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PROJECTIVELY SOLID SETS AND AN n–DIMENSIONAL PICCARD'S THEOREM

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ABSTRACT. We discuss functions $f : X \times Y \to Z$ such that sets of the form $f(A \times B)$ have non-empty interiors provided that A and B are non-empty sets of second category and have the Baire property.

1. Introduction. In 1939, S. Piccard [8] showed that if A and B are non-empty subsets of the space of reals that are of second category and have the Baire property, then the set $A + B = \{a + b : a \in A \text{ and } b \in B\}$ contains a non-empty open interval. We analyze this phenomenon more generally by viewing the operation $+$ as a function on the (Tychonoff) product of two topological spaces and the set $A + B$ as the image of the set $A \times B$ by this function. Our goal is to discover properties of the product spaces which cause such an effect. The key ingredients in our study are projectively solid sets and Baire-open maps, which we introduce and discuss in Section 2. Along the way, we get an n-dimensional analogue of Piccard's theorem and some applications to topological semigroups.

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W. Sanders [9] and E. J. McShane [6] undertook a similar general approach; however, most of our results are new and different from those obtained by Sanders and McShane.

Before Piccard, H. Steinhaus [10] showed that if A and B are subsets of the space of reals which are Lebesgue measurable and of positive measure, then $A+B$ contains a non-empty open interval. Piccard's result is a topological analogue of Steinhaus's measure-theoretic one. But there is no need to consider topological and measurable cases separately, for the measurable version can be reduced to the topological one. To see how, assume that f is a function from the product $X \times Y$ into a topological space Z and that X and Y are endowed with $\sigma - finite$ and complete measure structures $(\mathcal{M}, m)$ and $(\mathcal{S}, s)$, respectively. There exist topologies on X and Y such that sets with the Baire property coincide with measurable ones and first category sets with those of measure zero. To get such topologies, apply the Maharam – von Neuman theorem on the existence of lifting for $\sigma - finite$ complete measures and then generate the density topology by means of a lifting (cf. [3] for details). Now, Piccard's theorem holds for f if and only if Steinhaus's theorem does.

2. Baire-open maps and projectively solid sets. Let us recall that if $X_1, X_2, \ldots, X_n$ are topological spaces, then their (Tychonoff) product, denoted by $\Pi_{i=1}^n X_i$, is the topological space with underlying set $X_1 \times X_2 \times \cdots \times X_n$ and the topology generated by the "open rectangles" $U_1 \times U_2 \times \cdots \times U_n$, where U_i is an open subset of the space $X_i,\ i = 1, \ldots, n$. The symbol π_{X_i} is reserved for the projection of the product onto X_i; i.e., $\pi_{X_i}(x_1, x_2, \ldots, x_n) = x_i$.

Definition 1. *A subset A of the product $X \times Y$ is said to be* projectively somewhere dense *provided that for each open set $W \subseteq X \times Y$ intersecting A, both $\pi_X(A \cap W)$ and $\pi_Y(A \cap W)$ are somewhere dense.*

Definition 2. *A subset A of the product $X \times Y$ is said to be* projectively of second category *provided that for each open set $W \subseteq X \times Y$ intersecting A, both $\pi_X(A \cap W)$ and $\pi_Y(A \cap W)$ are of second category.*

Definition 3. *A subset A of the product $X \times Y$ is said to be* projectively solid *provided that for arbitrary sets of first category, E in X and F in Y, both $A - (E \times Y)$ and $A - (X \times F)$ are projectively of second category.*

Clearly, each projectively solid set is projectively of second category which in turn is projectively somewhere dense. The examples and lemmas below shed more light on some possible relationships between these three types of subsets of $X \times Y$.

Example 4. It is obvious that any subset of the product $X \times Y$ which is of second category at each of its points or any open subset of a set which is projectively of second category is an instance of a set which is projectively of second category.

Example 5. Let X and Y be Baire spaces and let C, D be dense subsets of X and Y, respectively. Then the set $A = (C \times Y) \cup (X \times D)$ is projectively of second category.

Example 6. Let X and Y be Baire spaces such that $X \times Y$ is a space of first category (cf. [7] or [2]). Then $X \times Y$ is of first category and projectively of second category.

Example 7. It is obvious that any subset of the product $X \times Y$ that is of second category at each of its point or any open subset of a set which is projectively solid is projectively solid. Also, if X is a Baire space, then the diagonal in the space $X \times X$ is a projectively solid set.

Example 8. Let X and Y be separable Baire spaces. Let C and D be countable dense subsets of X and Y, respectively. By virtue of Example 5, the set $A = (C \times Y) \cup (X \times D)$ is projectively of second category, but it is not projectively solid. Notice also that Example 7 provides an instance of a set which is projectively solid and nowhere dense.

Lemma 9. *Let A be a projectively of second category subset of $X \times Y$. Then $\pi_X(A)$ is of second category at each of its points, and $\pi_Y(A)$ is of second category at each of its points.*

Proof. Let $x \in \pi_X(A)$ and let U be an open neighborhood of x. Then $U \cap \pi_X(A)$ is of second category because $U \cap \pi_X(A) = \pi_X((U \times Y) \cap A)$. A similar argument works for the set $\pi_Y(A)$. $\square$

Let (!) denote the following condition:

(!) If U and V are open subsets of X and Y, respectively, such that $(U \times V) \cap A \neq \emptyset$, then for any sets of first category $E \subseteq X$ and $F \subseteq Y$, $((U - E) \times (V - F)) \cap A \neq \emptyset$.

Lemma 10. *Let A be a subset of $X \times Y$. Then A is projectively solid if and only if the condition* (!) *holds for A.*

Proof. Suppose that A is projectively solid and let us take suitable U, V, E and F to check (!). Since A is projectively solid, $\pi_X((U \times V) \cap A)$ is of second category. Hence $\pi_X((U \times V) \cap A) - E$ is non-empty. This means that $U \times V$ intersects $A - (E \times Y)$. Therefore, $\pi_Y((U \times V) \cap (A - (E \times Y))) - F$

is non-empty. This means that $(U \times V) \cap (A - (E \times Y)) - (X \times F) \neq \emptyset$. As elementary calculations show, the last set equals $((U - E) \times (V - F)) \cap A$ and thus A satisfies (!).

Let us assume that the condition (!) holds for A and show that A is projectively solid. Suppose otherwise. We may assume that there exist a set of first category $E \subseteq X$ and an open set $W \subseteq X \times Y$ such that the projection of the set $B = W \cap (A - (E \times Y))$ onto either X or Y is a non-empty set of first category. Since W intersects A, there exist open sets, $U \subseteq X$ and $V \subseteq Y$, such that $U \times V \subseteq W$ and $(U \times V) \cap A \neq \emptyset$. If $\pi_X(B)$ were of first category, A would be disjoint from $(U - (E \cup \pi_X(B)) \times V$; if $F = \pi_Y(B)$ were of first category, A would be disjoint from $(U - E) \times (V - F)$. To see the last fact notice that $B \subseteq X \times F$; i.e., $(W \cap (A - (E \times X)) - (X \times F) = \emptyset$. Since $U \times V \subseteq W$, elementary calculations show the required disjointedness.

In both cases we get a contradiction to (!); the lemma has been proved. $\square$

We are now in a position to show that some of the above three types of sets, generally different, may coincide in the presence of completeness.

Proposition 11. *Let A be a G_δ subset of the product of two Čech-complete spaces X and Y. Then the following conditions* (1), (2) *and* (3) *are pairwise equivalent:* (1) *A is projectively solid;* (2) *A is projectively of second category;* (3) *A is projectively somewhere dense.*

P r o o f. Only the implication that (1) follows from (3) needs a proof. For this purpose we will use the characterization from Lemma 10. Let U, V, E and F be as in (!). Let $E = \cup\{E_n : n = 1, 2, \ldots\}$ and $F = \cup\{F_n : n = 1, 2, \ldots\}$, where E_n and F_n are nowhere dense sets for each n. Let $A = \cap\{G_n : n = 1, 2, \ldots\}$, where G_n is open for each $n = 1, 2, \ldots$ By induction, one can construct decreasing sequences $\{U_n\}$ and $\{V_n\}$ of open sets of "suitably small" diameter such that $\operatorname{cl} U_{n+1} \subseteq U_n - E_{n+1} \subseteq U$, $\operatorname{cl} V_{n+1} \subseteq V_n - F_{n+1} \subseteq V$ and $U_n \times V_n \subseteq G_n$ for each $n = 1, 2, \ldots$ By completeness, there exist points $p \in \cap\{U_n : n = 1, 2, \ldots\}$ and $q \in \cap\{V_n : n = 1, 2, \ldots\}$. Hence $(p, q) \in ((U - E) \times (V - F)) \cap A$. $\square$

Definition 12. *A function $f : X \times Y \to Z$ is said to be* Baire-open *provided that each set of the form $f(A \times B)$ has non-empty interior whenever A and B are non-empty sets of second category with the Baire property in X and Y, respectively.*

Definition 13. *A function $f : X \to Y$ is* quasi-continuous *provided that for each open set $V \subseteq Y$, the set* $\operatorname{Int} f^{-1}(V)$ *is dense in $f^{-1}(V)$.*

Definition 14. *A function $f : X \to Y$ is* quasi-open *provided that for each non-empty open set $U \subseteq X$, the set* $\operatorname{Int} f(U)$ *is non-empty.*

Definition 15. *A function $f : X \to Y$ is a* quasi-homeomorphism *provided that f is both quasi-continuous and quasi-open.*

Lemma 16. *Let $f : X \to Y$ be a quasi-homeomorphism. If F is a nowhere dense subset of Y, then $f^{-1}(F)$ is a nowhere dense subset of X.*

Proof. Let U be a non-empty open subset of X. Since f is quasi-open and F is nowhere dense, there exists a non-empty open set V such that $V \subseteq f(U) - F$. Since f is quasi-continuous, $G = \operatorname{Int} f^{-1}(V) \cap U$ is a non-empty open subset of U disjoint from $f^{-1}(F)$. $\square$

Lemma 17. *Let $f : X \to Y$ be a quasi-homeomorphism. If $A \subseteq X$ is of second category, then $f(A)$ is of second category.*

Lemma 18. *Let $f : X \to Y$ be a quasi-homeomorphism. If B is a dense subset of an open set $V \subseteq Y$, then $f^{-1}(B)$ is a dense subset of $f^{-1}(V)$.*

Proof. Let U be an open set intersecting the set $f^{-1}(V)$. Since f is quasi-continuous, $G = U \cap \operatorname{Int} f^{-1}(V) \neq \emptyset$. Hence $U \cap f^{-1}(B) \neq \emptyset$. $\square$

Let $(*)$ denote the following condition:

$(*)$ For any dense subset S of an arbitrary non-empty open subset of Z, $f^{-1}(S)$ is projectively solid.

Theorem 19. *Let $f : X \times Y \to Z$, be a quasi-continuous function. Then f is Baire-open if and only if it is quasi-open and satisfies the condition $(*)$.*

Proof. Suppose that f is Baire-open. Clearly, f must be quasi-open, so that f is a quasi-homeomorphism. To prove that $(*)$ holds for f take a non-empty open subset G of Z and a dense subset S of G. To show that $f^{-1}(S)$ is projectively solid, we will check (!) of Lemma 10. Let U and V be open subsets of X and Y, respectively, such that $(U \times V) \cap f^{-1}(S) \neq \emptyset$. Since f is a quasi-homeomorphism, $f^{-1}(S)$ is dense in $f^{-1}(G)$ and therefore $(U \times V) \cap f^{-1}(S) \cap \operatorname{Int} f^{-1}(G) \neq \emptyset$. So that we may additionally assume that $(U \times V) \subseteq f^{-1}(G)$. Since f is Baire-open, $\operatorname{Int} f((U-E)\times(V-F)) \neq \emptyset$. Because S is dense in G, the set $f((U-E)\times(V-F))$ must intersect S, that is $((U - E) \times (V - F)) \cap f^{-1}(S) \neq \emptyset$.

To prove that quasi-homeomorphisms satisfying $(*)$ are Baire-open, notice that X and Y must be Baire spaces. For observe that since Z is a dense subset of itself, $X \times Y = f^{-1}(Z)$ is projectively solid. By virtue of Lemma 9, X and Y are Baire spaces.

Suppose to the contrary that f is not Baire-open. There are non-empty open sets U, V and sets of first category E, F such that $f((U-E)\times(V-F))$ is a boundary subset of Z. We set $G = \operatorname{Int}\operatorname{cl} f((U-E)\times(V-F))$ and $B = G - f((U-E)\times(V-F))$. Since $(U-E)\times(V-F)$ is a dense subset of the non-empty open set $U\times V$ and because of Lemma 16, $f((U-E)\times(V-F))$ cannot be nowhere dense. Hence G is a non-empty open set and B is a boundary and dense subset of G. By virtue of Lemma 18, $f^{-1}(B)$ is dense in $f^{-1}(G)$. By virtue of the condition $(*)$, $f^{-1}(B)$ is projectively solid. There exist non-empty open sets $U_1 \subseteq U$ and $V_1 \subseteq V$ such that $U_1 \times V_1 \subseteq f^{-1}(G)$; the existence of such sets follows from the quasicontinuity of f and the fact that $U\times V$ intersects $f^{-1}(G)$. Hence $U_1\times V_1$ intersects $f^{-1}(B)$. By virtue of Lemma 10, $(U_1-E)\times(V_1-F)$ intersects $f^{-1}(B)$ as well. This is impossible because $f((U_1-E)\times(V_1-F)$ is a subset of $G\cap f(\,(U-E)\times(V-F))$ which is disjoint from B. $\square$

Proposition 20. *Let $f : X\times Y \to Z$ be a quasi-homeomorphism such that the set $PS(f) = \{z\in Z : f^{-1}(z)$ is projectively solid$\}$ contains a dense subset which is open in Z. Then f satisfies the condition $(*)$ of Theorem* 19.

P r o o f. Let S be a dense subset of a non-empty open set $W\subseteq Z$ and let $U\subseteq X$ and $V\subseteq Y$ be open sets such that $(U\times V)\cap f^{-1}(S)\neq\emptyset$. By virtue of Lemma 18, $G = (U\times V)\cap \operatorname{Int} f^{-1}(W)$ is a non-empty open subset of $X\times Y$. Hence $f(G)\cap W\cap PS(f)\neq\emptyset$. If z is an element of the former set, then $(U\times V)\cap f^{-1}(z)\neq\emptyset$. So that, if $E\subseteq X$ and $F\subseteq Y$ are sets of first category, then $((U-E)\times(V-F))\cap f^{-1}(z)\neq\emptyset$ and therefore $((U-E)\times(V-F))\cap f^{-1}(S)\neq\emptyset$. By virtue of Lemma 10, $f^{-1}(S)$ is projectively solid. $\square$

Definition 21. *A function $f : X\times Y\to Z$ is said to be* locally solvable at z *provided that for each open set W with $z\in f(W)$ there exist non-empty open sets U, V and a quasi-homeomorphism $h : U\to V$ such that $U\times V\subseteq W$ and $f(x, h(x)) = z$ for each $x\in U$.*

Proposition 22. *Let X, Y be Baire spaces and let $f : X\times Y\to Z$. If f is locally solvable at z, then $f^{-1}(z)$ is projectively solid.*

P r o o f. Let W be an open subset of $X\times Y$ such that $z\in f(W)$. Let h, U and V witness the local solvability of f at z for the set W. Assume that E is a first category subset of X and that W intersects the set $f^{-1}(z)-(E\times Y)$. We shall show that $h(U-E)\subseteq \pi_Y(W\cap(f^{-1}(z)-(E\times Y)))$. To this end, let $y\in h(U-E)$. There exists $x\in U-E$ with $y = h(x)$. Hence $(x,y) = (x,h(x))\in f^{-1}(z)-(E\times Y)$ and $(x,y)\in U\times V\subseteq W$. It follows that $y\in\pi_Y(W\cap(f^{-1}(z)-(E\times Y)))$. This inclusion shows that the last set is of second category since $h(U-E)$ is a second category set, according to Lemma 17.

Assume that F is a first category subset of Y and that W intersects the set $f^{-1}(z)-(X\times F)$. We shall show that $U-h^{-1}(F)\subseteq \pi_X(W\cap(f^{-1}(z)-(X\times F)))$. To this end, let $x\in U-h^{-1}(F)$. Hence $(x,h(x))\in f^{-1}(z)-(X\times F)$ and $(x,h(x))\in U\times V\subseteq W$. It follows that $x\in\pi_X(W\cap(f^{-1}(z)-(X\times F)))$. This inclusion shows that the last set is of second category since $h^{-1}(F)$ is a first category set, according to Lemma 16. Thus $f^{-1}(z)$ is projectively solid. □

3. Baire-openness of differentiable functions on $\boldsymbol{R^n}$. To facilitate the discussion that follows, we define the most important concepts and quote well-known theorems about functions on the Euclidean spaces R^n from advanced calculus.

Let $f : D\subseteq R^m\to R^n$ be a differentiable function at $x\in D$. Then the $n\times m$ matrix of partial derivatives of f at x is the Jacobian of f at x and is denoted by $J_{f,x}$. The rank of f at x is the rank of the Jacobian of f at x. If all partial derivatives of f (at x) are continuous, then f is a C^1 function (at x). If f and its inverse are C^1 functions, then f is a diffeomorphism.

Theorem 23 (The Inverse Function Theorem). *Let $f : D\subseteq R^n\to R^n$ be a C^1 function on an open set D in R^n and let x be a point in D such that* $\det\, J_{f,x}\neq 0$. *Then there exists a neighborhood U of x such that* $\det\, J_{f,y}\neq 0$ *for all $y\in U$, and the restriction of f to U is a diffeomorphism.*

Theorem 24 (The Implicit Function Theorem). *Let $U\subseteq R^m$ and $V\subseteq R^n$ be open sets and let $f : U\times V\to R^n$ be a C^1 function. Suppose further that $(p,q)\in U\times V$ and that* $\det J_{g,q}\neq 0$, *where g is a function on V defined by the rule $g(y)=f(p,y)$. Then there exist an open neighborhood U_1 of p and a unique function h from U_1 into some open neighborhood V_1 of q such that $h(p)=q$ and $f(x,h(x))=f(p,q)$ for each $x\in U_1$; moreover, h is a C^1 function.*

Theorem 25 (The Rank Theorem). *Let $f : D\subseteq R^m\to R^n$ be a C^1 function on an open set D in R^m such that rank $J_{f,x}=k$ for each $x\in D$. Then for each $p\in D$ there exist open neighborhoods, U of p and V of $f(p)$, and diffeomorphisms, $g : V\to R^n$ and $h : R^m\to U$, such that the composite function $g\circ f\circ h$ has its values in the space R^k.*

Theorem 26 (Brouwer's Invariance Theorem). *If $h : U\to h(U)\subseteq R^n$ is a homeomorphism on an open subset U of R^n, then $h(U)$ is also open.*

Lemma 27. *Let U and V be non-empty open subsets of the space R^n and let $f : U\times V\to R^n$ be a C^1 function. For fixed points p of U and q*

of V let $g_p : V \to R^n$ and $g^q : U \to R^n$ be functions given by the formulas: $g_p(y) = f(p, y)$ and $g^q(x) = f(x, q)$. If $\det J_{g_p,q} \neq 0 \neq \det J_{g^q,p}$, then f is locally solvable at $f(p, q)$.

Proof. Let $z = f(p, q)$ and let W be an open neighborhood of the point (p, q). It follows from the Implicit Function Theorem that there exists a unique C^1 function h from an open neighborhood U_1 of p into an open neighborhood V_1 of q such that $U_1 \times V_1 \subseteq W$, $h(p) = q$ and $f(p, q) = f(x, h(x))$ for each $x \in U_1$. By the Implicit Function Theorem again, there exists a unique C^1 function d from an open neighborhood V_2 of q into an open neighborhood U_2 of p such that $U_2 \times V_2 \subseteq U_1 \times V_1$, $d(q) = p$ and $f(p, q) = f(d(y), y)$ for each $y \in V_2$. It follows from the uniqueness of h that d is the inverse to h on U_2. Hence h is a diffeomorphism on U_2, and so f is locally solvable at z. □

Lemma 28. *Let $f : D \subseteq R^m \to R^n$ be a C^1 function on an open set D in R^m, where $m \geq n$. Then f is quasi-open if and only if the set $M = \{x \in R^m : \operatorname{rank} J_{f,x} = n\}$ is dense in D.*

Proof. Suppose that $\operatorname{rank} J_{f,x} < n$ for each x in a non-empty open subset D_1 of D. Since f is a C^1 function, there exists a non-empty open subset D_2 of D_1 such that $\operatorname{rank} J_{f,x} = k$ for each $x \in D_2$. Let us fix a point x in D_2. It follows from the Rank Theorem that there exist open sets, $U \subseteq R^m$ and $V \subseteq R^n$, and diffeomorphisms, g of V onto R^n and h of R^m onto U, such that $x \in U$, $f(x) \in V$ and the composite function $g \circ f \circ h$ has its values in R^k, where $k < n$. Since diffeomorphisms are topological homeomorphisms, it follows from Brouwer's Invariance Theorem that $\operatorname{Int}(U \cap D_2) = \emptyset$. Since $U \cap D_2$ is a non-empty open set, f is not quasi-open.

Let W be a non-empty open subset of D and let $r \in W \cap M$. We may assume that $r = (p, q)$ with $p \in R^{m-n}$ and $q \in R^n$ and that $\det_{g,q} J \neq 0$, where g is the function given by $g(y) = f(p, y)$. Since f is a C^1 function, there exists an open set $V \subseteq R^n$ such that $(p, q) \in \{p\} \times V \subseteq W$ and $\det_{g,y} J \neq 0$ for each $y \in V$. By the Inverse Function Theorem, g is a diffeomorphism of V into R^n and therefore $g(V)$ is a non-empty open subset of R^n. But $g(V) = f(\{p\} \times V) \subseteq f(W)$. Hence $\operatorname{Int} f(W) \neq \emptyset$. □

The following corollary can be treated as an n-dimensional version of Piccard's theorem. It can also be viewed as a generalization of a theorem of M. E. Kuczma and M. Kuczma [4], because their differential condition is the same as our condition (%) below for $n = 1$. Let us add that the condition (%) means, roughly speaking, that functions $f_1, \ldots, f_n$ from R^m into R are functionally independent on any n-dimensional open section of R^m.

Let (%) denote the following condition:

(%) For each point $p = (p_1, \ldots, p_m) \in R^m$, for each n-tuple $1 \leq j_1 < \cdots < j_n \leq m$ of indices and for each family $\{U_1, \ldots, U_n\}$ of open subsets of R such that $p_{j_k} \in U_k,\ k = 1, \ldots, n$, the rank of the restriction of f to the set $G = \{(x_1, \ldots, x_m) : x_i = p_i$ if $i \neq j_k$ for each $k = 1, \ldots, n$ or $x_i \in U_k$ if $i = j_k\}$ is equal to n.

Corollary 29. *Let $f : R^m \to R^n$, $m \geq 2n$, be a C^1 function that satisfies the condition (%). If each of the sets B_i, $i = 1, 2, \ldots, m$, is a non-empty subset of R with the Baire property and of second category, then $f(B_1 \times \cdots \times B_m)$ has non-empty interior.*

Proof. Let $B_i = (V_i - E_i) \cup F_i$, where V_i is open and E_i, F_i are sets of first category in R for each $i = 1, \ldots, m$. We will prove the theorem for the case $m = 2n$ first.

The property (%) for f implies, in particular, that the set $M = \{x \in R^{2n} : \operatorname{rank} J_{f,x} = n\}$ is dense in R^{2n} and so f is quasi-open, by virtue of Lemma 28. Let $r \in V_1 \times \cdots \times V_{2n}$ be a point such that the rank of the Jacobian of f at r is maximal; i.e., rank $J_{f,r} = n$. Without loss of generality, we may assume that $r = (p, q)$, where $p \in V_1 \times \cdots \times V_n$, $q \in V_{n+1} \times \cdots \times V_{2n}$, and that $\det J_{g_p,q} \neq 0$. Since f is a C^1 function, there exist open sets $W_i \subseteq V_i$ such that $r \in W_1 \times \cdots \times W_{2n}$ and $\det J_{g_x,y} \neq 0$ for each $x \in W_1 \times \cdots \times W_n$ and $y \in W_{n+1} \times \cdots \times W_{2n}$. By virtue of (%), there exists a point $s \in W_{n+1} \times \cdots \times W_{2n}$ such that rank $J_{g^s,p} = n$. Using the C^1 property of f once more, one can find open sets $U_i \subseteq W_i$ such that $(p, s) \in U_1 \times \cdots \times U_{2n}$ and det $J_{g^y,x} \neq 0$ for each $x \in U = U_1 \times \cdots \times U_n$ and $y \in V = U_{n+1} \times \cdots \times U_{2n}$. Let us treat f as a function from $U \times V$ into R^n and show that it satisfies conditions of Theorem 19.

Clearly, f is a quasi-homeomorphism. By virtue of Lemma 27, f is locally solvable at each point of the set $f(U \times V)$. By virtue of Proposition 20, f also satisfies the condition $(*)$.

It follows from Theorem 19 that f is Baire open on $U \times V$. Hence $\operatorname{Int} f(B_1 \times \cdots \times B_{2n}) \neq \emptyset$.

To prove the theorem if $m > 2n$, let us fix a point p_i in B_i for $i = 1, \ldots, m - 2n$. We can apply the previous case to the function $g : R^{2n} \to R^n$, where $g_i(x_1,\ldots,x_{2n}) = f_i(p_1,\ldots,p_{m-2n},x_1,\ldots,x_{2n})$, $i=1,\ldots,n$. Hence

$$\begin{aligned}\emptyset &\neq \operatorname{Int} g(B_{m-2n+1}\times\cdots\times B_m)\\ &= \operatorname{Int} f(\{p_1\}\times\cdots\times\{p_{m-2n}\}\times B_{m-2n+1}\times\cdots\times B_m)\subseteq f(B_1\times\cdots\times B_m).\end{aligned}$$

□

Conjecture 30. *The above corollary holds for all m between n and $2n$.*

4. Baire-openness in topological semigroups. Let $(G;+)$ be an additive algebraic group endowed with a topology under which the operation $+$ is separately continuous and the inverse operation in G is continuous, i.e., the following three functions are continuous: $a_x(y) = x+y$, $a^y(x) = x+y$, $i(x) = -x$. Such a structure $(G;+)$ is called a topological semigroup. It is well known that the above three functions are autohomeomorphisms of the semigroup G.

Lemma 31. *Let $(G;+)$ be a topological semigroup. Then the operation $+ : G\times G\to G$ is (quasi)open and locally solvable at each $z\in G$.*

Proof. The openness of $+$ on $G\times G$ follows immediately from the fact that a_x is an autohomeomorphism.

Let $p+q=z$ for some $p,q,z\in G$. Then the function $h(x) = -x+z$ is an autohomeomorphism of G such that $h(p)=q$ and $x+h(x)=z$ for each $x\in G$. Hence the operation $+$ on G is locally solvable at z. □

Corollary 32. *Let $(G;+)$ be a topological semigroup such that the operation $+$ is a quasi-continuous function on $G\times G$ and G itself is a Baire space. Then the operation $+$ is Baire-open.*

Proof. It follows from Lemma 18 that $+$ is a quasi-homeomorphism on $G\times G$. It follows from Lemma 18 and Proposition 20 that $+$ satisfies the condition $(*)$. Thus $+$ is Baire-open, by virtue of Theorem 1. □

Corollary 33. *Let $(G;+)$ be a topological semigroup that is a regular first countable Baire space. Then the operation $+$ is Baire-open.*

Proof. It is enough to show that $+$ is quasi-continuous on $G\times G$. To this end, let W be an open subset of G and let $z=x+y\in W$. Let W_1 be an open neighborhood of z contained with its closure in W. Let $\{U_n : n=1,2,\ldots\}$ be a countable base at x. There exists an open neighborhood V of y such that $x+V\subseteq W_1$. Let $F_n=\{s\in V : U_n+s\subseteq W_1\}$. Clearly, $\cup\{F_n : n=1,2,\ldots\}=V$.

Since G is a Baire space, there exist a non-empty open subset D of V and n such that $D \subseteq \mathrm{cl}\, F_n$. Hence $U_n + D \subseteq \mathrm{cl}\, W_1 \subseteq W$. $\Box$

S. Banach's well known theorem [1] asserts that subgroups of topological groups that are of second category and have the Baire property must be open and closed. K. Kuratowski [5] gave an elegant and short proof of this theorem using Baire-openness of group operations. We use Kuratowski's technique to establish Banach's theorem for certain semigroups.

Corollary 34. *Let $(G;+)$ be a regular first countable topological semigroup. If H is a non-empty subset of G that is, topologically, of second category and has the Baire property, and algebraically, a subgroup of G, then H is open and closed.*

Proof. Let us recall that both the functions $a_x(y) = x + y,\ y \in G$, and $a^y(x) = x + y$, $x \in G$, are autohomeomorphisms on G. This along with the fact that H is of second category imply that G is a Baire space. By virtue of Corollary 33, $+$ is Baire-open and so $H = H + H$ contains a non-empty open set, say U. If $x \in U$, then $-x \in H$ and so $V = a_{-x}(U)$ is a non-empty open set containing 0 and contained in H. Hence H is an open subset of G being the union of open sets $a_z(V)$, where $z \in H$. If $z \notin H$, then $a_z(V)$ is an open neighborhood of z disjoint from H. Hence H is also closed. $\Box$

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