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GENERALIZATION OF A CONJECTURE IN THE GEOMETRY OF POLYNOMIALS

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Dedicated to the memory of my brilliant student and closed friend Vasil Popov January 14, 1942 – May 31, 1990

Abstract. In this paper we survey work on and around the following conjecture, which was first stated about 45 years ago: If all the zeros of an algebraic polynomial p (of degree $n \geq 2$) lie in a disk with radius r, then, for each zero z_1 of p, the disk with center z_1 and radius r contains at least one zero of the derivative p'. Until now, this conjecture has been proved for $n \leq 8$ only. We also put the conjecture in a more general framework involving higher order derivatives and sets defined by the zeros of the polynomials.

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1. Introduction. This paper is about a conjecture I formulated almost half a century ago. It is strikingly simple to state and can be explained to high-school students: Given a polynomial p of degree $n \geq 2$ with all zeros z_k in $\{z \in \mathcal{C} : |z| \leq r\}$, for each zero z_k at least one zero of the derivative p' is in $\{z \in \mathcal{C} : |z - z_k| \leq r\}$. Here and further $\mathcal{C}$ is the complex plane. As for now, the conjecture is proved by Brown and Xiang [12] for $n \leq 8$. My aim here is to give an overview to the work in more than 80 related papers that are known to me and also to put the conjecture in a more general perspective.

To place the conjecture in the context of the surrounding theory, we start with notation and definitions. Let $p(z) = (z - z_1)(z - z_2) \cdots (z - z_n)$ be an algebraic polynomial and $p'(z) = n(z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_{n-1})$ be its derivative. The operator of differentiation $\mathcal{D} = \partial/\partial z$ may be considered as a mapping of the set $A = \{z_1, z_2, \ldots, z_n\}$ of n points (the zeros of the polynomial p) from the complex plane $\mathcal{C}$ in to the set $A' = \{\zeta_1, \zeta_2, \ldots, \zeta_{n-1}\}$ of $n - 1$ points (the zeros of its derivative, called also *critical points* of p.). Some of the zeros $z_1, z_2, \ldots, z_n$ of p may coincide and they represent a multiple zero. A classical area of algebra, the so-called *Geometry of Polynomials*, studies the properties of the mapping $\mathcal{D}$; a basic reference for us is the celebrated book of M. Marden [29]. A milestone in this theory is the classical

Gauss-Lucas Theorem. *The convex hull of A contains A'.*

From this theorem we obtain:

Corrollary 1. *If all the zeros of an algebraic polynomial $p(z)$, of degree $n \geq 2$, lie in a disk with radius r and z_1 is a zero of $p(z)$, then the disk with center z_1 and radius $2r$ contains* **all zeros** *of the derivative $p'(z)$.*

In 1958 I was intrigued by the $2r$ in this result and started thinking of what would happen if we have just r there. My thoughts led me to a statement which I decided must be true, and I formulated it as a conjecture. The conjecture is as follows:

Conjercture 1. *If all the zeros of an algebraic polynomial $p(z)$, of degree $n \geq 2$, lie in a disk with radius r and z_1 is a zero of $p(z)$, then the disk with center z_1 and radius r contains* **at least one zero** *of the derivative $p'(z)$.*

If true, this conjecture is sharp in the sense that the polynomial $p(z) = z^n - 1$ with zeros in the unit circle and $p'(z) = nz^{n-1}$ with a zero of multiplicity $n - 1$ in the origin.

Conjecture 1 became first known as *Ilieff's conjecture* in the following way. L. Ilieff and W. K. Hayman attended the International Conference on the

Theory of analytic functions in Erevan, 6–13 September 1965. At this conference, Prof. Ilieff formulated Conjecture 1, mentioning my name as its author. Prof. Hayman remembered the conjecture as coming from Ilieff and included it in his book [21, Problem 4.5.] as *Ilieff's conjecture*.

M. Marden [31, p. 267] wrote the following for Conjecture 1: "*This conjecture was included in the collection of Research Problems in Function Theory, published in 1967 by Professor Hayman* [21]. *Since it had been brought to Hayman's attention by Professor Ilieff, it became known as "Ilieff's conjecture". Actually, Conjecture 1 was due to the Bulgarian mathematician Bl. Sendov who had acquainted me and probably others with it in 1962 at the International Congress of Mathematics held in Stockholm.*"

The interest of M. Marden, one of the masters in *Geometry of Polynomials*, in Conjecture 1 is mentioned by his sons in [28]: *"To give a flavor of his interests, we will end by stating a conjecture he was obsessed with over 25 years; it was repeated in most of his NSF grants, and most of his Monthly article of 1983 was devoted to it. This is the Ilieff Conjecture which Morrie asserts is due to Sendov who told Morrie about it in 1962: ..."*

Conjecture 1 is trivial for polynomials of degree 2. After it appeared in Hayman's book [21], a number of proofs have been published for polynomials of degree 3, see [7, 44, 46, 34, 22, 45, 15, 6, 53] and of degree 4, see [44, 46, 34, 22, 14, 6]. A simple proof of these cases is a part of Corollary 4 given in further lines. The proof for $n \leq 5$ was given by A. Meir and A. Sharma [34] in 1969, see also [27, 6]. More than 20 years later a proof for $n \leq 6$ was published by J. Brown [9] in 1991, see also [5, 6]. The case $n = 7$ was proved first by J. Borcea [6] in 1996 and by J. Brown [10] in 1997. In a recent paper J. E. Brown and G. Xiang (1999) [12] proved Conjecture 1 for $n \leq 8$. The proof is very elaborate and is based on obtaining good upper and lower estimates on the product of the moduli of the critical points of p. The method of proof in [12] could be probably extended to $n = 9$ but is becoming too laborious.

Conjecture 1 is proved for every polynomial with 3, 4, 5, 6, 7 and 8 distinct zeros, see [45, 8, 24, 25, 27, 12]. G. L. Cohen and G. H. Smith [14] proved that Conjecture 1 is true for polynomials of degree n with m distinct zeros, if $n \geq 2^{m-1}$. The general case is still open. Interestingly enough, it is not proved even for polynomials with real coefficients and only real critical points, see [11].

A conjecture, stronger than Conjecture 1 was announced in 1969 by A. G. Goodman, Q. I. Rahman and J. Ratti [19] and independently by G. Schmeisser [46]

Conjecture 2 (Ratti-Schmeisser)**.** *If all the zeros of an algebraic poly-*

nomial $p(z)$ of degree $n \geq 2$ lie in the unit disk $D(0,1) = \{z : |z| \leq 1\}$ and z_1 is a zero of $p(z)$, then the disk $D(z_1/2, 1 - |z_1|/2)$ contains at least one zero of $p'(z)$.

Conjecture 1 follows from Conjecture 2 inasmuch the disk $D(z_1/2, 1 - |z_1|/2)$ lies in the disk $D(z_1, 1)$. Conjecture 2 was proved by G. Gacs [18] for $2 \leq n \leq 5$. M. J. Miller [35], using computers, constructed counterexamples for Conjecture 2 for polynomials of degree 6, 8, 10 and 12. Counterexamples for polynomials of degree 7, 9 and 11 were found by S. Kumar and B. G. Shenoy [26]. Most probably, Conjecture 2 is not true for every natural $n \geq 6$. The motivation behind Conjecture 2 was that it is true for a zero z_1 if $|z_1| = 1$, see [19, 46] and Corollary 2. From this fact it follows that *Conjecture 1 is true for a polynomial p if all the corners of the convex hull of its zeros lie in a circle.*

By applying a linear transformation, it suffices to prove Conjecture 1 only for the polynomials p from **the set $\mathcal{P}_n$ of all monic polynomials** $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$, **for which the unit disk $D = D(0,1)$ is the smallest disc containing all zeros of p.** The substance of the Conjecture 1 is to determine **how large** may be the deviation $\rho(A(p); A(p'))$ (see Subsection 2.1) of the set $A(p)$ of the zeros of an algebraic polynomial p from the set $A(p')$ of the zeros of its derivative p'. In this notation, Conjecture 1 may be formulated as follows.

Conjecture 1. *If $p \in \mathcal{P}_n$, then $\rho(A(p); A(p')) \leq 1$.*

A problem in a sense inverse to Conjecture 1 was formulated and solved by A. Aziz [2].

Theorem 1 (A. Aziz). *If $p \in \mathcal{P}_n$, then $\rho(A(p'); A(p)) \leq 1$.*

In fact, A. Aziz proved a stronger statement, that *if $p \in \mathcal{P}_n$ and ζ_1 is a zero of $p'(z)$, then the disk with center $2\zeta_1$ and radius 1 contains at least one zero of the polynomial $p(z)$.* Theorem 1 follows directly from the Gauss–Lucas Theorem, as the following general statement is true. *Let the set of points A lie in a circle with radius r and the point b lies in the convex hull of A. Then the disk with center b and radius r contains at least one point from A.*

In what follows we review some results related to Conjecture 1 and consider several generalizations.

Conjecture 3. *If $p \in \mathcal{P}_n$ and $n \geq s + 1$, then*

$$\rho(A(p); A(p^{(s)})) \leq \frac{2s}{s+1}.$$

For $s = 1$, Conjecture 3 is Conjecture 1, and for $s = n - 1$ it is simple to prove, since $p^{(n-1)}(z)$ has only one zero. By Corollary 4, Conjecture 3 is proved also for: $s = n - 2$ and $n \geq 3$, for $s = n - 3$ and $n \geq 4$, for $s = n - 4$ and $n \geq 6$.

For every natural $n \geq 2$, the set $\mathcal{P}_n$ is compact. Therefore, for every $n \geq s + 1$ and $s = 1, 2, 3 \ldots$, there exists a polynomial $p_{n,s} \in \mathcal{P}_n$, such that

$$\rho_{n,s} = \rho(A(p_{n,s}); A(p_{n,s}^{(s)})) = \sup\{\rho(A(p); A(p^{(s)})) : \ p \in \mathcal{P}_n\}. \tag{1}$$

The polynomial $p_{n,s}$ is called **extremal for** $\rho(A(p); A(p^{(s)}))$ **in** $\mathcal{P}_n$ and a zero $z_1 \in A(p_{n,s})$ is called **extremal zero of** $p_{n,s}$, if $\rho(z_1; A(p_{n,s}^{(s)})) = \rho_{n,s}$.

In 1972 D. Phelps and R. S. Rodriguez [40] conjectured that, *if a polynomial p is extremal for $\rho(A(p); A(p'))$ in $\mathcal{P}_n$, then $p'(z) = nz^{n-1}$.* We generalize this conjecture as follows.

Conjecture 4. *If a polynomial p is extremal for $\rho(A(p); A(p^{(s)}))$ in $\mathcal{P}_n$, then*

$$p^{(s)}(z) = \frac{n!}{(n-s)!}(z - \lambda_{n,s})^{n-s},$$

where $\lambda_{n,s}$ is a constant.

By Theorem 10, an extremal polynomial for $\rho(A(p);\ A(p''))$ in the set of polynomials $p \in \mathcal{P}_4$ with real coefficients, is the polynomial

$$p(z) = (z-1)^2\left(z^2 + \frac{2}{3}z + 1\right) \quad \text{with} \quad p''(z) = 12\left(z - \frac{1}{3}\right)^2. \tag{2}$$

Until now, we fail to prove that the polynomial (2) is extremal for $\rho(A(p);$ $A(p''))$ in $\mathcal{P}_4$ and that $\rho_{4,2} = 2/\sqrt{3}$. Observe that, according to Conjecture 3, we have $\rho_{4,2} \leq 4/3 = \rho_{3,2}$.

In 1972, G. Schmeisser [47] formulated the following problem, which is a somewhat relaxed version of Conjecture 1:

Problem 1 (G. Schmeisser)**.** *Find a constant ρ (as small as possible), such that for every $p \in \mathcal{P}_n$, the inequality $\rho(A(p); A(p')) \leq \rho$ holds.*

In other words, the problem is to estimate from above $\rho_{n,1},\ n = 2, 3, \ldots$.

From Gauss–Lucas Theorem it follows that $\rho \leq 2$, see Corollary 1. In [47] it is proved that $\rho \leq 1.568$. The best estimate for ρ, till now, is given by B. Bojanov, Q. I. Rahman and J. Szynal [4], which is based on the result that *for every $p \in \mathcal{P}_n$, the inequality*

$$\rho(A(p); A(p')) \leq (1 + |z_1 z_2 \cdots z_n|)^{1/n} \tag{3}$$

holds.

After J. E. Brown and G. Xiang [12] proved Conjecture 1 for $n \leq 8$, from (3) it follows that

$$\rho(A(p); A(p')) \leq 1.08006\ldots\ .$$

This inequality also implies that $\lim_{n\to\infty} \rho_{n,1} = 1$, or that Conjecture 1 is asymptotically true. There are arguments, supporting a more general statement.

Conjecture 5. *For every fixed natural number s,*

$$\rho_{s+1,s} \geq \rho_{s+2,s} \geq \rho_{s+3,s} \geq \cdots \quad \text{and} \quad \lim_{n\to\infty} \rho_{n,s} = 1.$$

We may consider also deviations of sets defined by the zeros of a polynomial. Such sets, for example, are:

$H(p) = H(A(p))$ – the convex hull of the zeros of p and

$D(p) = D(c(p), r(p))$ – the smallest disk containing all zeros of p with center $c(p)$ and radius $r(p)$. By definition, $D(p)$ for a polynomial $p \in \mathcal{P}_n$ is the unit disk $D = D(0,1)$.

In 1977, G. Schmeisser [48] formulated

Conjecture 6 (G. Schmeisser)**.** *For every $p \in \mathcal{P}_n$, the inequality $\rho(H(p); A(p')) \leq 1$ holds.*

Conjecture 6 is stronger than Conjecture 1 inasmuch $A(p) \subset H(p)$. Conjecture 6 is true, if all the corners of $H(p)$ lie in the unit circle.

In the following sections we restate Conjecture 1 in the format of the set-valued metric topology. This new formulation naturally induces new related problems and generalizations. In Section 2 we start with an introduction to set-valued metric topology in the complex plane $\mathcal{C}$, which was first used in explicit way in the *Geometry of polynomials* in [38].

In Section 3, for a given polynomial p we consider, along with the set $A(p)$ of its zeros of this polynomial, the convex hull $H(p)$ of $A(p)$, containing the set $A(p)$. Then, for every polynomial p of degree n, we study the deviation of one of the sets $A(p)$ and $H(p)$ from one of the sets $A(p^{(s)})$ and $H(p^{(s)})$ for $s = 1, 2, \ldots, n-1$. Some of these deviations are trivial for calculation, but for most of them it is difficult even to conjecture an exact estimate.

2. Notation and basics.

2.1. Deviation of sets. Let A and B be two bounded and closed sets of points in the complex plane $\mathcal{C}$. We will use the notations:

1. $\rho(b; A) = \inf\{|b-a| : \ a \in A\}$,
2. $\rho(B; A) = \sup\{\rho(b; A) : \ b \in B\}$ – the **deviation** of B from A,
3. $D(A) = D(c(A), r(A))$ - the smallest closed disk, which contains A, with center $c(A)$ and radius $r(A)$.

A simple and useful statement is

Lemma 1. *If A, B, V are bounded point sets in $\mathcal{C}$ and $B \subset V$, then*

$$\rho(A; B) \geq \rho(A; V) \quad \text{and} \quad \rho(B; A) \leq \rho(V; A).$$

P r o o f. From the definition

$$\rho(A; B) = \sup\{\rho(a; B) : \ a \in A\} \geq \sup\{\rho(a; V) : \ a \in A\} = \rho(A; V),$$

as $B \subset V$ and

$$\rho(a; B) = \inf\{|a-b| : \ b \in B\} \geq \inf\{|a-v| : \ v \in V\} = \rho(a; V).$$

In the same way

$$\rho(V; A) = \sup\{\rho(v; A) : \ v \in V\} \geq \sup\{\rho(b; A) : \ b \in B\} = \rho(B; A),$$

as $B \subset V$. $\square$

In general $\rho(B; A) \neq \rho(A; B)$. The *Hausdorff distance* [20] between two sets A and B is

$$h(A, B) = \max\{\rho(A; B), \rho(B; A).$$

In connection with the Theorem 1, Conjecture 1 may be stated as follows: *If the zeros of an algebraic polynomial lie in a circle with radius r, then the Hausdorff distance between the set of its zeros and the set of the zeros of its derivative is not greater than r.*

It is natural to ask for the Hausdorff distance between the set of the zeros of a polynomial and the set of the zeros of its second and higher derivatives. We consider this question in the following section.

2.2. Basic statements. We list some classical theorems from the Geometry of polynomials, needed for the following. For more see [29].

Theorem 2. *Let z_1 and z_2 be two different zeros of the polynomial p and let l be the bisector of the segment from z_1 to z_2. Then in every two closed half-plane, defined by l, there is at least one zero of p'.*

The bisector property was noticed for the first time by G. Szegó [50].

Definition 1. *The polynomials*

$$p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

are called **apolar** *if*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k b_{n-k} = 0. \tag{4}$$

The equation (4) *is called the* **apolarity condition**.

Theorem 3 (Grace Apolarity Theorem)**.** *Let p and q be apolar, then any circular region containing all zeros of p or q contains at least a zero of the other.*

A circular region is a closed disk or a closed half-plain.

Lemma 2. *If $p(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m$ and $a_0 \neq 0$, then p has at least one zero in the disk $D(0, r)$, where $r = |a_m/a_0|^{1/m}$.*

Proof. The polynomial $q(z) = a_0 z^m - (-1)^m a_m$ is apolar to the polynomial p and all of the zeros of q lie in the disk $D(0, |a_m/a_0|^{1/m})$. The proof of the lemma follows from the Grace Apolarity Theorem 3. $\square$

Lemma 3. *If p is a monic polynomial of degree n, $p(a) = 0$ and*

$$|p^{(s)}(a)| \leq \frac{n!}{(n-s)!} r^{n-s}, \qquad s = 1, 2, \ldots, n-1,$$

then $\rho(a; A(p^{(s)})) \leq r$.

Proof. We have

$$p^{(s)}(a - z) = (-1)^{n-s} \frac{n!}{(n-s)!} z^{n-s} + \cdots + p^{(s)}(a).$$

By Lemma 2

$$\left| \frac{a_{n-s}}{a_0} \right| = \left| \frac{p^{(s)}(a)(n-s)!}{n!} \right| = r^{n-s},$$

or $r = |a_m/a_0|^{1/(n-s)}$. $\square$

2.3. Polynomial set $\mathcal{P}_n$. The linear transformation $z = au + b$, $a \neq 0$ transforms the zeros of a polynomial and the zeros of its derivatives in the same way. Therefore, we may consider only polynomials with zeros on the unit disk.

Let $\mathcal{P}_n$ be the set of all polynomials

$$p(z) = (z - z_1)(z - z_2) \cdots (z - z_n); \quad n \geq 2$$

for which the unit disk $D = D(0, 1) = \{z : \ |z| \leq 1\}$ is the smallest disk containing $z_1, z_2, \ldots, z_n$. With $A(p)$ we denote the set of all distinct zeros of the polynomial p and with $H(p) = H(A(p))$ the convex hull of $A(p)$. In the same way we use the notations

$$D(p) = D(A(p)) = D(c(A(p)), r(A(p))) = D(c(p), r(p)).$$

From the definition of $\mathcal{P}_n$, a simple and useful statement follows:

Lemma 4. *If $p \in \mathcal{P}_n$, then:*

1. *At least two zeros of p lie in the unit circle $C = \{z; \ |z| = 1\}$;*
2. *At least one zero of p lies in each subarc of C with length π.*

3. Deviations. For a polynomial $p \in \mathcal{P}_n$, we shall consider the deviation of one of the sets

$$A(p), H(p) \tag{5}$$

from one of the sets

$$A(p^{(s)}), H(p^{(s)}), \tag{6}$$

for $s = 1, 2, \ldots n - 1$.

By definition

$$A(p) \subset H(p) \quad \text{and} \quad A(p^{(s)}) \subset H(p^{(s)}). \tag{7}$$

If $\mathcal{U}$ is a compact set of polynomials, then the sets

$$A(\mathcal{U}) = \{A(p) : \ p \in \mathcal{U}\} \ \text{ and } \ H(\mathcal{U}) = \{H(p) : \ p \in \mathcal{U}\}$$

are also compact. Since for every deviation between a set (5) from a set (6) there exist extremal polynomials. The polynomial $\overline{p}$ is extremal for the deviation $\rho(A(p); A(p^{(s)}))$ in $\mathcal{P}_n$ if

$$\rho(A(\overline{p}); A(\overline{p}^{(s)})) = \sup\{\rho(A(p); A(p^{(s)})) : \ p \in \mathcal{P}_n\} = \overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(s)})). \tag{8}$$

3.1. Deviation of a particular zero. Let $p \in \mathcal{P}_n$ and $\{z_1, z_2, \ldots, z_n\}$ be the zeros of p. By definition, $|z_k| \le 1$ for $k = 1, 2, \ldots, n$. We call a zero z_k *peripheral*, if $|z_k| = 1$ and *internal*, if $|z_k| < 1$.

3.1.1. Peripheral zero. Following A. Meir and A. Sharma [34], we prove:

Lemma 5. *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then there exists a zero ζ_1 of $p^{(s)}(z)$ on the disk $D(z_1/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$.*

Proof. Without loss of generality, set $z_1 = 1$ and $p(z) = (z-1)q(z)$. Let $\zeta_1, \zeta_2, \ldots, \zeta_{n-s}$ be the zeros of $p^{(s)}$ and $\eta_1, \eta_2, \ldots, \eta_{n-s}$ be the zeros of $q^{(s-1)}$. By Gauss–Lucas Theorem $|\eta_k| \le 1;\ k = 1, 2, \ldots, n-s$. If $p^{(s)}(1) = 0$, we are done. Let $p^{(s)}(1) \ne 0$, then

$$\frac{p^{(s+1)}(1)}{p^{(s)}(1)} = \frac{s+1}{s} \cdot \frac{q^{(s)}(1)}{q^{(s-1)}(1)}$$

and

$$\sum_{j=1}^{n-s} \Re\left(\frac{1}{1-\zeta_j}\right) = \frac{s+1}{s} \sum_{k=1}^{n-s} \Re\left(\frac{1}{1-\eta_k}\right) \ge \frac{(s+1)(n-s)}{2s}.$$

If $\Re(1/(1-\zeta_1)) = \max\{\Re(1/(1-\zeta_j)) :\ j = 1, 2, \ldots, n-s\}$, then

$$\Re\left(\frac{1}{1-\zeta_1}\right) \ge \frac{s+1}{2s},$$

therefore

$$\zeta_1 \in D(1/(s+1), s/(s+1)) \subset D(1, 2s/(s+1)). \qquad \square$$

Corrolary 2. *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$ and $|z_1| = 1$, then there exists $\zeta_1 \in A(p^{(s)})$, such that*

$$|z_1 - \zeta_1| \le \frac{2s}{s+1}; \quad s = 1, 2, \ldots, n-1$$

and

$$\rho(z_1; A(p^{(s)})) \le \frac{2s}{s+1}; \quad s = 1, 2, \ldots, n-1.$$

3.1.2. Internal zero.

Lemma 6. *If $p \in \mathcal{P}_n$, $z_1 \in A(p)$, $|z_1| < 1$ and $s \in \{1, 2, \ldots, n-1\}$, then*

$$|p^{(s)}(z_1)| \le \frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4).$$

Proof. As $|z_1| < 1$, by Lemma 4, there are two zeros $z_2, z_3 \in A(p)$ such that $|z_2| = |z_3| = 1$,

$$\arg z_2 \leq \arg z_1 \leq \arg z_3 \quad \text{and} \quad 0 \leq \arg z_3 - \arg z_2 \leq \pi.$$

Then, it is geometrically obvious that

$$|z_1 - z_2||z_1 - z_3| \leq 2 \quad \text{and} \quad \left| z_1 - \frac{z_2 + z_3}{2} \right| \leq 1. \tag{9}$$

Using (9), we have

$$p^{(s)}(z_1) = s! \sum_{2 \leq k_1 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) =$$

$$s! \Bigg[(z_1 - z_2)(z_1 - z_3) \sum_{4 \leq k_3 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_3})(z_1 - z_{k_4}) \cdots (z_1 - z_{k_{n-s}}) +$$

$$2\left(z_1 - \frac{z_2 + z_3}{2}\right) \sum_{4 \leq k_2 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_2})(z_1 - z_{k_3}) \cdots (z_1 - z_{k_{n-s}}) +$$

$$\sum_{4 \leq k_1 \leq \cdots \leq k_{n-s} \leq n} (z_1 - z_{k_1})(z_1 - z_{k_2}) \cdots (z_1 - z_{k_{n-s}}) \Bigg]$$

and

$$|p^{(s)}(z_1)| \leq s! \left[2 \cdot 2^{n-s-2} \binom{n-3}{n-s-2} + 2 \cdot 2^{n-s-1} \binom{n-3}{n-s-1} + 2^{n-s} \binom{n-3}{n-s} \right] =$$

$$\frac{(n-3)!}{(n-s)!} 2^{n-s-1} s(n^2 + s^2 - 3n - 3s + 4). \qquad \square$$

Corrolary 3. *If $p \in \mathcal{P}_4$, $z_1 \in A(p)$ and $|z_1| < 1$, then*

$$|p''(z_1)| \leq 1 \quad \text{and} \quad \rho(z_1; A(p'')) \leq 1.$$

3.2. Deviation of $A(p)$. In this section we consider some easy cases of estimation of the deviation of $A(p)$ from $A(p^{(s)})$ and $H(p^{(s)})$; $n \geq s + 1$.

3.2.1.Deviation of $A(p)$ from $A(p^{(s)})$. We start with partial proofs of a generalization of Conjecture 1.

Conjecture 3. *For every $p \in \mathcal{P}_n$ and $s \in \{1, 2, \dots, n-1\}$, the inequality*

$$\rho(A(p); A(p^{(s)})) \le \frac{2s}{s+1}$$

holds.

For $n = 1$, Conjecture 3 is Conjecture 1. For $s = n-1$, Conjecture 3 is trivial, as $p^{(n-1)}(z)$ has only one zero $\zeta_1 = (z_1 + z_2 + \dots + z_n)/n$. Then

$$|z_1 - \zeta_1| \le \sup\left\{ z_1 - \frac{z_1 + z_2 + \dots + z_n}{n} \ : \ |z_k| \le 1, k = 1, 2, \dots, n \right\} = \frac{2(n-1)}{n}.$$

Theorem 4. *Conjecture 3 is true for a polynomial $p \in \mathcal{P}_n$ if all the corners of $H(p)$ are on the unit circle $C = \{z : \ |z| = 1\}$.*

P r o o f. If all the corners $z_{k_1}, z_{k_2}, \dots, z_{k_l}$ of $H(p)$ are on the unit circle (i. e., are peripheral zeros), then

$$H(p) \subset \bigcup_{m=1}^{l} D\left(\frac{z_{k_m}}{s+1}, \frac{s}{s+1} \right).$$

According to Lemma 5, if a zero $z_1 \in D(z_{k_1}/(s+1), s/(s+1)) \subset D(z_1, 2s/(s+1))$, then there exists a zero ζ_{k_1} of $p^{(s)}$, such that $|z_1 - \zeta_{k_1}| \le 2s/(s+1)$. $\square$

From Corollary 2 and Lemma 6, it follows that:

Theorem 5. *Conjecture 3 is true for a given pair (s, n) if*

$$s\frac{n^2 + s^2 - 3n - 3s + 4}{n(n-1)(n-2)} \le 2\left(\frac{s}{s+1}\right)^{n-s}. \tag{10}$$

Corrolary 4. *Conjecture 3 is true for:*

1) $s = n-1$ *and* $n \ge 2$*;*
2) $s = n-2$ *and* $n \ge 3$*;*
3) $s = n-3$ *and* $n \ge 4$*;*
4) $s = n-4$ *and* $n \ge 6$*.*

From Corollary 4, we have that Conjecture 1 is true for $n = 3$ and $n = 4$, which is a simple proof of the conjecture for this cases. As we already mentioned, Conjecture 1 is proved for every natural $n \le 8$ [12].

Theorem 6. *For every natural $s \geq 2$ and $n \geq s+1$, the inequality*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(s)})) > 1$$

holds, see (8).

Proof. Consider the polynomial

$$p(z) = z^n - z - 1 \quad \text{with} \quad p^{(s)}(z) = \frac{n!}{s!} z^{n-s}.$$

If $\rho(A(p); A(p^{(s)})) = 1$, then the center of the disk $D(p)$ is in the origin, together with all zeros of $p^{(s)}(z)$. Suppose that $D(p) = D(0, r)$. Then, by Lemma 4, there exist two zeros $z_k = re^{i\varphi_k}$; $k = 1, 2$ of $p(z)$ with $0 \leq \varphi_1 < \varphi_2 \leq \pi$. For $k = 1, 2$, we have

$$p(re^{i\varphi_k}) = r^n e^{in\varphi_k} - re^{i\varphi_k} - 1 = r^n \cos n\varphi_k - r\cos\varphi_k - 1 + i(r^n \sin n\varphi_k - r\sin\varphi_k) = 0,$$

or

$$\cos\varphi_k = \frac{r^{2n} - r^2 - 1}{2r} \quad \text{for} \quad k = 1, 2,$$

which is a contradiction, as $0 \leq \varphi_1 < \varphi_2 \leq \pi$. □

3.2.2. Deviation of $A(p)$ from $A(p')$. In this subsection we list some cases, when Conjecture 1 is proved.

In 1969 G. Schmeisser [46] proved the following statement.

Theorem 7. *Conjecture* 1 *is true for a polynomial $p \in \mathcal{P}_n$ if $p(0) = 0$.*

Proof. An elegant geometric proof is given in G. Gacs (1971) [18], based on Theorem 2. Let $z_1 \neq 0$ be a zero of p and l be the bisector of the segment from z_1 to 0. Let L be the half-plane defined by l and $0 \notin L$. Then by Theorem 2 and Gauss–Lucas Theorem it follows that $(L \bigcap D) \subset D(z_1, 1)$ contains at least one zero of p'. □

Remark 1. Conjecture 1 is trivially true for every $p \in \mathcal{P}_n$, if $p'(0) = 0$.

Let $p \in \mathcal{P}_n$ and have the form

$$p(z) = \sum_{k=0}^{m} a_k z^{n_k}; \quad n = n_m > n_{m-1} > \cdots > n_0. \tag{11}$$

If $m < n$, the polynomial (11) is lacunary.

Theorem 8 (G. Schmeisser)**.** *Conjecture* 1 *is true for the polynomial* (11), *if $n \geq 3m - 2$.*

Theorem 9 (G. Schmeisser). *Conjecture* 1 *is true for* $p \in \mathcal{P}_n$, *if*

$$p(z) = z^n + a_2 z^{n_2} + a_1 z^{n_1} + a_0 z^{n_0}, \quad \text{where} \quad n > n_2 > n_1 > n_0.$$

D. Phelps and R. S. Rodriguez [40] proved Conjecture 1 for all polynomials p with $H(p)$ a segment.

In 1977 G. Schmeisser [48] proved Conjecture 1 for polynomials p with a convex hull $H(p)$, which is a triangle. The proof is purely geometric.

Conjecture 7 (G. Schmeisser). *Conjecture* 1 *is true for all* $p \in \mathcal{P}_n$ *for which* $H(p)$ *is a quadrangle.*

3.2.3. Deviation of $A(p)$ from $A(p'')$. It is natural to expect that the problem to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}''_n))$ (see (8)) is much more difficult than to find $\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}'_n))$. Most probably, Conjecture 3 is not sharp for $s = 2, 3, \dots, n-2$.

It is trivial that $\overline{\rho}(A(\mathcal{P}_3); A(\mathcal{P}''_3)) = 4/3$.

Conjecture 8. $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}''_4)) = 2/\sqrt{3}$.

This conjecture is motivated by the polynomial $p(z) = \left(z^2 + \frac{2}{3}z + 1\right)(z-1)^2$, as for this polynomial $\rho(A(p); A(p'')) = 2/\sqrt{3}$, and by the following statement.

Theorem 10. *Conjecture* 8 *is true for a polynomial* $p \in \mathcal{P}_4$, *if* p *has real coefficients.*

P r o o f. Suppose the contrary, that $\overline{\rho}(A(\mathcal{P}_4); A(\mathcal{P}''_4)) > 2/\sqrt{3}$. Let p be an extremal polynomial for $\rho(A(p); A(p''))$ in $\mathcal{P}_4$ and z_1 be an extremal zero of p. Then, from Corollary 3 follows that $|z_1| = 1$.

Consider the two possible cases, when p has real coefficients:

1) z_1 is a real extremal zero of p, then $z_1 = -1$ or 1. We may suppose that $z_1 = 1$ and that p has the form

$$p(z) = (z^2 + 2\alpha z + 1)(z - \beta)(z - 1) = z^4 + (2\alpha - 1 - \beta)z^3 + (1 + \beta)(1 - 2\alpha)z^2 + \cdots,$$

where $\alpha \in [0, 1]$ and $\beta \in [-1, 1]$.

Then

$$\frac{1}{12}p''(z) = z^2 - \frac{x + y}{2}z + \frac{1}{6}y(1 + x),$$

with zeros

$$\zeta_{1,2} = \frac{x + y}{4} \pm \frac{1}{2}\sqrt{\Delta(x, y)}, \tag{12}$$

where $y = 1 - 2\alpha \in [-1, 1]$, $x = \beta \in [-1, 1]$ and

$$\Delta(x, y) = \left(\frac{x+y}{2}\right)^2 - \frac{2}{3}y(1+x).$$

From $\Delta(x, y) = 0$ and $-1 \leq y \leq 1$ follows that

$$y = y_1(x) = \frac{1}{3}\left(x + 4 - \sqrt{8(2-x)(1+x)}\right).$$

1.1) For

$$\Delta(x, y) \geq 0, \quad y \leq \frac{1}{3}\left(x + 4 - \sqrt{8(2-x)(1+x)}\right)$$

we have

$$1 - \zeta_1 = 1 - \frac{x+y}{4} - \frac{1}{2}\sqrt{\Delta(x, y)}.$$

Suppose that

$$1 - \zeta_1 > \frac{2}{\sqrt{3}}$$

or

$$-\frac{x+y}{4} - \frac{1}{2}\sqrt{\Delta(x, y)} > \frac{2}{\sqrt{3}} - 1 = \lambda,$$

hence

$$-\lambda - \frac{x+y}{4} > \frac{1}{2}\sqrt{\delta(x, y)} \geq 0 \tag{13}$$

and

$$y < -x - 4\lambda. \tag{14}$$

From (13) we have

$$\lambda^2 + \frac{x+y}{2}\lambda > -\frac{1}{6}y(1+x),$$

or

$$y > -\frac{3\lambda(2\lambda + x)}{1 + x + 3\lambda}. \tag{15}$$

From (14) and (15) it follows that

$$x + 4\lambda < \frac{3\lambda(2\lambda + x)}{1 + x + 3\lambda}$$

or

$$x^2 + (1 + 4\lambda)x + 4\lambda + 6\lambda^2 < 0,$$

which is impossible for $-1 \le x \le 1$.

1.2) For

$$\Delta(x,y) \le 0, \quad y \ge \frac{1}{3}\left(x + 4 - \sqrt{8(2-x)(1+x)}\right),$$

we have

$$|1-\zeta_1|^2 = \left(1 - \frac{x+y}{4}\right)^2 - \frac{1}{4}\Delta(x,y) = 1 - \frac{x+y}{2} + \frac{1}{6}y(1+x).$$

Suppose that

$$|1-\zeta_1|^2 > \frac{4}{3},$$

or

$$y < -\frac{2+3x}{2-x}.$$

Then

$$\frac{1}{3}(x + 4 - \sqrt{8(2-x)(1+x)}) < -\frac{2+3x}{2-x},$$

or

$$(2-x)(x + 4 - \sqrt{8(2-x)(1+x)}) > 14 + 11x - x^2,$$

which is impossible for $-1 \le x \le 1$.

2) z_1 is a complex extremal zero of p. In this case $z_2 = \overline{z}_1$ is also a zero of p. We may suppose that

$$z_1 = -\alpha + i\sqrt{1-\alpha^2}, \;\; z_2 = -\alpha - i\sqrt{1-\alpha^2}, \;\; \alpha \in [0,1).$$

Then, the circles $C(z_1, 2/\sqrt{3})$ and $C(z_2, 2/\sqrt{3})$ cross the positive real axis in the point

$$x = \alpha + \sqrt{\alpha^2 + 1/3} > (1-\alpha)/2,$$

therefore there is at least one zero in every one of the disks $D(z_1, 2/\sqrt{3})$ and $D(z_2, 2/\sqrt{3})$. This contradicts to the supposition that $\overline{\rho}(A\mathcal{P}_4); A\mathcal{P}_4'') > 2/\sqrt{3}$. $\square$

3.2.4. Deviation of $A(p)$ from $A(p^{(n-2)})$. It is trivial that

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-1)})) = 2 - \frac{2}{n}.$$

An extremal polynomial is $p(z) = (z-1)(z+1)^{n-1}$.

Lemma 7. *For $n \geq 5$, the inequalities*

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-2)})) \geq \begin{cases} 2\dfrac{n-1}{n+1} & \text{for} \quad n \ \text{odd}, \\ 1 + \sqrt{\dfrac{(n-2)(n-4)}{n(n+2)}} & \text{for} \quad n \ \text{even} \end{cases}$$

hold.

P r o o f. For $n = 2m+1$, consider the polynomial

$$p(z) = \left(z^2 + \frac{2m}{m+1}z + 1\right)^m (z-1) \tag{16}$$

with

$$p^{(2m-1)}(z) = \frac{(2m+1)!}{2}\left(z + \frac{m-1}{m+1}\right)^2.$$

$p^{(2m-1)}$ has a double zero $\zeta_1 = \zeta_2 = -(m-1)/(m+1)$ and

$$\rho(A(p); A(p^{(n-2)})) = |1 - \zeta_1| = \frac{2m}{m+1} = 2\frac{n-1}{n+1}.$$

For $n = 2m+2$, consider the polynomial

$$p(z) = \left(z^2 + 2\sqrt{\frac{m^2-1}{m(m+2)}}z + 1\right)^m (z^2 - 1) \tag{17}$$

with

$$p^{(2m)}(z) = \frac{(2m+2)!}{2}\left(z + \sqrt{\frac{m(m-1)}{(m+1)(m+2)}}\right)^2.$$

$p^{(2m)}$ has a double zero $\zeta_1 = \zeta_2 = -\sqrt{m(m-1)/(m+1)(m+2)}$ and

$$\rho(A(p); A(p^{(n-2)})) = |1 - \zeta_1| = 1 + \sqrt{\frac{(n-2)(n-4)}{n(n+2)}}. \qquad \square$$

We conjecture that

$$\overline{\rho}(A(\mathcal{P}_n); A(\mathcal{P}_n^{(n-2)})) = \begin{cases} 2\dfrac{n-1}{n+1} & \text{for} \quad n \ \text{odd}, \\ 1 + \sqrt{\dfrac{(n-2)(n-4)}{n(n+2)}} & \text{for} \quad n \ \text{even} \end{cases}$$

Observe that the polynomials (16) and (17) agree with Conjecture 4.

3.2.5. Deviation of $A(p)$ from $H(p^{(s)})$. From (7) and Lemma 1 it follows that for every $p \in \mathcal{P}_n$,

$$\rho(A(p); A(p^{(s)})) \geq \rho(A(p); H(p^{(s)})). \tag{18}$$

Then, we formulate a weaker conjecture than Conjecture 3.

Conjecture 9. *For every $n \geq s+1$, the following inequalities hold:*

$$\overline{\rho}(A(\mathcal{P}_n); H(\mathcal{P}_n^{(s)})) \leq \frac{2s}{s+1}, \tag{19}$$

$s = 1, 2, \ldots, n-1$.

Observe that Conjecture 9 follows from Conjecture 3 and Conjecture 4.

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