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THE AUTOMORPHISM GROUP OF THE FREE ALGEBRA OF RANK TWO

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Abstract. The theorem of Czerniakiewicz and Makar-Limanov, that all the automorphisms of a free algebra of rank two are tame is proved here by showing that the group of these automorphisms is the free product of two groups (amalgamating their intersection), the group of all affine automorphisms and the group of all triangular automorphisms. The method consists in finding a bipolar structure. As a consequence every finite subgroup of automorphisms (in characteristic zero) is shown to be conjugate to a group of linear automorphisms.

1. Introduction. Let $k\langle X\rangle$ be the free associative algebra on a set $X = \{x_1, \dots, x_d\}$ over a field k. Any invertible $d \times d$ matrix (α_{ij}) over k determines a unique k-algebra automorphism α of $k\langle X\rangle$ whose action on the free generators is given by $\alpha : x_i \mapsto \sum \alpha_{ij} x_j$. Such an automorphism will be called *linear* (relative

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to the given free generating set X). Kharchenko [6] has developed a Galois theory for free algebras, based on the theorem (proved in [6], and also by Lane [9]) that the fixed algebra of any finite group of linear automorphisms of $k\langle X\rangle$ is free.

Our object here is to show that for the algebra $k\langle x, y\rangle$ Kharchenko's results extend to any finite group of automorphisms whose order is invertible in k, by showing that such a group is actually conjugate (in the full automorphism group) to group of linear automorphisms (Theorem 5.1).

The proof depends on the observation (Theorem 3.2) that the full automorphism group of $k\langle x, y\rangle$ can be expressed as a free product (with amalgamation). This observation in turn is an easy consequence of the theorem of Czerniakiewicz [4] and Makar-Limanov [11, 12] that every automorphism of $k\langle x, y\rangle$ is tame (see §3 for definitions). However, the proofs of this latter theorem in [4, 11, 12] are not very perspicuous; we shall therefore give a direct proof of Theorem 3.2 which thus provides a new (and, it is hoped, more transparent) proof of the Czerniakiewicz-Makar-Limanov theorem.

The proof of Theorem 3.2 is in two steps. We firstly note (in §2) that it is enough to prove the result for the group of centred (augmentation-preserving) automorphisms, – the general case then follows by forming pullbacks. Now for centred automorphisms we can use the representation in $GL_2(k\langle x, y\rangle)$ described in [3], and invoke Nagao's theorem which expresses the latter group as a free product (cf. [10, 14]). Some care is needed here, since the representation is not a homomorphism; in fact the fastest route is to find a bipolar structure and then use Stallings' characterization of free products [15, 10]. Stallings' theorem and other results needed are recalled in §2, the proof is carried out in §4 and §5 brings the conjugacy theorem which was our original objective.

I am grateful to Warren Dicks for reading an earlier draft of a weaker result and whose suggestions helped to simplify its proof to a point where the present form became apparent.

2. Bipolar structures and the normal form for GE_2. Stallings [15] has given an axiomatic description of the free product of two groups with an amalgamated subgroup, in terms of bipolar structures. We briefly recall his result (in the form given by Lyndon and Schupp [10], p.207).

A *bipolar structure* on a group G is a partition of G into five disjoint sets F, EE, EE^*, E^*E, E^*E^*, satisfying the following axioms, where $X, Y, \ldots$ stand for E or E^* and $X^{**} = X$.

B.1. F is a subgroup of G.

B.2. If $f \in F$, $g \in XY$, then $fg \in XY$.

B.3. If $g \in XY$, then $g^{-1} \in YX$.

B.4. If $g \in XY$, $h \in Y^*Z$, then $gh \in XZ$.

B.5. If $g \in G$, then there is an integer $N(g)$ such that for any representation $g = g_1 \dots g_n$ ($g \in X_{i-1}^* X_i$) we have $n \leq N(g)$.

B.6. $EE^* \neq \emptyset$.

An element g of G is said to be *irreducible* if $g \in XY \cup F$ and g is not of the form $g = hk$, where $h \in XZ$, $k \in Z^*Y$. From B.5 it follows that G is generated by its irreducible elements. Now we have

Theorem A (Stallings). *Every non-trivial free product $P *_F Q$ with amalgamated subgroup has a bipolar structure, and conversely, if G has a bipolar structure such that EE^* has no irreducible elements, then G is a free product with amalgamation: $G = P *_F Q$, where P, Q consist of the irreducible elements of EE, E^*E^* respectively.*

We remark that the proof is quite straightforward (see [10], pp. 210–212). Stallings also shows that the bipolar structures for which EE^* has irreducible elements characterize HNN-extensions, but this fact will not be needed here.

To apply Theorem A we shall need a normal form for the elements of $GL_2(k\langle x, y\rangle)$. At first let R be any ring, denote its group of units by $U = U(R)$ and write $U_0 = U \cup \{0\}$. Further, for any $\alpha, \beta \in U$, $a \in R$, set

$$[\alpha, \beta] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad D(\alpha) = [\alpha, \alpha^{-1}], \quad E(a) = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}. \tag{1}$$

Clearly all these matrices are invertible; we denote by $GE_2(R)$ the group generated by them. We have

Theorem B ([2], Theorem 2.2, p. 9). *In any ring R, the generators $E(a)$, $[\alpha, \beta]$ of $GE_2(R)$ satisfy the relations*

$$E(x)E(0)E(y) = -E(x+y), \tag{2}$$

$$E(x)[\alpha, \beta] = [\beta, \alpha]E(\beta^{-1}x\alpha), \tag{3}$$

$$E(x)E(a)E(y) = E(x - \alpha^{-1})D(\alpha)E(y - \alpha^{-1}), \tag{4}$$

$$E(x)^{-1} = E(0)E(-x)E(0). \tag{5}$$

Moreover, every $A \in GE_2(R)$ can be expressed in the form

$$A = [\alpha, \beta]E(a_1) \dots E(a_r), \quad \alpha, \beta \in U(R), \quad a_i \in R, \tag{6}$$

where $a_i \notin U_0(R)$ for $1 < i < r$, and a_1, a_r are not both 0 when $r = 2$.

We shall mainly be interested in the application of this result to free algebras:

Theorem C ([2], Theorem 7.2, p. 25)**.** *Over the free algebra $k\langle X\rangle$ every invertible 2×2 matrix A can be written in the form (6) in just one way.*

This property is expressed by saying the $k\langle X\rangle$ is a GE_2-ring with a unique standard form for GE_2. For the case of one indeterminate this is of course well known and one has the result of Nagao [13]:

$$GL_2(k[x]) = B_2(k[x]) *_B GL_2(k), \tag{7}$$

where $B_2(k[x])$ is the subgroup of upper triangular matrices over $k[x]$ and $B = B_2(k[x]) \cap GL_2(k)$.

It is possible to prove (7) by defining a bipolar structure on $GL(k[x])$ (see [10], p. 213f.), but of course a direct proof is quicker. In the same way one has

Theorem 2.1. *For any free algebra $k\langle X\rangle$,*

$$GL_2(k\langle X\rangle) = B_2(k\langle X\rangle) *_B GL_2(k).$$

Again this is very easily proved directly, because one knows that the left-hand side is generated by $B_2(k\langle X\rangle)$ and $GL_2(k)$, by using the weak algorithm in $k\langle X\rangle$. A proof via bipolar structures is again possible, though more tedious, but in §4 we shall come to a situation where this is the only route open to us.

3. Reduction to the centred case. Let $R = k\langle x, y\rangle$ be the free k-algebra on x and y. Any automorphism f of R can be written

$$(x, y) \mapsto (x^f, y^f) = (p + \lambda, q + \mu),$$

where $\lambda, \mu \in k$ and p, q are polynomials in x, y with zero constant term. It follows that $f = f'f''$, where

$$f' : (x, y) \mapsto (x + \lambda, y + \mu), \quad f'' : (x, y) \mapsto (p, q).$$

An automorphism of the form f' is called a *translation* and one of the form f'' *centred* or *augmentation preserving*. Let us write $\mathrm{Aut}(R)$ for the group of all automorphisms of R, as k-algebra, and denote by T the group of all translations and by C the group of all centred automorphisms. It is easy to see (and of course well known) that T is normal in $\mathrm{Aut}(R)$ with quotient C, so that we have the split exact sequence (i.e. a representation of $\mathrm{Aut}(R)$ as a semidirect product)

$$1 \to T \to \mathrm{Aut}(R) \to C \to 1. \tag{8}$$

In what follows we shall also need the following subgroups of $\mathrm{Aut}(R)$:

1. The group A of all affine automorphisms of R:

$$(x,y) \mapsto (\alpha x + \beta y + \lambda, \gamma x + \delta y + \mu), \quad (\alpha, \ldots, \mu \in k,\ \alpha\delta - \beta\gamma \neq 0).$$

2. The group Δ of all triangular automorphisms of R (also known as de Jonquières transformations in the commutative case):

$$(x,y) \mapsto (\alpha x + p(y), \delta y + \mu), \quad (\alpha, \delta \neq 0,\ p \in k[y]).$$

An automorphism is called *tame* if it can be written as a product of affine and triangular automorphisms, *wild* otherwise.

We recall the following key result:

Theorem 3.1 (Czerniakiewicz [4], Makar-Limanov [11, 12]). *Every automorphism of $k\langle x, y\rangle$ is tame.*

Here k can be any field; the proof in [11] is for the complex number field, but those in [4, 12] are quite general. We shall obtain a proof of Theorem 3.1 as a corollary of

Theorem 3.2. *The group $\mathrm{Aut}(k\langle x,y\rangle)$ is the free product of the groups A and Δ, amalgamating their intersection.*

Again k can be any field, even skew, but of course all the elements of k commute with x and y.

We shall prove this result in §4. For the moment we note that in the proof we can replace $\mathrm{Aut}(R)$ by the subgroup of all centred automorphisms. This follows from

Proposition 3.3. *Let $G = P *_F Q$, be a free product with amalgamated subgroup and consider an extension E of G by a group T. Then $E = P' *_{F'} Q'$, where P', Q', F' are extensions of P, Q, F respectively by T.*

Proof. In the following diagram the bottom line is the given extension and the last vertical arrow is an isomorphism:

$$\begin{array}{ccccccccc} 1 & \to & T & \longrightarrow & P' *_{F'} Q' & \longrightarrow & P *_F Q & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & T & \longrightarrow & E & \longrightarrow & G & \to & 1 \end{array}$$

The pullback of $E \to G$ and $P *_F Q \to G$ is of the form $P' *_{F'} Q'$, as we see by forming the pullback with $P *_F Q$ replaced by P, Q, F in turn. Moreover, just as in an additive category we find that the map $P' *_{F'} Q' \to P *_F Q$ has kernel T, and we can complete the diagram as shown. Now a diagram chase shows that the pullback homomorphism $P' *_{F'} Q' \to E$ is an isomorphism and the result follows. $\square$

Suppose that we have a free product representation of the group C of all centred automorphisms; then by applying Proposition 3.3 to the exact sequence (8) we obtain a free product representation of $\mathrm{Aut}(R)$, so to prove Theorem 3.2 it only remains to show that

$$C = \Delta_0 *_S L, \tag{9}$$

where $\Delta_0 = \Delta \cap L$ is the group of centred triangular automorphisms, $L = A \cap C$ is the group of linear automorphisms and $S = \Delta_0 \cap L$ is the group of linear triangular automorphisms (generalized shears). If we use Theorem 3.1 and the standard form of Theorem C (or the standard form for automorphisms, described by Lane [8]), the representation (8) is immediate, but we shall proceed differently: with only a little more trouble we can find a bipolar structure on C to which Theorem A can be applied, so that we do not need to use Theorem 3.1.

4. The bipolar structure on the group of centred automorphisms. Any element $p \in R = k\langle x, y\rangle$ can be written uniquely in the form $p = p_1 x + p_2 y + \lambda$, where $p_1, p_2 \in R$ and $\lambda \in k$. Hence every centred automorphism g of R has the form

$$\begin{aligned} x^g &= ax + by, \\ y^g &= cx + dy, \end{aligned} \qquad (a, b, c, d \in R). \tag{10}$$

Writing $u = \begin{pmatrix} x \\ y \end{pmatrix}$, $T_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can express this in matrix form as

$$u^g = T_g u.$$

It follows that

$$T_{gh}u = u^{gh} = (T_g u)^h = T_g^h T_h u,$$

where T_g^h is the matrix obtained by letting h act on T_g. Hence

$$T_{gh} = T_g^h T_h. \tag{11}$$

Clearly $T_1 = I$, and it follows that each T_g is invertible, with inverse

$$T_g^{-1} = T_{g^{-1}}^g. \tag{12}$$

It follows that $T_g \in GL_2(R)$, so that we have a mapping $C \to GL_2(R)$, which however is not a homomorphism, in view of (11) (it is essentially a crossed homomorphism).

We now come to the result needed to complete the proof of Theorem 3.2.

Theorem 4.1. *The group C of all centred automorphisms of $R = k\langle x, y\rangle$ is the free product of the group Δ_0 of centred triangular automorphisms and the group L of all linear automorphisms, amalgamating $S = \Delta_0 \cap L$:*

$$C = \Delta_0 *_S L. \tag{13}$$

Proof. Given $g \in C$, we have $T_g \in GL_2(R)$ and by Theorem C of §2 we have a unique standard form for T_g:

$$T_g = [\alpha, \beta]E(a_1)\ldots E(a_r), \tag{14}$$

where $\alpha, \beta \in k$, $\alpha\beta \neq 0$, $a_i \in R$; moreover, $a_i \notin k$ for $1 < i < r$ and when $r = 2$, a_1, a_2 are not both 0. In particular, the matrices of Δ_0 have the form

$$T_g = [\alpha, \beta]E(a)E(0), \tag{15}$$

where $g \in S$ if and only if $a \in k$, while the matrices of L have the form

$$T_g = [\alpha, \beta]E(\lambda) \quad \text{or} \quad [\alpha, \beta]E(\lambda)E(\mu), \quad \alpha, \beta, \lambda, \mu \in k, \quad \alpha\beta \neq 0, \tag{16}$$

and here $g \in S$ if and only if the second form applies and $\mu = 0$.

We construct a bipolar form on C as follows. Put $F = S$; next suppose that $g \notin S$ and that T_g has the standard form (14). Then

(i) $g \in EE$ if $a_1 \in k$, $a_r \neq 0$,

(ii) $g \in EE^*$ if $a_1 \in k$, $a_r = 0$,

(iii) $g \in E^*E$ if $a_1 \notin k$, $a_r \neq 0$,

(iv) $g \in E^*E^*$ if $a_1 \notin k$, $a_r = 0$.

It remains to verify B.1–6. B.1 is clear; to prove B.2, let $h \in F$, $g \in XY$, say T_g is given by (14) and $T_h = [\alpha_1, \beta_1]E(\lambda)E(0)$; then $T_h^g = T_h$ because α_1, β_1, λ are fixed under the automorphism, so by (3), (2),

$$\begin{aligned} T_{hg} = T_h T_g &= [\alpha_1, \beta_1][\alpha, \beta]E(\alpha^{-1}\lambda\beta)E(0)E(a_1)\ldots E(a_r) \\ &= [-\alpha_1\alpha, -\beta_1\beta]E(a_1 + \alpha^{-1}\lambda\beta)E(a_2)\ldots E(a_r). \end{aligned}$$

Since $a_1 + \alpha^{-1}\lambda\beta \in k$ if and only if $a_1 \in k$, it follows that $hg \in XY$.

B.3. If T_g is given by (14), then

$$T_g^{-1} = [\alpha', \beta']E(0)E(a'_r)\ldots E(a'_1)E(0),$$

where α', β' are α, β in some order and a'_i is associated to a_i. Moreover, $T_{g^{-1}} = (T_g^{-1})^{g^{-1}}$ and $a''_i = (a'_i)^{g^{-1}}$ lies in k if and only if a_i does. Thus we have

$$T_{g^{-1}} = [\alpha', \beta']E(0)E(a''_r)\ldots E(a''_1)E(0). \tag{17}$$

Clearly $g \in XE^* \iff a_r = 0 \iff a''_r = 0$, and this is so if and only if the coefficient of the first factor $E(.)$ in the reduced form of (17) is not in k. Thus $g \in XE^* \iff g^{-1} \in E^*Y$; taking complements we find that $g \in XE \iff g^{-1} \in EY$, and combining these cases with those obtained by interchanging g and g^{-1} we find that $g \in XY$ if and only if $g^{-1} \in YX$.

B.4. We take T_g in the standard form (14) and

$$T_h = [\gamma, \delta]E(b_1)\ldots E(b_s).$$

Then

$$T_{gh} = T_g^h T_h = [\lambda, \mu]E(a''_1)\ldots E(a''_r)E(b_1)\ldots E(b_s), \tag{18}$$

where $\lambda, \mu \in k$, $\lambda\mu \neq 0$, $a''_i = (a'_i)^h$ and a'_i is an associate of a_i. Here (18) may not be in standard form, but we reach a standard form after finitely many applications of (2), (4) and (3). Any terms b_j that remain are unaffected by these changes, while the a''_i that remain, only change by at most a unit factor. Thus if a''_1 and b_s are still present after the reduction to standard form, then for $g \in XU$, $h \in VZ$ we have $gh \in XZ$. In particular, this will be the case if in the expression (18) for T_{gh} not all the factors $E(.)$ stemming from g, nor all those stemming from h, cancel.

We shall now show that if $g \in XY$, $h \in Y^*Z$, then there is no cancellation at all. For suppose that some cancellation takes place in (18) and write $gh = k$; then $k \neq 1$, because g, h cannot be mutually inverse, by B.3. Hence not all of g and h is cancelled, say not all of g. Write $g = kh^{-1}$; then in the expression for $T_{kh^{-1}}$ not all the factors $E(.)$ from k nor all those from h^{-1} are cancelled. Now by B.3, $h^{-1} \in ZY^*$, so if $k \in UV$, then by what has been shown, $g \in UY^*$, which contradicts the fact that $g \in XY$; so the assertion is proved.

Now B.4 is clear, and B.5 also follows, for if T_g has the form (14), and $g = g_1 \dots g_n$, where $g_i \in X^*_{i-1}X_i$, then in the expression for $T_{g_1 \dots g_n}$ there can be no cancellation. Hence $n \leq r$, and so B.5 holds with $N(g) = r$.

Finally B.6 is clear, since $E(\lambda)E(a)E(0) \in EE^*$, for $\lambda \in k$, $a \notin k$. Thus B.1–6 all hold; further any element of EE^* has the form g, where

$$T_g = [\alpha, \beta]E(\lambda)E(a_1)\dots E(a_r)E(0),$$

and $r \geq 1$ because $g \notin S$. Now $g = hk$, where

$$T_h = [\alpha, \beta]E(\lambda), \qquad T_k = E(a_1)\dots E(a_r)E(0).$$

Here $h \in EE$, $k \in E^*E^*$; thus EE^* contains no irreducible elements, and so the conclusion follows by applying Theorem A. It is easily seen that the irreducible elements in EE constitute L while those in E^*E^* constitute Δ_0. $\square$

Remarks 1. We have nowhere used the commutativity of k; the result therefore holds even when k is a skew field. However, it is of course necessary for the variables to centralize k; this fact was used in deriving (10).

2. Any automorphism of $k\langle x, y\rangle$ defines an automorphism of the polynomial ring $k[x, y]$, by allowing the variables to commute. Thus there is a natural homomorphism

$$\varphi : \mathrm{Aut}(k\langle x, y\rangle) \to \mathrm{Aut}(k[x, y]).$$

Since every automorphism of $k[x, y]$ is aiso tame (Jung's theorem, proved in [5] for characteristic 0 and by van der Kulk [7] generally), it follows that φ is surjective. In fact Czerniakiewicz [4] and Makar-Limanov [12] show that φ is an isomorphism. The injectivity also follows from the fact that (6) is a unique standard form for the elements of $GE_2(k[x, y])$ (see [2], Theorem 7.1, p. 24). We therefore have the

Corollary 4.2. *The group* $\mathrm{Aut}(k[x, y])$ *is the free product of its affine and triangular subgroups, amalgamating their intersection.*

Here we need Jung's theorem to ensure that φ is surjective; the method of proof of Theorem 4.1 is not at our disposal because $GL_2(k[x, y])$ is not generated by elementary matrices (i.e. $k[x,y]$ is not a GE_2-ring, see [2], p. 26). Nevertheless, it might be possible to describe a bipolar structure on $\mathrm{Aut}(k[x,y])$ (which we know exists, by Corollary 4.2), and so obtain an independent proof of Jung's theorem.

5. The conjugacy theorem. It is now an easy matter to prove the conjugacy theorem stated in the introduction.

Theorem 5.1. *Every finite subgroup of* $\mathrm{Aut}(k\langle x,y\rangle)$ *of order invertible in* k *has a conjugate in* L*, the subgroup of linear automorphisms.*

Proof. Let G be a finite subgroup of $\mathrm{Aut}(R)$; the representation in Theorem 4.1 shows that G has a conjugate in Δ or in A (see e.g. Serre [14], p. 54). By passing to a conjugate we may assume that G itself is a subgroup of Δ or of A. We treat these cases in turn.

(i) $G \subseteq \Delta$. Here the action of G leaves invariant the subspaces $kx + k[y]$ and $k[y]$, so by Maschke's theorem we may choose a G-invariant complement kx' of $k[y]$ in $kx + k[y]$. This provides an automorphism $\alpha : (x,y) \mapsto (x',y)$ such that $\alpha^{-1}G\alpha \subseteq T$.

Explicitly we write for each $g \in G$,

$$g : (x,y) \mapsto (\lambda_g x + f_g(y), \mu_g y + \nu_g).$$

Put $|G| = n$, $f(y) = n^{-1}\sum_g \lambda_g^{-1} f_g(y)$ and define $\alpha : (x,y) \mapsto (x - f(y), y)$. Then by comparing expressions for x^{gh} we have $\lambda_{gh} = \lambda_g\lambda_h$, $f_{gh} = \lambda_g f_h(y) + f_g(\mu_h y + \nu_h)$, so for any $h \in G$,

$$nf(y) = \textstyle\sum_g \lambda_{gh}^{-1} f_{gh}(y) = n\lambda_h^{-1} f_h(y) + n\lambda_h^{-1} f(\mu_h y + \nu_h).$$

It follows that $\alpha^{-1}h\alpha : (x,y) \mapsto (\lambda_h x, \mu_h x + \nu_h)$, so $\alpha^{-1}G\alpha \subseteq T$. We are thus reduced to the case where $G \subseteq A$.

(ii) $G \subseteq A$. Now G acts on the space $kx + ky + k$ with invariant subspace k. Again we can find a complement $kx' + ky'$ of k in $kx + ky + k$; now $\alpha : (x,y) \mapsto (x',y')$ is an automorphism such that $\alpha^{-1}G\alpha \subseteq L$. Explicitly, write for $g \in G$,

$$g : (x,y) \mapsto (\lambda_g x + \mu_g y + \nu_g, \lambda'_g x + \mu'_g y + \nu'_g),$$

and define $\alpha : (x,y) \mapsto (x + \nu, y + \nu')$, where $\nu = n^{-1}\sum \nu_g$, $\nu' = n^{-1}\sum \nu'_g$; then $n\nu = \sum_g \nu_{hg} = \sum(\lambda_h\nu_g + \mu_h\nu'_g + \nu_h) = n(\lambda_h\nu + \mu_h\nu'_g + \nu_h)$ and similarly for ν', therefore $\alpha^{-1}h\alpha : (x,y) \mapsto (\lambda_h x + \mu_h y, \lambda'_h x + \mu'_h y)$, and this shows that $\alpha^{-1}G\alpha \subseteq L$ as required. $\square$

Clearly we cannot omit the hypothesis that $|G|$ is prime to char k. E.g. if k is finite, then A is a finite subgroup of $\mathrm{Aut}(R)$, but no conjugate of A lies in L, a proper subgroup of A. More specifically it may be shown that any triangular automorphism of the form $\alpha : (x, y) \mapsto (x + y^2 f(y), y)$ is not conjugate to a linear automorphism unless $\alpha = 1$, although $\alpha^p = 1$ in characteristic p.

With the help of the Kharchenko-Lane theorem we have

Corollary 5.2. *Let G be a finite group of automorphisms of $k\langle x, y\rangle$ of order invertible in k. Then the fixed algebra $k\langle x, y\rangle^G$ of G is free.*

G. M. Bergman [1] has shown that for any subgroup G of $\mathrm{Aut}(k\langle X\rangle)$ the fixed ring $k\langle X\rangle^G$ is a 2-fir (i.e. all 2-generator left or right ideals are free, of unique rank). Whether $k\langle X\rangle^G$ is always free is not known, but if G is of finite order invertibie in k, then $k\langle X\rangle^G$ is hereditary, by a result of Bergman ([1], Proposition 1.4). Together with an unpublished result of Dicks, that a homogeneous hereditary subalgebra of $k\langle X\rangle$ is free, this leads to another proof of the Kharchenko-Lane theorem. However, Corollary 5.2 depends for its proof on methods where the condition $|X| = 2$ is used in an essential way.

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