Serdica
Mathematical Journal
Сердика
Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **28** (2002), 219-232

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON A CLASS OF VERTEX FOLKMAN NUMBERS

Nedyalko Dimov Nenov

Communicated by R. Hill

Abstract. Let a_1, ..., a_r be positive integers, $m = \sum_{i=1}^{r}(a_i - 1) + 1$ and $p = \max\{a_1, \ldots, a_r\}$. For a graph G the symbol $G \to (a_1, \ldots, a_r)$ means that in every r-coloring of the vertices of G there exists a monochromatic a_i-clique of color i for some $i \in \{1, \ldots, r\}$. In this paper we consider the vertex Folkman numbers

$$F(a_1, \ldots, a_r; m-1) = \min\{|V(G)| : G \to (a_1, \ldots, a_r) \text{ and } K_{m-1} \not\subset G\}$$

We prove that $F(a_1, \ldots, a_r; m-1) = m+6$, if $p = 3$ and $m \geqq 6$ (Theorem 3) and $F(a_1, \ldots, a_r; m-1) = m+7$, if $p = 4$ and $m \geqq 6$ (Theorem 4).

1. Notations. We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call p-clique of G any set of p vertices, each two of which are adjacent. The largest natural number p, such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$ (the clique number of G).

2000 *Mathematics Subject Classification:* 05C55.
Key words: vertex Folkman graph, vertex Folkman number.

If $W \subseteq V(G)$ then $G[W]$ is the subgraph of G induced by W and $G - W$ is the subgraph induced by $V(G) \setminus W$. We shall use also the following notations:

$\overline{G}$ — the complement of graph G;

$\alpha(G)$ — the vertex independence number of G;

$N(v)$, $v \in V(G)$ — the set of all vertices of G adjacent to v;

$\chi(G)$ — the chromatic number of G;

K_n — complete graph of n vertices;

C_n — simple cycle of n vertices.

$K_n - C_m$, $m \leqq n$ — the graph obtained from K_n by deleting all edges of some cycle C_m.

The equality $C_n = v_1, v_2, \dots, v_n$ means that

$$V(C_n) = \{v_1, \dots, v_n\} \quad \text{and} \quad E(C_n) = \{[v_i, v_{i+1}],\ i = 1, \dots, n-1,\ [v_1, v_n]\}.$$

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1),\ y \in V(G_2)\}$.

The Ramsey number $R(p, q)$ is the smallest natural number n, such that for arbitrary n-vertex graph G, either $\mathrm{cl}(G) \geqq p$ or $\alpha(G) \geqq q$. We need the identities $R(3, 4) = R(4, 3) = 9$, [3].

2. The vertex Folkman graphs and vertex Folkman numbers.

Definition. *Let $a_1, \dots, a_r$ be positive integers. An r-coloring*

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \emptyset,\ i \neq j,$$

of the vertices of a graph G is said to be $(a_1, \dots, a_r)$-free if for all $i \in \{1, \dots, r\}$ the graph G does not contain a monochromatic a_i-clique of color i. The symbol $G \to (a_1, \dots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \dots, a_r)$-free.

A graph G such that $G \to (a_1, \dots, a_r)$ is called a *vertex Folkman graph.* Obviously, it is true that:

Proposition 1. *Let a_1, …, a_r be positive integers, $r \geqq 2$ and $a_i = 1$ for some $i \in \{1, \dots, r\}$. Then*

$$G \to (a_1, \dots, a_r) \quad \Leftrightarrow \quad G \to (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r)$$

Proposition 2. *For each permutation φ of the symmetric group S_r*

$$G \to (a_1, \dots, a_r) \quad \Leftrightarrow \quad G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)}).$$

Define:

$F(a_1, \dots, a_r; q) = \min\{|V(G)| : G \to (a_1, \dots, a_r) \text{ and } \operatorname{cl}(G) < q\}.$

Clearly, $G \to (a_1, \dots, a_r) \;\Rightarrow\; \operatorname{cl}(G) \geqq \max\{a_1, \dots, a_r\}$.

Folkman [2] proved that there exists a graph G, such that $G \to (a_1, \dots, a_r)$ and $\operatorname{cl}(G) = \max\{a_1, \dots, a_r\}$. Therefore,

(1) $$F(a_1, \dots, a_r; q) \text{ exist} \quad \Leftrightarrow \quad q > \max\{a_1, \dots, a_r\}.$$

and they are called *vertex Folkman numbers*. For every positive integers $a_1, \dots, a_r$ we define

(2) $$m = \sum_{i=1}^{r}(a_i - 1) + 1, \quad p = \max\{a_1, \dots, a_r\}.$$

Obviously, $K_m \to (a_1, \dots, a_r)$ and $K_{m-1} \nrightarrow (a_1, \dots, a_r)$. Therefore, if $q \geqq m+1$, then $F(a_1, \dots, a_r; q) = m$. It is true that:

Proposition 3 ([13] and [14])**.** *Let G be a graph, such that $G \to (a_1, \dots, a_r)$. Then $\chi(G) \geqq m$.*

By (1), the numbers $F(a_1, \dots, a_r; m)$ exist only if $m \geqq p + 1$. For these numbers the following theorem is known:

Theorem A ([4])**.** *Let $a_1, \dots, a_r$ be positive integers and let m and p satisfy* (2)*, where $m \geqq p+1$. Then $F(a_1, \dots, a_r; m) = m+p$. If $G \to (a_1, \dots, a_r)$, $\operatorname{cl}(G) < m$ and $|V(G)| = m + p$, then $G = K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$.*

The proof of this Theorem given in [4] is based on Lemma 1, [4, p. 251]. But the proof of this Lemma is not correct because the sentence "If we delete both endpoints of any of its edges not adjacent to $\{x, y\}$, then $\alpha(G)$ decreases again." is not true (see p. 252). Correct proofs of theorem A are given in [13] and [14].

According to (1), the numbers $F(a_1, \dots, a_r; m-1)$ exist only if $m \geqq p+2$. Very little is known about these numbers. It is true that:

Theorem B ([13]). *Let $a_1, \ldots, a_r$ be positive integers. Let m and p satisfy* (2)*, where $m \geqq p+2$. Then $F(a_1, \ldots, a_r; m-1) \geqq m+p+2$.*

Theorem C ([15]). *Let $a_1, \ldots, a_r$ be positive integers. Let m and p satisfy* (2)*, where $m \geqq p+2$. If G is a graph such that $G \to (a_1, \ldots, a_r)$ and* $\mathrm{cl}(G) < m-1$*, then:*

(a) $|V(G)| \geqq m+p+\alpha(G)-1$;

(b) if $|V(G)| = m+p+\alpha(G)-1$, then $|V(G)| \geqq m+3p$.

According to Proposition 2, $F(a_1, \ldots, a_r; q)$ is a symmetric function and thus we may assume that $a_1 \leqq a_2 \leqq \cdots \leqq a_r$. By Proposition 1, we may assume also that $a_i \geqq 2$, $i = 1, \ldots, r$. Next theorem implies that in special situation $a_1 = a_2 = \cdots = a_r = 2$, $r \geqq 5$, the inequality from Theorem B is exact.

Theorem D.

$$F(\underbrace{2, \ldots, 2}_{r}; r) = \begin{cases} 11, & r = 3 \text{ or } r = 4 \\ r+5, & r \geqq 5. \end{cases}$$

Obviously, $G \to (\underbrace{2, \ldots, 2}_{r}) \iff \chi(G) \geqq r+1.$

Mycielski in [5] presented an 11-vertex graph G, such that $G \to (2,2,2)$ and $\mathrm{cl}(G) = 2$, proving that $F(2,2,2;3) \leqq 11$. Chvátal [1], proved that the Mycielski's graph is the smallest such graph and hence $F(2,2,2;3) = 11$. The inequality $F(2,2,2,2;4) \geqq 11$ was proved in [8] and inequality $F(2,2,2,2;4) \leqq 11$ was proved in [7] and [12] (see also [9]). The equality

$$F(\underbrace{2, \ldots, 2}_{r}; r) = r+5, \quad r \geqq 5$$

was proved in [7, 12] and later in [4].

It is true also that:

Theorem E. $F(3,3;4) = 14.$

The inequality $F(3,3;4) \leqq 14$ was proved in [6] and the opposite inequality was verified by means of computer in [20].

Theorem F ([17]). $F(2,2,2,3;5) = F(2,3,3;5) = 12.$

Only a few more numbers of the type $F(a_1, \dots, a_r; m-1)$ are known, namely: $F(3,4;5) = 13$, [10]; $F(2,2,4;5) = 13$, [11]; $F(4,4;6) = 14$, [19]; $F(2,2,2,4;6) = F(2,3,4;6) = 14$, [18].

3. Main results.

Theorem 1. *Let $p \geqq 3$ be integer, such that $F(2,2,p;p+1) \geqq 2p+5$. Then for each $t \geqq 2$ we have $F(\underbrace{2,\dots,2}_{t}, p; t+p-1) \geqq t+2p+3$.*

Theorem 2. *Let $a_1, \dots, a_r$ be positive integers. Let m and p satisfy* (2)*, where $p \geqq 3$ and $m \geqq p+2$. If $F(2,2,p;p+1) \geqq 2p+5$, then $F(a_1, \dots, a_r; m-1) \geqq m+p+3$.*

Theorem 3. *Let $a_1, \dots, a_r$ be positive integers. Let m and p satisfy* (2)*, where $p = 3$ and $m \geqq 6$. Then $F(a_1, \dots, a_r; m-1) = m+6$.*

Remark 1. If $m = 5$, $p = 3$ and $2 \leqq a_1 \leqq \dots \leqq a_r$ then $r = 2$, $a_1 = a_2 = 3$ or $r = 3$, $a_1 = a_2 = 2$, $a_3 = 3$. According to Theorem E, $F(3,3;4) = 14 > 11$. The equality $F(3,3;4) = 14$ implies $F(2,2,3;4) \leqq 14$ (see Lemma 4), but the exact value of $F(2,2,3;4)$ is unknown.

Remark 2. The special situation $a_1 = \dots = a_r = 3$, $r \geqq 3$ of Theorem 3 was proved in [16].

Theorem 4. *Let $a_1, \dots, a_r$ be positive integers. Let m and p satisfy* (2)*, where $p = 4$ and $m \geqq 6$. Then $F(a_1, \dots, a_r; m-1) = m+7$.*

4. Lemmas.

Lemma 1. *Let $a_1, \dots, a_r$ be positive integers and m and p satisfy* (2)*. Let G be a graph, such that $\mathrm{cl}(G) < m-1$, $G \to (a_1, \dots, a_r)$ and $N(u) \subseteq N(v)$ for some $u, v \in V(G)$. Then $|V(G)| \geqq m+p+3$.*

P r o o f. Obviously, $[u,v] \notin E(G)$. It is clear from $G \to (a_1, \dots, a_r)$ and $N(u) \subseteq N(v)$ that $G - u \to (a_1, \dots, a_r)$. By Theorem B, $|V(G-u)| \geqq m+p+2$. Therefore $|V(G)| \geqq m+p+3$. □

Lemma 2. *Let $a_1, \ldots, a_r$ be positive integers and m and p satisfy* (2). *Let G be a graph, such that* $\mathrm{cl}(G) < m-1$, $G \to (a_1, \ldots, a_r)$ *and* $\alpha(G) \neq 2$. *Then* $|V(G)| \geqq m+p+3$.

Proof. Since G cannot be complete we know that $\alpha(G) \geqq 3$. If $\alpha(G) \geqq 4$, the inequality $|V(G)| \geqq m+p+3$ it follows from Theorem C (a). Let $\alpha(G) = 3$. Suppose that $|V(G)| \leqq m+p+2$. Then, according to Theorem B, $|V(G)| = m+p+2 = m+p+\alpha(G)-1$. From Theorem C (b), $|V(G)| \geqq m+3p > m+p+2$, a contradiction. $\square$

Lemma 3. *Let n and p be fixed positive integers and $p \geqq 2$. Let G be a graph, such that*

$$\left.\begin{array}{l} b_1, \ldots, b_s \in \mathbb{Z} \\ 1 \leqq b_1 \leqq \cdots \leqq b_s \leqq p \\ \sum\limits_{i=1}^{s}(b_i - 1) + 1 = n \end{array}\right\} \quad \Longrightarrow \quad G \to (b_1, \ldots, b_s). \tag{3}$$

Then for every positive integer $a_1, \ldots, a_r$, such that $\max\{a_1, \ldots, a_r\} \leqq p$ *and* $\sum\limits_{i=1}^{r}(a_i - 1) + 1 = m \geqq n$, *we have* $K_{m-n} + G \to (a_1, \ldots, a_r)$.

Proof. We prove Lemma 3 by induction on $t = m - n$. Let $t = 0$, i.e. $m = n$. According to Proposition 2, we may assume that $1 \leqq a_1 \leqq \cdots \leqq a_r$. By (3), $G \to (a_1, \ldots, a_r)$.

Let $t \geqq 1$ and $\widetilde{G} = K_t + G = K_{m-n} + G$. Let $w \in V(K_t)$ and $G' = \widetilde{G} - w = K_{t-1} + G$. Consider an arbitrary r-coloring $V_1 \cup \ldots \cup V_r$ of $V(\widetilde{G})$. Suppose that $w \in V_i$ and let V_j, $j \neq i$ contains no an a_j-clique. We prove that V_i contains an a_i-clique. Since $w \in V_i$, if $a_i = 1$ this is clear. Let $a_i \geqq 2$. By the inductive hypothesis,

$$G' = K_{t-1} + G \to (a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_r) \tag{4}$$

Consider the r-coloring

$$V(G') = V_1 \cup \ldots \cup (V_i \setminus \{w\}) \cup \ldots \cup V_r.$$

From (4) it follows that $V_i \setminus \{w\}$ contains an $(a_i - 1)$-clique. Hence, V_i contains an a_i-clique. So, every r-coloring of $V(\widetilde{G})$ is not $(a_1, \ldots, a_r)$-free. Therefore, $\widetilde{G} = K_{m-n} + G \to (a_1, \ldots, a_r)$. $\square$

Lemma 4. *Let* $G \to (a_1, \ldots, a_r)$ *and let for some* i, $a_i \geqq 2$. *Then*

$$G \to (a_1, \ldots, a_{i-1}, 2, a_i - 1, a_{i+1}, \ldots, a_r).$$

Proof. Consider an $(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1}, \ldots, a_r)$-free $(r+1)$-coloring $V(G) = V_1 \cup \ldots \cup V_{r+1}$. If we color the vertices of V_i with the same color as the vertices of V_{r+1}, we obtain an $(a_1, \ldots, a_r)$-free coloring of $V(G)$, a contradiction. □

5. Proof of Theorem 1. We prove Theorem 1 by induction on t.

I. $t = 3$. If $p = 3$, the inequality follows from Theorem F. Therefore, we may assume that $p \geqq 4$. Let $G \to (2, 2, 2, p)$ and $\mathrm{cl}(G) < p + 2$. We need to prove that $|V(G)| \geqq 2p + 6$. Suppose that $|N(v)| = |V(G)| - 1$ for some $v \in V(G)$. Then $G - v \to (2, 2, p)$ and $\mathrm{cl}(G - v) < p + 1$. By $F(2, 2, p; p + 1) \geqq 2p + 5$, $|V(G - v)| \geqq 2p + 5$. Hence, $|V(G)| \geqq 2p + 6$. Therefore, we will assume that

$$|N(v)| \neq |V(G)| - 1, \quad \forall v \in V(G). \tag{5}$$

According to Theorem B, $|V(G)| \geqq 2p + 5$. Hence, it is sufficient to prove that $|V(G)| \neq 2p + 5$. Assume the contrary. Then, by Lemma 1, $N(u) \nsubseteq N(v)$, $\forall u, v \in V(G)$. Therefore, $|N(v)| \neq |V(G)| - 2$. This, thogether with (5), implies that

$$|N(v)| \leqq |V(G)| - 3, \quad \forall v \in V(G). \tag{6}$$

It follows from Lemma 2 that

$$\alpha(G) = 2. \tag{7}$$

According to Theorem B, $F(2, 2, p + 1; p + 2) \geqq 2p + 6$. Hence, $G \nrightarrow (2, 2, p + 1)$. Let $V(G) = X \cup Y \cup Z$ be a $(2, 2, p+1)$-free 3-coloring. According to (7), $|X| \leqq 2$, $|Y| \leqq 2$. From (6) and (7) it follows that we may assume that $|X| = 2$, $|Y| = 2$. Let $X = \{a, b\}$, $Y = \{c, d\}$, $G_1 = G[a, b, c, d]$ and $G_2 = G[Z]$. Obviously,

$$G \to (2, 2, 2, p) \;\Rightarrow\; G_2 \to (2, p).$$

Since Z contains no $(p + 1)$-cliques, $\mathrm{cl}(G_2) < p + 1$. From Theorem A it follows that $G_2 = \overline{C}_{2p+1}$. Let $C_{2p+1} = v_1, \ldots, v_{2p+1}$. We define

$$Q = \{v_{2i-1} : i = 1, \ldots, p - 2\} \cup \{v_{2p}\},$$
$$Q_1 = Q \cup \{v_{2p-3}\} \quad \text{and} \quad Q_2 = Q \cup \{v_{2p-2}\}.$$

Obviously, Q_1 and Q_2 are p-cliques of $\overline{C}_{2p+1}$. From (7) it follows that $E(G_1)$ contains two independent edges. Without loss of generality we can assume that $[a,c]$, $[b,d] \in E(G_1)$. From $\mathrm{cl}(G) < p+2$ it follows that one of the vertices a, c is not adjacent to at least one of the vertices $v_1, \ldots, v_{2p+1}$, say $[a, v_1] \notin E(G)$. Consider the 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_{2p}, v_{2p+1}\}$, $V_2 = \{v_{2p-1}, v_{2p-2}\}$, $V_3 = \{c, d\}$. Since V_1, V_2, V_3 are independent sets, it follows from $G \to (2,2,2,p)$ that V_4 contains a p-clique. Since $Q' = Q_1 \setminus \{v_{2p}\}$ is the unique $(p-1)$-clique in $V_4 \setminus \{a,b\}$ then this p-clique is either $Q' \cup \{a\}$ or $Q' \cup \{b\}$. Since $v_1 \in Q'$ and $[a, v_1] \notin E(G)$, $Q' \cup \{a\}$ is not a clique. Hence, $Q' \cup \{b\}$ is a p-clique and thus

$$Q' = Q_1 \setminus \{v_{2p}\} \subseteq N(b). \tag{8}$$

Let $Q'' = Q_2 \setminus \{v_{2p-5}\}$. Similarly from the 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_{2p-3}, v_{2p-4}\}$, $V_2 = \{v_{2p-5}, v_{2p-6}\}$, $V_3 = \{c, d\}$ it follows that either $Q'' \cup \{a\}$ or $Q'' \cup \{b\}$ is a p-clique. Since $p \geqq 4$, we have $2p - 6 \geqq 2$ and thus $v_1 \in Q''$. From $[a, v_1] \notin E(G)$ it follows that $Q'' \cup \{b\}$ is a p-clique. Therefore,

$$Q'' = Q_2 \setminus \{v_{2p-5}\} \subseteq N(b). \tag{9}$$

By (8) and (9),

$$Q_1 \subseteq N(b). \tag{10}$$

$$Q_2 \subseteq N(b). \tag{11}$$

Case 1. $[b,c] \in E(G)$. Consider the 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_{2p-1}, v_{2p-2}\}$, $V_2 = \{v_{2p-3}, v_{2p-4}\}$, $V_3 = \{a, b\}$. Since V_1, V_2, V_3 are independent sets, then it follows from $G \to (2,2,2,p)$ that V_4 contains a p-clique L. Since Q is the unique $(p-1)$-clique in $V_4 \setminus \{a,b\}$, either $Q \cup \{c\} = L$ or $Q \cup \{d\} = L$. If $Q \cup \{c\} = L$, then from $\mathrm{cl}(G) < p+2$, (10) and (11) it follows that $\{c, v_{2p-2}, v_{2p-3}\}$ is an independent set, contradicting equality (7). The case $L = Q \cup \{d\}$ similarly leads to a contradiction.

Case 2. $[b,c] \notin E(G)$. Consider the 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_{2p}, v_{2p+1}\}$, $V_2 = \{v_{2p-1}, v_{2p-2}\}$, $V_3 = \{v_{2p-3}, v_{2p-4}\}$. Since V_1, V_2, V_3 are independent sets, then it follows from $G \to (2,2,2,p)$ that V_4 contains a p-clique L. Since $\mathrm{cl}(G_1) = 2$, $|L \cap V(\overline{C}_{2p+1})| \geqq p-2$. Observe that $\widetilde{Q} = Q \setminus \{v_{2p}\}$ is the unique $(p-2)$-clique in $V_4 \setminus \{a,b,c,d\}$. Therefore, $L \cap V(\overline{C}_{2p+1}) = \widetilde{Q}$. From

$v_1 \in \widetilde{Q}$ and $[a, v_1] \notin E(G)$ it follows that $b \in L$. By $[b, c] \notin E(G)$, $L = \widetilde{Q} \cup \{b, d\}$. Thus,

$$\widetilde{Q} = Q \setminus \{v_{2p}\} \subseteq N(d). \tag{12}$$

Similarly, from the 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_{2p-1}, v_{2p-2}\}$, $V_2 = \{v_{2p-3}, v_{2p-4}\}$, $V_4 = \{v_{2p-5}, v_{2p-6}\}$, it follows that

$$Q \setminus \{v_{2p-5}\} \subseteq N(d). \tag{13}$$

By (12) and (13),

$$Q \subseteq N(d). \tag{14}$$

From $\mathrm{cl}(G) < p + 2$, (10) and (14) it follows that $[d, v_{2p-3}] \notin E(G)$. By $\mathrm{cl}(G) < p + 2$, (11) and (14), $[d, v_{2p-2}] \notin E(G)$. So, $\{d, v_{2p-3}, v_{2p-2}\}$ is an independent set, contradicting equality (7).

II. $t = 4$. Let $G \to (2, 2, 2, 2, p)$ and $\mathrm{cl}(G) < p + 3$. We need to prove that $|V(G)| \geqq 2p + 7$. According to Theorem B, $|V(G)| \geqq 2p + 6$. Hence, it is sufficien to prove that $|V(G)| \neq 2p + 6$. Assume the contrary. As in the previous situation $t = 3$, we may assume that the graph G satisfies the conditions (6) and (7). According to Theorem B, $F(2, 2, p + 2; p + 3) \geqq 2p + 8$. Hence, $G \nrightarrow (2, 2, p + 2)$. Let $V(G) = X \cup Y \cup Z$ be a $(2, 2, p + 2)$-free 3-coloring. From (6) and (7) it follows that we may assume that $|X| = 2$, $|Y| = 2$. Let $X = \{a, b\}$, $Y = \{c, d\}$ and $G_1 = G[Z]$. Observe that

$$G \to (2, 2, 2, 2, p) \;\Rightarrow\; G_1 \to (2, 2, p).$$

Since Z contains no $(p + 2)$-cliques, $\mathrm{cl}(G_1) < p + 2$. According to Theorem A, $G_1 = K_1 + \overline{C}_{2p+1}$. Let $V(K_1) = \{w\}$ and $C_{2p+1} = v_1, \dots, v_{2p+1}$. From (7) it follows that either $[a, w] \in E(G)$ or $[b, w] \in E(G)$, say $[a, w] \in E(G)$. Similarly, we may assume also that $[c, w] \in E(G)$. From (6) it follows that $[w, b] \notin E(G)$ and $[w, d] \notin E(G)$.

Case 1. $[a, c] \notin E(G)$. Obviously, $G[w, a, b, c, d]$ contains no 3-cliques. Since $\overline{C}_{2p+1} - \{v_1, \dots, v_7\}$ contains no $(p - 2)$-cliques, the set $M = V(G) \setminus \{v_1, \dots, v_7\}$ contains no p-cliques. Thus, the 5-coloring

$$V(G) = \{v_1, v_2\} \cup \{v_3, v_4\} \cup \{v_5, v_6\} \cup \{v_7\} \cup M$$

is $(2, 2, 2, 2, p)$-free, a contradiction.

Case 2. $[a, c] \in E(G)$. From $\mathrm{cl}(G) < p+3$ it follows that one of the vertices a, c is not adjacent to at least of the vertices $v_1, \ldots, v_{2p+1}$, say $[a, v_1] \notin E(G)$. Since $G[w, b, c, d]$ contains no 3-cliques, then $N = V(G) \setminus \{a, v_1, \ldots, v_7\}$ contains no p-cliques. Thus, the 5-coloring

$$V(G) = \{v_1, a\} \cup \{v_2, v_3\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup N$$

is $(2, 2, 2, 2, p)$-free, a contradiction.

III. $t \geqq 5$. Let

$$G \to (\underbrace{2, \ldots, 2}_{t}, p) \quad \text{and} \quad \mathrm{cl}(G) < p + t - 1.$$

Then, according to Proposition 3,

$$\chi(G) \geqq t + p. \tag{15}$$

We need to prove that $|V(G)| \geqq t + 2p + 3$.

Case 1. $G \to (2, t+p-2)$. Obviously, $\chi(\overline{C}_{2t+2p-3}) = t+p-1$. Thus, from (15) it follows that $G \neq \overline{C}_{2t+2p-3}$. According to Theorem A, $|V(G)| \geqq 2t+2p-2$. Observe that if $t \geqq 5$, then $2t+2p-2 \geqq t+2p+3$. Therefore, $|V(G)| \geqq t+2p+3$.

Case 2. $G \nrightarrow (2, t + p - 2)$. Let $V(G) = X \cup Y$ be $(2, t + p - 2)$-free 2-coloring and $G_1 = G[Y]$. Clearly, we may assume that $X \neq \emptyset$. It is clear also that

$$G \to (\underbrace{2, \ldots, 2}_{t}, p) \ \Rightarrow \ G_1 \to (\underbrace{2, \ldots, 2}_{t-1}, p)$$

Since Y contains no $(t + p - 2)$-cliques, $\mathrm{cl}(G_1) < t + p - 2$. By the inductive hypothesis, $|V(G_1)| \geqq t + 2p + 2$. Since $X \neq \emptyset$, $|V(G)| \geqq t + 2p + 3$. $\square$

6. Proof of Theorem 2. Consider the set $M \subseteq \{a_1, \ldots, a_r\}$, where $a_i \in M \iff a_i = 2$. We prove Theorem 2 by induction on $n = m - |M| - 1$. Obviously, $n = \sum\limits_{a_i \geqq 3} (a_i - 1) \geqq p - 1$. The base of the induction is then $n = p - 1$. According to Proposition 1 and Proposition 2 we may assume that $2 \leqq a_1 \leqq \cdots \leqq a_r = p$. From these inequalities and $n = p-1$ it follows that $a_1 = \cdots = a_{r-1} = 2$. Therefore, if $n = p - 1$, Theorem 2 follows from Theorem 1. Let $n \geqq p$. Then from some $i \in \{1, \ldots, r - 1\}$, $a_i \geqq 3$. By Lemma 4,

$$F(a_1, \ldots, a_r; m - 1) \geqq F(a_1, \ldots, a_{i-1}, 2, a_i - 1, a_{i+1}, \ldots, a_r; m - 1).$$

By the inductive hypothesis,

$$F(a_1, \ldots, a_{i-1}, 2, a_i - 1, \ldots, a_r; m-1) \geqq m + p + 3.$$

Hence, $F(a_1, \ldots, a_r; m-1) \geqq m + p + 3$. □

7. Proof of Theorem 3.

I. Proof of the inequality $F(a_1, \ldots, a_r; m-1) \geqq m+6$. Let G be a graph such that $G \to (2,2,3)$ and $\mathrm{cl}(G) < 4$. By Theorem B, $|V(G)| \geqq 10$. From $R(4,3) = 9$ and $\mathrm{cl}(G) < 4$ it follows that $\alpha(G) \geqq 3$. According to Lemma 2, $|V(G)| \geqq 11$. Hence, $F(2,2,3;4) \geqq 11$. From Theorem 2, it follows that $F(a_1, \ldots, a_r; m-1) \geqq m+6$.

II. Proof of the inequality $F(a_1, \ldots, a_r; m-1) \leqq m+6$. Consider the graph P_1, whose complementary graph $\overline{P}_1$ is given in Fig. 1. We prove that this graph satisfies the conditions of Lemma 3 with $p = 3$ and $n = 6$. Obviously, from

$$\begin{cases} b_i \in \mathbb{Z}, \ i = 1, \ldots, s \\ 2 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_s \leqq 3 \\ \sum\limits_{i=1}^{s} (b_i - 1) + 1 = 6 \end{cases}$$

it follows that:

1. $s = 3$, $b_1 = 2$, $b_2 = b_3 = 3$;
2. $s = 4$, $b_1 = b_2 = b_3 = 2$, $b_4 = 3$;
3. $s = 5$, $b_1 = b_2 = b_3 = b_4 = b_5 = 2$.

It is proved in [17] that $P_1 \to (2,3,3)$. From Lemma 4 it follows that $P_1 \to (2,2,2,3)$ and $P_1 \to (2,2,2,2,2)$. By Proposition 1 and Lemma 3, $K_{m-6} + P_1 \to (a_1, \ldots, a_r)$. Since $\mathrm{cl}(P_1) = 4$, $\mathrm{cl}(K_{m-6} + P_1) = m-2$. Hence, $F(a_1, \ldots, a_r; m-1) \leqq |V(K_{m-6} + P_1)| = m+6$.

8. Proof of Theorem 4.

I. Proof of the inequality $F(a_1, \ldots, a_r; m-1) \geqq m+7$. Since $F(2,2,4;5) = 13$, [11], from Theorem 2 it follows that $F(a_1, \ldots, a_r; m-1) \geqq m+7$.

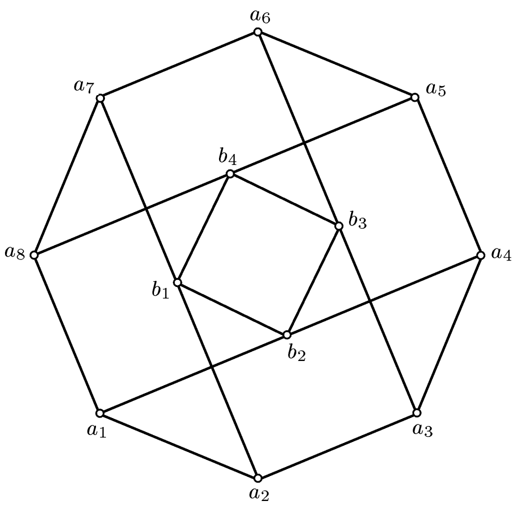

Fig. 1. Graph $\overline{P}_1$.

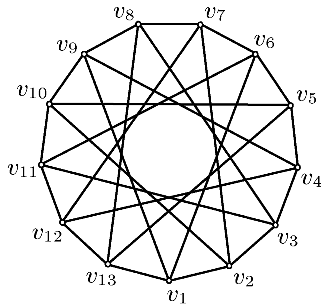

Fig. 2. Graph $\overline{P}_2$.

II. Proof of the inequality $F(a_1, \ldots, a_r; m-1) \leqq m+7$. Consider the graph P_2, whose complementary graph $\overline{P}_2$ is given in Fig. 2. This is well known construction of Greenwood and Gleason [3], which shows that the Ramsey number $R(3,5) \geqq 14$. We prove that this graph satisfies the conditions of Lemma 3 with $p = 4$ and $m = 6$. Obviously, from

$$\begin{cases} b_i \in \mathbb{Z},\ i = 1, \ldots, s \\ 2 \leqq b_1 \leqq \cdots \leqq b_s \leqq 4 \\ \sum_{i=1}^{s} (b_i - 1) + 1 = 6 \end{cases}$$

it follows that:

1. $s = 2,\ b_1 = 3,\ b_2 = 4;$
2. $s = 3,\ b_1 = b_2 = 2,\ b_3 = 4;$
3. $s = 3,\ b_1 = 2,\ b_3 = b_4 = 3;$
4. $s = 4,\ b_1 = b_2 = b_3 = 2,\ b_4 = 3;$
5. $s = 5,\ b_1 = b_2 = b_3 = b_4 = b_5 = 2.$

It is proved in [10] that $P_2 \to (3,4)$. From Lemma 4 it follows that $P_2 \to (2,2,4)$, $P_2 \to (2,3,3)$, $P_2 \to (2,2,2,3)$, $P_2 \to (2,2,2,2,2)$. By Proposition 1 and Lemma 3, $K_{m-6} + P_2 \to (a_1, \ldots, a_r)$. Since $\mathrm{cl}(P_2) = 4$, $\mathrm{cl}(K_{m-6} + P_2) = m - 2$. Hence, $F(a_1, \ldots, a_r; m-1) \leqq |V(K_{m-6} + P_2)| = m + 7$.

REFERENCES

[1] V. Chvátal. The minimality of the Mycielski graph. *Lecture Notes in Math.* **406** (1974), 243–246.

[2] J. Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* **18** (1970), 19–24.

[3] R. Greenwood, A. Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.* **7** (1955), 1–7.

[4] T. Łuczak, A. Ruciński, S. Urbański. On minimal vertex Folkman graphs. *Discrete Math.* **236** (2001), 245–262.

[5] J. Mycielski. Sur le coloriage des graphes. *Colloq. Math.* **3** (1955), 161–162.

[6] N. Nenov. An example of a 15-vertex $(3,3)$-Ramsey graph with clique number 4. *C. R. Acad. Bulgare Sci.* **34** (1981), 1487–1489 (in Russian).

[7] N. Nenov. On the Zykov numbers and some its applications to Ramsey theory. *Serdica Bulg. Math. Publ.* **9** (1983), 161–167 (in Russian).

[8] N. Nenov. The chromatic number of any 10-vertex graph without 4-cliques is at most 4. *C. R. Acad. Bulgare Sci.* **37** (1984), 301–304 (in Russian).

[9] N. Nenov. On the small graphs with chromatic number 5 without 4-cliques. *Discrete Math.* **188** (1998), 297–298.

[10] N. Nenov. On the vertex Folkman number $F(3,4)$. *C. R. Acad. Bulgare Sci.* **54**, *2* (2001), 23–26.

[11] N. Nenov. On the 3-colouring vertex Folkman number $F(2,2,4)$. *Serdica Math. J.* **27** (2001), 131–136.

[12] N. Nenov. Ramsey graphs and some constants related to them. Ph. D. Thesis, University of Sofia, Sofia, 1980.

[13] N. Nenov. On a class of vertex Folkman graphs. *Annuaire Univ. Sofia, Fac. Math. Inform.* **94** (2000), 15–25.

[14] N. Nenov. A generalization of a result of Dirac. *Annuaire Univ. Sofia Fac. Math. Inform.* **95** (2001), 7–16.

[15] N. Nenov. Lower bound for a number of vertices of some vertex Folkman graphs. *C. R. Acad. Bulgare Sci.* **55**, *4* (2002), 33–36.

[16] N. Nenov. On the triangle vertex Folkman numbers. *Discrete Math.*, to appear.

[17] N. Nenov. Computation of the vertex Folkman numbers $F(2,2,2,3;5)$ and $F(2,3,3;5)$. *Annuaire Univ. Sofia, Fac. Math. Inform.* **95** (2001), 17–27.

[18] E. Nedialkov, N. Nenov. Computation of the vertex Folkman numbers $F(2,2,2,4;6)$ and $F(2,3,4;6)$. *Electron. J. Combin.* **9** (2002), # R9.

[19] E. Nedialkov, N. Nenov. Computation of the vertex Folkman number $F(4,4;6)$. Proceedings of the Third Euro Workshop on Optimal Codes and related topics, Sunny Beach, Bulgaria, 11–16 June 2001, 123–128.

[20] K. Piwakowski, S. Radziszowski, S. Urbański. Computation of the Folkman number $F_e(3,3;5)$. *J. Graph Theory*, **32** (1999), 41–49.

Faculty of Mathematics and Informatics
"St. Kliment Ohridski" University of Sofia
5, James Bourchier Blvd.
1164 Sofia, Bulgaria
e-mail: `nenov@fmi.uni-sofia.bg`

Received April 29, 2002