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RELIABILITY FOR BETA MODELS

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Abstract. In the area of stress-strength models there has been a large amount of work as regards estimation of the reliability $R = \Pr(X_2 < X_1)$ when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. The algebraic form for $R = \Pr(X_2 < X_1)$ has been worked out for the majority of the well-known distributions including Normal, uniform, exponential, gamma, weibull and pareto. However, there are still many other distributions for which the form of R is not known. We have identified at least some 30 distributions with no known form for R. In this paper we consider some of these distributions and derive the corresponding forms for the reliability R. The calculations involve the use of various special functions.

1. Introduction. In the context of reliability the stress–strength model describes the life of a component which has a random strength X_1 and is subjected to random stress X_2. The component fails at the instant that the stress applied to

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it exceeds the strength and the component will function satisfactorily whenever $X_1 > X_2$. Thus $R = \Pr(X_2 < X_1)$ is a measure of component reliability. It has applications in many areas. For example, if X_2 represents the maximum chamber pressure generated by ignition of a solid propellant and X_1 represents the strength of the rocket chamber then R is the probability of successful firing of the engine. Another example is when X_2 represents the diameter of a shaft and X_1 represents the diameter of a bearing that is to be mounted on the shaft – here R is the probability that the bearing fits without interference. Because of these applications, the calculation and the estimation of $R = \Pr(X_2 < X_1)$ is important. This has been investigated extensively in the literature when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions. The algebraic form for R has been worked out for the majority of the well-known distributions in their standard forms. These include Normal, uniform, exponential, gamma, weibull and the pareto distributions. However, we have identified many other distributions including extensions of the above distributions for which the form of R is not known. In this paper we consider the class of beta distributions and derive the corresponding forms for R.

We shall assume throughout this paper that X_1 and X_2 are continuous and independent random variables. Let f_i and F_i denote, respectively, the probability density function (pdf) and the cumulative distribution function (cdf) of X_i. With this notation, we can write

$$\begin{aligned} R &= \Pr\left(X_2 < X_1\right) \\ &= \int_{-\infty}^{\infty} F_2(x) f_1(x) dx. \end{aligned} \tag{1}$$

The calculations of (1) will make use of the following special functions: the gamma function defined by

$$\Gamma(a) = \int_0^{\infty} z^{a-1} \exp(-z) dz;$$

the beta function defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)};$$

the incomplete beta function ratio defined by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^{x} w^{a-1} (1-w)^{b-1} dw; \tag{2}$$

and, the generalized hypergeometric function defined by

$$(3) \qquad {}_pF_q\left(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x\right) = \sum_{k=0}^{\infty} \frac{\left(\alpha_1\right)_k \left(\alpha_2\right)_k \cdots \left(\alpha_p\right)_k}{\left(\beta_1\right)_k \left(\beta_2\right)_k \cdots \left(\beta_q\right)_k} \frac{x^k}{k!}.$$

When $p = 1$ and $q = 1$, (3) is known as the confluent hypergeometric function. When $p = 2$ and $q = 1$, (3) is known as the Gauss hypergeometric function. Some properties of the incomplete beta function ratio that we shall need are:

$$(4) \qquad I_x(a, b) = \frac{x^a}{aB(a, b)} \, {}_2F_1(a, 1-b; 1+a; x),$$

$$(5) \qquad I_x(a, n-a+1) = \sum_{k=a}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{if } a \text{ is an integer},$$

$$(6) \qquad I_x(a, b) = 1 - \sum_{k=1}^{a} \frac{\Gamma(b+k-1)}{\Gamma(b)\Gamma(k)} x^{k-1} (1-x)^b, \quad \text{if } a \text{ is an integer},$$

$$(7) \qquad I_x(a, b) = \sum_{k=1}^{b} \frac{\Gamma(a+k-1)}{\Gamma(a)\Gamma(k)} x^a (1-x)^{k-1}, \quad \text{if } b \text{ is an integer},$$

and

$$(8) \qquad I_x\left(k - \frac{1}{2}, j - \frac{1}{2}\right) = \frac{2}{\pi} \arctan \sqrt{\frac{x}{1-x}} + \sum_{l=1}^{j-1} c_l - \sqrt{x(1-x)} \sum_{l=1}^{k-1} d_l,$$

where

$$c_l = \frac{\Gamma(k+l-1)}{\Gamma(k-1/2)\Gamma(l+1/2)} x^{k-1/2} (1-x)^{l-1/2},$$

$$d_l = \frac{\Gamma(l)}{\Gamma(l+1/2)\Gamma(1/2)} x^{l-1}.$$

Some properties of the generalized hypergeometric function that we shall need are:

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c > a+b, \tag{9}$$

$$ {}_3F_2(a,b,c;a+1,b+1;1) = \frac{ab}{a-b}\Gamma(1-c)\left\{\frac{\Gamma(b)}{\Gamma(1+b-c)} - \frac{\Gamma(a)}{\Gamma(1+a-c)}\right\}, \quad a \neq b, \quad c \neq 1, \quad c < 2, \tag{10}$$

$$ \int_0^1 z^{a-1}(1-z)^{b-1}\, {}_2F_1(c,d;e;z)dz = B(a,b)\, {}_3F_2(c,d,a;e,a+b;1), \quad a > 0, \quad b > 0, \quad e+b > c+d \tag{11}$$

and

$$ \int_0^x z^{a-1}(x-z)^{b-1}\, {}_2F_1(c,d;e;z)dz = B(a,b)x^{a+b-1}\, {}_3F_2(c,d,a;e,a+b;x), \quad x > 0, \quad a > 0, \quad b > 0. \tag{12}$$

We shall also need the following properties:

$$ \int_0^\infty \frac{\arctan z}{z}\left(\frac{z^p}{1+z^{2p}}\right)^{2q} dz = \frac{\pi^{3/2}}{2^{2q+2}p}\frac{\Gamma(q)}{\Gamma(q+1/2)}, \quad q > 0, \tag{13}$$

$$ \int_0^x z^{d-1}(z+a)^c dz = \frac{a^c x^d}{d}\, {}_2F_1\left(-c,d;1+d;-\frac{x}{a}\right), \quad d > 0 \tag{14}$$

and

$$ \int_0^x z^{b-1}(x-z)^{a-1}\exp(cz)dz = B(a,b)x^{a+b-1}\, {}_1F_1(b;a+b;cx), \quad a > 0, \quad b > 0. \tag{15}$$

Further properties of the special functions being used can be found in Prudnikov et al. [9] and Gradshteyn and Ryzhik [3].

2. Standard Beta Distribution

For the standard form of the beta distribution, the pdf and the cdf of X_i are

$$f_i(x) = \frac{1}{B(a_i, b_i)} x^{a_i-1}(1-x)^{b_i-1} \tag{16}$$

and

$$F_i(x) = I_x(a_i, b_i), \tag{17}$$

(where $0 \leq x \leq 1$), respectively. These distributions are very versatile and a variety of uncertainties can be usefully modeled by them. In recent years beta distributions have attracted applications in activity in PERT analysis, breakage models, communication theory, gas absorption, hydrology, particle size distributions, photovoltaic system analysis, sea-state reflectivity, solar radiation and other areas. For (16) and (17), the reliability given by (1) takes the form:

$$R = \int_0^1 \frac{I_x(a_2, b_2)}{B(a_1, b_1)} x^{a_1-1}(1-x)^{b_1-1} dx. \tag{18}$$

This can be evaluated by applying (4) to re-express $I_x(a_2, b_2)$ and then using (11) to calculate the integral:

$$\begin{aligned} R &= \frac{1}{a_2 B(a_1, b_1) B(a_2, b_2)} \int_0^1 x^{a_1+a_2-1}(1-x)^{b_1-1} {}_2F_1(a_2, 1-b_2; 1+a_2; x)\, dx \\ &= \frac{B(a_1+a_2, b_1)\ {}_3F_2(a_2, 1-b_2, a_1+a_2; 1+a_2, a_1+a_2+b_1; 1)}{a_2 B(a_1, b_1) B(a_2, b_2)}. \end{aligned} \tag{19}$$

This expression for R can be reduced to elementary forms for the particular choices of the parameters a_i and b_i considered below.

Case 1: If $a_1 + a_2 + b_1 + b_2 = 1$ then, using (9), we have

$$\begin{aligned} & {}_3F_2(a_2, 1-b_2, a_1+a_2; 1+a_2, a_1+a_2+b_1; 1) \\ = & {}_2F_1(a_2, a_1+a_2; 1+a_2; 1) \\ = & \frac{\Gamma(1+a_2)\Gamma(1-a_1-a_2)}{\Gamma(1-a_1)} \\ = & \frac{1-a_1}{B(1+a_2, 1-a_1-a_2)}; \end{aligned}$$

so, (19) reduces to:

$$R = \frac{(1-a_1) B(a_1+a_2, b_1)}{a_2 B(a_1, b_1) B(a_2, b_2) B(1+a_2, 1-a_1-a_2)}.$$

Case 2: If $a_1 = 1$ then, again using (9), we have

$$\begin{aligned} & {}_3F_2(a_2, 1-b_2, a_1+a_2; 1+a_2, a_1+a_2+b_1; 1) \\ = & {}_2F_1(a_2, 1-b_2; 1+a_2+b_1; 1) \\ = & \frac{\Gamma(1+a_2+b_1)\Gamma(b_1+b_2)}{\Gamma(1+b_1)\Gamma(a_2+b_1+b_2)} \\ = & \frac{B(1+a_2+b_1, b_1+b_2)}{B(1+b_1, a_2+b_1+b_2)}; \end{aligned}$$

so, (19) reduces to:

$$R = \frac{B(a_1+a_2, b_1) B(1+a_2+b_1, b_1+b_2)}{a_2 B(a_1, b_1) B(a_2, b_2) B(1+b_1, a_2+b_1+b_2)}.$$

Case 3: If $b_1 = 1$ then, using (10), we have

$$\begin{aligned} & {}_3F_2(a_2, 1-b_2, a_1+a_2; 1+a_2, a_1+a_2+b_1; 1) \\ = & \frac{a_2(a_1+a_2)}{(-a_1)}\Gamma(b_2)\left\{\frac{\Gamma(a_1+a_2)}{\Gamma(a_1+a_2+b_2)} - \frac{\Gamma(a_2)}{\Gamma(a_2+b_2)}\right\} \\ = & \frac{a_2(a_1+a_2)}{a_1}\left\{B(a_2, b_2) - B(a_1+a_2, b_2)\right\}; \end{aligned}$$

so, (19) reduces to:

$$R = \frac{(a_1+a_2) B(a_1+a_2, b_1)\left\{B(a_2, b_2) - B(a_1+a_2, b_2)\right\}}{a_1 B(a_1, b_1) B(a_2, b_2)}.$$

Case 4: If $a_2 = a$ and $b_2 = n-a+1$ (where a is an integer) then, substituting (5) into (18), we have

$$\begin{aligned} R &= \sum_{k=a}^{n} \frac{1}{B(a_1, b_1)} \binom{n}{k} \int_0^1 x^{a_1+k-1}(1-x)^{b_1+n-k-1} dx \\ &= \sum_{k=a}^{n} \binom{n}{k} \frac{B(a_1+k, b_1+n-k)}{B(a_1, b_1)}. \end{aligned}$$

<u>Case 5</u>: If a_2 is an integer then, substituting (6) into (18), we have

$$R = 1 - \sum_{k=1}^{a_2} \frac{1}{B(a_1, b_1)} \frac{\Gamma(b_2 + k - 1)}{\Gamma(b_2)\Gamma(k)} \int_0^1 x^{a_1+k-2}(1-x)^{b_1+b_2-1} dx$$

$$= 1 - \sum_{k=1}^{a_2} \frac{B(a_1 + k - 1, b_1 + b_2)}{(b_2 + k - 1) B(a_1, b_1) B(b_2, k)}.$$

In the particular case $a_2 = 1$,

$$R = 1 - \frac{B(a_1, b_1 + b_2)}{B(a_1, b_1)}.$$

<u>Case 6</u>: Similarly, if b_2 is an integer, substituting (7) into (18), we have

$$R = \sum_{k=1}^{b_2} \frac{B(a_1 + a_2, b_1 + k - 1)}{(a_2 + k - 1) B(a_1, b_1) B(a_2, k)}.$$

In the particular case $b_2 = 1$,

$$R = \frac{B(a_1 + a_2, b_1)}{B(a_1, b_1)}.$$

<u>Case 7</u>: Finally, if $a_2 = k - 1/2$ and $b_2 = j - 1/2$, substituting (8) into (18), we have

$$R = \frac{2}{\pi B(a_1, b_1)} \int_0^1 \arctan \sqrt{\frac{x}{1-x}} x^{a_1-1}(1-x)^{b_1-1} dx$$
$$+ \sum_{l=1}^{j-1} \frac{B(a_1 + k - 1/2, b_1 + l - 1/2)}{(k + l - 1) B(k - 1/2, l + 1/2) B(a_1, b_1)}$$
$$- \sum_{l=1}^{k-1} \frac{B(a_1 + l - 1/2, b_1 + 1/2)}{l B(l + 1/2, 1/2) B(a_1, b_1)}. \tag{20}$$

On substituting $y = \sqrt{x/(1-x)}$, the integral term reduces to

$$2 \int_0^\infty \frac{y^{2a_1-1} \arctan y}{(1 + y^2)^{a_1+b_1}} dy. \tag{21}$$

This integral cannot be simplified further for general a_1 and b_1. However, in the particular case $a_1 = b_1 = c$ (say), using (13) with $p = 1$ and $q = c$, (21) can be reduced to

$$\frac{\pi^{3/2}}{2^{2c+1}} \frac{\Gamma(c)}{\Gamma(c+1/2)}.$$

Substituting this into (20), we get

$$R = \frac{\sqrt{\pi}\Gamma(2c)}{4^c\Gamma(c)\Gamma(c+1/2)} + \sum_{l=1}^{j-1} \frac{B(c+k-1/2, c+l-1/2)}{(k+l-1)B(k-1/2, l+1/2)B(c,c)} - \sum_{l=1}^{k-1} \frac{B(c+l-1/2, c+1/2)}{lB(l+1/2, 1/2)B(c,c)}$$

for the particular case $a_1 = b_1 = c$.

3. Uniform distribution. If X_i have the uniform distribution then the pdf and the cdf are

$$f_i(x) = \frac{1}{d_i - c_i} \tag{22}$$

and

$$F_i(x) = \frac{x - c_i}{d_i - c_i}, \tag{23}$$

(where $c_i \leq x \leq d_i$), respectively. These distributions have found extensive applications in life testing and traffic flow modeling. The standard forms of (22)–(23) when $c_i = 0$ and $d_i = 1$ are particular cases of the beta distribution in (16)–(17). The form of the reliability for (22)–(23) is easy to calculate as shown below:

$$\begin{aligned} R &= \int_{\max(c_1,c_2)}^{\min(d_1,d_2)} \frac{x - c_2}{d_2 - c_2} \frac{1}{d_1 - c_1} dx + \int_{\min(d_1,d_2)}^{d_1} \frac{1}{d_1 - c_1} dx \\ &= \frac{\{\min(d_1,d_2)\}^2 - \{\max(c_1,c_2)\}^2 + 2c_2\{\max(c_1,c_2) - \min(d_1,d_2)\}}{2(d_1 - c_1)(d_2 - c_2)} \\ &\quad + \frac{d_1 - \min(d_1,d_2)}{d_1 - c_1}. \end{aligned}$$

In the standard case $c_1 = c_2 = 0$ and $d_1 = d_2 = 1$, we have $R = 1/2$ as expected (this is also true if $c_1 = c_2$ and $d_1 = d_2$).

4. Power Function Distribution. For a power function distribution the pdf and the cdf of X_i are

$$f_i(x) = a_i \left(\frac{x - c_i}{d_i - c_i} \right)^{a_i - 1} \tag{24}$$

and

$$F_i(x) = (d_i - c_i) \left(\frac{x - c_i}{d_i - c_i} \right)^{a_i}, \tag{25}$$

(where $c_i \le x \le d_i$), respectively. These distributions have found recent applications in electrical component reliability (Meniconi and Barry, [6]) and mineral resource prediction (Shen and Zhao, [11]). The standard forms of (24)–(25) when $c_i = 0$ and $d_i = 1$ are particular cases of the beta distribution in (16)–(17). The reliability for (24)–(25) takes the form:

$$\begin{aligned} R = {} & a_1 \int_{\max(c_1,c_2)}^{\min(d_1,d_2)} (d_2 - c_2) \left(\frac{x - c_2}{d_2 - c_2} \right)^{a_2} \left(\frac{x - c_1}{d_1 - c_1} \right)^{a_1 - 1} dx \\ & + a_1 \int_{\min(d_1,d_2)}^{d_1} \left(\frac{x - c_1}{d_1 - c_1} \right)^{a_1 - 1} dx. \end{aligned} \tag{26}$$

The second term on the right of (26) is:

$$1 - F_1\left(\min\left(d_1, d_2\right)\right) = 1 - (d_1 - c_1) \left(\frac{\min\left(d_1, d_2\right) - c_1}{d_1 - c_1} \right)^{a_1}.$$

The first term on the right of (26), say J, can be evaluated using (14). If $c_1 > c_2$ then, substituting $y = x - c_1$, we see

$$J = \frac{a_1}{(d_1 - c_1)^{a_1 - 1} (d_2 - c_2)^{a_2 - 1}} \int_0^{\min(d_1,d_2) - c_1} (y + c_1 - c_2)^{a_2} y^{a_1 - 1} dy$$

which reduces to

$$J = \frac{(c_1 - c_2)^{a_2} \left\{\min\left(d_1, d_2\right) - c_1\right\}^{a_1}}{(d_1 - c_1)^{a_1 - 1} (d_2 - c_2)^{a_2 - 1}} \, {}_2F_1 \left(-a_2, a_1; 1 + a_1; \frac{\min\left(d_1, d_2\right) - c_1}{c_2 - c_1} \right)$$

on applying (14). Similarly, if $c_2 > c_1$ then

$$J = \frac{a_1(c_2-c_1)^{a_1-1}\{\min(d_1,d_2)-c_2\}^{1+a_2}}{(1+a_2)(d_1-c_1)^{a_1-1}(d_2-c_2)^{a_2-1}} {}_2F_1\left(1-a_1,1+a_2;2+a_2;\frac{\min(d_1,d_2)-c_2}{c_1-c_2}\right).$$

If $c_1 = c_2 = c$ (say) then it is easy to see

$$J = \frac{a_1\left\{\min\left(d_1,d_2\right)-c\right\}^{a_1+a_2}}{\left(a_1+a_2\right)\left(d_1-c\right)^{a_1-1}\left(d_2-c\right)^{a_2-1}}.$$

5. Arc-sine distribution. For an arc-sine distribution the pdf and the cdf of X_i are

$$f_i(x) = \frac{1}{\pi\sqrt{\left(4/c_i^2\right)-x^2}} \tag{27}$$

and

$$F_i(x) = \frac{1}{2}+\frac{1}{\pi}\arcsin\left(\frac{\mid c_i\mid x}{2}\right), \tag{28}$$

(where $-2/\mid c_i\mid \leq x \leq 2/\mid c_i\mid$), respectively. The arc-sine distribution arises naturally in statistical communication theory; see Lee [5, Chapter 6] and Middleton [7, Chapter 14], where (27) is used as a model for the amplitude of a periodic signal in thermal noise and the limiting spectral density function of a high-index-angle modulated carrier, respectively. The arc-sine distribution arises also in the study of the simple random walk. The standard forms of (27) and (28) when $c_i = 2$ are particular cases of the beta distribution in (16)–(17). The reliability, (1), for (27)–(28) can be written as:

$$R = \frac{1}{2\pi}\int_{-\delta}^{\delta}\left(\frac{4}{c_1^2}-x^2\right)^{-1/2}dx + \frac{1}{\pi^2}\int_{-\delta}^{\delta}\arcsin\left(\frac{\mid c_2\mid x}{2}\right)\left(\frac{4}{c_1^2}-x^2\right)^{-1/2}dx$$

$$+\frac{1}{\pi}\int_{\delta}^{2/|c_1|}\left(\frac{4}{c_1^2}-x^2\right)^{-1/2}dx, \tag{29}$$

where $\delta = 2\min(1/\mid c_1\mid, 1/\mid c_2\mid)$. The first and the third terms on the right of (29) are equal to:

$$\frac{1}{2}\left\{F_1\left(\delta\right)-F_1\left(-\delta\right)\right\} = \frac{1}{\pi}\arcsin\left[\frac{\mid c_1\mid}{\max\left(\mid c_1\mid,\mid c_2\mid\right)}\right]$$

and

$$1 - F_1\left(\delta\right) = \frac{1}{2} - \frac{1}{\pi} \arcsin\left[\frac{\mid c_1 \mid}{\max\left(\mid c_1 \mid, \mid c_2 \mid\right)}\right],$$

respectively. The second term on the right of (29) is zero because its integrand is an odd function. Thus for the arc-sine distribution $R = 1/2$ irrespective of what c_1 and c_2 are.

6. Generalized Beta distribution. A four-parameter generalization of the standard beta distribution in (16) and (17) is given by

$$f_i(x) = \frac{1}{\left(d_i - c_i\right) B\left(a_i, b_i\right)} \left(\frac{x - c_i}{d_i - c_i}\right)^{a_i - 1} \left(1 - \frac{x - c_i}{d_i - c_i}\right)^{b_i - 1} \tag{30}$$

and

$$F_i(x) = I_{\frac{x - c_i}{d_i - c_i}}\left(a_i, b_i\right), \tag{31}$$

where $c_i \leq x \leq d_i$. These contain the uniform and the power function distributions (discussed above) as special cases. Setting $Y_i = (X_i - c_i)/(d_i - c_i)$, one can see (30) and (31) reduce to the standard forms. So, if $c_1 = c_2$ and $d_1 = d_2$ then $R = \Pr(X_2 < X_1) = \Pr(Y_2 < Y_1)$ takes the expression given in (19). In the general case, R can be written as

$$\begin{aligned} R = {} & \frac{1}{(d_1-c_1)B(a_1,b_1)} \int_{\max(c_1,c_2}^{\min(d_1,d_2)} I_{\frac{x-c_2}{d_2-c_2}}(a_2,b_2) \left(\frac{x-c_1}{d_1-c_1}\right)^{a_1-1} \left(1-\frac{x-c_1}{d_1-c_1}\right)^{b_1-1} dx \\ & + \frac{1}{(d_1-c_1)B(a_1,b_1)} \int_{\min(d_1,d_2)}^{d_1} \left(\frac{x-c_1}{d_1-c_1}\right)^{a_1-1} \left(1-\frac{x-c_1}{d_1-c_1}\right)^{b_1-1} dx. \end{aligned} \tag{32}$$

The second term on the right of (32) is

$$1 - F_1\left(\min\left(d_1, d_2\right)\right) = 1 - I_{\frac{\min(d_1,d_2)-c_1}{d_1-c_1}}\left(a_1, b_1\right).$$

On using (4) and substituting $y = (x - a_2)/(b_2 - a_2)$, the first term on the right of (32) can be re-written as

$$\frac{\left(c_2 - c_1\right)^{a_1 - 1} \left(d_1 - c_2\right)^{b_1 - 1} \left(d_2 - c_2\right)}{a_2 \left(d_1 - c_1\right)^{a_1 + b_1 - 1} B\left(a_1, b_1\right) B\left(a_2, b_2\right)} J,$$

where

$$(33)\quad J = \int_{\delta_1}^{\delta_2} y^{a_2} \left(1+\frac{d_2-c_2}{c_2-c_1}y\right)^{a_1-1} \left(1-\frac{d_2-c_2}{d_1-c_2}y\right)^{b_1-1} {}_2F_1(a_2, 1-b_2; 1+a_2; y) dy$$

with $\delta_1 = \{\max(c_1, c_2) - c_2\}/(d_2 - c_2)$ and $\delta_2 = \{\min(d_1, d_2) - c_2\}/(d_2 - c_2)$. This integral cannot be simplified further for general a_1 and b_1. However, if we assume that $a_1 > 1$ and $b_1 > 1$ are both integers then using binomial expansion we can re-write (33) as

$$J = \sum_{k=0}^{a_1-1} \sum_{l=0}^{b_1-1} (-1)^l \binom{a_1-1}{k} \binom{b_1-1}{l} \left(\frac{d_2-c_2}{c_2-c_1}\right)^k \left(\frac{d_2-c_2}{d_1-c_2}\right)^l J(k,l),$$

where $J(k,l)$ is the simpler integral

$$(34)\quad J(k,l) = \int_{\delta_1}^{\delta_2} y^{k+l+a_2}\, {}_2F_1\left(a_2, 1-b_2; 1+a_2; y\right) dy.$$

On applying (12), (34) can be evaluated as

$$\begin{aligned} J(k,l) = {} & \frac{\delta_2^{1+k+l+a_2}}{1+k+l+a_2}\, {}_3F_2(a_2, 1-b_2, 1+k+l+a_2; 1+a_2, 2+k+l+a_2; \delta_2) \\ & - \frac{\delta_1^{1+k+l+a_2}}{1+k+l+a_2}\, {}_3F_2(a_2, 1-b_2, 1+k+l+a_2; 1+a_2, 2+k+l+a_2; \delta_1). \end{aligned}$$

7. Non-central Beta distribution. There are three types of non-central-beta distributions. The one that is most commonly known and studied is the Type I noncentral-beta distribution. For this, the pdf and the cdf of X_i are

$$(35)\quad f_i(x) = \sum_{k=0}^{\infty} \frac{(\lambda_i/2)^k \exp\left(-\lambda_i/2\right)}{k!} \frac{x^{a_i+k-1}(1-x)^{b_i-1}}{B\left(a_i+k, b_i\right)}$$

and

$$(36)\quad F_i(x) = \sum_{k=0}^{\infty} \frac{(\lambda_i/2)^k \exp\left(-\lambda_i/2\right)}{k!} I_x\left(a_i+k, b_i\right),$$

(where $0 \leq x \leq 1$), respectively. These distributions have been recently utilized as models in geophysics (Kimball and Scheibner, [4]) and psychophysics (Rousseau and Ennis, [10]). For (35) and (36), the reliability in (1) can be written as:

$$\text{(37) } R = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\lambda_1^k \lambda_2^l}{2^{k+l} k! l!} \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) \int_0^1 \frac{I_x(l+a_2, b_2)}{B(k+a_1, b_1)} x^{k+a_1-1}(1-x)^{b_1-1} dx.$$

The integral on the right has the form of the reliability expression (18) for the standard beta distribution. Thus, using (19), (37) can be reduced to:

$$R = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\lambda_1^k \lambda_2^l B\left(k+l+a_1+a_2, b_1\right)}{2^{k+l} k! l! \left(l+a_2\right) B\left(k+a_1, b_1\right) B\left(l+a_2, b_2\right)} \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right)$$
$$\times\ {}_3F_2\left(l+a_2, 1-b_2, k+l+a_1+a_2; 1+l+a_2, k+l+a_1+a_2+b_1; 1\right).$$

Clearly this is a very complicated expression. A simpler expression can be found if we assume that the b_i in (35) and (36) are integers. In this case, (35) and (36) reduce to

$$f_i(x) = \exp\left\{-\lambda_i(1-x)\right\} \sum_{k=0}^{b_i-1}\sum_{l=1}^{b_i-k} \frac{\lambda_i^k}{k!} \binom{a_i+b_i-l-1}{b_i-k-l} x^{a_i+k-1}(1-x)^{b_i-l-1}$$
$$\times \left\{-\lambda_i x^2 + \left(l-k+\lambda_i-a_i-b_i\right)x + k + a_i\right\}$$

and

$$F_i(x) = \exp\left\{-\lambda_i(1-x)\right\} \sum_{k=0}^{b_i-1}\sum_{l=1}^{b_i-k} \frac{\lambda_i^k}{k!} \binom{a_i+b_i-l-1}{b_i-k-l} x^{a_i+k}(1-x)^{b_i-l},$$

respectively (see Nicholson [8]). Thus the corresponding form for the reliability can be written as

$$R = \exp\{-(\lambda_1+\lambda_2)\} \sum_{k=0}^{b_2-1}\sum_{l=1}^{b_2-k}\sum_{m=0}^{b_1-1}\sum_{n=1}^{b_1-m} \frac{\lambda_1^m \lambda_2^k}{m! k!} \binom{a_2+b_2-l-1}{b_2-k-l} \binom{a_1+b_1-n-1}{b_1-m-n}$$
$$\text{(38)} \quad \times \left\{-\lambda_1 I_1 + \left(n-m+\lambda_1-a_1-b_1\right) I_2 + \left(m+a_1\right) I_3\right\},$$

where

$$I_1 = \int_0^1 x^{1+k+m+a_1+a_2}(1-x)^{b_1+b_2-l-n-1} \exp\left\{\left(\lambda_1+\lambda_2\right)x\right\} dx,$$

$$I_2 = \int_0^1 x^{k+m+a_1+a_2} (1-x)^{b_1+b_2-l-n-1} \exp\{(\lambda_1+\lambda_2)x\}\, dx$$

and

$$I_3 = \int_0^1 x^{k+m+a_1+a_2-1} (1-x)^{b_1+b_2-l-n-1} \exp\{(\lambda_1+\lambda_2)x\}\, dx.$$

On applying (15), we can reduce these three integrals to:

$$\begin{aligned} I_1 &= B(b_1+b_2-n-l, 2+m+k+a_1+a_2) \\ &\quad \times {}_1F_1(2+m+k+a_1+a_2; 2+k-l+m-n+a_1+b_1+a_2+b_2; \lambda_1+\lambda_2), \end{aligned}$$

$$\begin{aligned} I_2 &= B(b_1+b_2-n-l, 1+m+k+a_1+a_2) \\ &\quad \times {}_1F_1(1+m+k+a_1+a_2; 1+k-l+m-n+a_1+b_1+a_2+b_2; \lambda_1+\lambda_2) \end{aligned}$$

and

$$\begin{aligned} I_3 &= B(b_1+b_2-n-l, m+k+a_1+a_2) \\ &\quad \times {}_1F_1(m+k+a_1+a_2; k-l+m-n+a_1+b_1+a_2+b_2; \lambda_1+\lambda_2), \end{aligned}$$

respectively. Substituting these into (38), we obtain an expression for R that is a finite sum of confluent hypergeometric functions. In the particular case $b_1 = b_2 = 1$, (38) reduces to the simple expression:

$$\begin{aligned} R = \exp\{-(\lambda_1+\lambda_2)\} & \left[\frac{\lambda_1}{1+a_1+a_2}\, {}_1F_1(1+a_1+a_2; 2+a_1+a_2; \lambda_1+\lambda_2) \right. \\ & \left. + \frac{a_1}{a_1+a_2}\, {}_1F_1(a_1+a_2; 1+a_1+a_2; \lambda_1+\lambda_2) \right]. \end{aligned}$$

8. Log Beta distribution. As the name indicates, X_i are said to have the log beta distribution if $\log X_i$ have the standard beta distribution. Clearly the support of X_i must be a finite positive interval, say $0 < e_i \le X_i \le f_i$. Then $Y_i = (\log X_i - \log e_i)/(\log f_i - \log e_i)$ will have the standard beta distribution for some parameters, say a_i and b_i. Using this relationship, the reliability in (1) can be written as

$$\begin{aligned} R &= \Pr(X_2 < X_1) \\ &= \Pr\left(\frac{\log X_2 - \log e_2}{\log f_2 - \log e_2} < \frac{\log X_1 - \log e_2}{\log f_2 - \log e_2}\right) \\ &= \Pr\left(\frac{\log X_2 - \log e_2}{\log f_2 - \log e_2} < \frac{\log f_1 - \log e_1}{\log f_2 - \log e_2}\frac{\log X_1 - \log e_1}{\log f_1 - \log e_1} + \frac{\log e_1 - \log e_2}{\log f_2 - \log e_2}\right) \\ &= \Pr\left(Y_2 < \frac{\log f_1 - \log e_1}{\log f_2 - \log e_2} Y_1 + \frac{\log e_1 - \log e_2}{\log f_2 - \log e_2}\right). \end{aligned}$$

Hence, the reliability of the log beta distribution is the same as that of the generalized beta distribution given in (32) for the particular choices $c_1 = (\log e_1 - \log e_2)/(\log f_2 - \log e_2)$, $d_1 = (\log f_1 - \log e_2)/(\log f_2 - \log e_2)$, $c_2 = 0$ and $d_2 = 1$. The log beta distribution has applications to the evolution of aerosol growth (Bunz et al., [1]; Chang et al., [2]).

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