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ON PARABOLIC SUBGROUPS AND HECKE ALGEBRAS OF SOME FRACTAL GROUPS

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Abstract. We study the subgroup structure, Hecke algebras, quasi-regular representations, and asymptotic properties of some fractal groups of branch type.

We introduce parabolic subgroups, show that they are weakly maximal, and that the corresponding quasi-regular representations are irreducible. These (infinite-dimensional) representations are approximated by finite-dimensional quasi-regular representations. The Hecke algebras associated to these parabolic subgroups are commutative, so the decomposition in irreducible components of the finite quasi-regular representations is given by the double cosets of the parabolic subgroup. Since our results derive from considerations on finite-index subgroups, they also hold for the profinite completions $\widehat{G}$ of the groups G.

The representations involved have interesting spectral properties investigated in [6]. This paper serves as a group-theoretic counterpart to the studies in the mentioned paper.

We study more carefully a few examples of fractal groups, and in doing so exhibit the first example of a torsion-free branch just-infinite group.

We also produce a new example of branch just-infinite group of intermediate growth, and provide for it an L-type presentation by generators and relators.

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1. Introduction. Fractal groups entered recently in the *avant-scène* of group theory, and are related to diverse areas such as the theory of branch groups [20], automata groups [6] and so on.

Fractal groups of branch type have many interesting properties. Namely, the first examples of groups of intermediate growth were found in this class of groups [18]; the simplest examples of infinite finitely-generated torsion groups too [16, 22] (thus contributing to the general Burnside problem); fractal groups provide sporadic examples of groups of finite width with unusual associated Lie algebra [5], thus answering a question by Efim Zel'manov; etc.

It is therefore of utmost interest to pursue the study of the algebraic, geometric and analytic properties of these groups, and in particular their subgroup structure.

Fractal groups and branch groups are defined in the category of profinite groups as well. These new classes of profinite groups already started to play an important role. For instance, they gave an answer to a question of Efim Zel'manov about groups of finite width [5], they were used by Dan Segal [31] to solve in the negative a conjecture by Alex Lubotzky, Laci Pyber and Aner Shalev [25] about a gap in the range of subgroup growths, these groups have an universal embedding property [21], and it is believed that branch groups may play an important role in Galois theory [10].

Fractal groups are groups acting on regular rooted trees and have self-similarity properties inspired by those of the tree they act on. Branch groups are groups acting on regular (or, more generally, spherically homogeneous [20]) rooted trees, and having a *branch structure* that endows them with properties similar to those of the full tree automorphism group.

The action of a fractal group G extends to an action on the boundary of the tree. A *parabolic* subgroup P of G is the stabilizer of an element in the boundary of the tree — or, equivalently, the stabilizer of an infinite geodesic Al path starting at the root vertex. Parabolic subgroups can be defined for any group acting on a tree, but in the case of branch groups they have the remarkable *weak maximality* property, and the quotient spaces G/P typically have polynomial growth, usually of non-integer degree.

Viewing P as the stabilizer of an infinite path $e = (e_1, e_2, \ldots)$, it is approximated by the stabilizers P_n of finite paths $(e_1, \ldots, e_n)$, in the sense that $P = \bigcap P_n$. The homogeneous space G/P is then also approximated by the finite spaces G/P_n. These finite spaces have a limit in the Gromov sense, which is a compact finite-dimensional space; in case its Hausdorff dimension is not an integer, we obtain a fractal set of a new nature, as we observed in [6]. The study of such spaces is promising.

The present paper contains several new results concerning properties of branch fractal groups, in particular a part of their subgroup lattice and the structure of their parabolic subgroups.

One of the main fruits of this research is the first example of torsion-free branch just-infinite group (see Section 7). This paper also serves as a companion to [6], in that it studies the structure of parabolic subgroups P of fractal groups, and the decomposition of the associated quasi-regular representations $\rho_{G/P}$.

These representations are irreducible, and that there are uncountably many different (pairwise non-equivalent) among them. They are infinite-dimensional, but are approximated by finite-dimensional representations ρ_{G/P_n} where $\{P_n\}$ is a sequence of subgroups of finite index such that $P = \bigcap_{n\in\mathbb{N}} P_n$. For these finite-dimensional representations we describe a decomposition in irreducible components. This decomposition is obtained by a complete description of the structure of the Hecke algebra associated to the pair (G, P). These Hecke algebra turn out to be abelian.

We believe branch fractal groups have good "analytical properties" in the sense that a sufficiently rich representation theory for these groups, their finite images, and the corresponding profinite completions can be developed in order to answer the main questions about harmonic analysis on these groups — their spectrum, the structure of various completions of their group algebra etc.

The set $\{\rho_{G/P} | P \text{ is a parabolic subgroup of } G\}$ is probably sufficiently large for this purpose, since parabolic subgroups have the property that $\bigcap_{g\in G} P^g = 1$ (for the finite-dimensional analogue, this implies that the regular representation ρ_{G_n} is a subrepresentation of the tensor product $\bigotimes_{|\sigma|=n} \rho_{G/\mathsf{Stab}_G(\sigma)}$.)

We are following the first steps along this direction in the present paper. The results given further are already used for the computation of spectra related to fractal groups [6], where we show that in some cases these spectra are simple transformations of Julia sets of quadratic maps of the complex plane.

The paper is organized as follows: in Section 2 we give general definitions concerning groups acting on rooted trees, introduce the congruence property, parabolic subgroups, portraits of elements an Hausdorff dimension of closed subgroups of $\mathsf{Aut}(\mathcal{T})$.

In Section 3 we recall the definition of branch group, weakly branch group and regular branch group. We prove the weak maximality of parabolic subgroups and provide a criterion evaluating the congruence property for regular branch groups.

In Sections 4, 5, 6, 7, 8 we define groups G, $\tilde{G}$, Γ, $\overline{\Gamma}$, $\overline{\overline{\Gamma}}$, and study some properties of these groups. We prove that Γ and $\overline{\Gamma}$ are virtually torsion-free, in contrast to G and $\overline{\overline{\Gamma}}$ which are torsion, and $\tilde{G}$ which is neither torsion nor virtually torsion-free.

We prove that $\tilde{G}$ is (like G [18] and Γ, $\overline{\Gamma}$ and $\overline{\overline{\Gamma}}$ [3]) of intermediate growth, and produce a presentation of $\tilde{G}$ which is of L-type, that is, which involves finitely many relators, along with their iterates under a word substitution. An analogous representation for G was found by Igor Lysionok [26].

For each of the involved groups we draw a part of their subgroup lattice, and provide a tree-like decomposition of their parabolic subgroup.

We prove that G, $\tilde{G}$, Γ and $\overline{\overline{\Gamma}}$ are just-infinite branch groups, while $\overline{\Gamma}$ is a just-nonsolvable weakly branch group (the first example of such a group was given in [11]).

Finally in Section 9 we study, for a branch group G, the quasi-regular representations corresponding to parabolic subgroups P and stabilizers P_n of vertices at level n. We show that the quasi-regular representations $\rho_{G/P}$ are irreducible, and for the finite-dimensional representations ρ_{G/P_n} we describe their decomposition in irreducible components, which we explicit in the case of our examples. The Hecke algebra $\mathcal{L}(G, P_n)$, which controls the decomposition of ρ_{G/P_n} in irreducible components, is abelian. As a consequence, the orbit structure of P_n on the homogeneous G-space G/P_n is closely related to that decomposition.

Note that all these results — structure of the parabolic subgroup, lattice of finite-index subgroups, weak maximality of P, abelian Hecke algebra — hold also for the closures $\overline{G}$ of the groups we consider in $\mathsf{Aut}\,\mathcal{T}$, which are branch fractal profinite groups. For instance the statements about the structure of parabolic subgroups are valid for them as well if one replaces the restricted tree-like decomposition by an unrestricted one. Also, in our situation, a group and its closure have the same sequences of (finite) Hecke algebras so one can consider these algebras as associated to profinite groups as well. Four of our groups satisfy the congruence property, so their closures are isomorphic to their profinite completions.

It will be important in the future to develop the theory of representations of profinite branch groups. The results of Section 9 are a first step in this direction. As we obtained a simple description of the double coset decomposition with respect to a parabolic subgroup there is a hope that the classical methods (described for instance in [13]) as well as more recent developments [28] will lead to a complete theory of the representations of the considered groups as well as of other groups of this type.

The results in this paper are used in [6], and are announced in the two notes [8, 7].

1.1. Notation. The following conventional notations shall be used: for g, h in a group G,

$$g^h = hgh^{-1}, \qquad [g,h] = ghg^{-1}h^{-1};$$

for elements or subsets $g_1, \dots, g_n$ in G, the subgroup they generate is written $\langle g_1, \dots, g_n\rangle$ and its normal closure $\langle g_1, \dots, g_n\rangle^G$.

The symmetric subgroup on a set Σ is written $\mathfrak{S}_\Sigma$.

We also introduce a notation for 'subsemidirect products', as follows:

Definition 1.1. *Let A and B be two subgroups of a group G, with $A \cap B = C$, and assume that B is in the normalizer $N_G(A)$ and thus acts on A by conjugation. We write $A \rtimes_C B$ for the subgroup of G generated by A and B, and call it the* subsemidirect product *of A and B.*

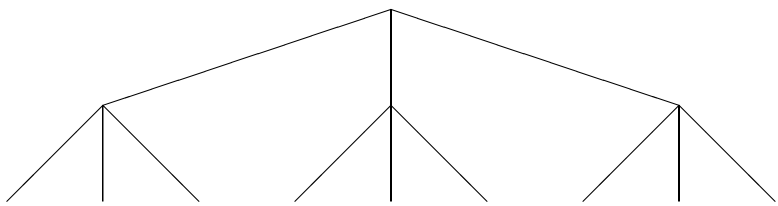
If for some prime p we have $C = \langle B^p, B'\rangle$, then we write $A \rtimes_{p-ab} B$ for the subsemidirect product of A and B, and call it the elementary abelian subsemidirect product *of A and B.*

The motivation for this name is that $A \rtimes_C B$ is naturally a subgroup of the semidirect product $A \rtimes B$. Note, however, that $A \rtimes_C B$ is in general not a split extension.

In particular, the elementary abelian subsemidirect product is an extension of A by the elementary abelian p-group $B/\langle B^p, B'\rangle$, that is in general not split.

2. Groups acting on rooted trees. The groups we shall consider will all be subgroups of the group $\mathsf{Aut}(\mathcal{T})$ of automorphisms of a regular rooted tree $\mathcal{T}$. Let Σ be a finite alphabet. The vertex set of the tree $\mathcal{T}_\Sigma$ is the set of finite sequences over Σ; two sequences are connected by an edge when one can be obtained from the other by right-adjunction of a letter in Σ. The top node is the empty sequence $\emptyset$, and the children of σ are all the σs, for $s \in \Sigma$. We shall also consider the *boundary* $\partial\mathcal{T}_\Sigma$ of $\mathcal{T}_\Sigma$ consisting of the semi-infinite sequences over Σ. In most cases we shall write $\mathcal{T}$ for the rooted tree involved, when there is no ambiguity on Σ.

We suppose $\Sigma = \mathbb{Z}/d\mathbb{Z}$, with the operation $\overline{s} = s + 1 \bmod d$. Let a, called the *rooted automorphism* of $\mathcal{T}_\Sigma$, be the automorphism of $\mathcal{T}_\Sigma$ defined by $a(s\sigma) = \overline{s}\sigma$: it acts nontrivially on the first symbol only, and geometrically is realized as a cyclic permutation of the d subtrees just below the root.



Fix some Σ, and consider any subgroup $G < \mathsf{Aut}(\mathcal{T})$. Let $\mathsf{Stab}_G(\sigma)$, the *vertex stabilizer* of σ, denote the subgroup of G consisting of the automorphisms that fix the sequence σ, and let $\mathsf{Stab}_G(n)$, the *level stabilizer*, denote the subgroup of G consisting of the automorphisms that fix all sequences of length n:

$$\mathop{\mathsf{Stab}}_G(\sigma) = \{g \in G \mid g\sigma = \sigma\}, \qquad \mathop{\mathsf{Stab}}_G(n) = \bigcap_{\sigma \in \Sigma^n} \mathop{\mathsf{Stab}}_G(\sigma).$$

The $\mathsf{Stab}_G(n)$ are normal subgroups of finite index of G; in particular $\mathsf{Stab}_G(1)$ is of index at most $d!$. Let G_n be the quotient $G/\mathsf{Stab}_G(n)$. If $g \in \mathsf{Aut}(\mathcal{T})$ is an automorphism fixing the sequence σ, we denote by $g_{|\sigma}$ the element of $\mathsf{Aut}(\mathcal{T})$ corresponding to the restriction to sequences starting by σ:

$$\sigma g_{|\sigma}(\tau) = g(\sigma\tau).$$

As the subtree starting from any vertex is isomorphic to the initial tree $\mathcal{T}_\Sigma$, we obtain this way a map

$$\phi : \begin{cases} \mathop{\mathsf{Stab}}_{\mathsf{Aut}(\mathcal{T})}(1) \to \mathsf{Aut}(\mathcal{T})^\Sigma \\ h \mapsto (h_{|0}, \dots, h_{|d-1}) \end{cases} \tag{1}$$

which is an embedding. For a sequence σ and an automorphism $g \in \mathsf{Aut}(\mathcal{T})$, we denote by g^σ the element of $\mathsf{Aut}(\mathcal{T})$ acting as g on the sequences starting by σ, and trivially on the others:

$$g^\sigma(\sigma\tau) = \sigma g(\tau), \qquad g^\sigma(\tau) = \tau \text{ if } \tau \text{ doesn't start by } \sigma.$$

The *rigid stabilizer* of σ is $\mathsf{Rist}_G(\sigma) = \{g^\sigma \mid g \in G\} \cap G$. We also set

$$\mathop{\mathsf{Rist}}_G(n) = \langle \mathop{\mathsf{Rist}}_G(\sigma) \mid \sigma \in \Sigma^n \rangle = \prod_{|\sigma|=n} \mathop{\mathsf{Rist}}_G(\sigma)$$

and call it the *rigid level stabilizer* ($\prod$ denotes direct product). We say G has *infinite rigid stabilizers* if all the $\mathsf{Rist}_G(\sigma)$ are infinite.

Definition 2.1. *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is* spherically transitive *if the action of G on Σ^n is transitive for all $n \in \mathbb{N}$.*

G is fractal *if for every vertex σ of $\mathcal{T}_\Sigma$ one has $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$, where the isomorphism is given by identification of $\mathcal{T}_\Sigma$ with its subtree rooted at σ.*

G has the congruence property *if every finite-index subgroup of G contains $\mathsf{Stab}_G(n)$ for some n large enough.*

Lemma 2.2. *The group $G < \mathsf{Aut}(\mathcal{T})$ is fractal if and only if $\phi_{|\mathsf{Stab}_G(1)}$: $\mathsf{Stab}_G(1) \to \mathsf{Aut}(\mathcal{T})^\Sigma$ is a subdirect embedding into $G \times \dots \times G$, i.e. if it is an embedding that is surjective on each factor.*

Proof. If $p_i\phi_{|G} \neq G$ for some projection p_i on the vertex i, then $\mathsf{Stab}_G(i)_{|i} \neq G$ so G is not fractal. We now suppose $\phi_{|G}$ is a subdirect embedding and prove by induction that $\mathsf{Stab}_G(\sigma)_{|\sigma} \cong G$ for all σ.

The induction basis, for $|\sigma| = 1$, is equivalent to the hypothesis. Now by induction $G \to G^{\Sigma^{n-1}}$ is a subdirect embedding, and each factor G maps to G^{Σ} by $\phi_{|G}$. The composition of two subdirect embeddings is again subdirect, so $G \to G^{\Sigma^n}$ is subdirect. $\square$

For fractal groups, we usually shall write ϕ instead of $\phi_{|G}$.

Lemma 2.3. *A fractal group is spherically transitive if and only if it acts transitively on the first level* Σ.

Proof. Assume by induction that G is fractal and transitive on Σ^{n-1}, the induction starting at $n = 2$. Since ϕ is subdirect, G is transitive on $i\Sigma^{n-1}$ for all $i \in \Sigma$, and since it is transitive on Σ, it is also transitive on Σ^n. $\square$

The full automorphism group $\mathsf{Aut}(\mathcal{T})$ has the structure of a profinite group: it is approximated by the finite groups $\mathsf{Aut}(\mathcal{T})_n = \mathsf{Aut}(\mathcal{T})/\mathsf{Stab}(n)$, and we have

$$\mathsf{Aut}(\mathcal{T}) = \varprojlim_{n\to\infty} \mathsf{Aut}(\mathcal{T})_n.$$

More on the topology of $\mathsf{Aut}(\mathcal{T})$ is said in [6]. The following lemma follows directly from the definition of a profinite completion:

Lemma 2.4. *Let* $G < \mathsf{Aut}(\mathcal{T})$ *have the congruence property. Then its profinite completion* $\widehat{G}$ *is isomorphic (as a profinite group) to its closure* $\overline{G}$ *in* $\mathsf{Aut}(\mathcal{T})$. *If moreover* G *is contained in a Sylow pro-p-subgroup* $\mathsf{Aut}_p(\mathcal{T})$, *then* $\overline{G}$ *is isomorphic to the pro-p completion* $\widehat{G}_p$ *of* G.

Proof. By the congruence property, $\{\overline{\mathsf{Stab}_G(n)}\}$ is a basis of neighbourhoods of the identity in $\widehat{G}$. $\square$

Definition 2.5. *Assume* $G < \mathsf{Aut}(\mathcal{T})$ *is given, with a subset* $S \subset G$. *The* portrait *of* $g \in G$ *with respect to* S *is a subtree of* $\mathcal{T}$, *with inner vertices labeled by* $\mathfrak{S}_\Sigma$ *and leaf vertices labeled by* $S \cup \{1\}$. *It is defined recursively as follows: if* $g \in S \cup \{1\}$, *the portrait of* g *is the subtree reduced to the root vertex, labeled by* g *itself. Otherwise, let* $\alpha \in \mathfrak{S}_\Sigma$ *be the permutation of the top branches of* $\mathcal{T}$ *such that* $g\alpha^{-1} \in \mathsf{Stab}_G(1)$; *let* $(g_0, \dots, g_{d-1}) = \phi(g\alpha^{-1})$ *and let* $\mathcal{T}_i$ *be the portrait of* g_i. *Then the portrait of* g *is the subtree of* $\mathcal{T}$ *with* α *labeling the root vertex and subtrees* $\mathcal{T}_0, \dots, \mathcal{T}_{d-1}$ *connected to the root.*

The portrait *of* $g \in G$ *is its portrait with respect to* $\emptyset$. *The element* g *is called* finitary *if its portrait is finite.*

The depth *of $g \in G$ is the height (length of a maximal path starting at the root vertex) $\partial(g) \in \mathbb{N} \cup \{\infty\}$ of the portrait of g.*

We now suppose $d = p$ is prime, and consider $\mathsf{Aut}_*(\mathcal{T})$, the Sylow pro-$p$ subgroup of $\mathsf{Aut}(\mathcal{T})$ consisting of all elements g whose portrait is labeled by powers of the cycle $(0, 1, \dots, d-1)$. It has the structure of an infinitely iterated wreath product

$$\mathsf{Aut}_*(\mathcal{T}) = \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z} \wr \dots.$$

For a closed subgroup G of $\mathsf{Aut}_*(\mathcal{T})$, its *Hausdorff dimension* $\dim_*(G)$ is defined in [1] as

$$\dim_*(G) = \liminf_{n\to\infty} \frac{p-1}{p^n} \log_p |G_n| = \liminf_{n\to\infty} \frac{\log_p |G_n|}{\log_p |\mathsf{Aut}_*(\mathcal{T})_n|}.$$

In particular, the Hausdorff dimension of $\mathsf{Aut}_*(\mathcal{T})$ is 1, and $\dim_*$ is invariant upon taking finite-index subgroups.

3. Branch groups. We consider now a special class of groups acting on rooted trees. We shall always implicitly assume they act spherically transitively.

Definition 3.1.

(1) *G is a* regular branch *group if it has a finite-index normal subgroup $K < \mathsf{Stab}_G(1)$ such that*

$$K^\Sigma < \phi(K).$$

It is then said to be regular branch over *K.*

(2) *A subgroup $G < \mathsf{Aut}(\mathcal{T})$ is a* branch group *if for every $n \geq 1$ the subgroup $\mathsf{Rist}_G(n)$ has finite index in G.*

(3) *G is a* weakly branch *group if all of its rigid stabilizers $\mathsf{Rist}_G(\sigma)$ are nontrivial.*

Note that the definition of a branch group admits an even more general setting — see [20]. Four of our examples will be regular branch groups, and the last one will not be a branch, but rather a weakly branch group. The following lemma shows that, for fractal groups, implies implies in Definition 3.1.

Lemma 3.2. *If G is a fractal, regular branch group, then it is a branch group. If G is a branch group, then it is a weakly branch group.*

P r o o f. Assume G is a regular branch group over its subgroup K. Clearly K^{Σ^n} can be viewed, through ϕ^n, as a subgroup of $\mathsf{Rist}_G(n)$, and is of finite index in G^{Σ^n}, so $\mathsf{Rist}_G(n)$ is of finite index in G. The second implication holds because branch groups are infinite, and 'finite index in an infinite group' is stronger than 'non-trivial'. □

Note also that if all rigid stabilizers are non-trivial, then they are all infinite; moreover,

Lemma 3.3. *Let G be a weakly branch group, $\sigma \in \Sigma^n$ a vertex, and $\overline{\sigma} = \sigma_1 \dots \sigma_n \sigma_{n+1} \dots \in \Sigma^{\mathbb{N}}$ an infinite ray extending σ. Then the $\mathsf{Rist}_G(\sigma)$-orbit of $\overline{\sigma}$ is infinite.*

Proof. It suffices to show that the orbit of $v_k = \sigma_1 \dots \sigma_{n+k}$ becomes arbitrarily large as k increases. Since $\mathsf{Rist}_G(\sigma)$ is non-trivial, it contains g moving $\sigma\tau$ to $\sigma\tau'$ for some $\tau, \tau' \in \Sigma^k$. Since G is spherically transitive, it contains h moving v_k to $\sigma\tau$. Consider now g^h: it belongs to $\mathsf{Rist}_G(\sigma)$, and does not fix v_k, whence v_k's orbit contains at least 2 points.

The argument applied to v_k shows that some $v_{k+k'}$ has at least 2 points in its $\mathsf{Stab}_G(v_k)$-orbit, so at least 4 points in its $\mathsf{Stab}_G(\sigma)$-orbit; and this process can be repeated an arbitrary number of times to produce vertices $v_{k+\dots+k^{(j)}}$ with at least 2^{j+1} points in their orbit. □

If G is a regular branch group over its subgroup K, the following notations are also introduced: given a subgroup L of K, we write $L_{(0)} = L$ and inductively $L_{(n)} = \phi^{-1}(L^{\Sigma}_{(n-1)})$. These $L_{(i)}$ form a sequence of subgroups of L with $\bigcap_{n \geq 0} L_{(n)} = \{1\}$.

Definition 3.4. *A group G is* just-infinite *if it is infinite, and for any non-trivial normal subgroup N the quotient G/N is finite.*

Note that in checking just-infiniteness one may restrict one's attention to subgroups $N = \langle g \rangle^G$, i.e. normal closures of a non-trivial element of G. We will use the following criterion:

Proposition 3.5. *Let G be regular branch over K. Then G is just infinite if and only if $|K : K'| < \infty$.*

Proof. Clearly if K' is of infinite index in K then $\langle K' \rangle^G$ is of infinite index in G, and is not trivial (K clearly cannot be abelian) so G is not just infinite.

Conversely, assume $|K : K'| < \infty$ and let $G \ni g \neq 1$. Let $N = \langle g \rangle^G$; we will show that N is of finite index. Determine n such that $g \in \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(n-1) \setminus \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(n)$. Then there is a sequence σ of length $n-1$ such that $g_{|\sigma} \notin \mathsf{Stab}_{\mathsf{Aut}(\mathcal{T})}(1)$. Choose now two elements k_1, k_2 of K. Because G is branched on K, it contains for $i = 1, 2$ the elements $\xi_i = k_i^{\sigma 0}$. Let $\eta = [\xi_1, g] \in N$. It fixes all sequences except: those starting by $\sigma 0$, upon which it acts as k_1, and possibly those starting by σx for $x \geq 1$. Consider $\zeta = [\eta, \xi_2] \in N$. Clearly

$\zeta = [k_0, k_1]^{\sigma 0}$; as the commutator $[k_0, k_1]$ was chosen arbitrarily, it follows that N contains $K'^{\sigma 0}$; and as N is normal, it contains $K' \times \ldots \times K'$, the product having d^n factors. Now K' is of finite index in K which is of finite index in G, so $K' \times \ldots \times K'$ is of finite index in G and the same holds for N. $\square$

Definition 3.6. *A group G is* just-non-solvable *if it is not solvable, but all its non-trivial quotients are solvable.*

Proposition 3.7. *Let G be regular branch over K. Then G is just-non-solvable if and only if $G/K_{(1)}$ is solvable. In particular, if d is prime, every regular branch subgroup of $\mathsf{Aut}_*(\mathcal{T})$ is just-non-solvable.*

Proof. Let G be just-non-solvable. Then $K_{(1)}$ is a non-trivial normal subgroup, so $G/K_{(1)}$ is solvable.

Let now N be a non-trivial normal subgroup of G. It is shown in [20, Theorem 4] that N contains $K'_{(n)}$ for some $n \in \mathbb{N}$, so it suffices to show that all $G/K'_{(n)}$ are solvable. Consider the chain

$$G/K'_{(n)} \rhd K/K'_{(n)} \rhd \ldots \rhd K_{(n)}/K'_{(n)} \cong K/K' \times \ldots \times K/K'.$$

The last group is abelian, hence solvable ; successive quotients in the sequence are also solvable because

$$(K_{(i)}/K'_{(n)})/(K_{(i+1)}/K'_{(n)}) = K_{(i)}/K_{(i+1)} \cong K/K_{(1)} \times \ldots \times K/K_{(1)},$$

and $K/K_{(1)}$ is solvable by assumption. Also, $(G/K'_{(n)})/(K/K'_{(n)}) \cong G/K$ is solvable; therefore $G/K'_{(n)}$ is an extension of solvable groups, so is solvable.

In case d is prime and G is a branch subgroup of $\mathsf{Aut}_*(\mathcal{T})$, the quotient $G/K_{(1)}$ is a finite d-group, so is solvable. $\square$

The following criterion describes which branch groups enjoy the congruence property:

Proposition 3.8. *Let G be regular branch over K. Then G has the congruence property if K' contains $\mathsf{Stab}_G(m)$ for some $m \in \mathbb{N}$.*

Proof. Let N be a finite-index subgroup of G. By replacing N with its core $\bigcap_{g \in G} N^g$, still of finite index, we may suppose N is normal in G. By [20, Theorem 4], N contains $K'_{(n)}$ for some $n \in \mathbb{N}$, so

$$N > K'_{(n)} > \mathop{\mathsf{Stab}}_G(m)_{(n)} > \mathop{\mathsf{Stab}}_G(m+n). \qquad \square$$

As an example of regular branch group not enjoying the congruence property, consider $G = \mathsf{Aut}_f(\mathcal{T})$, the automorphisms of $\mathcal{T}$ whose action $\phi_v \in \mathfrak{S}_\Sigma$

is non-trivial only at finitely many vertices v, and its subgroup $H = \{g \in G | \prod_{v \in \Sigma^*} \phi_v \in \mathfrak{A}_\Sigma\}$, where $\mathfrak{A}_\Sigma$ is the alternate subgroup of $\mathfrak{S}_\Sigma$. Here G is regular branch, with $K = G$, but H does not contain any level stabilizer.

When furthermore G is finitely generated, the following 'quantitative congruence property' shall be useful to prove equalities among subgroups:

Proposition 3.9 (Quantitative Congruence Property)**.** *Let G be regular branch over K, finitely generated by the set S and with the congruence property. Let n be minimal such that $\langle s\rangle^G \geq \mathsf{Stab}_G(n)$ for all generators $s \in S$. Let m be minimal such that K' contains $\mathsf{Stab}_G(m)$.*

Let N be any non-trivial normal subgroup of G and $1 \neq g \in N$. Then N contains $\mathsf{Stab}_G(\partial(g) + m + n)$.

Proof. This follows again from Theorem 4 in [20]. □

3.1. Parabolic subgroups. In the context of groups acting on the hyperbolic space, a parabolic subgroup is the stabilizer of a point on the boundary. We give here a few general facts concerning parabolic subgroups of fractal or branched groups, and recall some results on growth of groups and sets on which they act.

Definition 3.10. *Let $\mathcal{T} = \Sigma^*$ be a rooted tree. A* ray *e in $\mathcal{T}$ is an infinite geodesic starting at the root of $\mathcal{T}$, or equivalently an element of $\partial\mathcal{T} = \Sigma^{\mathbb{N}}$.*

Let $G < \mathsf{Aut}(\mathcal{T})$ be any subgroup and e be a ray. The associated parabolic subgroup *is $P_e = \mathsf{Stab}_G(e)$.*

The following important facts are easy to prove:

- $\bigcap_{e \in \partial\mathcal{T}} P_e = \bigcap_{g \in G} P^g = 1$.
- Let e be an infinite ray and define the subgroups $P_n = \mathsf{Stab}_G(e_1 \dots e_n)$. Then the P_n have index d^n in G (since G acts transitively) and satisfy
$$P_e = \bigcap_{n \in \mathbb{N}} P_n.$$
- P has infinite index in G, and has the same image as P_n in the quotient $G_n = G/\mathsf{Stab}_G(n)$.

Definition 3.11. *Let G be a group generated by a finite set S, let X be a set upon which G acts transitively, and choose $x \in X$. The* growth *of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ defined by*

$$\gamma(n) = |\{gx \in X | \, |g| \leq n\}|,$$

where $|g|$ denotes the minimal length of g when written as a word over S. By the growth *of G we mean the growth of the action of G on itself by left-multiplication.*

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there is a constant $C \in \mathbb{N}$ such that $f(n) < Cg(Cn + C) + C$ for all $n \in \mathbb{N}$, and $f \sim g$ if $f \preceq g$ and $g \preceq f$. The equivalence class of the growth of X is independent of the choice of S and of x.

X is of polynomial growth *if $\gamma(n) \preceq n^d$ for some d. It is of* exponential growth *if $\gamma(n) \succeq e^n$. It is of* intermediate growth *in the remaining cases. This trichotomy does not depend on the choice of x or S.*

Definition 3.12. *Two infinite sequences $\sigma, \tau : \mathbb{N} \to \Sigma$ are* confinal *if there is an $N \in \mathbb{N}$ such that $\sigma_n = \tau_n$ for all $n \geq N$.*

Confinality is an equivalence relation, and equivalence classes are called confinality classes.

The following result is due to Volodymyr Nekrashevych and Vitaly Sushchansky.

Proposition 3.13. *Let G be a group acting on a regular rooted tree $\mathcal{T}$, and assume that for any generator $g \in G$ and infinite sequence σ, the sequences σ and $g\sigma$ differ only in finitely many places. Then the confinality classes of the action of G on $\partial\mathcal{T}$ are unions of orbits. If moreover $\mathsf{Stab}_G(\sigma)$ contains the rooted automorphism a for all $\sigma \in \mathcal{T}$, the orbits of the action are confinality classes.*

The proof of the following result appears in [6].

Proposition 3.14. *Let $G < \mathsf{Aut}(\mathcal{T})$ satisfy the conditions of Proposition* 3.13*, and suppose that for the map $\phi : g \mapsto (g_1, \dots, g_d)$ defined in (1) there are constants λ, μ such that $|g_i| \leq \lambda|g| + \mu$ for all i. Let P be a parabolic subgroup. Then G/P, as a G-set, is of polynomial growth of degree at most $\log_{1/\lambda}(d)$. If moreover G is spherically transitive, then G/P's asymptotical growth is polynomial of degree $\log_{1/\lambda'}(d)$, with λ' the infimum of the λ as above.*

Definition 3.15. *Let G be a branch group, and H any subgroup. We say H is* weakly maximal *if H is of infinite index in G, but all subgroups of G strictly containing H are of finite index in G.*

Note that every infinite finitely generated group admits maximal subgroups, by Zorn's lemma.

Note also that some branch groups may not contain any infinite-index maximal subgroup; this is the case for G, as was shown by Ekaterina Pervova.

Proposition 3.16. *Let P be a parabolic subgroup of a regular branch group G. Then P is weakly maximal.*

Proof. Assume G is regular branch over K, and $P = \mathsf{Stab}_G(e)$. Recall that G contains a product of d^n copies of K at level n, and clearly P contains a product of $d^n - 1$ copies of K at level n, namely all but the one indexed by the vertex $e_1 \dots e_n$.

Take $g \in G \setminus P$. There is then an $n \in \mathbb{N}$ such that $g(e_n) \neq e_n$, so $\langle P, P^g \rangle$ contains the product of d^n copies of K at level n, hence is of finite index in G. □

4. The group G. We give here the basic facts we will use about the first of Grigorchuk's examples, the group G [16, 24]. We apply the discussion of the previous section to $\Sigma = \{0, 1\}$. Recall a is the automorphism permuting the top two branches of $\mathcal{T}_2$. Define recursively by b the automorphism acting as a on the right branch and c on the left, by c the automorphism acting as a on the right branch and d on the left, and by d the automorphism acting as 1 on the right branch and b on the left. In formulæ,

$$\begin{aligned} b(0x\sigma) &= 0\overline{x}\sigma, & b(1\sigma) &= 1c(\sigma), \\ c(0x\sigma) &= 0\overline{x}\sigma, & c(1\sigma) &= 1d(\sigma), \\ d(0x\sigma) &= 0x\sigma, & d(1\sigma) &= 1b(\sigma). \end{aligned}$$

G is the group generated by $\{a, b, c, d\}$. It is readily checked that these generators are of order 2 and that $\{1, b, c, d\}$ constitute a Klein group; one of the generators $\{b, c, d\}$ can thus be omitted.

We write $H_n = \mathsf{Stab}_G(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \begin{cases} H \to G \times G \\ b \mapsto (a, c), & b^a \mapsto (c, a) \\ c \mapsto (a, d), & c^a \mapsto (d, a) \\ d \mapsto (1, b), & d^a \mapsto (b, 1). \end{cases}$$

Define also the following subgroups of G:

$$\begin{aligned} B &= \langle b \rangle^G = \langle b, b^a, b^{da}, b^{ada} \rangle, \\ K &= \langle (ab)^2 \rangle^G = \langle (ab)^2, (abad)^2, (adab)^2 \rangle. \end{aligned}$$

The group G

- is an infinite torsion 2-group.
- is of intermediate growth.
- is amenable.
- is fractal and branched on its subgroup K.
- is just-infinite.

- is residually finite.
- has an infinite recursive presentation [26] of L-type

$$G = \langle a, b, c, d |\, a^2, b^2, c^2, d^2, bcd, \sigma^i(ad)^4, \sigma^i(adacac)^4\ (i \in \mathbb{N})\rangle,$$

  where σ is the substitution on $\{a, b, c, d\}^*$ defined by

$$\sigma(a) = aca, \quad \sigma(b) = d, \sigma(c) = b, \sigma(d) = c,$$

  which induces an injective expanding endomorphism of G of infinite-index image. Moreover none of the relators of G are superfluous [19].
- The subgroup B is of index 8 in G and K is of index 16. Also, K contains $\mathsf{Stab}_G(3)$ and K' contains $K_{(2)}$, so G has the congruence property.
- The quotients $G_n = G/\mathsf{Stab}_G(n)$ have order $2^{5\cdot 2^{n-3}+2}$ for $n \geq 3$ (and order 2^{2^n-1} for $n \leq 3$). Therefore the closure $\overline{G}$ of G in $\mathsf{Aut}(\mathcal{T})$ is isomorphic to the profinite completion $\widehat{G}$ and is a pro-2-group. It has Hausdorff dimension $5/8$ [20].

Most of these facts are known, and appear in the extensive reference [24] and in [20]. One can then check by direct computation that K is of index 16. To prove that G is regular branch, set $x = (ab)^2$. Then one has $\phi([x, d]) = (x, 1)$ so by conjugation $\phi(K) > K \times 1$ and thus $\phi(K) > K \times K$. By direct computation, K' is of index 64 in K, so G is just-infinite.

For all other computations, we propose an alternate method of proof, based on the following

Lemma 4.1. *G satisfies the Quantitative Congruence Property for $m = 4$ and $n = 3$.*

Proof. This follows from the above description of K. □

Proposition 4.2. *We have*

$$\begin{aligned}
\phi(H) &= (B \times B) \rtimes \langle \phi(c), \phi(c^a)\rangle,\\
\phi(B) &= (K \times K) \rtimes_{\langle \phi(ab)^8\rangle} \langle \phi(b), \phi(b^a)\rangle,\\
\phi(K) &= (K \times K) \rtimes_{\langle \phi(ab)^8\rangle} \langle \phi(ab)^2\rangle,
\end{aligned}$$

with the notation introduced in Definition 1.1.

Proof. Each of these subgroups H, B, K are normal finite-index subgroups of G. By the Quantitative Congruence Property, they are all contained in some $\mathsf{Stab}_G(n)$. It is therefore equivalent to study them directly or to study their images in $G_n = G/\mathsf{Stab}_G(n)$, which is a finite group. A computer algebra system, like [30], can then be used to derive their structure.

4.1. Low-index subgroups. G has 7 subgroups of index 2:

$$\begin{array}{rl} \langle b, ac\rangle, & \langle c, ad\rangle, \quad \langle d, ab\rangle, \\ \langle b, a, a^c\rangle, & \langle c, a, a^d\rangle, \langle d, a, a^b\rangle, \\ & H = \langle b, c, b^a, c^a\rangle. \end{array}$$

As can be computed from its presentation [26] and a computer algebra system [30], G has the following subgroup count:

Index	Subgroups	Normal	In H	Normal
1	1	1	0	0
2	7	7	1	1
4	19	7	9	4
8	61	7	41	7
16	237	5	169	5
32	843	3	609	3

4.2. Normal closures of generators. They are as follows:

$$\begin{array}{ll} A = \langle a\rangle^G = \langle a, a^b, a^c, a^d\rangle, & G/A \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \\ B = \langle b\rangle^G = \langle b, b^a, b^{ad}, b^{ada}\rangle, & G/B \cong D_8, \\ C = \langle c\rangle^G = \langle c, c^a, c^{ad}, c^{ada}\rangle, & G/C \cong D_8, \\ D = \langle d\rangle^G = \langle d, d^a, d^{ac}, d^{aca}\rangle, & G/D \cong D_{16}. \end{array}$$

4.3. Some other subgroups. To complete the picture, we introduce the following subgroups of G:

$$\begin{array}{lll} K = \langle (ab)^2\rangle^G, & L = \langle (ac)^2\rangle^G, & M = \langle (ad)^2\rangle^G, \\ \overline{B} = \langle B, L\rangle, & \overline{C} = \langle C, K\rangle, & \overline{D} = \langle D, K\rangle, \\ & T = K^2 = \langle (ab)^4\rangle^G, & \\ T_{(m)} = \underbrace{T \times \ldots \times T}_{2^m}, & K_{(m)} = \underbrace{K \times \ldots \times K}_{2^m}, & N_{(m)} = T_{(m-1)} K_{(m)}. \end{array}$$

Theorem 4.3.

- *In the Lower Central Series,* $\gamma_{2^m+1}(G) = N_{(m)}$ *for all* $m \geq 1$.
- *In the Derived Series,* $K^{(n)} = \mathsf{Rist}_G(2n)$ *for all* $n \geq 2$ *and* $G^{(n)} = \mathsf{Rist}_G(2n-3)$ *for all* $n \geq 3$.

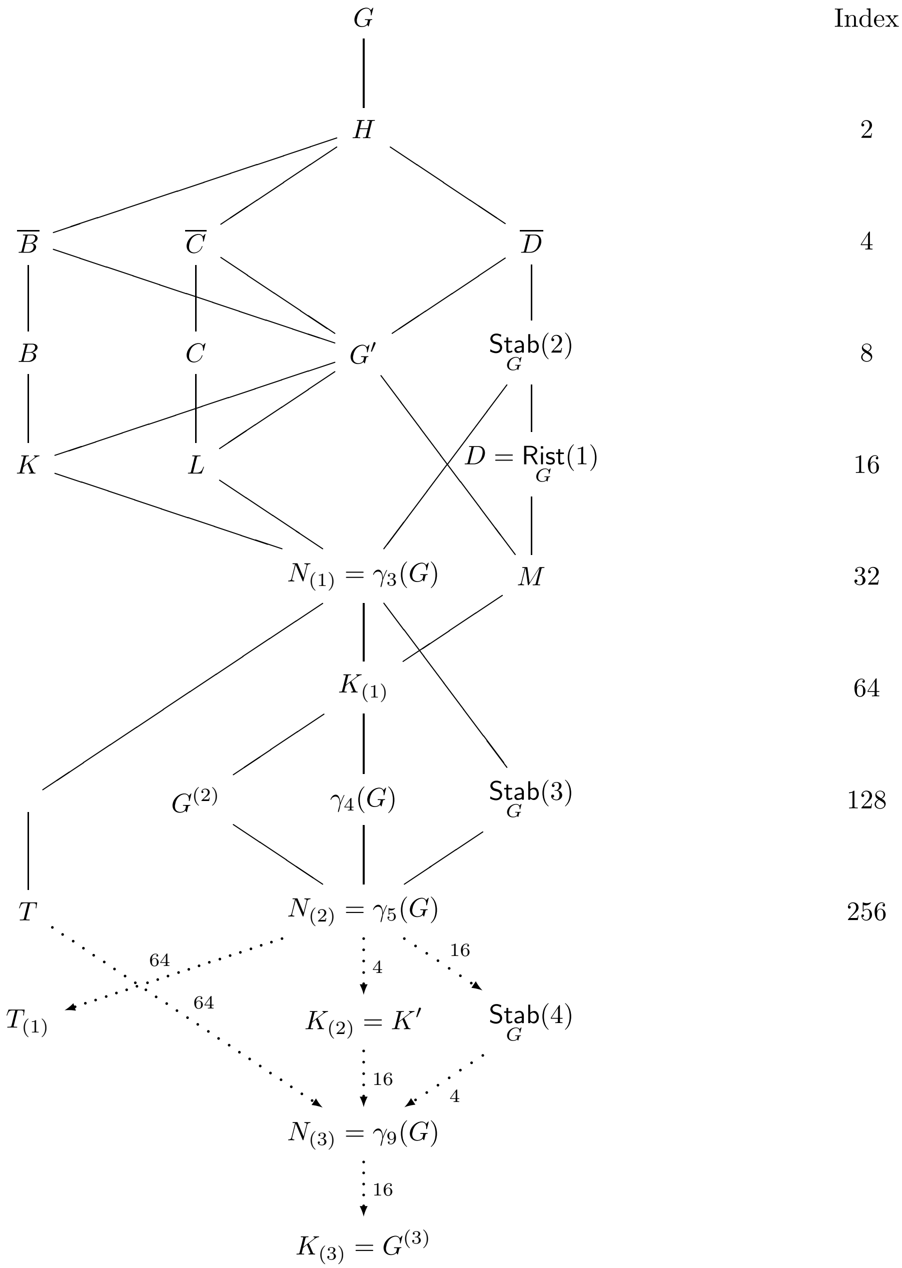


Fig. 1. The top of the lattice of normal subgroups of G below H. The index of the inclusions are indicated next to the edges

- *The rigid stabilizers satisfy*

$$\underset{G}{\mathsf{Rist}}(n) = \begin{cases} D & \text{if } n = 1, \\ K_{(n)} & \text{if } n \geq 2. \end{cases}$$

- *The level stabilizers satisfy*

$$\underset{G}{\mathsf{Stab}}(n) = \begin{cases} H & \text{if } n = 1, \\ \langle D, T \rangle & \text{if } n = 2, \\ \langle N_{(2)}, (ab)^4(adabac)^2 \rangle & \text{if } n = 3, \\ \underbrace{\underset{G}{\mathsf{Stab}}(3) \times \ldots \times \underset{G}{\mathsf{Stab}}(3)}_{2^{n-3}} & \text{if } n \geq 4. \end{cases}$$

- *There is for all $\sigma \in \Sigma^n$ a surjection $\cdot_{|\sigma} : \mathsf{Stab}_G(n) \twoheadrightarrow G$ given by projection on the factor indexed by σ.*

The top of the lattice of normal subgroups of G below H is given in Figure 1.

P r o o f. The first three points are proven by Alexander Rozhkov in [29]. To prove the fourth assertion, we apply Lemma 4.1 to determine the structure of $\mathsf{Stab}_G(n)$ for $n \leq 4$, and note that $\mathsf{Stab}_G(4) = \mathsf{Stab}_G(3) \times \mathsf{Stab}_G(3)$.

For the last statement, note that $s = (ab)^4(adabac)^2$ belongs to $\mathsf{Stab}_G(3))$, and that $\phi^n(\sigma^{n-3}(s)) = (1, \ldots, 1, ba, d, d, ba, a, c, a, c)$ giving, after conjugation, any generator of G in any position on any level n. $\square$

4.4. The subgroup P. Let e be the ray 1^∞ and let P be the corresponding parabolic subgroup.

Theorem 4.4. *P/P' is an infinite elementary 2-group generated by the images of c, $d = (1, b)$ and of all elements of the form $(1, \ldots, 1, (ac)^4)$ in $\mathsf{Rist}_G(n)$ for $n \in \mathbb{N}$. The following decomposition holds:*

$$P = \Big(B \times \Big((K \times ((K \times \ldots) \rtimes \langle (ac)^4 \rangle)) \rtimes \langle b, (ac)^4 \rangle\Big)\Big) \rtimes \langle c, (ac)^4 \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Note that we use the same notation for a subgroup B or K acting on a subtree, keeping in mind the identification of a subtree with the original tree. The same convention will hold for Theorems 5.14, 6.7, 7.8, 8.6, and all related propositions. Note also that ϕ is omitted when it would make the notations too heavy.

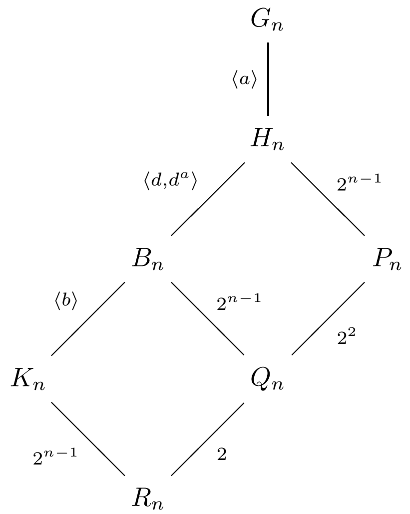


Fig. 2. The finite group G_n and its subgroups

P r o o f. Define the following subgroups of G_n:

$$\begin{aligned} B_n &= \langle b\rangle^{G_n}; & K_{(n)} &= \langle (ab)^2\rangle^{G_n};\\ Q_n &= B_n \cap P_n; & R_n &= K_{(n)} \cap P_n. \end{aligned}$$

Then the theorem follows from the following proposition. $\square$

Proposition 4.5. *These subgroups have the following structure:*

$$\begin{aligned} P_n &= (B_{n-1} \times Q_{n-1}) \rtimes \langle c, (ac)^4\rangle;\\ Q_n &= (K_{n-1} \times R_{n-1}) \rtimes \langle b, (ac)^4\rangle;\\ R_n &= (K_{n-1} \times R_{n-1}) \rtimes \langle (ac)^4\rangle. \end{aligned}$$

P r o o f. A priori, P_n, as a subgroup of H_n, maps in $(B_{n-1} \times B_{n-1}) \rtimes \langle (a,d),(d,a)\rangle$. Restricting to those pairs that fix e_n gives the result. Similarly, Q_n, as a subgroup of B_n, maps in $(K_{n-1} \times K_{n-1}) \rtimes \langle (a,c),(c,a)\rangle$, and R_n, as a subgroup of K_n, maps in $(K_{n-1} \times K_{n-1}) \rtimes \langle (ac,ca),(ca,ac)\rangle$. $\square$

The group G_n and its subgroups $H_n, B_n, K_n, P_n, R_n, Q_n$ are arranged in the lattice of Figure 2, with the quotients or the indices are represented next to the arrows.

5. The group $\tilde{G}$. We describe another fractal group, acting on the same tree $\mathcal{T}_2$ as G. We denote again by a the automorphism permuting the top two branches, and recursively by $\tilde{b}$ the automorphism acting as a on the right branch and $\tilde{c}$ on the left, by $\tilde{c}$ the automorphism acting as 1 on the right branch and $\tilde{d}$ on the left, and by $\tilde{d}$ the automorphism acting as 1 on the right branch and $\tilde{b}$ on the left. In formulæ,

$$\begin{aligned}
\tilde{b}(0x\sigma) &= 0\overline{x}\sigma, & \tilde{b}(1\sigma) &= 1\tilde{c}(\sigma),\\
\tilde{c}(0\sigma) &= 0\sigma, & \tilde{c}(1\sigma) &= 1\tilde{d}(\sigma),\\
\tilde{d}(0\sigma) &= 0\sigma, & \tilde{d}(1\sigma) &= 1\tilde{b}(\sigma).
\end{aligned}$$

Then $\tilde{G}$ is the group generated by $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$. Clearly all these generators are of order 2, and $\{\tilde{b}, \tilde{c}, \tilde{d}\}$ is elementary abelian of order 8.

We write $\tilde{H}_n = \mathsf{Stab}_{\tilde{G}}(n)$ and $\tilde{H} = \tilde{H}_1$. Explicitly, the map ϕ restricts to

$$\phi : \begin{cases} \tilde{H} \to \tilde{H} \times \tilde{H} \\ \tilde{b} \mapsto (a, \tilde{c}), & \tilde{b}^a \mapsto (\tilde{c}, a) \\ \tilde{c} \mapsto (1, \tilde{d}), & \tilde{c}^a \mapsto (d, 1) \\ \tilde{d} \mapsto (1, \tilde{b}), & \tilde{d}^a \mapsto (\tilde{b}, 1). \end{cases}$$

Proposition 5.1. *$\tilde{G}$ contains elements of finite and infinite order.*

Proof. Consider the element $x = a\tilde{b}\tilde{c}\tilde{d}$ of $\tilde{G}$. Then $x^2 \in \tilde{H}$ satisfies $\phi(x^2) = (x, x)$, so $x^{2^n} \neq 1$ for all n; as $\tilde{G} < \mathsf{Aut}(\mathcal{T}_2)$ has only 2-torsion, it follows that x is of infinite order. □

Note that x acts on $\partial\mathcal{T}_2$ like an 'adding machine' (see [9]). More generally, every spherically transitive automorphism of $\mathcal{T}_p$ is conjugated in $\mathsf{Aut}(\mathcal{T}_p)$ to a standard one, called the *adding machine*, that can be written $z \mapsto z + 1$ after identification of $\partial\mathcal{T}_p$ with $\mathbb{Z}_p$.

Proposition 5.2. *$\tilde{G}$ contains G as a subgroup of infinite index.*

Proof. The embedding is given by $a \mapsto a$, $b \mapsto \tilde{b}\tilde{d}$, $c \mapsto \tilde{c}\tilde{b}$, $d \mapsto \tilde{d}\tilde{c}$. The index is infinite because the subgroups G and $\langle a\tilde{b}\tilde{c}\tilde{d}\rangle$ do not intersect, one being torsion and the other torsion-free. □

Define the elements $u = (a\tilde{b})^2$ and $v = (a\tilde{d})^2$ in $\tilde{G}$, and consider its following subgroups:

$$\tilde{H} = \langle \tilde{b}, \tilde{c}, \tilde{d}\rangle^{\tilde{G}}, \qquad \tilde{B} = \langle \tilde{b}, \tilde{d}\rangle^{\tilde{G}}, \qquad \tilde{C} = \langle \tilde{b}, v\rangle^{\tilde{G}}, \qquad \tilde{K} = \langle u, v\rangle^{\tilde{G}}.$$

Proposition 5.3. *They have the following structure:*

$\tilde{H} = \langle \tilde{b}, \tilde{c}, \tilde{d}, \tilde{b}^a, \tilde{c}^a, \tilde{d}^a \rangle$ *is normal of index* 2 *in* $\tilde{G}$.
$\tilde{B} = \langle \tilde{b}, \tilde{d}, \tilde{b}^a, \tilde{d}^a, \tilde{b}^{\tilde{c}a}, \tilde{b}^{a\tilde{c}a} \rangle$ *is normal of index* 8 *in* $\tilde{G}$.
$\tilde{C} = \langle \tilde{b}, v, \tilde{b}^a, \tilde{b}^{\tilde{c}a}, \tilde{b}^{a\tilde{c}a} \rangle$ *is normal of index* 16 *in* $\tilde{G}$.
$\tilde{K} = \langle u, v, (a\tilde{b}\tilde{d})^2, u^a, u^{a\tilde{c}} \rangle$ *is normal of index* 32 *in* $\tilde{G}$.

Furthermore,

$$\begin{aligned}
\phi(\tilde{H}) &= (\tilde{B} \times \tilde{B}) \rtimes \langle \phi(\tilde{b}), \phi(\tilde{b}^a) \rangle,\\
\phi(\tilde{B}) &= (\tilde{C} \times \tilde{C}) \rtimes \langle \phi(\tilde{b}), \phi(\tilde{b}^a) \rangle,\\
\phi(\tilde{C}) &= (\tilde{K} \times \tilde{K}) \rtimes_{\phi\langle([\tilde{b},v],[\tilde{b}^a,v])\rangle} \langle \phi(\tilde{b}), \phi(\tilde{b}^a), v \rangle,\\
\phi(\tilde{K}) &= (\tilde{K} \times \tilde{K}) \rtimes_{\phi\langle [u,v]\rangle^{\tilde{G}}} \langle u, v \rangle.
\end{aligned}$$

Proposition 5.4. $\tilde{G}$ *is spherically transitive, fractal and regular branch over its subgroup* $\tilde{K}$.

Proof. $\tilde{G}$ is fractal by Lemma 2.2 and the nature of the map ϕ. As $\tilde{K}$ is normal, $\phi(\tilde{K})$ contains $\phi[u, \tilde{d}] = (1, u)$ and $\phi[u, \tilde{c}] = (1, v)$, so by conjugation it contains $1 \times \tilde{K}$ and $\tilde{K} \times 1$, so finally it contains $\tilde{K} \times \tilde{K}$. □

Proposition 5.5. $\tilde{G}$ *is just-infinite.*

Proof. By direct computation, $[\tilde{K} : \tilde{K}'] = 64$. Apply Proposition 3.5. □

Proposition 5.6. *Define the substitution* $\tilde{\sigma}$ *on* $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}^*$ *by*

$$\tilde{\sigma} : \begin{cases} a \mapsto a\tilde{b}a, & \tilde{b} \mapsto \tilde{d}, \\ \tilde{c} \mapsto \tilde{b}, & \tilde{d} \mapsto \tilde{c}. \end{cases}$$

Then $\tilde{G}$ *has a recursive presentation of L-type*

$$\begin{aligned}
(2)\quad \tilde{G} = \Big\langle a, \tilde{b}, \tilde{c}, \tilde{d} \Big|\, & a^2, \tilde{b}^2, \tilde{c}^2, \tilde{d}^2, [\tilde{b}, \tilde{c}], [\tilde{b}, \tilde{d}], [\tilde{c}, \tilde{d}],\\
& \tilde{\sigma}^i(a\tilde{c})^4, \tilde{\sigma}^i(a\tilde{d})^4, \tilde{\sigma}^i(a\tilde{c}a\tilde{d})^2, \tilde{\sigma}^i(a\tilde{b})^8, \tilde{\sigma}^i(a\tilde{b}a\tilde{b}a\tilde{c})^4, \tilde{\sigma}^i(a\tilde{b}a\tilde{b}a\tilde{d})^4,\\
& \tilde{\sigma}^i(a\tilde{b}a\tilde{b}a\tilde{c}a\tilde{b}a\tilde{b}a\tilde{d})^2 \quad (i \geq 0) \Big\rangle,
\end{aligned}$$

and $\tilde{\sigma}$ *induces an injective expanding endomorphism of* $\tilde{G}$ *of infinite-index image.*

Proof. Consider the groups

$$\begin{aligned}
\Gamma &= \langle \alpha, \beta, \gamma, \delta |\, \alpha^2, \beta^2, \gamma^2, \delta^2, [\beta, \gamma], [\beta, \delta], [\gamma, \delta], (\alpha\gamma)^4 \rangle,\\
\Xi &= \langle \beta, \gamma, \delta, \beta^\alpha, \gamma^\alpha, \delta^\alpha \rangle <_2 \Gamma.
\end{aligned}$$

Then $\tilde{G}$ is a quotient of Γ, written $\tilde{G} = \Gamma/\Omega$, via the map $\alpha \mapsto a, \beta \mapsto b, \gamma \mapsto c, \delta \mapsto d$, and the map ϕ lifts to a map $\theta : \Xi \to \Gamma \times \Gamma$. Define

$$\Omega_n = \{g \in \Gamma |\, \theta^n \text{ is applicable and } \theta^n(g) = (1, \dots, 1) \quad (2^n \text{ copies})\},$$

where the notation implies that $g \in \Xi, \theta(g) \in \Xi \times \Xi, \dots$. For any word w in $\{\alpha, \beta, \gamma, \delta\}$ of length at least 2 representing an element of Ξ, the corresponding words $\theta(w)_{1,2}$ will be strictly shorter; thus every $g \in \Omega$ eventually gives 1 through iterated application of θ, and thus $\Omega = \cup_{n \geq 0} \Omega_n$. We will obtain an explicit set of generators for Ω_n: let $\omega_0 = (\alpha\gamma)^4$ and

$$\{\omega_1, \dots, \omega_6\} = \{(\alpha\delta)^4, (\alpha\gamma\alpha\delta)^2, (\alpha\beta)^8, (\alpha\beta\alpha\beta\alpha\gamma)^4, (\alpha\beta\alpha\beta\alpha\delta)^4, (\alpha\beta\alpha\beta\alpha\gamma\alpha\beta\alpha\beta\alpha\delta)^2\}.$$

Then we claim that for all $n \geq 0$

$$\Omega_n = \langle \tilde{\sigma}^{j+1}(\omega_0), \tilde{\sigma}^j(\omega_i) \quad (0 \leq j \leq n-1, 1 \leq i \leq 6)\rangle^\Gamma.$$

By direct application of the Todd-Coxeter algorithm [30], we obtain the presentation

$$\Xi = \left\langle \beta, \gamma, \delta, \overline{\beta}, \overline{\gamma}, \overline{\delta} \,\middle|\, \beta^2, \gamma^2, \delta^2, \overline{\beta}^2, \overline{\gamma}^2, \overline{\delta}^2, [\beta, \gamma], [\beta, \delta], [\gamma, \delta], [\overline{\beta}, \overline{\gamma}], [\overline{\beta}, \overline{\delta}], [\overline{\gamma}, \overline{\delta}], [\gamma, \overline{\gamma}] \right\rangle.$$

Computation shows that $\theta(\Xi)$ is of index 8 in $\Gamma \times \Gamma$. From this we obtain, again using Todd-Coxeter, the presentation

$$\begin{aligned}\theta(\Xi) = \Big\langle \beta, \gamma, \delta, \overline{\beta}, \overline{\gamma}, \overline{\delta} \,\Big|\, &\beta^2, \gamma^2, \delta^2, \overline{\beta}^2, \overline{\gamma}^2, \overline{\delta}^2, [\beta, \gamma], [\beta, \delta], [\gamma, \delta], [\overline{\beta}, \overline{\gamma}], [\overline{\beta}, \overline{\delta}], [\overline{\gamma}, \overline{\delta}], \\ &[\gamma, \overline{\gamma}], [\gamma, \overline{\delta}], [\delta, \overline{\gamma}], [\delta, \overline{\delta}], (\beta\overline{\beta})^4, (\beta\overline{\beta}x\beta\overline{\beta}\overline{y})^2 \, (x, y \in \{\gamma, \delta\}) \Big\rangle.\end{aligned}$$

As a consequence, we can write $\ker(\theta)$ as a normal subgroup of Γ by keeping only those relators of $\theta(\Xi)$ that do not appear in Ξ and rewriting them in $\{\alpha, \beta, \gamma, \delta\}$, namely

$$\Omega_1 = \ker(\theta) = \langle \omega_1, \dots, \omega_6 \rangle^\Gamma.$$

Then a direct computation shows that $\theta\tilde{\sigma}(\omega_i) = (1, \omega_i)$ for $i = 0, \dots, 6$. This proves that

$$\begin{aligned}\Omega_n &= \{g \in \Xi |\, \theta(g) \in \Omega_{n-1} \times \Omega_{n-1}\} \\ &= (\{\tilde{\sigma}(g) |\, g \in \Omega_{n-1}\} \cup \Omega_{n-1})^\Gamma \\ &= \langle \tilde{\sigma}^{j+1}(\omega_0), \tilde{\sigma}^j(\omega_i) \quad (0 \leq j \leq n-1, 1 \leq i \leq 6)\rangle^\Gamma. \qquad \square\end{aligned}$$

Corollary 5.7. *All relations of $\tilde G$ have even length. As a consequence, the Cayley graph of $\tilde G$ relative to the generating set $\{a, \tilde b, \tilde c, \tilde d\}$ is bipartite.*

We believe the relations given in the previous theorem are independent, and that the method used in [19] can be used to prove this.

Note that the relations of G can be obtained from those of $\tilde G$; in the following equalities we indicate by an underscore the letters affected by a relation in $\tilde G$.

$$\begin{aligned}(ad)^4 &= (a\tilde c\tilde d)^4 =_{\tilde G} (a\tilde c\tilde d a\tilde d\tilde c)^2 =_{\tilde G} (a\tilde c\tilde d a\tilde d(\tilde d a)^4\tilde c)(a\tilde c\tilde d a\tilde d\tilde c)\\ &=_{\tilde G} (\underline{a\tilde c a\tilde d}\ \underline{a\tilde d a\tilde c})(a\tilde d\tilde c a\tilde c\tilde d) =_{\tilde G} (\tilde d a\tilde c a\tilde c a\tilde d a)(a\tilde d\tilde c a\tilde c\tilde d)\\ &=_{\tilde G} \tilde d(a\tilde c)^4\tilde d =_{\tilde G} \tilde d^2 =_{\tilde G} 1,\end{aligned}$$

and

$$\begin{aligned}(adacac)^4 &= (a\tilde d\underline{\tilde c a\tilde c}\tilde b a\tilde b\tilde c)^4 =_{\tilde G} (\underline{a\tilde d a\tilde c}a\tilde c a\tilde b a\tilde b\tilde c)^4 =_{\tilde G} \tilde c a(\tilde d\underline{\tilde c a\tilde b a\tilde b a})(\tilde d\ tildeca\tilde b a\tilde b a)^3 a\tilde c\\ &=_{\tilde G} \tilde c a(\tilde d(a\tilde b a\tilde b a\tilde c)^3(\tilde d\tilde c a\tilde b a\tilde b a)^3)a\tilde c\\ &=_{\tilde G} \tilde c a(\tilde d(a\tilde b a\tilde b a\tilde c)\underline{(a\tilde b a\tilde b a\tilde c)(a\tilde b a\tilde b a\tilde d a\tilde b a\tilde b a)(\tilde c}\tilde d a\tilde b a\tilde b a)(\tilde d\tilde c a\tilde b a\tilde b a))a\tilde c\\ &=_{\tilde G} \tilde c a(\tilde d(a\tilde b a\tilde b a\tilde c)\underline{(\tilde d a\tilde b a\tilde b a)^2(}\tilde d\tilde c a\tilde b a\tilde b a))a\tilde c\\ &=_{\tilde G} \tilde c a\tilde d(a\tilde b a\tilde b a\tilde c a\tilde b a\tilde b a\tilde d)^2\tilde d a\tilde c =_{\tilde G} 1.\end{aligned}$$

Proposition 5.8. *The finite quotients $\tilde G_n = \tilde G/\mathsf{Stab}_{\tilde G}(n)$ of $\tilde G$ have order $2^{13\cdot 2^{n-4}+2}$ for $n \geq 4$, and 2^{2^n-1} for $n \leq 4$.*

Proof. For $n \geq 4$, $\phi(\tilde H)$ is a subgroup of index 8 in $\tilde G \times \tilde G$, so $\tilde G_n$ is a subgroup of index 8 in $\tilde G_{n-1} \wr \mathbb{Z}/2$ and $|\tilde G_n| = |\tilde G_{n-1}|^2/4$. For $n \leq 4$ one has $\tilde G_n = \mathsf{Aut}(\mathcal{T})_n = \mathbb{Z}/2 \wr \ldots \wr \mathbb{Z}/2$. $\square$

Proposition 5.9. *$\tilde K \geq \mathsf{Stab}_{\tilde G}(4)$ and $\tilde K' \geq \tilde K'_{(2)}$, so $\tilde G$ has the congruence property. Additionally, $\tilde K' \geq \mathsf{Stab}_{\tilde G}(5)$.*

Proof. The first and third assertions can be checked on a computer. For the second, K contains $y = [u, d]$ and $z = [u, c]$; these elements satisfy $\phi(y) = (1, u)$ and $\phi(z) = (1, v)$. Then K' contains $[y, v] = \phi^{-2}(1, 1, u, 1)$ and $[z, d] = \phi^{-2}(1, 1, v, 1)$, so it contains $\phi^{-2}(1 \times 1 \times K \times 1)$ and $K_{(2)}$. $\square$

Corollary 5.10. *The closure $\overline{\tilde G}$ of $\tilde G$ in $\mathsf{Aut}(\mathcal{T})$ is isomorphic to the profinite completion $\widehat{\tilde G}$ and is a pro-2-group. It has Hausdorff dimension 13/16.*

5.1. The growth of $\tilde G$. By the growth of a group one means the growth, in the sense of Definition 3.11, of the group acting on itself. We rephrase the definition of growth of a group in a slightly more general frame:

Definition 5.11. *Let G be a group generated by a finite set S, and let $\nu : S \to \mathbb{R}_+^*$ be any function. The* weight *of $g \in G$ is*

$$|g| = \min\{\nu(s_1) + \ldots + \nu(s_n) | \, s_1 \cdots s_n = g, s_i \in S\}.$$

The growth series *of G with respect to ν is*

$$F_\nu(\tau) = \sum_{g \in G} e^{\tau |g|}.$$

This series converges at least in the half-plane $\Re(\tau) < -\log(n)/\min_{s\in S}\nu(s)$. Let $\rho(\nu)$, the growth rate *of G with respect to ν, be the smallest non-positive value such that the series converges.*

Proposition 5.12. *If $\rho(\nu) < 0$, then G has exponential growth, while if $\rho(\nu) = 0$, then G has intermediate or polynomial growth.*

Proof. Let m and M be the minimum and maximum of the weight function ν, and set $R = \lim \sqrt[n]{\gamma_G^S(n)}$. By considering the series $F_S(\tau) = \sum_{n\geq 0} \gamma(n)\tau^n$, whose radius of convergence is $1/R$ and comparing it with $F_\nu(\tau)$, we obtain

$$M\rho(\nu) \leq \log(1/R) \leq m\rho(\nu),$$

so $R > 1$ is equivalent to $\rho(\nu) < 0$. □

The first examples of groups of intermediate growth were constructed in [17]; the group G is one of them.

Theorem 5.13. *$\tilde{G}$ has intermediate growth.*

Proof. First, note that $\tilde{G}$ cannot have polynomial growth, since it contains G whose growth function is greater than $e^{\sqrt{n}}$ [2].

Take as generators for $\tilde{G}$ the set $S = \{a, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{b}\tilde{c}, \tilde{b}\tilde{d}, \tilde{c}\tilde{d}, \tilde{b}\tilde{c}\tilde{d}\}$; let θ be strictly between the real root of the equation $-2 + X + X^2 + X^3 = 0$ and 1, for instance $\theta = 0.811$ and let ν be defined by

$$\nu(a) = 1,$$
$$\nu(\tilde{b}) = (\theta + \theta^3)/(1 - \theta^3) \approx 2.87,$$
$$\nu(\tilde{c}) = (-1 + \theta + \theta^2 + \theta^3)/(1 - \theta^3) \approx 2.14,$$
$$\nu(\tilde{d}) = (\theta^2 + \theta^3)/(1 - \theta^3) \approx 2.54,$$
$$\nu(\tilde{b}\tilde{c}) = (-1 + \theta + \theta^3)/(1 - \theta^3) \approx 0.73,$$
$$\nu(\tilde{b}\tilde{d}) = (\theta^3)/(1 - \theta^3) \approx 1.13,$$
$$\nu(\tilde{c}\tilde{d}) = (-1 + \theta^2 + \theta^3)/(1 - \theta^3) \approx 0.41,$$
$$\nu(\tilde{b}\tilde{c}\tilde{d}) = (1 + \theta^3)/(1 - \theta^3) \approx 3.28.$$

Clearly any element $g \in G$, when expressed as a minimal word in S, will have the form $[a]x_1ax_2\dots ax_n[a]$, where the first and last a are optional and $x_i \in S \setminus \{a\}$. Indeed the function ν satisfies the triangular inequalities $\nu(\tilde{b}) + \nu(\tilde{c}) < \nu(\tilde{b}\tilde{c})$, etc. Choose once and for all a minimal expression for every element of $\tilde{G}$.

Suppose now for contradiction that $\rho(\nu) < 0$. For some value $\eta \in (0,1)$ to be chosen later, partition $\tilde{G}$ in two sets: A containing those elements $g \in G$ whose minimal expression $s_1 \dots s_n$ contains at least ηn occurrences of the generator $x = \tilde{b}\tilde{c}\tilde{d}$, and B the other elements. Define two generating series

$$F_A(\tau) = \sum_{g\in A} e^{\tau|g|}, \qquad F_B(\tau) = \sum_{g\in A} e^{\tau|g|}.$$

Clearly $F_\nu = F_A + F_B$. We will show that for an appropriate value of η both F_A and F_B will converge up to some σ with $\rho(\nu) < \sigma < 0$.

We bound F_A by replacing A by a larger set, namely the set of all words $s_1 \dots s_n$ containing at least ηn occurrences of x. Then

$$F_A(\tau) < \sum_{n\geq 0} \binom{n}{\eta n} \left(\sum_{s\in S} e^{\tau\nu(s)} \right)^{(1-\eta)n} \left(e^{\tau\nu(x)} \right)^{\eta n}.$$

By Stirling's formula,

$$\binom{n}{\eta n} \approx \frac{\sqrt{2\pi\eta(1-\eta)}\sqrt{n}}{(\eta^\eta + (1-\eta)^{1-\eta})^n}.$$

Putting these together, we conclude that F_A converges up to any $\sigma > \rho(\nu)$ if

$$\frac{\left(\sum_{s\in S} e^{\sigma\nu(s)}\right)^{1-\eta} \left(e^{\sigma\nu(x)}\right)^\eta}{\eta^\eta(1-\eta)^{1-\eta}} < 1,$$

and this will hold for η large enough, as both the first multiplicand and the denominator tend to 1 as η tends to 1, while the second multiplicand tends to $e^{\sigma\nu(x)} < 1$.

We then approximate F_B by considering the subset $B' \subset B$ of words $s_1 \dots s_n$ that either start or end by a, but not both; and further that contain an even number of as. The series $F_{B'}(\tau)$ obtained this way will satisfy $F_B \approx 4F_{B'}$. Now B' injects in $G \times G$ through the map ϕ, written $g \mapsto (g_{|0}, g_{|1})$. We will compare $|g|$ with $|g_{|0}| + |g_{|1}|$. Thanks to the choice of ν, every generator $s \neq x$ contributing $\nu(s)$ to $|g|$ will contribute at most $\theta\nu(s)$ to $|g_{|0}| + |g_{|1}|$, while every

x contributing $\nu(x)$ to $|g|$ will contribute $\nu(x)$ to $|g_{|0}| + |g_{|1}|$. We conclude that for all $g \in B'$ we have

$$\frac{|g_{|0}| + |g_{|1}|}{|g|} < \frac{\eta\nu(x) + (1-\eta)\min\nu}{\eta\nu(x) + (1-\eta)\theta\min\nu} =: \zeta < 1.$$

This means every element of weight n in B' can be written as a pair of elements of G with total weight at most ζn, or in formulæ

$$F_{B'}(\tau) \leq (F_\nu(\zeta\tau))^2.$$

The series $F_{B'}$ thus converges up to $\zeta\rho(\nu) > \rho(\nu)$; the same holds for F_B. Then the series F_ν converges up to $\min(\zeta\rho(\nu), \sigma) > \rho(\nu)$, a contradiction. □

5.2. The subgroup $\tilde{P}$. Let e be the ray 1^∞ and let $\tilde{P}$ be the corresponding parabolic subgroup.

Theorem 5.14. *$\tilde{P}/\tilde{P}'$ is an infinite elementary 2-group generated by the images of $\tilde{b}$, $\tilde{c} = (1, \tilde{d})$, $\tilde{d} = (1, \tilde{b})$ and of all elements of the form $(1, \dots, 1, u^2)$ or $(1, \dots, 1, v)$. The following decomposition holds:*

$$\tilde{P} = \left(\tilde{B} \times \left((\tilde{C} \times ((\tilde{K} \times \dots) \rtimes \langle u^2, v\rangle)) \rtimes \langle \tilde{b}, u^2, \tilde{d}, v\rangle\right)\right) \rtimes \langle \tilde{b}, u^2\rangle.$$

Define the following subgroups of $\tilde{G}_n$:

$$\begin{array}{lll} \tilde{B}_n = \langle \tilde{b}, \tilde{d}\rangle^{\tilde{G}_n}; & \tilde{C}_n = \langle \tilde{b}, \tilde{v}\rangle^{\tilde{G}_n}; & \tilde{K}_n = \langle u, v\rangle^{\tilde{G}_n}; \\ \tilde{Q}_n = \tilde{B}_n \cap \tilde{P}_n; & \tilde{R}_n = \tilde{C}_n \cap \tilde{P}_n; & S_n = \tilde{K}_n \cap \tilde{P}_n. \end{array}$$

Proposition 5.15. *These subgroups have the following structure:*

$$\begin{aligned} \tilde{P}_n &= (\tilde{B}_{n-1} \times \tilde{Q}_{n-1}) \rtimes \langle \tilde{b}, u^2\rangle; \\ \tilde{Q}_n &= (\tilde{C}_{n-1} \times \tilde{R}_{n-1}) \rtimes \langle \tilde{b}, u^2\rangle; \\ \tilde{R}_n &= (\tilde{K}_{n-1} \times \tilde{S}_{n-1}) \rtimes_{\langle [b,v]\rangle} \langle b, u^2, v\rangle; \\ \tilde{S}_n &= (\tilde{K}_{n-1} \times \tilde{S}_{n-1}) \rtimes_{\langle [u^2,v]\rangle} \langle u^2, v\rangle. \end{aligned}$$

P r o o f. The claims match those of Proposition 5.3, and are proved by restricting to elements preserving e_n the 'y' and 'z' in decompositions of the kind $(x \times y) \rtimes z$. □

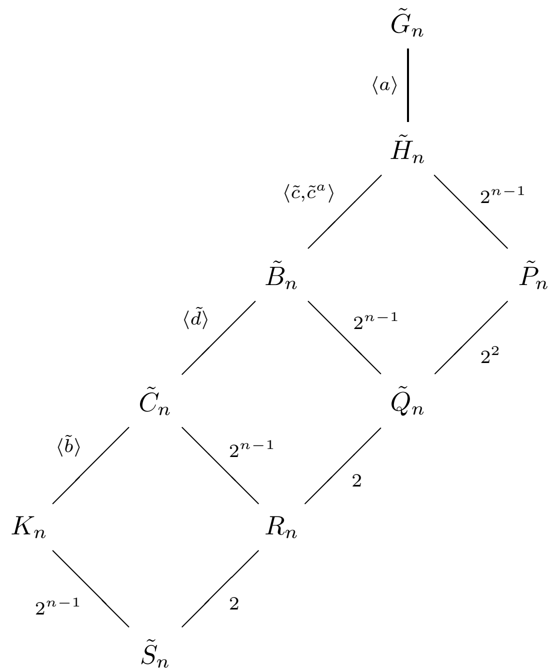


Fig. 3. The finite group $\tilde{G}_n$ and its subgroups

The group $\tilde{G}_n$ and its subgroups $\tilde{H}_n, \tilde{B}_n, \tilde{C}_n, \tilde{K}_n, \tilde{P}_n, \tilde{R}_n, \tilde{Q}_n, \tilde{S}_n$ are arranged in the lattice of Figure 3, with the quotients or the indices are represented next to the arrows.

6. The group Γ. The next three groups we study are subgroups of $\mathsf{Aut}(\mathcal{T}_3)$. Denote by a the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let t be the automorphisms of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1x\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then Γ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$; its growth was studied by Jacek Fabrykowski and Narain Gupta [15].

We write $H_n = \mathsf{Stab}_\Gamma(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \Big\{ t \to (a, 1, t), \quad t^a \to (t, a, 1), \quad t^{a^2} \to (1, t, a).$$

Define the elements $x = at$, $y = ta$ of Γ. Let K be the subgroup of Γ generated by x and y, and let L be the subgroup of K generated by K' and cubes in K.

Proposition 6.1. *We have the following diagram of normal subgroups:*

$$
\begin{tikzcd}
 & \Gamma \arrow[dl, dash, "\langle a|\, a^3\rangle"'] \arrow[dr, dash, "\langle a|\, a^3\rangle"] & \\
K \arrow[dr, dash, "\langle x|\, x^3\rangle"'] & & H = \underset{\Gamma}{\mathsf{Stab}}(1) \arrow[dl, dash, "\langle t|\, t^3\rangle"] \\
 & \Gamma' = K \cap H = [K, H] \arrow[d, dash] & \\
 & L = \langle K', K^3\rangle = \gamma_3(\Gamma) \arrow[dl, dash] \arrow[dr, dash] & \\
K' \arrow[d, dash] & & H' = \phi^{-1}(\Gamma' \times \Gamma' \times \Gamma') = \mathsf{Stab}_\Gamma(2) \\
\langle L \times L \times L, x^3y^{-3}, [x, y^3]\rangle = \gamma_4(\Gamma) \arrow[d, dash] & & \\
\langle L \times L \times L, [x, y^3]\rangle = \gamma_5(\Gamma) & &
\end{tikzcd}
$$

where the quotients are represented next to the arrows; all edges represent normal inclusions of index 3*. Furthermore* $L = K \cap \phi^{-1}(K \times K \times K)$*.*

P r o o f. First we prove K is normal in Γ, of index 3, by writing $y^t =$

$x^{-1}y^{-1}$, $y^{a^{-1}} = y^{-1}x^{-1}$, $y^{t^{-1}} = y^a = x$; similar relations hold for conjugates of x. A transversal of K in Γ is $\langle a\rangle$. All subgroups in the diagram are then normal.

Since $[a,t] = y^{-1}x = t^a t^{-1}$, we clearly have $\Gamma' < K \cap H$. Now as $\Gamma' \neq K$ and $\Gamma' \neq H$ and Γ' has index 3^2, we must have $\Gamma' = K \cap H$. Finally $[a,t] = [x,t]^{t^{-1}}$, so $\Gamma' = [K,H]$.

Next $x^3 = [a,t][t,a^{-1}][a^{-1},t^{-1}]$ and similarly for y, so $K^3 < \Gamma'$ and $L < \Gamma'$. Also, $\phi[x,y] = (y^{-1}, y^{-1}, x^{-1})$ and $\phi x^3 = (y,x,y)$ both belong to $K \times K \times K$, while $[a,t]$ does not; so L is a proper subgroup of Γ', of index 3 (since K/L is the elementary abelian group $(\mathbb{Z}/3\mathbb{Z})^2$ on x and y).

Consider now H'. It is in $\mathsf{Stab}_\Gamma(2)$ since $H = \mathsf{Stab}_\Gamma(1)$. Also, $[t,t^a] = y^3[y^{-1},x]$ and similarly for other conjugates of t, so $H' < L$, and $\phi[t,t^a] = ([a,t],1,1)$, so $\phi(H') = \Gamma' \times \Gamma' \times \Gamma'$. Finally H' it is of index 3 in L (since $H/H' = (\mathbb{Z}/3\mathbb{Z})^3$ on $t, t^a, t^{a^{-1}}$), and since $\mathsf{Stab}_\Gamma(2)$ is of index 3^4 in Γ (with quotient $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}$) we have all the claimed equalities. □

Proposition 6.2. *Γ is a just-infinite fractal group, is regular branch over Γ', and has the congruence property.*

Proof. Γ is fractal by Lemma 2.2 and the nature of the map ϕ. By direct computation, $[\Gamma : \Gamma'] = [\Gamma' : \phi^{-1}(\Gamma' \times \Gamma' \times \Gamma')] = [\phi^{-1}(\Gamma' \times \Gamma' \times \Gamma') : \Gamma''] = 3^2$, so Γ is branched on Γ'. Then $\Gamma'' = \gamma_5(\Gamma)$, as is shown in [4], so Γ'' has finite index and Γ is just-infinite by Proposition 3.5.

$\Gamma' \geq \mathsf{Stab}_\Gamma(2)$, so Γ has the congruence property. □

Proposition 6.3. *We have, with the notation introduced in Definition* 1.1*,*

$$\phi(H) = (\Gamma' \times \Gamma' \times \Gamma') \rtimes_{3-ab} \langle t, t^a, t^{a^2}\rangle,$$
$$\phi(\Gamma') = (\Gamma' \times \Gamma' \times \Gamma') \rtimes_{3-ab} \langle [a,t], [a^2,t]\rangle.$$

Theorem 6.4. *The subgroup K of Γ is torsion-free; thus Γ is virtually torsion-free.*

Proof. For $1 \neq g \in K$, let $|g|_t$, the t-length of g, denote the minimal number of $t^{\pm 1}$'s required to write g as a word over the alphabet $\{a^{\pm 1}, t^{\pm 1}\}$. We will show by induction on $|g|_t$ that g is of infinite order.

First, if $|g|_t = 1$, i.e. $g \in \{x^{\pm 1}, y^{\pm 1}\}$, we conclude from $\phi(x^3) = (*,*,x)$ and $\phi(y^3) = (*,*,y)$ that g is of infinite order.

Suppose now that $|g|_t > 1$. If $g \in L$, then $\phi(g) = (g_0, g_1, g_2) \in K \times K \times K$ and it suffices to show that one of the g_i is of infinite order—this follows by induction since $|g_i|_t < |g|_t$ and some $g_i \neq 1$. We may thus suppose that $g \in K \setminus L$. Up to symmetry, it suffices also to consider elements g of the form ℓx, ℓxy and ℓxy^{-1}, for $\ell \in L$. Write $\phi(\ell) = (\ell_0, \ell_1, \ell_2)$.

In the first case, we have $\phi(g^3) = \phi(\ell x)^3 = (a\ell_2 t\ell_1\ell_0, *, *)$. It suffices to show that the first coordinate of this expression is non-trivial, as K contains at worst only 3-torsion, being contained in the 3-Sylow of $\mathsf{Aut}(\mathcal{T}_3)$. Now map Γ to Γ/Γ', an elementary abelian group of order 9. One checks that $\ell_0\ell_1\ell_2 \equiv 1$ in the abelian quotient, so the first coordinate maps to $\overline{at} \not\equiv 1$ in Γ/Γ'.

The second case is handled in the same way. Finally, if $g = \ell xy^{-1}$, then $\phi(g^3) \in L \times L \times L$, so $\phi^2(g^3) \in K \times \ldots \times K$ (9 factors); each factor has strictly smaller t-length than g, and as before the projection in one of the coordinates onto the abelian quotient gives some $x \not\equiv 1$. $\square$

Proposition 6.5. *The finite quotients* $\Gamma_n = \Gamma/H_n$ *of* Γ *have order* $3^{3^{n-1}+1}$ *for* $n \geq 2$, *and* 3 *for* $n = 1$.

Proof. Follows immediately from $[\Gamma : \Gamma'] = 3^2$ and $[\Gamma' : \phi^{-1}(\Gamma' \times \Gamma' \times \Gamma')] = 3^2$. $\square$

Corollary 6.6. *The closure* $\overline{\Gamma}$ *of* Γ *in* $\mathsf{Aut}(\mathcal{T})$ *is isomorphic to the profinite completion* $\widehat{\Gamma}$ *and is a pro-3-group. It has Hausdorff dimension* $2/3$.

6.1. The subgroup *P*. Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

Theorem 6.7. P/P' *is an infinite elementary* 3*-group generated by* t, t^a *and all elements of the form* $(1, \ldots, 1, [a, t])$. *The following decomposition holds:*

$$P = \Big(\Gamma' \times \Gamma' \times \Big((\Gamma' \times \Gamma' \times ((\Gamma' \times \Gamma' \times \ldots) \rtimes_{3-ab} \langle [a, t] \rangle)) \rtimes_{3-ab} \langle [a, t] \rangle\Big)\Big) \rtimes_{3-ab} \langle t, t^a \rangle,$$

where each factor (of nesting n*) in the decomposition acts on the subtree just below some* e_n *but not containing* e_{n+1}.

Define the following subgroups of Γ_n:

$$\Gamma'_n = \langle [a, t] \rangle^{\Gamma_n}; \qquad Q_n = \Gamma'_n \cap P_n.$$

Proposition 6.8. *These subgroups have the following structure:*

$$P_n = (\Gamma'_{n-1} \times \Gamma'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle t, t^a \rangle;$$
$$Q_n = (\Gamma'_{n-1} \times \Gamma'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle [a, t] \rangle.$$

7. The group $\overline{\Gamma}$. Recall a denotes the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{x}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\Gamma}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$.

We write $H_n = \mathsf{Stab}_{\overline{\Gamma}}(n)$ and $H = H_1$. Explicitly, the map ϕ restricts to

$$\phi : \Big\{ t \to (a, a, t), \quad t^a \to (t, a, a), \quad t^{a^2} \to (a, t, a).$$

Define the elements $x = ta^{-1}$, $y = a^{-1}t$ of $\overline{\Gamma}$, and let K be the subgroup of $\overline{\Gamma}$ generated by x and y. Then K is normal in $\overline{\Gamma}$, because $x^t = y^{-1}x^{-1}$, $x^a = x^{-1}y^{-1}$, $x^{t^{-1}} = x^{a^{-1}} = y$, and similar relations hold for conjugates of y. Moreover K is of index 3 in $\overline{\Gamma}$, with transversal $\langle a \rangle$.

Lemma 7.1. *H and K are normal subgroups of index 3 in $\overline{\Gamma}$, and $\overline{\Gamma}' = \mathsf{Stab}_K(1) = H \cap K$ is of index 9; furthermore $\phi(H \cap K) \triangleleft K \times K \times K$. For any element $g = (u, v, w) \in \phi(H \cap K)$ one has $wvu \in H \cap K$.*

P r o o f. First note that $\mathsf{Stab}_K(1) = \langle x^3, y^3, xy^{-1}, y^{-1}x \rangle$, for every word in x and y whose number of a's is divisible by 3 can be written in these generators. Then compute

$$\begin{aligned} \phi(x^3) &= (y, x^{-1}y^{-1}, x), & \phi(y^3) &= (x^{-1}y^{-1}, x, y), \\ \phi(xy^{-1}) &= (1, x^{-1}, x), & \phi(y^{-1}x) &= (y, 1, y^{-1}). \end{aligned}$$

The last assertion is also checked on this computation. □

Proposition 7.2. *Writing $c = [a, t] = x^{-1}y^{-1}x^{-1}$ and $d = [x, y]$, we have the following diagram of normal subgroups:*

$$
\begin{tikzcd}
 & \overline{\Gamma} \arrow[dl, dash, "\langle a|\, a^3\rangle"'] \arrow[dr, dash, "\langle a|\, a^3\rangle"] & \\
K \arrow[dr, dash, "\langle x,y|\, x^3,y^3,x=y\rangle"'] & & H \arrow[dl, dash, "\langle t_0,t_1,t_2|\, t_0^3,t_0=t_1=t_2\rangle"] \\
 & \overline{\Gamma}' = \langle c, c^t, c^{a^{-1}}, c^{at}\rangle = K \cap H = [K,H] \arrow[dl, dash, "\mathbb{Z}^2"'] \arrow[dr, dash, "(\mathbb{Z}/3\mathbb{Z})^2"] & \\
K' = \langle d, d^t, d^{a^{-1}}, d^{at}\rangle \arrow[dr, dash, "\mathbb{Z}^2"'] & & H' \arrow[dl, dash] \\
 & \overline{\Gamma}'' = \phi^{-1}(K' \times K' \times K') \arrow[d, dash, "(\mathbb{Z}/3\mathbb{Z})^2"] & \\
 & K'' &
\end{tikzcd}
$$

where the quotients are represented next to the arrows; additionally,

$$
\begin{aligned}
K/K' &= \langle x, y|\, [x,y]\rangle \cong \mathbb{Z}^2, \\
\overline{\Gamma}'/\overline{\Gamma}'' &= \langle c, c^t, c^{a^{-1}}, c^{at}|\, [c, c^t], \ldots\rangle \cong \mathbb{Z}^4, \\
K'/K'' &= \langle d, d^t, d^{a^{-1}}, d^{at}|\, [d, d^t], \ldots, (d/d^{at})^3, (d^{a^{-1}}/d^t)^3\rangle \cong \mathbb{Z}^2 \times (\mathbb{Z}/3\mathbb{Z})^2.
\end{aligned}
$$

Writing each subgroup in the generators of the groups above it, we have

$$
\begin{aligned}
K &= \langle x = at^{-1}, y = a^{-1}t\rangle, \\
H &= \langle t, t_1 = t^a, t_2 = t^{a^{-1}}\rangle, \\
\overline{\Gamma}' &= \langle b_1 = xy^{-1}, b_2 = y^{-1}x, b_3 = x^3, b_4 = y^3\rangle \\
&= \langle c_1 = tt_1^{-1}, c_2 = tt_1t, c_3 = tt_2^{-1}, c_4 = tt_2t\rangle.
\end{aligned}
$$

Proposition 7.3. $\overline{\Gamma}$ *is a fractal group, is weakly branch, and just-nonsolvable; however it is not branch.*

P r o o f. $\overline{\Gamma}$ is fractal by Lemma 2.2 and the nature of the map ϕ. The subgroup K described above has an infinite-index derived subgroup K' (with infinite cyclic quotient), from which we conclude that $\overline{\Gamma}$ is not just-infinite; indeed K' is normal in $\overline{\Gamma}$ and $\overline{\Gamma}/K' \cong \mathbb{Z}^2 \rtimes \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ is infinite. □

Proposition 7.4. *The subgroup K of $\overline{\Gamma}$ is torsion-free; thus $\overline{\Gamma}$ is virtually torsion-free.*

P r o o f. For $1 \neq g \in K$, let $|g|_t$, the t-length of g, denote the minimal number of $t^{\pm 1}$'s required to write g as a word over the alphabet $\{a^{\pm 1}, t^{\pm 1}\}$. We will show by induction on $|g|_t$ that g is of infinite order.

First, if $|g|_t = 1$, i.e. $g \in \{x^{\pm 1}, y^{\pm 1}\}$, we conclude from $\phi(x^3) = (*, *, x)$ and $\phi(y^3) = (*, *, y)$ that g is of infinite order.

Suppose now that $|g|_t > 1$, and $g \in H_n \setminus H_{n+1}$. Then there is some sequence σ of length n that is fixed by g and such that $g_{|\sigma} \notin H$. By Lemma 7.1., $g_{|\sigma} \in K$, so it suffices to show that all $g \in K \setminus H$ are of infinite order.

Such a g can be written as $\phi^{-1}(u, v, w)z$ for some $(u, v, w) \in \phi(K \cap H)$ and $z \in \{x^{\pm 1}, y^{\pm 1}\}$; by symmetry let us suppose $z = x$. Then $g^3 = \phi^{-1}(uavawt, vawtua, wtuava) = \phi^{-1}(g_0, g_1, g_2)$, say. For any i, we have $|g_i|_t \leq |g|_t$, because all the components of $\phi(x)$ and $\phi(y)$ have t-length ≤ 1. We distinguish three cases:

1. $g_i = 1$ for some i. Then consider the image $\overline{g_i}$ of g_i in $\overline{\Gamma}/\overline{\Gamma}'$. By Lemma 7.1, $wvu \in G'$, so $\overline{g_i} = 1 = \overline{a^2 t}$. But this is a contradiction, because $\overline{\Gamma}/\overline{\Gamma}'$ is elementary abelian of order 9, generated by the independent images $\overline{a}$ and $\overline{t}$.
2. $0 < |g_i|_t < |g|_t$ for some i. Then by induction g_i is of infinite order, so g^3 too, and g too.
3. $|g_i|_t = |g|_t$ for all i. We repeat the argument with g_i substituted for g. As there are finitely many elements h with $|h|_t = |g|_t$, we will eventually reach either an element of shorter length or an element already considered. In the latter case we obtain a relation of the form $\phi^n(g^{3^n}) = (\dots, g, \dots)$ from which g is seen to be of infinite order. □

Proposition 7.5. *The finite quotients $\overline{\Gamma}_n = \overline{\Gamma}/H_n$ of $\overline{\Gamma}$ have order $3^{\frac{1}{4}(3^n+2n+3)}$ for $n \geq 2$, and $3^{\frac{1}{2}(3^n-1)}$ for $n \leq 2$.*

P r o o f. Define the following family of two-generated finite abelian groups:

$$A_n = \begin{cases} \langle x, y |\, x^{3^{n/2}}, y^{3^{n/2}}, [x, y]\rangle & \text{if } n \equiv 0[2], \\ \langle x, y |\, x^{3^{(n+1)/2}}, y^{3^{(n+1)/2}}, (xy^{-1})^{3^{(n-1)/2}}, [x, y]\rangle & \text{if } n \equiv 1[2]. \end{cases}$$

First suppose $n \geq 2$; Consider the diagram of groups described above, and quotient all the groups by H_n. Then the quotient K/K' is isomorphic to A_n, generated by x and y, and the quotient $K'/\overline{\Gamma}''$ is isomorphic to A_{n-1}, generated by $[x,y]$ and $[x,y]^t$. As $|A_n| = 3^n$, the index of K'_n in $\overline{\Gamma}_n$ is 3^{n+1} and the index of $\overline{\Gamma}''_n$ is 3^{2n}. Then as $\overline{\Gamma}''_n \cong K^3_{n-1}$ and $|\overline{\Gamma}''_2| = 1$ we deduce by induction that $|\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n-6n+3)}$ and $|K'_n| = 3^{\frac{1}{4}(3^n-2n-1)}$, from which $|\overline{\Gamma}_n| = 3^{2n} + |\overline{\Gamma}''_n| = 3^{\frac{1}{4}(3^n+2n+3)}$ follows.

For $n \leq 2$ we have $\overline{\Gamma}_n = \mathsf{Aut}(\mathcal{T})_n = \mathbb{Z}/3 \wr \ldots \wr \mathbb{Z}/3$. $\square$

Corollary 7.6. *The closure* $\overline{\overline{\Gamma}}$ *of* $\overline{\Gamma}$ *in* $\mathsf{Aut}(\mathcal{T})$ *has Hausdorff dimension* $1/2$.

Proposition 7.7. *We have*

$$\begin{aligned}\phi(H) &= (K' \times K' \times K') \rtimes_A \langle t_0, t_1, t_2\rangle,\\ \phi(K') &= (K' \times K' \times K') \rtimes_B \langle d, d^t\rangle,\end{aligned}$$

where A is such that $\langle t_0, t_1, t_2\rangle / A \cong \mathbb{Z}^4 \rtimes \mathbb{Z}/3\mathbb{Z}$ and B is such that $\langle d, d^t\rangle / B \cong \mathbb{Z}^2$.

7.1. The subgroup P.

Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

Theorem 7.8. *P/P' is the direct product of $(\mathbb{Z}/3\mathbb{Z})^2$ (generated by t and $atat^{-1}a$) and an infinitely-generated free abelian group, generated by $[tt_1t, tt_2t]$. The following decomposition holds:*

$$P = \Big(K' \times K' \times \Big(\big(K' \times K' \times ((K' \times K' \times \ldots) \rtimes \langle [tt_1t, tt_2t]\rangle)\big) \rtimes \langle [tt_1t, tt_2t]\rangle\Big)\Big) \rtimes \langle t, t_1t_2^{-1}\rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Define the following subgroups of Γ_n:

$$K'_n = \langle x, y\rangle'^{\Gamma_n}; \qquad Q_n = K'_n \cap P_n.$$

Proposition 7.9. *These subgroups have the following structure:*

$$\begin{aligned}P_n &= (K'_{n-1} \times K'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} (\mathbb{Z}^4 \ltimes \mathbb{Z}/3\mathbb{Z});\\ Q_n &= (K'_{n-1} \times K'_{n-1} \times Q_{n-1}) \rtimes \mathbb{Z}^2.\end{aligned}$$

8. The group $\overline{\overline{\Gamma}}$. Recall a denotes the automorphism of $\mathcal{T}_3$ permuting cyclically the top three branches. Let now t be the automorphism of $\mathcal{T}_3$ defined recursively by

$$t(0x\sigma) = 0\overline{x}\sigma, \qquad t(1x\sigma) = 1\overline{\overline{x}}\sigma, \qquad t(2\sigma) = 2t(\sigma).$$

Then $\overline{\overline{\Gamma}}$ is the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$; it was studied by Narain Gupta and Said Sidki [22, 23, 32, 33].

We will use the following known facts:

Theorem 8.1. *$\overline{\overline{\Gamma}}$ is a torsion 3-group.*

Proposition 8.2. *We have the following diagram of normal subgroups:*

$$\begin{tikzcd}
\overline{\overline{\Gamma}} \arrow[d, dash, "\langle a \mid a^3\rangle"] \\
H = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1) \arrow[d, dash, "\langle t \mid t^3\rangle"] \\
\overline{\overline{\Gamma}}' = [\overline{\overline{\Gamma}}, H] \arrow[d, dash, "{[a,t]}"] \\
\gamma_3(\overline{\overline{\Gamma}}) = \overline{\overline{\Gamma}}^3 = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(2) \arrow[d, dash, "(at)^3"] \\
H' = \phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}')
\end{tikzcd}$$

where the quotients are represented next to the arrows; all edges represent normal inclusions of index 3.

P r o o f. Clearly H is normal of index 3, being the kernel of the epimorphism $a \to a, t \to 1$. Then $\overline{\overline{\Gamma}}' \neq H$ (as can be checked in the finite quotient $\overline{\overline{\Gamma}}_2$) but is of index at most 3^2, so has precisely that index. Moreover, $\overline{\overline{\Gamma}}'$ is generated by the $[a^{\pm 1}, t^{\pm 1}]$: one has $[a,t]^a = [a^{-1}, t][a,t]^{-1}$, $[a,t]^t = [a,t]^{-1}[a,t^{-1}]$, etc.

$\gamma_3(\overline{\overline{\Gamma}}) < \overline{\overline{\Gamma}}^3$ holds in all 3-groups, and $\overline{\overline{\Gamma}}^3$ has index 3^3 because it is 2-generated 2-step nilpotent.

Now consider H'. It is in $\mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$ since $H = \mathsf{Stab}_{\overline{\overline{\Gamma}}}(1)$. Also, $[t, t^a] = (ta)^3(a^{-1}ta^{-1})^3$ and similarly for other conjugates, so $H' < \overline{\overline{\Gamma}}^3$, and $\phi[t^{-a^2}t^{-a}, t^{-a}t^{-1}] = ([a,t],1,1)$, so $\phi(H') = \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}'$. Finally H' it is of index 3 in $\overline{\overline{\Gamma}}^3$ (since $H/H' = (\mathbb{Z}/3\mathbb{Z})^3$ on $t, t^a, t^{a^{-1}}$), and since $\mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$ is of index 3^4 in Γ (with quotient $\mathbb{Z}/3\mathbb{Z} \wr \mathbb{Z}/3\mathbb{Z}$) we have all the claimed equalities. □

Proposition 8.3. *$\overline{\overline{\Gamma}}$ is a just-infinite fractal group, and is a regular branch group over $\overline{\overline{\Gamma}}'$.*

P r o o f. $\overline{\overline{\Gamma}}$ is fractal by Lemma 2.2 and the nature of the map ϕ. By direct computation, $[\overline{\overline{\Gamma}} : \overline{\overline{\Gamma}}'] = [\overline{\overline{\Gamma}}' : \phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}')] = [\phi^{-1}(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') : \overline{\overline{\Gamma}}''] = 3^2$, so $\overline{\overline{\Gamma}}$ is branched on $\overline{\overline{\Gamma}}'$ and is just-infinite by Proposition 3.5. □

Proposition 8.4. *$\overline{\overline{\Gamma}}' \geq \mathsf{Stab}_{\overline{\overline{\Gamma}}}(2)$, so $\overline{\overline{\Gamma}}$ has the congruence property.*

Proposition 8.5. *We have*

$$\phi(H) = (\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') \rtimes_{3-ab} \langle t, t^a, t^{a^2} \rangle,$$
$$\phi(\overline{\overline{\Gamma}}') = (\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}') \rtimes_{3-ab} \langle [a,t], [a^2,t] \rangle.$$

8.1. The subgroup P.

Let e be the infinite sequence 2^∞, and let P be the corresponding parabolic subgroup.

Theorem 8.6. *P/P' is an infinite elementary 3-group generated by t, $t^a t^{a^2}$ and all elements of the form $(1,\dots,1,tt^at^{a^2})$. The following decomposition holds:*

$$P = \Bigg(\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \bigg((\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times ((\overline{\overline{\Gamma}}' \times \overline{\overline{\Gamma}}' \times \dots) \rtimes_{3-ab} \langle tt^at^{a^2} \rangle)) \rtimes_{3-ab} \langle tt^at^{a^2} \rangle\bigg)\Bigg) \rtimes_{3-ab} \langle t, t^at^{a^2} \rangle,$$

where each factor (of nesting n) in the decomposition acts on the subtree just below some e_n but not containing e_{n+1}.

Define the following subgroups of $\overline{\overline{\Gamma}}_n$:

$$\overline{\overline{\Gamma}}'_n = \langle [a,t] \rangle^{\overline{\overline{\Gamma}}_n}; \qquad Q_n = \overline{\overline{\Gamma}}'_n \cap P_n.$$

Proposition 8.7. *These subgroups have the following structure:*

$$P_n = (\overline{\overline{\Gamma}}'_{n-1} \times \overline{\overline{\Gamma}}'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle t, t^a t^{a^2} \rangle;$$
$$Q_n = (\overline{\overline{\Gamma}}'_{n-1} \times \overline{\overline{\Gamma}}'_{n-1} \times Q_{n-1}) \rtimes_{3-ab} \langle t t^a t^{a^2} \rangle.$$

9. Quasi-regular representations. In this section we show how the information we gathered on the groups and their subgroups yields results on their representations. For G a group acting on a tree and P its parabolic subgroup, we let $\rho_{G/P}$ denote the quasi-regular representation of G on the space $\ell^2(G/P)$.

First of all consider the infinite-dimensional representations $\rho_{G/P}$. The criterion of irreducibility for quasi-regular representations was discovered by George Mackey and is as follows (the definition of commensurator is given after the theorem's statement):

Theorem 9.1 (Mackey [27, 12])**.** *Let G be an infinite group and let P be any subgroup of G. Then the quasi-regular representation $\rho_{G/P}$ is irreducible if and only if* $\mathrm{comm}_G(P) = P$.

Definitiom 9.2. *The* commensurator *(also called* quasi-normalizer*) of a subgroup H of G is*

$$\mathrm{comm}_G(H) = \{g \in G|\, H \cap H^g \textit{ is of finite index in } H \textit{ and } H^g\}.$$

Equivalently, letting H act on the left on the right cosets $\{gH\}$,

$$\mathrm{comm}_G(H) = \{g \in G|\, H \cdot (gH) \textit{ and } H \cdot (g^{-1}H) \textit{ are finite orbits}\}.$$

The equivalence follows, for T a finite transversal, from

$$H = \bigsqcup_{t \in T \subset H} t \cdot (H \cap H^g) \Longleftrightarrow HgH = \bigsqcup_{t \in T \subset H} t \cdot gH.$$

Proposition 9.3. *If G is weakly branch, then* $\mathrm{comm}_G(P) = P$.

P r o o f. Take $g \in G \setminus P$, with $P = \mathsf{Stab}_G(e)$ for some ray e; we will show that $P \cap P^g$ is of infinite index in P^g. Let n be such that $\sigma := e_1 \dots e_n \neq g(e_1 \dots e_n)$. Then $\mathsf{Rist}_{P^g}(\sigma) = \mathsf{Rist}_G(\sigma)$, while by Lemma 3.3 the index of $\mathsf{Rist}_{P \cap P^g}(\sigma) = \mathsf{Rist}_P(\sigma)$ in $\mathsf{Rist}_G(\sigma)$ is infinite. $\square$

Corollary 9.4. *If G is weakly branch, then $\rho_{G/P}$ is irreducible.*

The quasi-regular representations we consider are good approximants of the regular representation in the following sense:

Theorem 9.5. *ρ_G is a subrepresentation of $\bigotimes_{P \text{ parabolic}} \rho_{G/P}$.*

Proof. Since $\bigcap_{g\in G} P^g = 1$, it follows that the G-space G is a subspace of $\prod_{g\in G} G/P_g$. The representation on a product of spaces is the tensor product of the representation on the spaces. □

We have a continuum of parabolic subgroups $P_e = \mathsf{Stab}_G(e)$, where e runs through the boundary of a tree, so formally we also have a continuum of quasi-regular representations. If G is countable, there are uncountably many non-equivalent representations, because among the uncountably many P_e only countably many are conjugate. We therefore have the

Theorem 9.6. *There are uncountably many non-equivalent representations of the form $\rho_{G/P}$, where P is a parabolic subgroup.*

We now consider the finite-dimensional representations ρ_{G/P_n}, where P_n is the stabilizer of the vertex at level n in the ray defining P. These are permutational representations on the sets G/P_n. The ρ_{G/P_n} are factors of the representation $\rho_{G/P}$. Noting that $P = \bigcap_{n\geq 0} P_n$, it follows that

$$\rho_{G/P_n} \Rightarrow \rho_{G/P},$$

in the sense that for any non-trivial $g \in G$ there is an $n \in \mathbb{N}$ with $\rho_{G/P_n}(g) \neq 1$.

9.1. Hecke algebras. Corollary 9.4 showed that the quasi-regular representation $\rho_{G/P}$ is irreducible for all of our examples. We now describe the decomposition of the finite quasi-regular representations ρ_{G/P_n}. It turns out that it is closely related to the orbit structure of P_n on G/P_n, through the *Hecke algebra*. The result we shall prove is:

Theorem 9.7. *ρ_{G/P_n} and $\rho_{\tilde{G}/\tilde{P}_n}$ decompose as a direct sum of $n+1$ irreducible components, one of degree 2^i for each $i \in \{1,\dots,n-1\}$ and two of degree 1.*

$\rho_{\Gamma/P}$, $\rho_{\overline{\Gamma}/P}$ and $\rho_{\overline{\overline{\Gamma}}/P}$ decompose as a direct sum of $2n+1$ irreducible components, two of degree 2^i for each $i \in \{1,\dots,n-1\}$ and three of degree 1.

The proof of this theorem will appear after the following definitions and lemmata.

Definition 9.8. *Let G be a group and P a subgroup. Set $Q = \mathrm{comm}_G(P)$, and define*

$$\mathbb{C}[G,P] = \Big\{ f : Q \to \mathbb{C} \Big| f(pqp') = f(q) \forall p,p' \in P \textit{ and } \mathrm{supp}(f) \subset \bigcup_{\textit{finite}} PqP \Big\},$$

i.e. those (P,P)-invariant functions on Q whose support is a finite union of (P,P)-double cosets. $\mathbb{C}[G,P]$ is an algebra for the convolution product

$$(f \cdot g)(x) = \sum_{y \in G/P} f(xy) g(y^{-1}).$$

The Hecke algebra *(also called the* intersection algebra*) $\mathcal{L}(G,P)$ is the weak closure of $\mathbb{C}[G,P]$ in $\mathcal{L}(\ell^2(G/P))$.*

A few remarks are in order. First, the convolution product is well defined on $\mathbb{C}[G,P]$, since every double coset PqP is a finite union of left (or right) cosets. Second, $\mathcal{L}(G,P)$ coincides with the commutant $\rho_{G/P}(G)'$ of the right-regular representation of G in $\mathcal{L}(\ell^2(G/P))$. That $\mathcal{L}(G,P)$ commutes with ρ' is obvious, since these two operators derive from left- and right-actions on G. That $\mathcal{L}(G,P)$ is the full commutant requires an argument, based on approximation of functions in $\mathcal{L}(G,P)$ by finite-support functions.

Third, the whole theory of Hecke algebra can be extended to locally compact G and compact-open P — see for instance [34]. One then defines $\mathbb{C}[G,P]$ as those bi-P-invariant continuous maps $G \to \mathbb{C}$ whose support is contained in a finite union of PJP, where the J are compact-open subgroups of G. This algebra is represented in $\mathcal{L}(L^2(G/P),\mu)$, where μ is the projection of the Haar measure to G/P (which, beware, need not be G-invariant!). We shall not make use of this theory.

A variant of this notion, which we will use, is obtained by taking G profinite and P closed. Then $\mathbb{C}[G,P]$ consists of those bi-P-invariant continuous maps $G \to \mathbb{C}$ whose support is contained in a finite union of PJP, where the J are neighbourhoods of the identity in G.

$\mathcal{L}(G,P)$ is topologically spanned by compactly supported $(P-P)$-biinvariant functions on G. The following result stresses the importance of the Hecke algebra in the study of representation decomposition:

Theorem 9.9 ([13, Section 11D])**.** *Suppose $[G : P]$ is finite. Then $\mathcal{L}(G,P)$ is a semi-simple algebra. There is a canonical bijection between isotypical components of $\rho_{G/P}$ and simple factors of $\mathcal{L}(G,P)$, which maps χ^n (for χ simple) to $M_n(\mathbb{C})$.*

Then, if $\mathcal{L}(G,P)$ is abelian, its decomposition in simple modules is has as many components as there are double cosets PgP in G.

In our examples, the spaces have the following order of magnitude: the core of P_n is the normal subgroup $H_n = \bigcap_{g\in G} P_n^g$, of index $\sim e^{e^n}$. The subgroup P_n is of index $\sim e^n$. The number of double cosets is $\sim n$. We give the precise results for our five examples.

9.2. Orbits in G/P_n. As the double cosets P_ngP_n are in one-to-one correspondence with the orbits of P_n on G/P_n we shall now describe the orbits for this action.

Lemma 9.10. *There are two K_n-orbits on Σ^n: those sequences starting with* 0 *and those starting with* 1.

P_n has $n+1$ orbits in Σ^n; they are 1^n and the $1^i0\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $1^i0\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. As K_n contains $K_{n-1} \times K_{n-1}$, it follows by induction that K_n acts transitively on the sets $00\Sigma^{n-2}$ and $01\Sigma^{n-2}$. As K_n contains $(ab)^2 = (ca, ac)$, it also permutes $00\Sigma^{n-2}$ and $01\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$.

The last assertion follows from Theorem 4.4.

Lemma 9.11. *There are two $\tilde{K}_n$-orbits on Σ^n: those sequences starting with* 0 *and those starting with* 1.

$\tilde{P}_n$ has $n+1$ orbits in Σ^n; they are 1^n and the $1^i0\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of $\tilde{P}$ in $\mathcal{T}_\Sigma$ are the $1^i0\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. Completely similar to Lemma 9.10. □

Lemma 9.12. *There are three Γ'_n-orbits on Σ^n: those sequences starting with* 0*, those starting with* 1 *and those starting with* 2.

P_n has $2n+1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. As Γ'_n contains $\Gamma'_{n-1} \times \Gamma'_{n-1} \times \Gamma'_{n-1}$, it follows by induction that Γ'_n acts transitively on the sets $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$. As Γ'_n contains $[a,t] = (ta^{-1}, a, t^{-1})$, it also permutes $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$ and $2\Sigma^{n-1}$.

The last assertion follows from Theorem 6.7. □

Lemma 9.13. *For the group $\overline{\Gamma}$, there are three K'_n-orbits on Σ^n: those sequences starting with* 0*, those starting with* 1 *and those starting with* 2.

P_n has $2n+1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0 \le i < n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i \in \mathbb{N}$.

Proof. As K'_n contains $K'_{n-1}\times K'_{n-1}\times K'_{n-1}$, it follows by induction that K'_n acts transitively on the sets $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$. As K'_n contains $[x,y]=(at,at,ta)$, it also permutes $00\Sigma^{n-2}$, $01\Sigma^{n-2}$ and $02\Sigma^{n-2}$, so it acts transitively on $0\Sigma^{n-1}$. The same holds for $1\Sigma^{n-1}$ and $2\Sigma^{n-1}$.

The last assertion follows from Theorem 7.8. □

Lemma 9.14. *There are three $\overline{\overline{\Gamma}}'_n$-orbits on Σ^n: those sequences starting with 0, those starting with 1 and those starting with 2.*

P_n has $2n+1$ orbits in Σ^n; they are 2^n and the $2^i0\Sigma^{n-1-i}$ and $2^i1\Sigma^{n-1-i}$ for $0\le i<n$. The orbits of P in $\mathcal{T}_\Sigma$ are the $2^i0\Sigma^*$ and $2^i1\Sigma^*$ for all $i\in\mathbb{N}$.

Proof. Completely similar to Lemma 9.12. □

9.3. Gelfand pairs. We have seen the Hecke algebra $\mathcal{L}(G,P_n)$ is roughly of dimension n. Its structure is further simplified by the following consideration:

Definition 9.15 ([14])**.** *Let G be a group and P any subgroup. The pair (G,P) is a* Gelfand pair *if all irreducible subrepresentations of $\rho_{G/P}$ have multiplicity* 1.

Lemma 9.16 ([13, Exercise 18, page 306], [27, Theorem 1.20])**.** *(G,P) is a Gelfand pair if and only if $\mathcal{L}(G,P)$ is abelian.*

Proposition 9.17. *In our five examples the pairs (G,P_n) form a Gelfand pair for all $n\in\mathbb{N}$.*

Proof. Clearly $P_0=G$ so $\mathcal{L}(G,P_0)=\mathbb{C}$ is abelian. Furthermore, P_{n+1} is a subgroup of P_n, and the natural map $G/H_{n+1}\twoheadrightarrow G/H_n$ induces a map $P_{n+1}/H_{n+1}\twoheadrightarrow P_n/H_n$, so $\mathcal{L}(G,P_n)\cong\mathcal{L}(G/H_n,P_n/H_n)$ is a direct summand of $\mathcal{L}(G,P_{n+1})$, and their dimensions differ by $d-1$, which is 1 or 2 (recall d is the degree of the regular tree on which G acts). Now writing

$$\mathcal{L}(G,P_{n+1})=\mathcal{L}(G,P_n)\oplus A,$$

we see that A is semi-simple and of dimension $d-1<4$. All such semisimple algebras are abelian, $A\cong\mathbb{C}^{d-1}$, so $\mathcal{L}(G,P_{n+1})$ is abelian too. □

Proof of Theorem 9.7. By Proposition 9.17, the Hecke algebra $\mathcal{L}(G,P_n)$ is abelian, so it is isomorphic to $\mathbb{C}^{N_n}$, where N_n is its dimension. This N_n in turn is equal to the number of double cosets P_ngP_n. These numbers N_n are computed in the corollaries in Subsection 9.2. By Theorem 9.9, the number of irreducible subrepresentations of ρ_{G/P_n} is N_n. Finally, $\rho_{G/P_n}=\rho_{G/P_{n-1}}\oplus A_{n,1}\oplus\ldots\oplus A_{n,d-1}$, where the $A_{n,i}$ are irreducible representations.

Since $\dim \rho_{G/P_n} = d^n$ and $\dim A_i$ is a power of d, the only possibility is that $\dim A_{n,i} = d^{n-1}$ for all $i \in \{1, \dots, d-1\}$, and

$$\rho_{G/P_n} = \rho_{G/P_0} \oplus A_{1,1} \oplus \dots \oplus A_{1,d-1} \oplus \dots \oplus A_{n,1} \oplus \dots \oplus A_{n,d-1}. \qquad \square$$

It may well be that for all *GGS* groups the Hecke algebra associated to a parabolic subgroup is commutative.

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