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CERAMI (C) CONDITION AND MOUNTAIN PASS THEOREM FOR MULTIVALUED MAPPINGS

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Abstract. We prove a general minimax result for multivalued mapping. As application, we give existence results of critical point of this mapping which satisfies the Cerami (C) condition.

1. Introduction. Many papers has been devoted to prove the existence of the critical points, providing that the respective functional satisfies a suitable compactness condition: Palais-Smale (PS) or Cerami (C). For example, let us consider a functional $f : X \to \mathbb{R}$ of class C^1, X being a Banach space.
a) f satisfies the (PS)-condition at level c (shortly $(PS)_c$), if every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $f(x_n) \to c$ and $\|df(x_n)\| \to 0$, has a convergent subsequence;

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b) f satisfies the (C)-condition at level c (shortly $(C)_c$), if every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $f(x_n) \to c$ and $(1+\|x_n\|)\|df(x_n)\| \to 0$, has a convergent subsequence.

It is clear, that $(PS)_c$ implies the $(C)_c$.

Of course, if the class of the functional is different than C^1, the above conditions are modified. Instead of differential, if f is locally Lipschitz resp. continuous, we use the Clarke subdifferential (see [4]) and the weak slope (see [7]) respectively.

Several authors obtained deformation lemmas (and mountain pass theorems), corresponding to the class of the considered functional and to the compactness assumption; Brézis and Nirenberg in [2], Willem in [20], Ghoussoub in [12] for functionals of class C^1 with (PS) condition; Bartolo, Benci and Fortunato in [1], Ghoussoub and Preiss in [13] for functionals of class C^1 with (C) condition; Chang in [4], Ribarska, Tsachev and Krastanov in [17], [18] for locally Lipschitz functionals with (PS); Kourogenis and Papageorgiou in [15] for locally Lipschitz functionals with (C); Corvellec, Degiovanni and Marzocchi in [5], Fang in [9], Ioffe and Schwartzmann in [14] for continuous functionals with (PS) condition. Ribarska, Tsachev and Krastanov in [19] and Frigon in [11] obtained similar results for multivalued mappings, using the suitable (PS) condition. We follow the paper of Frigon [11], working on the graph of the functional instead of its domain. The aim of this paper is to obtain a mountain pass-type result for multivalued mappings with closed graph, using the Cerami condition (see Definition 5.1.). Our main theorem is a natural multivalued version of the above results.

The paper is organized as follows. In Section 2 we recall the notions which will be used in the paper. In Section 3 we prove a general deformation lemma for continuous functionals and we recall the notion of homotopy stability with boundary on the graph. We also state a localization theorem. In Section 4 we extend for multivalued mappings a result from [6] stating a relation between the weak slopes if we change the metric of the graph. Finally, in Section 5, using the Cerami (C) conditon for our case, we state the main result of this paper, i.e. the mountain pass-type result for multivalued mappings.

2. Preliminaries.

First, we recall some definitions and results.

Definition 2.1 (see [7]). *Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a continuous function and $u \in X$ a fixed element. We denote by $|df|(u)$ the supremum of the $\sigma \in [0,\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathcal{H} : B(u,\delta) \times [0,\delta] \to X$$

such that for all $v \in B(u,\delta)$ *and* $t \in [0,\delta]$

(a) $$d(\mathcal{H}(v,t),v) \le t;$$

(b) $$f(\mathcal{H}(v,t)) \le f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called *weak slope* of f at u.

Proposition 2.1 (see [6, Theorem 2.3]). *Let* (X,d) *be a complete metric space and let* $f : X \to \mathbb{R}$ *be a continuous function,* A *a closed subset of* X, $c \in \mathbb{R}$ *and* $\delta, \sigma > 0$ *such that*

$$c - 2\delta \le f(u) \le c + 2\delta,\ d(u,A) \le \delta/\sigma \Longrightarrow |df|(u) > 2\sigma.$$

Then there exists a continuous map $\eta : X \times [0,1] \to X$ *such that*
1) $d(\eta(u,t),u)) \le (\delta/\sigma)t$,
2) $\eta(u,t) \ne u \Longrightarrow f(\eta(u,t)) < f(u)$,
3) $u \in A,\ c - \delta \le f(u) \le c + \delta \ \Longrightarrow f(\eta(u,t)) \le f(u) - (f(u) - c + \delta)t$.

In the sequel, let (X,d) be a metric space and let $F : X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with nonempty values. We denote by

$$\operatorname{graph} F = \{(u,c) \in X \times \mathbb{R} \ \mid \ c \in F(u)\}.$$

We suppose that $\operatorname{graph} F$ is endowed with a metric d_g.

Definition 2.2. *Let* $F : X \to \mathbb{R} \cup \{\infty\}$ *be a multivalued mapping with non-empty values and let* $(u,b) \in \operatorname{graph} F$ *be a point. We denote by* $|d_g F|(u,b)$ *the supremum of* $\sigma \in [0,\infty[$ *such that there exists* $\delta > 0$ *and a continuous function*

$$\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : B_g((u,b),\delta) \times [0,\delta] \to \operatorname{graph} F,$$

(where $B_g((u,b),\delta)$ *is the open ball in* $\operatorname{graph} F$ *centered at* (u,b) *of radius* δ*) such that*

(a) $$d_g(\mathcal{H}((v,c),t),(v,c)) \le t\sqrt{1+\sigma^2};$$

(b) $$\mathcal{H}_2((v,c),t) \le c - \sigma t.$$

The extended real number $|d_g F|(u,b)$ is called *weak slope* of F at (u,b).

Remark 2.1. If $F : X \to \mathbb{R} \cup \{\infty\}$ is a multivalued mapping with closed graph, and the metric d_g is defined by

$$d_g((u,b),(v,c)) = \sqrt{d(u,v)^2 + |b-c|^2} \tag{2.0}$$

then the above definition reduces to the definition of Frigon, see [11, Definition 2.1]. In the case where $F(u) = \{f(u)\}$ is a continuous single-valued function then $|d_g F|(u, f(u)) = |df|(u)$, see [11, p. 737], and it coincides with the norm of the derivative when f is of class C^1 defined on a Finsler manifold of class C^1.

Let $\mathcal{G}_F$: graph $F \to \mathbb{R}$ defined by $\mathcal{G}_F(u,b) = b$. Clearly, this function is continuous. Now, we compare the above two weak slopes, supposing that graph F is endowed with an arbitrary metric d_g, not necessarily given by (2.0).

Proposition 2.2. *Let* $(u,b) \in \operatorname{graph} F$. *Then*

$$|d_g \mathcal{G}_F|(u,b) = \frac{|d_g F|(u,b)}{\sqrt{1 + |d_g F|^2(u,b)}}\,, \quad \text{if } |d_g F|(u,b) < \infty;$$

$$|d_g \mathcal{G}_F|(u,b) \geq 1\,, \quad \text{if } |d_g F|(u,b) < \infty.$$

Proof. We first demonstrate that

$$|d_g \mathcal{G}_F|(u,b) \geq \begin{cases} \dfrac{|d_g F|(u,b)}{\sqrt{1 + |d_g F|^2(u,b)}}, & \text{if } |d_g F|(u,b) < \infty; \\ 1, & \text{if } |d_g F|(u,b) = \infty. \end{cases} \tag{2.1}$$

If $|d_g F|(u,b) = 0$, the relation is obvious. Otherwise, let $0 < \sigma < |d_g F|(u,b)$ and $H = (H_1, H_2) : B_g((u,b),\delta) \times [0,\delta] \to \operatorname{graph} F$ as in Definition 2.2, i.e.

$$d_g(H((v,c),t),(v,c)) \leq t\sqrt{1+\sigma^2}; \tag{2.2}$$

$$H_2((v,c),t) \leq c - \sigma t. \tag{2.3}$$

Let $\mathcal{H} : B_g((u,b),\delta) \times [0,\delta] \to \operatorname{graph} F$ be a function defined by

$$\mathcal{H}((v,c),t) = H\left((v,c), \frac{t}{\sqrt{1+\sigma^2}}\right).$$

The function $\mathcal{H}$ is well defined and continuous. From the relation (2.2) we get

$$d_g(\mathcal{H}((v,c),t),(v,c)) = d_g(H((v,c), \frac{t}{\sqrt{1+\sigma^2}}),(v,c)) \leq \frac{t}{\sqrt{1+\sigma^2}}\sqrt{1+\sigma^2} = t.$$

Using the relation (2.3), we get

$$\mathcal{G}_F(\mathcal{H}((v,c),t)) = \mathcal{G}_F(H((v,c),\frac{t}{\sqrt{1+\sigma^2}})) = H_2((v,c),\frac{t}{\sqrt{1+\sigma^2}}) \leq$$

$$\leq c - \sigma\frac{t}{\sqrt{1+\sigma^2}} = \mathcal{G}_F(v,c) - \frac{\sigma}{\sqrt{1+\sigma^2}}t.$$

Using the above relations and Definition 2.1, we have $|d_g\mathcal{G}_F|(u,b) \geq \frac{\sigma}{\sqrt{1+\sigma^2}}$, from which we obtain easily (2.1).

Finally, we prove that

$$|d_g\mathcal{G}_F|(u,b) \leq \frac{|d_gF|(u,b)}{\sqrt{1+|d_gF|^2(u,b)}}, \quad \text{if } |d_gF|(u,b) < \infty. \tag{2.4}$$

On can suppose that $|d_g\mathcal{G}_F|(u,b) > 0$ and take any $\sigma > 0$ with

$$\frac{\sigma}{\sqrt{1+\sigma^2}} < |d_g\mathcal{G}_F|(u,b).$$

If one chooses $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : B_g((u,b),\delta) \times [0,\delta] \to \operatorname{graph} F$ according to Definition 2.1, with

$$H_2((v,c),t) \leq c - \frac{\sigma}{\sqrt{1+\sigma^2}}t,$$

then $H = (H_1, H_2) : B_g((u,b),\delta') \times [0,\delta'] \to \operatorname{graph} F$, (with $\delta'\sqrt{1+\sigma^2} \leq \delta$), defined by

$$H((v,c),t) = \mathcal{H}((v,c),t\sqrt{1+\sigma^2})$$

satisfies the requirements of Definition 2.2, whence $|d_gF|(u,b) \geq \sigma$. Since $|d_gF|(u,b) < \infty$, σ cannot be arbitrarily large. This means that $|d_g\mathcal{G}_F|(u,b) < 1$, and by the arbitrariness of σ the relation (2.4) follows. □

Remark 2.2. On the contrary, if $|d_gF|(u,b) = \infty$ one can only conclude that $|d_g\mathcal{G}_F|(u,b) \geq 1$.

3. Deformation lemma and minmax principle.

In this section, we prove a "Quantitative deformation lemma" for continuous functionals and a general minmax result, which will be used in the last section. Similar results were obtained by Corvellec in [6] and by the authors in [16]. We use the following notations:

$$f^c = \{x \in X| \ \ f(x) \leq c\}$$
$$C_\delta = \{x \in X| \ \ d(x, C) \leq \delta\}, \ \delta > 0$$

Theorem 3.1. *Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X, $c \in \mathbb{R}$ and $\varepsilon, \lambda > 0$. Suppose that*

$$\forall u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon} \Longrightarrow |df|(u) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}.$$

Then there exists a continuous map $\eta : X \times [0, 1] \to X$ such that:

a) $d(\eta(u, t), u) \leq \lambda t, \quad \forall t \in [0, 1], \ \forall u \in X,$

b) $f(\eta(u, t)) \leq f(u), \quad \forall t \in [0, 1], \ \forall u \in X,$

c) *if* $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon} : \eta(u, t) = u, \quad \forall t \in [0, 1],$

d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$, *where* $\varepsilon' = \dfrac{\varepsilon \min\{\varepsilon, \lambda\}}{2\sqrt{1+\varepsilon^2}}$,

e) $\forall \ t \in]0, 1]$ *and* $\forall \ u \in f^c \cap C$ *we have* $f(\eta(u, t)) < c$.

Proof. With the choice $A := C$, $\delta := \varepsilon'$, $\sigma := \dfrac{\varepsilon'}{\min\{\varepsilon, \lambda\}}$ the implication from Proposition 2.1 holds, since $2\delta \leq 2\varepsilon$, $\dfrac{\delta}{\sigma} \leq 2\varepsilon$, $\dfrac{\varepsilon}{\sqrt{1+\varepsilon^2}} = 2\sigma$. Therefore, there exists a continuous application $\eta' : X \times [0, 1] \to X$ which satisfies the properties from the above Proposition 2.1. Now, let $\vartheta : X \to [0, 1]$ be a continuous function such that $\vartheta(u) = 0$ for $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\varepsilon}$ and $\vartheta(u) = 1$ for $u \in f^{-1}([c - \varepsilon', c + \varepsilon']) \cap C$. Defining $\eta : X \times [0, 1] \to X$ by $\eta(u, t) = \eta'(u, \vartheta(u)t)$, it's easy to verify the above properties, using that $\dfrac{\delta}{\sigma} \leq \lambda$. □

Let $F : X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with non-empty values, supposing that graph F is endowed with a metric d_g.

Definition 3.1. *Let $c \in \mathbb{R}$. We say that u is a critical point of F at level c, if $c \in F(u)$ and $|d_g F|(u, c) = 0$.*

Under some assumption, we can localize the critical points of F. First of all, we need some notions.

Definition 3.2. *Let B be a subset of* graph F. *We shall say that the class $\mathcal{F}$ of subsets of* graph F *is homotopy stable with boundary B, if the following conditions hold:*

a) every set of $\mathcal{F}$ contains B;

b) for every set $A \in \mathcal{F}$ and for every continuous function $\eta : \operatorname{graph} F \times [0,1] \to \operatorname{graph} F$ satisfying $\eta = id$ on $(\operatorname{graph} F \times \{0\}) \cup (B \times [0,1])$ we have that $\eta(A,1) \in \mathcal{F}$.

Definition 3.3. *We say that the set $G \subset \operatorname{graph} F$ is dual to $\mathcal{F}$ if G verifies the following conditions:*

1) $d_g(G,B) > 0$ *(here, the d_g means the distance between G and B)*
2) $G \cap A \neq \emptyset$, *for all* $A \in \mathcal{F}$.

Now, we are in the position to prove a location result.

Theorem 3.2. *Let (X, d) be a metric space, $F : X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with nonempty values and d_g a metric on $\operatorname{graph} F$ such that $(\operatorname{graph} F, d_g)$ is complete. Let B be a subset of $\operatorname{graph} F$. We consider a homotopy stable family $\mathcal{F}$ with boundary B, and let G be a dual set to $\mathcal{F}$ which verifies the following condition*

$$\inf \mathcal{G}_F(G) \geq \inf_{A \in \mathcal{F}} \sup \mathcal{G}_F(A) = c, \tag{3.1}$$

where the number c is finite.

Let $\varepsilon \in \left(0, \dfrac{d_g(B,G)}{3}\right)$. Then for every $A \in \mathcal{F}$ which verifies the relation

$$\sup \mathcal{G}_F(A \cap G_\varepsilon) \leq c + \frac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}, \tag{3.2}$$

there exists $(x_\varepsilon, c_\varepsilon) \in \operatorname{graph} F$ such that the following assertions hold:

a) $c - 2\varepsilon \leq c_\varepsilon \leq c + 2\varepsilon$;

b) $|d_g F|(x_\varepsilon, c_\varepsilon) \leq \varepsilon$;

c) $d_g((x_\varepsilon, c_\varepsilon), A) \leq 2\varepsilon$;

d) $d_g((x_\varepsilon, c_\varepsilon), G) \leq 3\varepsilon$.

Proof. From the definition of the number c, it follows that there exists $A \in \mathcal{F}$ such that:

$$\sup \mathcal{G}_F(A \cap G_\varepsilon) \leq c + \frac{\varepsilon^2}{2\sqrt{1+\varepsilon^2}}.$$

The properties a)–d) are equivalent with the following implication:

$$\exists (x_\varepsilon, c_\varepsilon) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{G_{3\varepsilon}} \cap \overline{A_{2\varepsilon}} \text{ such that } |d_g F|(x_\varepsilon, c_\varepsilon) \leq \varepsilon.$$

We suppose the contrary, i.e.

$$\forall (v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{G_{3\varepsilon}} \cap \overline{A_{2\varepsilon}} \text{ we have } |d_g F|(v,b) > \varepsilon.$$

We consider the set $C := \overline{A \cap G_\varepsilon}$. We have $C_{2\varepsilon} \subset \overline{A_{2\varepsilon}} \cap \overline{G_{3\varepsilon}}$, therefore

$$\forall\ (v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \quad |d_g F|(v,b) > \varepsilon. \tag{3.3}$$

Using the relation (3.3) and Proposition 2.2, we obtain that for all

$$(v,b) \in \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \quad |d_g \mathcal{G}_F|(v,b) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}.$$

Since $(\operatorname{graph} F, d_g)$ is a complete metric space, we can apply Theorem 3.1, obtaining a continuous function $\eta := (\eta_1, \eta_2) : \operatorname{graph} F \times [0,1] \to \operatorname{graph} F$ which satisfies the following properties:

a$'$) $d_g(\eta((v,b),t),(v,b)) \le \varepsilon t$;

b$'$) $\eta_2((v,b),t) \le b$ for every $(v,b) \in \operatorname{graph} F$ and $t \in [0,1]$;

c$'$) if $(v,b) \notin \mathcal{G}_F^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$ then we have $\eta((v,b),t) = (v,b)$;

d$'$) $\eta(\mathcal{G}_F^{c+\varepsilon'} \cap C, 1) \subset \mathcal{G}_F^{c-\varepsilon'}$.

Let $A_1 = \eta(A,1)$. From a$'$) we have $\eta((v,b),0) = (v,b)$ for every $(v,b) \in \operatorname{graph} F$. From the relation c$'$) we have $\eta((v,b),t) = (v,b)$ for all $t \in [0,1]$ and $(v,b) \in B$, since $B \cap C_{2\varepsilon} = \emptyset$. Indeed, if we have the contrary, we obtain

$$2\varepsilon = 3\varepsilon - \varepsilon < d_g(B,G) - \varepsilon \le d_g(B,G_\varepsilon) \le d_g(B, A \cap G_\varepsilon) = d_g(B, \overline{A \cap G_\varepsilon}) \le 2\varepsilon,$$

which is a contradiction. Using the definition of homotopy stable family, we have $A_1 = \eta(A,1) \in \mathcal{F}$.

Moreover, we have $\eta(A,1) \cap G \subset \eta(A \cap G_\varepsilon, 1)$. In fact, if $x \in \eta(A,1) \cap G$, then there exists $y \in A$ such that $x = \eta(y,1) \in G$. But from the relation a$'$) we have that $d_g(\eta(y,1),y) \le \varepsilon$, therefore $y \in G_\varepsilon$. Hence, $x = \eta(y,1) \in \eta(A \cap G_\varepsilon, 1)$. Using the relations d$'$) and (3.2) we have

$$A_1 \cap G = \eta(A,1) \cap G \subset \eta(A \cap G_\varepsilon, 1) \subset \eta(\overline{A \cap G_\varepsilon} \cap \mathcal{G}_F^{c+\varepsilon'}, 1) \subset \mathcal{G}_F^{c-\varepsilon'} \tag{3.4}$$

From the relations (3.1) and (3.4) we get

$$c \le \inf \mathcal{G}_F(G) \le \inf \mathcal{G}_F(A_1 \cap G) \le c - \varepsilon',$$

which is a contradiction. The proof is complete. □

An imediate consequence of this result is the following, which will be used in the next section.

Theorem 3.3. *Let X, F, $\mathcal{F}$, B, G and c be as in the Theorem 3.2. Then there exists a sequence $(u_n, c_n) \in \operatorname{graph} F$, such that the following assertions hold:*

a) $c_n \to c$;

b) $|d_g F|(u_n, c_n) \to 0$;

c) $d_g((u_n, c_n), G) \to 0$.

In particular, if $F : X \to \mathbb{R} \cup \{\infty\}$ is with closed graph, and the metric d_g of graph F comes from the relation (2.0), we obtain a minimax result, using the Palais-Smale condition introduced in [11].

Definition 3.4 (see [11, Definition 2.5]). *Let $F : X \to \mathbb{R} \cup \{\infty\}$ be a set-valued mapping with closed graph, and let $c \in \mathbb{R}$ a real number. We say that the function F satisfies the Palais-Smale condition at level c (shortly $(PS)_c$) if every sequence $(u_n) \subset X$ for which there exists $c_n \in F(u_n)$ with $c_n \to c$ and $|d_g F|(u_n, c_n) \to 0$, has a convergent subsequence in X.*

Corollary 3.1. *Let (X,d) be a complete metric space, $F : X \to \mathbb{R} \cup \{\infty\}$ be a set-valued mapping with closed graph and $\mathcal{F}$, B, G and c as in the Theorem 3.2. If F satisfies the $(PS)_c$ then there exists a critical point of F at level c, i.e. $c \in F(u)$ and $|d_g F|(u, c) = 0$. Moreover, $(u, c) \in \overline{G}$.*

4. Changing the metric on the graph.

In [6], Corvellec established a result for continuous functionals concerning the weak slopes, if the metric of the space is changed. Here, we state a similar result for multivalued case, changing a given metric of the graph F to the so-called *Cerami metric* (see [3]). The main result of this section is the following

Theorem 4.1. *Let (X, d) be a complete metric space, $F : X \to \mathbb{R} \cup \{\infty\}$ a multivalued map with closed graph, $\tilde{A}$ a non-empty subset of graph F and $\beta : [0, \infty[\to]0, 1]$ be a continuous function. Then there exists a metric $\tilde{d}$ on graph F which is topologically equivalent to d_g (d_g from (2.0)) and the following assertions are true:*

For every $B \subset \operatorname{graph} F$ we have $\tilde{d}(\tilde{A}, B) \geq \int_0^{d_g(B,\tilde{A})} \beta(t)dt$.

Therefore, if $\int_0^{\infty} \beta(t)dt = \infty$, *then* $(\operatorname{graph} F, \tilde{d})$ *is complete.*

If we denote by $|\tilde{d}F|$ the weak slope of F with respect to the metric $\tilde{d}$, then we have $|\tilde{d}F|(u, c) \geq \dfrac{|d_g F|(u, c)}{\beta(d_g((u, c), \tilde{A}))}$, *for every $(u, c) \in \operatorname{graph} F$.*

P r o o f. The first part can be deduced by applying [6, Theorem 4.1 (a)] to the metric space $(X, d) := (\operatorname{graph} F, d_g)$ and the continuous functional $f := \mathcal{G}_F$. Let us prove the second part.

Let $(u,c) \in \operatorname{graph} F$ be a fixed element. One can suppose that $|\tilde{d}F|(u,c) < \infty$. Applying Proposition 2.2 to the metric $\tilde{d}$ we obtain

$$|\tilde{d}\mathcal{G}_F|(u,c) = \frac{|\tilde{d}F|(u,c)}{\sqrt{1+|\tilde{d}F|^2(u,c)}}. \tag{4.1}$$

Moreover, from [6, Theorem 4.1 (b)] one has

$$|\tilde{d}\mathcal{G}_F|(u,c) = \frac{|d_g\mathcal{G}_F|(u,c)}{\beta(d_g((u,c),\widetilde{A}))}. \tag{4.2}$$

Since $\beta \le 1$, the above relations and Proposition 2.2 imply that $|d_gF|(u,c) < \infty$. Therefore, using the relations (4.1), (4.2) and the first part of Proposition 2.2 for the metric d_g, an easy calculation shows that

$$|\tilde{d}F|(u,c) = \frac{|d_gF|(u,c)}{\sqrt{\beta(d_g((u,c),\widetilde{A}))^2 - (1-\beta(d_g((u,c),\widetilde{A}))^2)|d_gF|^2(u,c)}}.$$

Since again $\beta \le 1$, it follows

$$|\tilde{d}F|(u,c) \ge \frac{|d_gF|(u,c)}{\beta(d_g((u,c),\tilde{A}))}. \qquad \square$$

5. Mountain Pass Theorem. In this section we establish the main result of this paper, using the location theorem (Theorem 3.3) and the change of the metric. In the sequel, X will be a Banach space, the metric d on X coming from the norm.

Theorem 5.1. *Let X be a Banach space, and $F : X \to \mathbb{R} \cup \{\infty\}$ a multivalued mapping with closed graph and nonempty values. Let $\mathcal{F}$ be a homotopy stable family of* graph F *with boundary B, B being compact and G a closed set, dual to $\mathcal{F}$. We suppose that the condition* (3.1) *holds and the number c is finite. Then there exists a sequence $(u_n, c_n) \in \operatorname{graph} F$, such that:*

a) $c_n \to c$;

b) $(1+\|u_n\|) \cdot |d_gF|(u_n,c_n) \to 0$.

Proof. We apply Theorem 4.1 for $\beta(t) = \dfrac{1}{1+t}$, $t \ge 0$ and $\tilde{A} = (0_X, F(0_X))$, obtaining a metric $\tilde{d}$ on graph F such that $(\operatorname{graph} F, \tilde{d})$ is complete (because $\int_0^\infty \beta(t)dt = \infty$) and

$$|\tilde{d}F|(u,c) \geq \frac{|d_gF|(u,c)}{\beta(d_g((u,c),\tilde{A}))} = (1 + d_g((u,c),\tilde{A})) \cdot |d_gF|(u,c). \tag{5.1}$$

From hypotheses concerning the sets B and G, we obtain that $\tilde{d}(B,G) > 0$. Therefore, the hypotheses from Theorem 3.3 are verified for the metric space $(\operatorname{graph} F, \tilde{d})$. Then there exists a sequence $(u_n, c_n) \in \operatorname{graph} F$ such that $c_n \to c$, $|\tilde{d}F|(u_n, c_n) \to 0$.

Using the inequality $d_g((u,c),\tilde{A}) = \sqrt{\|u\|^2 + \inf\{|c-a|^2 : a \in F(0_X)\}} \geq \|u\|$ and the relation (5.1), we obtain that $(1 + \|u_n\|) \cdot |d_gF|(u_n, c_n) \to 0$. □

Definition 5.1 (Condition (C))**.** *Let $F : X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph. We say that F satisfies the condition (C) at level $c \in \mathbb{R}$ (shortly $(C)_c$) if every sequence (x_n) in X for which there exists $c_n \in F(x_n)$ with $c_n \to c$ and $(1 + \|x_n\|) \cdot |d_gF|(x_n, c_n) \to 0$, has a convergent subsequence on X, which converge to a critical point of F on the level c.*

Of course, condition $(PS)_c$ implies condition $(C)_c$. As an immediate consequence, we obtain the following result.

Theorem 5.2. *Let X, F, $\mathcal{F}$, B, G and c be as in the Theorem* 5.1*. If F satisfies the $(C)_c$ condition, then there exists a critical point of F at level c.*

Definition 5.2. *A closed subset $F_1 \subset \operatorname{graph} F$ separates two points (u_0, c_0) and (u_1, c_1) in* graph F*, if (u_0, c_0) and (u_1, c_1) belong to disjoint connected components in* graph $F \setminus F_1$*.*

The main result of this paper is the following theorem, which is the multivalued version of the well-known result of Ghoussoub-Preiss, see [13] and [8, Theorem 6, p. 140].

Theorem 5.3. *Let X be a Banach space, $F : X \to \mathbb{R} \cup \{\infty\}$ be a multivalued mapping with closed graph and nonempty values, and $(u_0, c_0), (u_1, c_1)$ two points in the same path-connected component of* graph F*. Let*

$$\Gamma = \{g \in \mathcal{C}([0,1], \operatorname{graph} F) \mid g(0) = (u_0, c_0),\ g(1) = (u_1, c_1)\ \}$$

and we consider the number $\gamma := \inf_{g \in \Gamma} \max_{t \in [0,1]} \mathcal{G}_F(g(t))$ which is finite. Suppose that the set $D_\gamma = \{\ (x,b) \in \operatorname{graph} F |\ b \geq \gamma\ \}$ separates the points $(u_0, c_0), (u_1, c_1)$.

Then there exists a sequence $(x_n, b_n) \in \operatorname{graph} F$ *such that the following assertions hold:*

a) $b_n \to \gamma$;

b) $(1 + \|x_n\|) \cdot |d_g F|(x_n, b_n) \to 0$;

In particular, if F *satisfies the condition* $(C)_\gamma$, *then there exists a critical point* $x \in X$ *at level* γ.

Proof. We observe that $\mathcal{F} = \Gamma$ is homotopy stable with boundary $B = \{(u_0, c_0), (u_1, c_1)\}$. Indeed, for every $g \in \Gamma$ we have from the definition that $B \subset g([0,1])$ and for any $g \in \Gamma$ and $\eta \in \mathcal{C}(\operatorname{graph} F \times [0,1], \operatorname{graph} F)$, verifying $\eta = id$ on $(\operatorname{graph} F \times \{0\}) \cup (B \times [0,1])$ it follows that $g_1(t) = \eta(g(t), 1)$ is continuous and $g_1(0) = \eta(g(0), 1) = \eta((u_0, c_0), 1) = (u_0, c_0)$ and $g_1(1) = (u_1, c_1)$. Therefore $g_1 \in \Gamma$.

Moreover, the set D_γ is dual to $\mathcal{F}$. Indeed, since D_γ separates the points (u_0, c_0) and (u_1, c_1), we have that $D_\gamma \cap B = \emptyset$. Using the fact that the set $D_\gamma = \mathcal{G}_F^{-1}([\gamma, \infty))$ is closed and B is compact, we have $d_g(B, D_\gamma) > 0$. From the definition of γ we get that $D_\gamma \cap g([0,1]) \neq \emptyset$ for all $g \in \Gamma$. In conclusion we have that D_γ is dual to $\mathcal{F}$.

Since $\inf \mathcal{G}_F(D_\gamma) \geq \gamma$, we can apply Theorem 5.1 and Theorem 5.2 respectively, obtaining the desired relations. □

We give an immediate consequence of the above results, which is the multivalued version of the classical "Mountain Pass" theorem of Ambrosetti-Rabinowitz, see [8, Corollary 9, p. 145].

Corollary 5.1. *Let* X, F, (u_0, c_0), (u_1, c_1), Γ *and* γ *be as in Theorem 5.3. Assume that* F *satisfies the condition* $(C)_\gamma$, *and* $\gamma > \max\{c_0, c_1\}$. *Then there exists a critical point of* F *at level* γ.

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