{N}(x)$ doesn't meet $X \setminus A$. Hence $A \in \mathcal{N}(x)$. Since Σ is a T_3-system, there exists a $B \in \mathcal{N}(x)$ which is Σ-surrounded by A. Consequently, if $\mathcal{F} \in \Sigma$ and $\mathcal{F}$ meets B then $A \in \mathcal{F}$, and hence $\mathcal{F}$ doesn't meet $X \setminus A$. This shows that $B\overline{\delta}(X \setminus A)$, i.e. that $B < A$. Since X is a T_1-space and $\mathcal{N}(x)$ doesn't meet $X \setminus B$, we obtain that $x\overline{\delta}(X \setminus B)$, i.e. that $x < B$. Thus the condition (P5) is fulfilled as well. Hence, δ is an R-proximity.

Further, the condition 2.5(ii) can be read now as follows: every element $\mathcal{F}$ of Σ is a rond filter in (X, δ). Therefore, $\alpha = (\delta, \Sigma)$ is an SR-proximity.

It remains to show that $\tau_\delta = \tau$, i.e. to prove that $cl_{(X,\tau)}(A) = \{x \in X : x\delta A\}$ for every $A \subseteq X$. Let $x \in cl_X(A)$. Then $\mathcal{N}(x)$ meets A. Hence $x\delta A$. Conversely, if $x\delta A$ then some element $\mathcal{F}$ of Σ meets both $\{x\}$ and A. Since X is a T_1-space, the note in 2.5 shows that $\mathcal{F} = \mathcal{N}(x)$. Hence $\mathcal{N}(x)$ meets A. Thus $x \in cl_X(A)$. So, we have shown that α is an SR-proximity on the space X. □

Theorem 3.8. *Let X be a regular space. Then the ordered sets $(\boldsymbol{R}(X), \leq_0)$ (resp. $(\boldsymbol{R}(X), \leq)$) and $(\boldsymbol{SRProx}(X), \leq_0)$ (resp. $(\boldsymbol{SRProx}(X), \leq)$) are isomorphic. The isomorphism between these ordered sets is constructed as follows: to every SR-proximity $\alpha = (\delta, \Sigma)$ on the space X corresponds the strict extension of X with filter trace Σ. (This extension will be denoted by $(r_\alpha X, r_\alpha)$.)*

Proof. Define a function $\varphi : \boldsymbol{SRProx}(X) \longrightarrow \boldsymbol{T}_3(X)$ by letting $\varphi(\delta, \Sigma) = \Sigma$ (see 2.5 and Definition 3.1 for the notations). Then the correctness of the definition of φ follows from Lemma 3.6, and Lemma 3.7 implies that φ is a bijection. Now Definitions 3.4 and 2.6 imply that φ is an isomorphism between the corresponding ordered sets. Applying Theorem 2.7, we complete the proof. □

Remark 3.9. Let's note that Theorem 3.8 could be proved without the help of Theorem 2.7 and the notion of T_3-system, but the proof then is much longer than the given one here.

We are now going to show that Theorem 3.8 implies Harris Theorem on regular-closed extensions [17, Theorem H]. For doing this we need (the first part of) the following lemma (note that Lemma 3.10(a) is only a slight generalization of Lemma 3 of [17]; let's note also that if X is a regular space then a regular filter $\mathcal{F}$ in X is a $\delta_w(X)$-end iff, for $A, B \subseteq X$, $cl_X(A) \subseteq \mathrm{Int}_X(B)$ implies that $(X \setminus A) \in \mathcal{F}$ or $B \in \mathcal{F}$):

Lemma 3.10. *Let (rX, r) be a regular extension of a regular space X and let δ be the R-proximity on X induced by (rX, r). Then:*

(a) a filter in X is a maximal round filter in (X, δ) iff it is the trace of a maximal regular filter in rX;

(b) the trace of every $\delta_w(rX)$-end in rX is a δ-end in X, but it is not true, in general, that any δ-end in X is a trace of a $\delta_w(rX)$-end in rX.

Proof. (a). In [17] D. Harris proved that a filter in X is a round filter in (X, δ) iff it is the trace of a regular filter in rX.

Let $\mathcal{F}$ be a maximal round filter in (X, δ). Then there exists a regular filter $\mathcal{F}'$ in rX whose trace is the filter $\mathcal{F}$. We will show that $\mathcal{F}'$ is a maximal regular filter in rX. Let $\mathcal{G}'$ be a regular filter in rX containing $\mathcal{F}'$. Then the trace $\mathcal{G}$ of $\mathcal{G}'$ on X is a round filter in (X,δ) containing $\mathcal{F}$. Hence $\mathcal{F}=\mathcal{G}$. Let $U'\in\mathcal{G}'$. Then there exists an open in rX set $V'\in\mathcal{G}'$ such that $cl_{rX}(V')\subseteq U'$. If V is the trace of V' on X then there exists an open in rX set $W'\in\mathcal{F}'$ whose trace on X is V. Since $cl_{rX}(W')=cl_{rX}(r(V))=cl_{rX}(V')\subseteq U'$, we obtain that $U'\in\mathcal{F}'$. Thus $\mathcal{G}'=\mathcal{F}'$, i.e. $\mathcal{F}'$ is a maximal regular filter in rX.

Conversely, let $\mathcal{F}'$ be a maximal regular filter in rX and $\mathcal{F}$ be its trace on X. Then $\mathcal{F}$ is a round filter in (X,δ). We will prove that $\mathcal{F}$ is a maximal round filter. Let $\mathcal{G}$ be a round filter in (X,δ) containing $\mathcal{F}$. Then there exists a regular filter $\mathcal{G}'$ in rX whose trace on X is $\mathcal{G}$. For proving that $\mathcal{G}=\mathcal{F}$ it is enough to show that $\mathcal{G}'\supseteq\mathcal{F}'$. So, let $U'\in\mathcal{F}'$. Then there exists an open in rX set $V'\in\mathcal{F}'$ such that $cl_{rX}(V')\subseteq U'$. Since the trace V of V' on X belongs to $\mathcal{F}$ and hence to $\mathcal{G}$, there exists an open in rX set $W'\in\mathcal{G}'$ whose trace on X is V. Then $cl_{rX}(W')=cl_{rX}(r(V))=cl_{rX}(V')\subseteq U'$. Thus $U'\in\mathcal{G}'$ and the proof of (a) is complete.

(*b*). Put $Y=rX$, for short. We may suppose without loss of generality that X is a subset of Y. Let $\mathcal{F}'$ be a $\delta_w(Y)$-end in Y and $\mathcal{F}$ be its trace on X. Then, by (a), $\mathcal{F}$ is a (maximal) round filter in (X,δ). Let $A,B\subseteq X$ and $A<B$. Then $A\overline{\delta}(X\setminus B)$ and hence $cl_Y(A)\cap cl_Y(X\setminus B)=\emptyset$. Thus $cl_Y(A)\subseteq Ex(B)$. Hence $cl_Y(A)\ \overline{\delta_w(Y)}\ (Y\setminus Ex(B))$. Since $\mathcal{F}'$ is a $\delta_w(Y)$-end in Y, we obtain that $(Y\setminus cl_Y(A))\in\mathcal{F}'$ or $Ex(B)\in\mathcal{F}'$. This implies that $(X\setminus cl_X(A))\in\mathcal{F}$ or $\mathrm{Int}_X(B)\in\mathcal{F}$. Hence, $(X\setminus A)\in\mathcal{F}$ or $B\in\mathcal{F}$. Therefore, $\mathcal{F}$ is an end in (X,δ).

For the second part of (b) see the last paragraph of the proof of Example 4.2 below. $\square$

Theorem 3.11. *Let X be a regular space, $\boldsymbol{RC}(X)$ be the set of all (up to equivalence) regular-closed extensions of X and $\boldsymbol{RCProx}(X)$ be the set of all SR-proximities α on the space X which have the form $\alpha=(\delta,\Sigma(\delta))$. Then the ordered sets $(\boldsymbol{RC}(X),\leq)$ and $(\boldsymbol{RCProx}(X),\leq)$ are isomorphic.*

P r o o f. It is well-known (see [3]) that a regular space X is regular-closed iff every maximal regular filter in X converges. Let's note also that the orders $\leq_0$ and $\leq$ coincide for the regular-closed extensions.

Let (rX,r) be a regular-closed extension of X, δ be the R-proximity on X induced by rX and Σ be the filter trace of (rX,r) on X. Then, by Theorem 3.8, $\alpha=(\delta,\Sigma)$ is the SR-proximity on X corresponding to the extension (rX,r). We know that $\Sigma\subseteq\Sigma(\delta)$ (see Proposition 3.3(d)). If $\mathcal{F}\in\Sigma(\delta)$ then, by Lemma 3.10(a), $\mathcal{F}$ is the trace of a maximal regular filter $\mathcal{F}'$ in rX. Since $\mathcal{F}'$ converges, we obtain that there exists an $y\in rX$ such that $\mathcal{F}'=\mathcal{N}(y)$. Thus $\mathcal{F}\in\Sigma$. So, $\Sigma=\Sigma(\delta)$.

Conversely, let $\alpha=(\delta,\Sigma(\delta))$ be an SR-proximity and let (rX,r) be the regular extension of X corresponding, by Theorem 3.8, to α. If $\mathcal{F}'$ is a maximal regular filter in rX then, by Lemma 3.10(a), its trace $\mathcal{F}$ on X is a maximal round filter in (X,δ), i.e. $\mathcal{F}\in\Sigma(\delta)$. Since, by its construction, (rX,r) is the strict extension of X with filter trace $\Sigma(\delta)$, there exists an $y\in rX$ such that $\mathcal{F}$ is the trace on X of the neighbourhood filter $\mathcal{N}(y)$ of y in rX. This implies that $\mathcal{F}'=\mathcal{N}(y)$, because every two distinct maximal regular filters in rX contain disjoint open members and, hence, they cannot have equal traces on X. Thus $\mathcal{F}'$ converges. This implies that rX is a regular-closed space.

Applying Theorem 3.8, we complete the proof. $\square$

Proposition 3.12. *Let (X,δ) be an R-proximity space. Then $\alpha=(\delta,\Sigma(\delta))$ is an SR-proximity iff δ is an RC-proximity.*

P r o o f. It is easy to see that the axiom (SR2) is equivalent to the following one:
(SR2$''$) $A<B$ iff every element $\mathcal{F}$ of Σ which meets A contains B.
Since the neighbourhood filters of the points of (X,τ_δ) are maximal round filters (see [17]), we have that $\alpha=(\delta,\Sigma(\delta))$ is an SR-proximity iff (SR2$''$) holds with $\Sigma=\Sigma(\delta)$. Thus $\alpha=(\delta,\Sigma(\delta))$ is an SR-proximity iff δ is an RC-proximity. $\square$

Remark 3.13. Theorem 3.11 and Proposition 3.12 imply Harris' result that there exists a bijection between the set of all RC-proximities on a regular space X and the set $\boldsymbol{RC}(X)$ [17, Theorem H]. In [17] D. Harris posed the question if comparable RC-proximities give rise to comparable regular-closed extensions. P. Sharma and S. Naimpally [30] settled this problem in negative. Ours Theorem 3.11 and Proposition 3.12 show that the "right" order on the set of all RC-proximities on a regular space X can be obtained through the interpretation of RC-proximities as special SR-proximities. Then an isomorphism (not only a bijection) between the corresponding ordered sets can be established. The example given in [30] demonstrates also that, in general, the converse implication in our Proposition 3.5(ii) doesn't hold, i.e. if $\alpha_i=(\delta_i,\Sigma_i),\ i=1,2$, are two SR-proximities on a space X then $\delta_1\le\delta_2$ does not imply, in general, that $\alpha_1\le_0\alpha_2$. Hence, by Proposition 3.5(i), in general, the converse implication in Proposition 3.3(c) doesn't hold as well. However, if $\delta_i,\ i=1,2$, are Efremovič proximities, then $\delta_1\le\delta_2$ iff $\alpha_1\le_0\alpha_2$ (see Proposition 5.5 below).

3.14. M. Lodato [23, 24] characterized those proximities δ on a space X which are induced on X by such spaces (resp. Hausdorff spaces) Y in which X is embedded as a regularly dense (resp. dense) subset, i.e., for $A,B\subseteq X$, $A\delta B$ iff $cl_Y(A)\cap cl_Y(B)\neq\emptyset$. (A subset Z of a space Y is *regularly dense in* Y if given U open in Y and p a point in U there exists a subset E of Z with $p\in cl_Y(E)\subseteq U$. "Regularly dense" implies "dense", and if Y is regular then the converse is also true.) We are now going to characterize those proximities δ on a regular space X which are induced by the regular extensions of X. It follows from the results of Harris [17] and Lodato [23] that any such proximity has to be simultaneously an R-proximity and a LO-proximity (i.e., a LR-proximity). However, this is not a characterization because, as it follows from the results of D. Harris (see [18, Theorem A]), there exist spaces X and LR-proximities on them which cannot be induced by any regular extension of X.

Definition 3.15. *Let X be a set and δ be a LR-proximity on X. δ is called a* LOR-proximity *if for every two subsets A and B of X, such that $A\delta B$, there exists an end in (X,δ) which does not contain the sets $X\setminus A$ and $X\setminus B$.*

Theorem 3.16. *Let (X,δ) be a basic proximity space. Then the following conditions are equivalent:*
(i) δ is a LOR-proximity;
(ii) $(\delta,\Sigma_{end}(\delta))$ is an SR-proximity;
(iii) There exists $\Sigma\subseteq\Sigma(\delta)$ such that (δ,Σ) is an SR-proximity;
(iv) The proximity δ is induced by a regular extension (rX,r) of (X,τ_δ).

Proof. $(i)\Rightarrow(ii)$. Let δ be a LOR-proximity, $A,B\subseteq X$, $\mathcal{F}\in\Sigma_{end}(\delta)$ and $\mathcal{F}$ does not contain the sets $X\setminus A$ and $X\setminus B$. Suppose that $A\overline{\delta}B$. Then $A<(X\setminus B)$. Since $\mathcal{F}$ is an end in (X,δ) and $X\setminus A\not\in\mathcal{F}$, we obtain that $(X\setminus B)\in\mathcal{F}$. This contradicts our assumption. Hence $A\delta B$. Thus, for $A,B\subseteq X$, the following holds: $A\delta B$ iff there exists an $\mathcal{F}\in\Sigma_{end}(\delta)$ which does not contain the sets $X\setminus A$ and $X\setminus B$. So, in order to show that $(\delta,\Sigma_{end}(\delta))$ is an SR-proximity, it remains to prove that every neighbourhood filter $\mathcal{N}(x)$ in (X,τ_δ) is an end in (X,δ). Since δ is an R-proximity, we obtain immediately from (P5) (see 2.4) that all $\mathcal{N}(x)$ are round filters. Further, let $x\in X$, $A,B\subseteq X$ and $A<B$. Then $A\overline{\delta}(X\setminus B)$. Since δ is a LO-proximity, we obtain that $cl_\delta(A)\ \overline{\delta}\ cl_\delta(X\setminus B)$. Hence $cl_\delta(A)<\mathrm{Int}(B)$. Thus $cl_\delta(A)\subseteq\mathrm{Int}(B)$. It is now easy to see that $(X\setminus A)\in\mathcal{N}(x)$ or $B\in\mathcal{N}(x)$. So, $(\delta,\Sigma_{end}(\delta))$ is an SR-proximity.

$(ii)\Rightarrow(iii)$. This is obvious.

$(iii)\Rightarrow(iv)$. Let $\alpha=(\delta,\Sigma)$ be an SR-proximity. Then, by Theorem 3.8, the strict extension $(r_\alpha X,r_\alpha)$ of (X,τ_δ) with filter trace Σ is a regular extension. Put, for short, $rX=r_\alpha X$ and $r=r_\alpha$. It is clear that if $A,B\subseteq X$ then $\mathcal{F}\in cl_{rX}(r(A))\cap cl_{rX}(r(B))$ (where $\mathcal{F}\in\Sigma=rX$) if and only if $\mathcal{F}$ meets both A and B. Hence the condition (SR2′) (see Proposition 3.3) implies that $A\delta B$ iff $cl_{rX}(r(A))\cap cl_{rX}(r(B))\neq\emptyset$.

$(iv)\Rightarrow(i)$. Let (rX,r) be a regular extension of (X,τ_δ) such that, for $A,B\subseteq X$, $A\delta B$ iff $cl_{rX}(r(A))\cap cl_{rX}(r(B))\neq\emptyset$. Then δ is a LO-proximity (see

[23]) and an R-proximity (see [17]). Let, for every $y \in rX$, $\mathcal{F}_y$ be the trace on X of the neighbourhood filter of y in the space rX. Obviously, $\mathcal{N}(x) = \mathcal{F}_{r(x)}$ for every $x \in X$. By [17], every $\mathcal{F}_y$, $y \in rX$, is a δ-round filter. Put $\Sigma = \{\mathcal{F}_y : y \in rX\}$. Then for $A, B \subseteq X$, we have: $(A\delta B)$ iff (there exists a point y of rX such that $y \in cl_{rX}(r(A)) \cap cl_{rX}(r(B))$) iff (there exists a filter $\mathcal{F} \in \Sigma$ which meets both A and B). Hence (SR2′) (see Proposition 3.3) is fulfilled. Thus $\alpha = (\delta, \Sigma)$ is an SR-proximity. Then, by Proposition 3.3(d), every element of Σ is an end in (X, δ). All this shows that δ is a LOR-proximity. □

Corollary 3.17. *Let X be a regular space and δ be a LOR-proximity on the space X. Then the following are equivalent:*
(a) δ is an RC-proximity;
(b) $\Sigma_{end}(\delta) = \Sigma(\delta)$.

Proof. $(a) \Rightarrow (b)$. This follows from Propositions 3.12, 3.3(d) and 2.4(∗).

$(b) \Rightarrow (a)$. Since δ is a LOR-proximity, Theorem 3.16 implies that $(\delta, \Sigma_{end}(\delta))$ is an SR-proximity on the space X. Hence $(\delta, \Sigma(\delta))$ is an SR-proximity on the space X. Thus, by Proposition 3.12, δ is an RC-proximity. □

The last theorem in this section is an immediate corollary of Theorems 3.8 and 2.8. For more general nearness-type theorems of this kind see [20, 4, 27].

Theorem 3.18. *Let (r_1X_1, r_1) and (r_2X_2, r_2), be regular extensions of the regular spaces X_1 and X_2 respectively, α_i be the SR-proximities on X_i, $i = 1, 2$, corresponding to these extensions (see Theorem 3.8) and $f : X_1 \longrightarrow X_2$ be a continuous function. Then the following conditions are equivalent:*
(i) There exists a continuous function $F : r_1X_1 \longrightarrow r_2X_2$ such that $F \circ r_1 = r_2 \circ f$;
(ii) $f : (X_1, \alpha_1) \longrightarrow (X_2, \alpha_2)$ is an SR-proximally continuous function.

Corollary 3.19. *Let (X, δ) be an RC-proximity space, Y be a regular-closed space and $f : (X, \tau_\delta) \longrightarrow Y$ be a continuous function. Let (rX, r) be the regular-closed extension of (X, τ_δ) corresponding to the RC-proximity δ (see [17, Theorem H] or Theorem 3.11 and Proposition 3.12 here). Then the following conditions are equivalent:*
(i) There exists a continuous function $F : rX \longrightarrow Y$ such that $F \circ r = f$;
(ii) If $\mathcal{F}$ is a maximal round filter in (X, δ) then the filter-base $f(\mathcal{F})$ converges.

Note that with Corollary 3.19 we obtain, in fact, the solution to Problem II of D. Harris [17], given by W. Hunsaker and P. Sharma (see [27]).

Corollary 3.20. *Let (X, α), where $\alpha = (\delta, \Sigma)$, be an SR-proximity space, Y be a compact Hausdorff space and $f : (X, \tau_\delta) \longrightarrow Y$ be a continuous function. Let (rX, r) be the regular extension of (X, τ_δ) corresponding to the SR-proximity α (see Theorem 3.8). Then the following conditions are equivalent:*

(i) There exists a continuous function $F : rX \longrightarrow Y$ *such that* $F \circ r = f$*;*
(ii) $f : (X, \delta) \longrightarrow (Y, \delta_w)$ *is a proximally continuous function.*

Proof. Using the compactness of the space Y, we infer easily our assertion from Theorem 3.18. □

Note that our Corollary 3.20 is, actually, a special case of the well-known Taĭmanov Theorem (see [16, Theorem 3.2.1]).

4. CE-regular extensions.

Definition 4.1 (E. Čech, J. Novák [9]). *Let* (rX, r) *be an extension of a space* X*. Then* X *is said to be* c-embedded *in* rX *if, for* $A, B \subseteq X$, $cl_X(A) \cap cl_X(B) = \emptyset$ *implies* $cl_{rX}(r(A)) \cap cl_{rX}(r(B)) = \emptyset$*. (We have to note that our "c-embedding" is introduced in* [9] *as "2-combinatorial embedding".)*

Theorem 4.2. *Let* X *be a regular space. Then there exists a regular extension* $(\kappa X, \kappa)$ *of* X *such that:*
(a) X *is c-embedded in* $(\kappa X, \kappa)$*;*
(b) $(\kappa X, \kappa)$ *is the largest element of the injectively ordered set of all (up to equivalence) regular extensions of* X *in which* X *is c-embedded;*
(c) $\kappa\kappa X$ *is isomorphic to* κX*.*

Proof. Theorem 3.16 implies that the Wallman R-proximity δ_w (see Example 3.2) is a LOR-proximity, because it is induced by the trivial extension (X, id_X) of X. Hence, by Theorem 3.16, $\alpha_\kappa = (\delta_w, \Sigma_{end}(\delta_w))$ is an SR-proximity. Let $(\kappa X, \kappa)$ be the regular extension of X corresponding to α_κ (see Theorem 3.8), i.e. $(\kappa X, \kappa)$ is the strict extension of X with filter trace $\Sigma_{end}(\delta_w)$. We will show that $(\kappa X, \kappa)$ is the desired extension.

(a). This follows from the fact that the induced by $(\kappa X, \kappa)$ R-proximity on X is precisely the Wallman R-proximity δ_w (see Theorem 3.8).

(b). Let (rX, r) be a regular extension of X in which X is c-embedded. Then, obviously, the induced by (rX, r) R-proximity δ on X coincides with the Wallman R-proximity δ_w. Let Σ be the filter trace of the extension (rX, r). Then, by Theorem 3.8, (δ_w, Σ) is an SR-proximity. Thus Proposition 3.3(d) implies that $\Sigma \subseteq \Sigma_{end}(\delta_w)$. Since, by Theorem 3.8, the extension (rX, r) is isomorphic to the strict extension of X with filter trace Σ, we obtain easily that $(\kappa X, \kappa)$ is injectively larger than (rX, r).

(c). We have, by (a), that $\kappa\kappa X$ is an extension of X in which X is c-embedded. Now, using (b), we complete the proof. □

Definition 4.3. *(a) A regular space is called* CE-regular *if it has no proper regular extension in which it is c-embedded. An extension* (rX, r) *of a* *space X is called* CE-regular extension *if rX is a CE-regular space.*

(*b*) *Let δ_w be the Wallman proximity on a regular space X. Then the elements of the set $\Sigma_{end}(\delta_w)$ will be called* regular ends on X.

Proposition 4.4. *Let X be a regular space. Then the following are equivalent:*
(*a*) *X is a CE-regular space;*
(*b*) *$\kappa X = X$;*
(*c*) *Every regular end on X converges.*

P r o o f. Obvious. □

4.5. We have to recall now some definitions from [29]. Let (Y, e) be an extension of a space X. We say that the space X is *open combinatorially embedded in* (Y, e) if, for open subsets U and V of X, $cl_X(U) \cap cl_X(V) = \emptyset$ implies $cl_Y(e(U)) \cap cl_Y(e(V)) = \emptyset$. A regular space X is called *OCE-regular* if X has no proper regular extension in which X is open combinatorially embedded. An extension (Y, e) of a space X is called *OCE-regular extension of* X if Y is an OCE-regular space.

Remark 4.6. Theorem 4.2(c) and Proposition 4.4 imply that if X is a regular space then the extension $(\kappa X, \kappa)$ is a CE-regular extension. Hence, every regular space has a CE-regular extension. The regular-closed extensions and the OCE-regular extensions are, obviously, CE-regular extensions, but, as we will see below, in general the converse doesn't hold.

Theorem 4.7. *Let X be a regular space. Then for every LOR-proximity δ on the space X there exists a CE-regular extension $(\kappa_\delta X, \kappa_\delta)$ of X which is the largest element of the injectively ordered set of all (up to equivalence) regular extensions of X inducing the proximity δ on X.*

P r o o f. Let δ be a LOR-proximity on the space X. Then, by Theorem 3.16, $\alpha = (\delta, \Sigma_{end}(\delta))$ is an SR-proximity. Let $(\kappa_\delta X, \kappa_\delta)$ be the regular extension of X corresponding to α (see Theorem 3.8), i.e. $(\kappa_\delta X, \kappa_\delta)$ is the strict extension of X with filter trace $\Sigma_{end}(\delta)$. We will show that $(\kappa_\delta X, \kappa_\delta)$ is the desired extension. Indeed, if a regular extension (rX, r) of X induces the proximity δ then, by Theorem 3.8, its corresponding SR-proximity is of the form (δ, Σ) and (rX, r) is isomorphic to the strict extension of X with filter trace Σ. Since, by Proposition 3.3(d), $\Sigma \subseteq \Sigma_{end}(\delta)$, we obtain that $(\kappa_\delta X, \kappa_\delta)$ is injectively larger than (rX, r). Further, if (Y, e) is a regular extension of $\kappa_\delta X$ in which $\kappa_\delta X$ is c-embedded, then, obviously, $(Y, e \circ \kappa_\delta)$ will be a regular extension of X inducing the proximity δ on X. Thus, as we have already proved, $(\kappa_\delta X, \kappa_\delta)$ will be injectively larger than $(Y, e \circ \kappa_\delta)$. Since, evidently, the converse also holds, we obtain that $(Y, e \circ \kappa_\delta)$

and $(\kappa_\delta X, \kappa_\delta)$ are isomorphic extensions of X. Hence (Y, e) is the trivial regular extension of $\kappa_\delta X$. Therefore, $(\kappa_\delta X, \kappa_\delta)$ is a CE-regular extension of X. □

4.8. Recall that a regular space is called: (a) *RC-regular* (D. Harris [17]) if it has a regular-closed extension, and (b) *RC-normal* (D. Harris [18]) if no maximal regular filter meets both of two disjoint closed sets. It is proved in [18] that: (i) a space X is RC-normal iff the Wallman proximity on X is an RC-proximity, and (ii) the Tychonoff plank is an RC-regular but not RC-normal space.

Proposition 4.9. *Let X be a regular space. Then:*

(a) *κX is a subspace of αX and $(\alpha X, \alpha)$ is injectively larger than $(\kappa X, \kappa)$.*

(b) *The following are equivalent:*

(i) *$(\kappa X, \kappa)$ is isomorphic to the Alexandroff extension $(\alpha X, \alpha)$ (see 2.3. for αX);*

(ii) *αX is a regular space and X is c-embedded in αX;*

(iii) *κX is a regular-closed space;*

(iv) *The Wallman proximity δ_w on X is an RC-proximity;*

(v) *X is RC-normal.*

Proof. (a). Obviously, a filter in X is a δ_w-round filter iff it is a regular filter. Then the constructions of κX and αX together with $(*)$ (see 2.4) imply our assertion.

(b). (i) $\Rightarrow$ (ii). This follows from Theorem 4.2.

(ii) $\Rightarrow$ (i). Since X is c-embedded in αX, Theorem 4.2(b) implies that $(\kappa X, \kappa)$ is injectively larger than $(\alpha X, \alpha)$. Now, using (a), we complete the proof.

(i) $\Rightarrow$ (iii). This folows from the P. S. Alexandroff's result (see [1]) that αX is regular-closed if it is regular. A proof, based only on the facts presented here, can be easily obtained as well. Indeed, since $\Sigma_{end}(\delta_w) = \Sigma(\delta_w)$, Corollary 3.17 implies that δ_w is an RC-proximity. Hence, applying Harris Theorem [17, Theorem H] or ours Proposition 3.12 and Theorem 3.11, we obtain that κX is regular-closed.

(iii) $\Rightarrow$ (i). Since κX is regular-closed, Theorem 3.11 implies that $(\delta_w, \Sigma(\delta_w))$ is an SR-proximity. Hence, by Proposition 3.3(d), $\Sigma(\delta_w) = \Sigma_{end}(\delta_w)$. Therefore, the constructions of αX and κX show that $(\alpha X, \alpha)$ and $(\kappa X, \kappa)$ are equivalent extensions of X.

(iv) $\Rightarrow$ (i). If δ_w is an RC-proximity then, by Proposition 3.12, $(\delta_w, \Sigma(\delta_w))$ is an SR-proximity. Hence, by Proposition 3.3(d), $\Sigma(\delta_w) = \Sigma_{end}(\delta_w)$ and, as above, we obtain that $(\alpha X, \alpha)$ and $(\kappa X, \kappa)$ are equivalent extensions of X.

(iii) $\Rightarrow$ (iv). Since (iii) implies that $\Sigma(\delta_w) = \Sigma_{end}(\delta_w)$ (see the proof of (iii) $\Rightarrow$ (i)), we obtain, by Corollary 3.17, that δ_w is an RC-proximity.

(iv) $\iff$ (v). This was proved by D. Harris (see 4.8(i)). □

Proposition 4.10. *If X is a completely regular space then $(\kappa X, \kappa)$ is equivalent to a compactification (cX, c) of X iff X is a normal space and (cX, c) is equivalent to the Stone-Čech compactification $(\beta X, \beta)$ of X.*

Proof. If $(\kappa X, \kappa)$ is isomorphic to a compactification (cX, c) of X then X is c-embedded in cX. This implies (see e.g. [16, Cor. 3.6.4]) that X is normal and (cX, c) is equivalent to $(\beta X, \beta)$. Conversely, if X is normal then X is c-embedded in $(\beta X, \beta)$. Hence $(\beta X, \beta)$ induces the Wallman proximity δ_w on X. Now the compactness of βX and Theorem 4.7 (or Theorem 4.2) imply that $(\kappa X, \kappa)$ is equivalent to $(\beta X, \beta)$. □

Example 4.11. (a). There exists a regular space X such that no one of its CE-regular extensions is regular-closed.

(b). There exists a completely regular space X with $(\kappa X, \kappa)$ not regular-closed.

Proof. (*a*). In [19, Beispiel 8], H. Herrlich constructed a regular space R which has no regular-closed extensions. By Remark 4.6 or Theorem 4.7, every regular space X has a CE-regular extension, e.g., $(\kappa X, \kappa)$ is such. Hence, the space R has at least one CE-regular extension and any such extension is not regular-closed.

(*b*). It follows from 4.8 and Proposition 4.9(b) that the Tychonoff plank is such an example. Another one is the constructed by H. Tong [34] completely regular space X such that the Alexandroff extension $(\alpha X, \alpha)$ (see [1]) of X is not regular. Indeed, then Proposition 4.9(b) implies that the Wallman proximity δ_w on X is not an RC-proximity. Hence, by Proposition 4.9(b), we obtain that κX is not regular-closed. Yet another example is the space Y from Example 4.12 below. □

Example 4.12. (a) There exists a normal space X with non-isomorphic CE-regular extensions inducing equal R-proximities on X.

(b) There exists a normal space X which has a CE-regular non-OCE-regular extension.

Proof. Let x be a point of $\beta\boldsymbol{N} \setminus \boldsymbol{N}$ such that the space $(\beta\boldsymbol{N} \setminus \boldsymbol{N}) \setminus \{x\}$ is not normal. (Such a point x exists (see [6]) and this is a theorem in ZFC. Under CH or MA all points of $\beta\boldsymbol{N} \setminus \boldsymbol{N}$ have this property (see, e.g., [35, 25])). Put $Y = \beta\boldsymbol{N} \setminus \{x\}$. Then Y is a nonnormal space and $\beta Y = \beta\boldsymbol{N}$. Thus κY is not equivalent (as an extension of Y) to $\beta\boldsymbol{N}$ (see Proposition 4.10). Put $X = \boldsymbol{N}$. We have, by Proposition 4.10, that $\kappa X = \beta X$. Since, obviously, βX, Y and κY are regular extensions of X which induce the Wallman proximity $\delta_w(X)$ on X, Theorem 4.7 implies that βX is injectively larger than κY (both regarded as extensions of X). Thus $\kappa Y = Y$. Hence Y is a CE-regular space. So, Y and

βX are two non-equivalent CE-regular extensions of X which both induce the Wallman proximity δ_w on X. Hence, (a) is proved. Since, obviously, βY is a proper extension of Y in which Y is open combinatorially embedded (because X is c-embedded in βY), we obtain that Y is not an OCE-regular space. Hence, Y is a non-OCE-regular CE-regular extension of X. Thus (b) is also proved.

We can now complete the proof of Lemma 3.10(b). Indeed, in the above-used notations, let $\mathcal{F}$ be the trace on X of the neighbourhood filter $\mathcal{N}_{\beta X}(x)$ of the point x in βX. Then $\mathcal{F}$ is, obviously, a $\delta_w(X)$-end in X and $\mathcal{F}$ doesn't belong to the filter trace of Y on X (because the traces on X of $\mathcal{N}_Y(y)$ and $\mathcal{N}_{\beta X}(y)$ coincide for every $y \in Y$). Since $\kappa Y = Y$, every $\delta_w(Y)$-end in Y is of the form $\mathcal{N}_Y(y)$ for some $y \in Y$ and, hence, its trace is different from $\mathcal{F}$. This completes the proof of Lemma 3.10(b). □

5. Completely regular, compact and locally compact extensions.

Definition 5.1. *Let (X, δ) be an R-proximity space. A filter $\mathcal{F}$ in X is called* CR-filter *in (X, δ) if for every $F \in \mathcal{F}$ there exist an element G of $\mathcal{F}$ and a proximally continuous function $f : (X, \delta) \longrightarrow (\boldsymbol{I}, \delta_w)$ such that $f(G) = 0$ and $f(X \setminus F) = 1$. The set of all CR-filters in (X, δ) will be denoted by $CRF(X, \delta)$.*

Theorem 5.2. *Let X be a completely regular space and*

$$\boldsymbol{CRProx}(X) = \{(\delta, \Sigma) \in \boldsymbol{SRProx}(X) : \Sigma \subseteq CRF(X, \delta)\}.$$

Then the ordered sets $(\boldsymbol{CR}(X), \leq_0)$ (resp. $(\boldsymbol{CR}(X), \leq)$) and $(\boldsymbol{CRProx}(X), \leq_0)$ (resp. $(\boldsymbol{CRProx}(X), \leq)$) are isomorphic.

P r o o f. Let (cX, c) be a completely regular extension of X and $\alpha = (\delta, \Sigma)$ be the SR-proximity on X corresponding to it (see Theorem 3.8). We have to show that $\Sigma \subseteq CRF(X, \delta)$. Let $\mathcal{F} \in \Sigma$ and $F \in \mathcal{F}$. Then there exists a point $y \in cX$ such that $\mathcal{F}$ is the trace on X of the neighbourhood filter $\mathcal{N}_{cX}(y)$ of y in cX. Hence there exists an $F' \in \mathcal{N}_{cX}(y)$ whose trace on X is F. Since cX is a completely regular space, there exist an open in cX set $G' \in \mathcal{N}_{cX}(y)$ and a continuous function $f' : cX \longrightarrow \boldsymbol{I}$ such that $f'(G') = 0$ and $f'(cX \setminus F') = 1$. Let G be the trace of G' on X and $f : X \longrightarrow \boldsymbol{I}$ be the "restriction" of f' on X (i.e. $f = f' \circ c$). Then $G \in \mathcal{F}$, $f : (X, \delta) \longrightarrow (\boldsymbol{I}, \delta_w)$ is a proximally continuous function and $f(G) = 0$, $f(X \setminus F) = 1$. Hence $\mathcal{F}$ is a CR-filter on (X, δ).

Conversely, let $\alpha = (\delta, \Sigma) \in \boldsymbol{CRProx}(X)$ and (cX, c) be the regular extension of X corresponding to α (see Theorem 3.8). We have to show that cX is a completely regular space. Let $y \in cX$. Then, by Theorem 3.8, there exists an $\mathcal{F}_y \in \Sigma$ which is the trace on X of the neighbourhood filter $\mathcal{N}_{cX}(y)$ of y in cX.

Let $U' \in \mathcal{N}_{cX}(y)$. There exists a $V' \in \mathcal{N}_{cX}(y)$ such that $cl_{cX}(V') \subseteq U'$. We have that the trace V on X of V' is an element of $\mathcal{F}_y$. Since $\mathcal{F}_y \in \Sigma$ (and hence it is a CR-filter in (X, δ)), there exist a $G \in \mathcal{F}_y$ and a proximally continuous function $f : (X, \delta) \longrightarrow (\boldsymbol{I}, \delta_w)$ such that $f(G) = 0$ and $f(X \setminus V) = 1$. Then, by Corollary 3.20, there exists a continuous function $f' : cX \longrightarrow \boldsymbol{I}$ such that $f' \circ c = f$. Obviously, the function f' separates y and U'. Hence, cX is completely regular.

It is clear now that the two paragraphs above together with Theorem 3.8 imply our assertion. □

Remark 5.3. Some non-proximity-type descriptions of the completely regular extensions of completely regular spaces were given by G. Dimov [10] (by using special families of open filters, called *CR-systems* (i.e. in the spirit of Doitchinov's description of the regular extensions)) and by H. Bentley, H. Herrlich and R. Ori [5] (by means of the nearness structures).

The next proposition is some variant of the Urysohn Lemma. We will use it for obtaining a new proof of the celebrated Smirnov Compactification Theorem. It will be based only on our results presented here.

Proposition 5.4. *Let (X, δ) be an R-proximity space and D be the set of all dyadic numbers in $\boldsymbol{I}$. A filter $\mathcal{F}$ in X is a CR-filter in (X, δ) iff for every $F \in \mathcal{F}$ there exists a family $\mathcal{G}(F) = \{G_d \in \mathcal{F} : d \in D\}$ such that $G_1 = F$ and $G_d < G_e$ if $d, e \in D$ and $d < e$.*

P r o o f. The necessity is clear. For the sufficiency, let $\mathcal{F}$ be a filter in X, $F \in \mathcal{F}$ and $\mathcal{G}(F) = \{G_d \in \mathcal{F} : d \in D\}$ be such that $G_1 = F$ and $G_d < G_e$ if $d, e \in D$ and $d < e$. We define a function $f : X \longrightarrow \boldsymbol{I}$ as follows: we put $f(x) = 1$ for every $x \in X \setminus F$ and, for $x \in F$, we set $f(x) = \inf\{d \in D : x \in G_d\}$. Then, letting $G = G_0$, we obtain that $f(G) = 0$. So, for proving that $\mathcal{F}$ is a CR-filter in (X, δ), it remains to show that $f : (X, \delta) \longrightarrow (\boldsymbol{I}, \delta_w)$ is proximally continuous. Let us examine first the following case: $A, B \subseteq F$, $A \delta B$ and $A \cap B = \emptyset$. Suppose that $cl(f(A)) \cap cl(f(B) = \emptyset$. Then there exist open sets $U = \bigcup\{(a_i, b_i) : i = 1, \ldots, n\}$ and $V = \bigcup\{(c_j, d_j) : j = 1, \ldots, m\}$ (where by (a, b) we mean the open interval (a, b) in $\boldsymbol{I}$ with $a < b$) such that $cl(U) \cap cl(V) = \emptyset$, $b_i < a_{i+1}$, $d_j < c_{j+1}$, $cl(f(A)) \subseteq U$, $cl(f(B)) \subseteq V$ and $(a_i, b_i) \cap f(A) \neq \emptyset$, $(c_j, d_j) \cap f(B) \neq \emptyset$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Put $A_i = A \cap f^{-1}((a_i, b_i))$ and $B_j = B \cap f^{-1}((c_j, d_j))$. Since $A \delta B$ and $A = \bigcup\{A_i : i = 1, \ldots, n\}$, $B = \bigcup\{B_j : j = 1, \ldots, m\}$, there exist i and j such that $A_i \delta B_j$. Since $cl(U) \cap cl(V) = \emptyset$, we have that either $b_i < c_j$ or $d_j < a_i$. Let, e.g., $b_i < c_j$. There exist $p, q \in D$ such that $b_i < p < q < c_j$. Then $A_i \subseteq G_p$ and $B_j \subseteq X \setminus G_q$. Since $G_p < G_q$, we obtain that $A_i \overline{\delta} B_j$. This contradicts our assumption. So, $f(A)\ \delta_w\ f(B)$. In all other possible cases for A and B we come easily to the same conclusion. Hence, f is proximally continuous. $\square$

Proposition 5.5. *Let X be a completely regular space. Then:*
(a) $\alpha = (\delta, \Sigma(\delta)) \in \boldsymbol{RCProx}(X) \cap \boldsymbol{CRProx}(X)$ iff δ is an Efremovič proximity;
(b) If $\alpha_i = (\delta_i, \Sigma(\delta_i)) \in \boldsymbol{RCProx}(X) \cap \boldsymbol{CRProx}(X)$, $i = 1, 2$, then $\alpha_1 \leq_0 \alpha_2$ iff $\delta_1 \leq \delta_2$.

Proof. (a). Let $\alpha = (\delta, \Sigma(\delta)) \in \boldsymbol{RCProx}(X) \cap \boldsymbol{CRProx}(X)$ and let (cX, c) be the regular extension of X corresponding to α (see Theorem 3.8). Then, by Theorems 3.11 and 5.2, cX is a completely regular regular-closed space. Hence cX is a compact Hausdorff space. Since δ is induced by (cX, c) (see Theorem 3.8), the normality of cX implies that δ satisfies the axiom (EF) (see 2.4). Hence, δ is an Efremovič proximity.

Conversely, let δ be an Efremovič proximity. Then, by Theorem 6.9 from [28], $\Sigma(\delta) = \Sigma_{end}(\delta)$. Further, Theorem 6.15 from [28] together with our Corollary 3.17 imply that δ is an RC-proximity (for another proof of this fact see [18, 5.1–5.3]). Hence $\alpha = (\delta, \Sigma(\delta))$ is an SR-proximity (see Proposition 3.12). Let $\mathcal{F}$ be a round filter in (X, δ). Then the axiom (EF) and Proposition 5.4 imply that $\mathcal{F}$ is a CR-filter in (X, δ). Hence $\Sigma(\delta) \subseteq CRF(X, \delta)$. So, $\alpha \in \boldsymbol{RCProx}(X) \cap \boldsymbol{CRProx}(X)$.

(b). By Proposition 3.5, we have that $\alpha_1 \leq_0 \alpha_2$ implies $\delta_1 \leq \delta_2$. So, it remains to prove the converse implication. Let $\delta_1 \leq \delta_2$ and let $\mathcal{F} \in \Sigma(\delta_2)$. Then there exists an ultrafilter $\mathcal{G}$ in X containing $\mathcal{F}$. Put $\mathcal{G}^0 = \{E \subseteq X :$ there exists a $G \in \mathcal{G}$ such that $G <_1 E\}$ (where $G <_1 E$ means that $G\overline{\delta}_1(X \setminus E)$). Then, by Theorem 6.14 from [28], $\mathcal{G}^0$ is an end in (X, δ_1). Hence $\mathcal{G}^0 \in \Sigma(\delta_1)$. We will show that $\mathcal{G}^0 \subseteq \mathcal{F}$. Indeed, if $E \in \mathcal{G}^0$ then there exists a $G \in \mathcal{G}$ such that $G <_1 E$. Now, the axiom (EF) implies that there exists a sequence $(E_i)_{i\in\omega}$ of subsets of X such that $E_1 = E$ and $E_{i+1} <_1 E_i$, $G <_1 E_i$, for $i \in \omega$. Then $E_i \in \mathcal{G}$ and $E_{i+1} <_2 E_i$, for $i \in \omega$. Hence $\mathcal{F}$ meets any E_i, $i \in \omega$, and $\mathcal{F} \cup \{E_i : i \in \omega\}$ is a filter-base of a round filter $\mathcal{F}'$ in (X, δ_2). Now the maximality of $\mathcal{F}$ implies that $\mathcal{F}' = \mathcal{F}$. Hence $E = E_1 \in \mathcal{F}$. So, $\mathcal{G}^0 \subseteq \mathcal{F}$. Thus $\alpha_1 \leq_0 \alpha_2$. $\square$

Theorem 5.6 (J. M. Smirnov [31])**.** *Let X be a completely regular space. Denote by $\boldsymbol{EFProx}(X)$ the set of all EF-proximities on the space X. Then the ordered sets $(\boldsymbol{C}(X), \leq)$ and $(\boldsymbol{EFProx}(X), \leq)$ are isomorphic.*

Proof. Since a Hausdorff space is compact iff it is regular-closed and completely regular, the theorem follows from the Theorems 5.2, 3.11 and Proposition 5.5. $\square$

Remark 5.7. Let's note that we give not only a new proof of the celebrated Smirnov Compactification Theorem, but, as Proposition 5.5 shows, our investigations lead automatically to the description of the class of those proximities which correspond to the compact extensions. Hence, we arrive in a natural way even to the *notion* of Efremovič proximity and to the *formulation* of the Smirnov Compactification Theorem.

Definition 5.8. *An SR-proximity $\alpha = (\delta, \Sigma)$ on a set X is called a* LC-proximity *if for every $\mathcal{F} \in \Sigma$ there exists an $U \in \mathcal{F}$ with the following two properties:*
(i) the restriction δ_U of δ to U is an EF-proximity;
(ii) if $\mathcal{G} \in \Sigma_{end}(\delta)$ and $U \in \mathcal{G}$ then $\mathcal{G} \in \Sigma$.

Theorem 5.9. *Let X be a completely regular space and $\boldsymbol{LCProx}(X)$ be the set of all LC-proximities on the space X. Then the ordered sets $(\boldsymbol{LC}(X), \leq_0)$ (resp. $(\boldsymbol{LC}(X), \leq)$) and $(\boldsymbol{LCProx}(X), \leq_0)$ (resp. $(\boldsymbol{LCProx}(X), \leq)$) are isomorphic.*

Proof. Let (eX, e) be a Hausdorff locally compact extension of X and $\alpha = (\delta, \Sigma)$ be the SR-proximity on X corresponding to (eX, e) (see Theorem 3.8). We will show that $\alpha \in \boldsymbol{LCProx}(X)$. Let $\mathcal{F} \in \Sigma$. Then there is an $y \in eX$ such that $\mathcal{F}$ is the trace on X of the neighbourhood filter $\mathcal{N}_{eX}(y)$ of y in eX. Since eX is locally compact, there exists an open $U' \in \mathcal{N}_{eX}(y)$ having a compact closure in eX. Let U be the trace of U' on X. Then $U \in \mathcal{F}$. We will show that U satisfies the conditions (i) and (ii) from Definition 5.8. Indeed, since $bU = cl_{eX}(e(U)) = cl_{eX}(U')$ is compact, Theorem 5.6 implies that the restriction δ_U of δ to U is an EF-proximity. So, U has the property (i). For (ii), note that, by Theorem 3.16, $\beta = (\delta, \Sigma_{end}(\delta))$ is an SR-proximity. Let (cX, c) be the regular extension of X corresponding to β (see Theorem 3.8). Then, by Theorem 4.7 and its proof, (cX, c) is injectively larger than (eX, e). Hence we can assume, without loss of generality (= w.l.o.g.), that $X \subseteq eX \subseteq cX = \Sigma_{end}(\delta)$. We now have that $cl_{cX}(Ex_{cX}(U)) = cl_{cX}(U) = cl_{cX}(cl_{eX}(U)) = cl_{cX}(cl_{eX}(U')) = cl_{eX}(U') \subseteq eX$. Hence $\{\mathcal{G} \in \Sigma_{end}(\delta) : U \in \mathcal{G}\} = Ex_{cX}(U) \subseteq eX = \Sigma$. Thus U satisfies (ii). Therefore, $\alpha \in \boldsymbol{LCProx}(X)$.

Conversely, let $\alpha = (\delta, \Sigma) \in \boldsymbol{LCProx}(X)$ and (eX, e) be the regular extension of X corresponding to α (see Theorem 3.8). We will show that the space eX is locally compact. Indeed, let $y \in eX$ and $\mathcal{F} \in \Sigma$ be the trace on X of the neighbourhood filter $\mathcal{N}_{eX}(y)$ of y in eX. Then there exists an $U \in \mathcal{F}$ satisfying the conditions (i) and (ii) from Definition 5.8. Since the filter $\mathcal{F}$ is round, there exist $V, W \in \mathcal{F}$ such that $W < V < U$. Put $C = cl_{eX}(e(W))$. We will show that C is a compact neighbourhood of y in eX. By (i), the proximity δ_U induced by δ on the subset U of X is an EF-proximity. Let (cU, c) be the

Smirnov compactification of U corresponding to δ_U (see Theorem 5.6). Put $C' = cl_{cU}(c(W))$. We will construct a homeomorphism between C and C'. Obviously, we can assume, w.l.o.g., that $eX = \Sigma$, $C = \{\mathcal{F} \in \Sigma : \mathcal{F} \text{ meets } W\}$, $cU = \Sigma(\delta_U)$, $C' = \{\mathcal{G} \in \Sigma(\delta_U) : \mathcal{G} \text{ meets } W\}$. Recall also that eX and cU are the strict extensions of X and U with filter traces Σ and $\Sigma(\delta_U)$ respectively. It is easy to see that if $\mathcal{F} \in C$ then $\mathcal{F} \cap U$ is an end in (U, δ_U). Indeed, it is obvious that $\mathcal{F} \cap U$ is a round filter in (U, δ_U). Further, if $A, B \subseteq U$ and $A <_U B$ (i.e. $A\ \overline{\delta_U}\ (U \setminus B)$) then $A\ \overline{\delta}\ (U \setminus B)$ and, hence, $(U \setminus B) < (X \setminus A)$. Since $\mathcal{F}$ is a δ-end, we obtain that $((X \setminus U) \cup B) \in \mathcal{F}$ or $(X \setminus A) \in \mathcal{F}$. Thus, $(U \setminus A) \in \mathcal{F} \cap U$ or $B \in \mathcal{F} \cap U$. Therefore, $\mathcal{F} \cap U \in \Sigma(\delta_U) = cU$. So, letting $\varphi(\mathcal{F}) = \mathcal{F} \cap U$, for $\mathcal{F} \in C$, we define a function from C to cU. Since distinct elements of C contain disjoint open members, we obtain immediately that φ is an injection. Obviously, $\varphi(C) \subseteq C'$. For showing that $\varphi(C) = C'$, take a $\mathcal{G} \in C'$. Evidently, $C' = cl_{cU}(c(W)) = cl_{cU}(W_c^*) \subseteq V_c^*$ (where $A_c^* = \{\mathcal{G}' \in cU : A \in \mathcal{G}'\}$, for any $A \subseteq U$). Hence $V \in \mathcal{G}$. We now obtain easily that $\mathcal{G}$ is a round filter-base in (X, δ). Indeed, let $F \in \mathcal{G}$. Then $F' = F \cap V \in \mathcal{G}$. Hence there exists a $G \in \mathcal{G}$ such that $G <_U F'$ (i.e. $G\ \overline{\delta}\ (U \setminus F')$). Obviously, $G \subseteq V$. Since $V < U$, we obtain that $G < U$. So, $G\ \overline{\delta}\ (X \setminus U)$. This implies that $G\ \overline{\delta}\ ((X \setminus U) \cup (U \setminus F'))$. Hence $G\ \overline{\delta}\ (X \setminus F')$, i.e. $G < F'$. Therefore, $G < F$. So, $\mathcal{G}$ is a round filter-base in (X, δ). Let $\mathcal{G}'$ be the filter in X with filter-base $\mathcal{G}$. Then $\mathcal{G}' \in \Sigma_{end}(\delta)$. Indeed, let $A, B \subseteq X$ and $A < B$. Suppose $(X \setminus A) \notin \mathcal{G}'$, i.e. $\mathcal{G}'$ meets A. We have to show that $B \in \mathcal{G}'$. Put $A' = A \cap V$ and $B' = B \cap U$. Now, $V < U$ and $A < B$ imply that $A' < B'$ and, hence, $A'\ \overline{\delta_U}\ (U \setminus B')$. Then $(U \setminus A') \in \mathcal{G}$ or $B' \in \mathcal{G}$, because $\mathcal{G}$ is an end in (U, δ_U). If $(U \setminus A') \in \mathcal{G}$ then $(V \cap (U \setminus A')) \in \mathcal{G}$, i.e. $(V \setminus A) \in \mathcal{G}$, and we obtain that $(X \setminus A) \in \mathcal{G}'$. This contradiction shows that $(U \setminus A') \notin \mathcal{G}$. Thus $B' \in \mathcal{G}$, which implies that $B \in \mathcal{G}'$. So, $\mathcal{G}'$ is an end in (X, δ). Since $\mathcal{G} \subseteq \mathcal{G}'$ and $U \in \mathcal{G}$, we get that $U \in \mathcal{G}'$. Then, by (ii), $\mathcal{G}' \in \Sigma$. Hence $\mathcal{G}' \in C$. Obviously, $\mathcal{G}' \cap U = \mathcal{G}$, i.e. $\varphi(\mathcal{G}') = \mathcal{G}$. Consequently, $\varphi(C) = C'$. This implies easily that the restriction of φ to C is a homeomorphism between C and C'. Since C' is compact and C is, evidently, a neighbourhood of y in eX, we obtain that eX is locally compact.

It is clear now that the two paragraphs above together with Theorem 3.8 imply our assertion. □

Remark 5.10. Some other descriptions of the ordered set of all (up to equivalence) locally compact extensions of a completely regular space were given by S. Leader [22] (on the basis of the notion of *local proximity* in which both the *boundedness* and the proximity are primitive terms), by V. Zaharov [36] (by means of some special vector lattices of functions) and by G. Dimov and D.

Doitchinov [12] (by using the notion of *supertopological space*).

6. Nearness structures and SR-proximities. For the sake of brevity, we assume the reader is familiar with Herrlich's paper [21] on nearness spaces and with Bentley and Herrlich's paper [4] on extensions of spaces. For the same reason, we omit all proofs in this section, since they are long but enough straightforward.

If X is a set and $\mathcal{G} \subseteq Exp(X)$ then we set, as in [28], $\mathcal{G}^* = \{E \subseteq X : X \setminus E \notin \mathcal{G}\}$.

In [4], H. Bentley and H. Herrlich proved that if (X, τ) is a regular space then the ordered set $(\boldsymbol{R}(X), \leq_0)$ is isomorphic to the ordered set $(\boldsymbol{NR}(X), \leq)$ of all nearness spaces (X, ξ) on (X, τ) which are subtopological and regular. (The order in $\boldsymbol{NR}(X)$ is defined as follows: $(X, \xi_1) \leq (X, \xi_2)$ iff $\xi_2 \subseteq \xi_1$.) The regular extension $(r_\xi X, r_\xi)$ of (X, τ) corresponding to an element $\underline{X} = (X, \xi)$ of $\boldsymbol{NR}(X)$ is constructed in [4] by means of the completion (X^*, ξ^*) of (X, ξ). It can be shown that the extension $(r_\xi X, r_\xi)$ is equivalent to the strict extension of (X, τ) with filter trace $\Sigma = \{\mathcal{G}^* : \mathcal{G}$ is a $\underline{X}$-cluster$\}$. This observation is useful for finding the relationships between the nearness structures and SR-proximities.

Let (X, α), where $\alpha = (\delta, \Sigma)$, be an SR-proximity space. We will say that a collection $\mathcal{G}$ of subsets of X is *near* iff there exists an $\mathcal{F} \in \Sigma$ which meets every element of $\mathcal{G}$. Then we affirm that:

(a) $\xi_\alpha = \{\mathcal{G} \subseteq Exp(X) : \mathcal{G}$ is near$\}$ is a nearness structure on (X, τ_δ);
(b) $\Sigma \subseteq \xi_\alpha$;
(c) the nearness space $\underline{X}_\alpha = (X, \xi_\alpha)$ is subtopological and regular;
(d) $\{\mathcal{F}^* : \mathcal{F} \in \Sigma\}$ is the family of all $\underline{X}_\alpha$-clusters;
(e) the extension $(r_{\xi_\alpha} X, r_{\xi_\alpha})$ of (X, τ_δ) is equivalent to the extension $(r_\alpha X, r_\alpha)$ defined in Theorem 3.8;
(f) if α' and α'' are two SR-proximities on the space (X, τ_δ) then $(X, \alpha') \leq_0 (X, \alpha'')$ iff $\underline{X}_{\alpha'} \leq \underline{X}_{\alpha''}$ (i.e. iff $\xi_{\alpha''} \subseteq \xi_{\alpha'}$).

We will refer to the nearness ξ_α as to the *nearness generated by* α. The proofs (which we omit here) of these assertions are direct, i.e. the theories of the regular extensions developed in [4] and here are not used in them. Of course, on the basis of these theories, some short proofs of the statements listed above could be given.

Conversely, let $\underline{X} = (X, \xi)$ be a subtopological and regular nearness space. For $A, B \subseteq X$, put $A \delta_\xi B$ iff $\{A, B\} \in \xi$ and set $\Sigma_\xi = \{\mathcal{G}^* : \mathcal{G}$ is a $\underline{X}$-cluster$\}$. Then we assert that:

(i) $\alpha_\xi = (\delta_\xi, \Sigma_\xi)$ is an SR-proximity on the space (X, τ_ξ);
(ii) $\Sigma_\xi \subseteq \xi$;

(iii) the extension $(r_{\alpha_\xi}X, r_{\alpha_\xi})$ of (X, τ_ξ) is equivalent to the extension $(r_\xi X, r_\xi)$;
(iv) the nearness generated by α_ξ coincides with ξ.
The proofs of these facts are again direct (in the above sense).

All this shows that our SR-proximities could be interpreted as bases (or generators) of the subtopological and regular nearness spaces. In addition, they contain the whole information, necessary for the construction of the extensions, which could be extracted from the generated by them nearness structure, i.e. they constitute that part of the nearness structures which is enough for obtaining a theory of the regular extensions (and their subclasses).

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