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COMPACTNESS IN THE FIRST BAIRE CLASS AND BAIRE-1 OPERATORS

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Abstract. For a polish space M and a Banach space E let $B_1(M,E)$ be the space of first Baire class functions from M to E, endowed with the pointwise weak topology. We study the compact subsets of $B_1(M,E)$ and show that the fundamental results proved by Rosenthal, Bourgain, Fremlin, Talagrand and Godefroy, in case $E=\mathbb{R}$, also hold true in the general case. For instance: a subset of $B_1(M,E)$ is compact iff it is sequentially (resp. countably) compact, the convex hull of a compact bounded subset of $B_1(M,E)$ is relatively compact, etc. We also show that our class includes Gulko compact.

In the second part of the paper we examine under which conditions a bounded linear operator $T : X^* \to Y$ so that $T|_{B_{X^*}} : (B_{X^*}, w^*) \to Y$ is a Baire-1 function, is a pointwise limit of a sequence (T_n) of operators with $T|_{B_{X^*}} : (B_{X^*}, w^*) \to (Y, \|\cdot\|)$ continuous for all $n \in \mathbb{N}$. Our results in this case are connected with classical results of Choquet, Odell and Rosenthal.

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Introduction. In this article we are concerned with first Baire class functions from a polish space M with values in a Banach space E. These functions $f : M \to E$ are defined as pointwise-norm (resp. pointwise-weak) limits of sequences of continuous functions $f_n : M \to (E, \|\cdot\|)$, $n \in \mathbb{N}$ and will be called Baire-1 (resp. weak-Baire-1) functions.

Our work is divided into two parts. In the first part we focus on pointwise-weak compact subsets of the space $B_1(M,E)$ of Baire-1 functions from M to E. We call these sets Rosenthal-Banach compact sets. Our interest (and the terminology) for these compact is justified from the fact that in case $E = \mathbb{R}$, the compact subsets of $B_1(M,\mathbb{R}) \equiv B_1(M)$ (called Rosenthal compact) have been studied extensively by Rosenthal $[R]_1, [R]_2$, Bourgain, Fremlin and Talagrand [2], Godefroy [7] and others. These authors have proved a number of important properties for Rosenthal compact sets: If $\Omega \subseteq B_1(M)$ is a relatively countably compact, then Ω is relatively compact and every point in $\overline{\Omega}$ is the limit of a sequence in Ω, the convex hull of a bounded compact subset of $B_1(M)$ is a relatively compact subset of $B_1(M)$, etc. We notice that the same properties has (another important class of compact sets) the class of weakly compact subsets of Banach spaces (that is, of Eberlein compact sets). We also notice that the class of Rosenthal compact is wide enough to include (Eberlein and more generally) Gulko compact of topological weight at most 2^ω (= the cardinality of continuum) (see [12], Theorem 3.5]).

We show in this article that Rosenthal-Banach compact is a natural common extension both of the classes of (classical) Rosenthal compact and of Gulko compact, without the restriction on the topological weight (obviously, each Rosenthal compact has weight at most 2^ω). So we prove that the fundamental results of Rosenthal, Bourgain, Fremlin, Talagrand and Godefroy, also hold true for the Rosenthal-Banach compact (Theorems 1.8, 1.9, 1.16 and 1.18). We also prove that every Gulko compact is a Rosenthal-Banach compact (Theorem 1.22). We close this section with some interesting open questions.

In the second part of our work, we study those bounded linear operators $T : X^* \to Y$, where X, Y are separable Banach spaces, so that their restriction $T|_{B_{X^*}}$ is a Baire-1 (resp. weak-Baire-1) function from (B_{X^*}, w^*) to $(Y, \|\cdot\|)$. We give the name Baire-1 (resp. weak-Baire-1) to these operators. The main question here is under which conditions on X, Y a Baire-1 (resp. weak-Baire-1) operator T is a pointwise-norm (resp. pointwise-weak) limit of a sequence of operators $T_n : X^* \to Y$, $n \in \mathbb{N}$, so that each $T_n|_{B_{X^*}}$ is a weak*-to norm continuous function. If an operator satisfies this property it will be called affine Baire-1 or affine weak-Baire-1. It should be mentioned here that this question was suggested by the following classical result of Choquet, if K is a compact convex metrizable subset of a locally convex space and $f : K \to \mathbb{R}$ is an affine Baire-1 function, then f is the pointwise limit of a sequence of continuous affine functions [3] and of a similar result of Odell and Rosenthal [17]. We give partial answers to the above question assuming the bounded approximation property (B.A.P.) for X^* or Y with Theorems 2.20 and 2.21. We also separate the classes of Baire-1 and weak-Baire-1 operators with two examples stated in Theorems 2.29 and 2.30.

We state here the characterization (due essentially to Baire) of Baire-1 functions from a complete metric space to a Banach space (see [4], Theorem 4.1 p. 18, [24] and [6]).

Theorem. *Let $f : M \to E$ be a function from the complete metric space M to the Banach space E. The following are equivalent:*

(*i*) *f is in the first Baire class;*

(*ii*) *$f|_C$ has a point of continuity for each closed subset C of M;*

(*iii*) *$f|_C$ has a point of continuity for each compact subset C of M.*

(*iv*) *$h \circ f$ is in the first Baire class for each continuous function $h : E \to \mathbb{R}$;*

(*v*) *$f^{-1}(C)$ is G_δ for each closed subset C of E.*

We also state (a version of) the well known result of Namioka on separate continuity [14].

Theorem. *Let X be a complete metric space, Y a compact Hausdorff space and $f : X \times Y \to \mathbb{R}$ a separately continuous function. Then there exists a dense G_δ subset D of X such that f is jointly continuous at each point of $D \times Y$.*

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1. Rosenthal-Banach compact sets. In this section we define and investigate the class of Rosenthal-Banach compact sets, which extends in a natural way the class of Rosenthal compact sets. In our case the Baire-1 functions have values in a Banach space. The basic theory still stands. Another important thing we also notice is that the classes of Eberlein, Talagrand and Gulko compact sets are included here, that is, in the class of Rosenthal-Banach compact sets.

We start with some basic definitions.

Definitions and conventions 1.0. *Let X be a topological space and E a Banach space.*

(*i*) *A function $f : X \to E$ is called **Baire-1** if there is a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n : X \to (E, \|\cdot\|)$ continuous for every $n \in \mathbb{N}$, such that if $t \in X$ then*

$$\lim_{n\to p} \|f_n(t) - f(t)\| = 0.$$

The set of these functions will be denoted by $B_1(X, E)$. If $E = \mathbb{R}$ we set, $B_1(M) = B_1(M, \mathbb{R})$.

(*ii*) *A function $f : X \to E$ is called **weak Baire-1** if there is a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n : X \to (E, \|\cdot\|)$ continuous for every $n \in \mathbb{N}$, such that if $t \in X$ then*

$$w - \lim_{n\to p} f_n(t) = f(t).$$

The set of these functions will be denoted by $B_{1,w}(X, E)$. It is clear that the sets $B_1(X, E)$, $B_{1,w}(X, E)$ are (with the usual pointwise operations) linear spaces over $\mathbb{R}$.

(*iii*) *We consider the space E^X of all functions from X to E endowed with the topology of pointwise-weak convergence. That is, a net $(f_i)_{i\in I}$ of functions from X to E converges pointwise-weak to the function f of the same space if for every $t \in X$, $w - \lim_{i\in I} f_i(t) = f(t)$.*

We notice that if $E = C(K)$ where K is a compact Hausdorff space, we also consider the space E^X with the pointwise to pointwise topology (pw to pw); this topology is defined in the same way as the pointwise to weak topology.

Definition 1.1. *A compact set K is called **Rosenthal-Banach compact**, if there is a polish space X and a Banach space E so that K is homeomorphic with a compact subset of $B_1(X, E)$ in the topology of pointwise-weak convergence defined above.*

We notice that:

(a) Every classical Rosenthal compact is Rosenthal-Banach compact (put $E=\mathbb{R}$).

(b) Every Eberlein compact is Rosenthal-Banach compact (take for M a one point set).

The next proposition (the proof of which is omitted because it's a part of the folklore) gives us the relation between Baire-1 and weak-Baire-1 functions defined above with appropriate classes of Baire-1 real valued functions (cf. Lemma 2.1 of [14]).

Let X, Y be Hausdorff topological spaces with Y a compact one. Then it is clear that there exists a one-to-one correspondence between the space of all functions $f : X \times Y \to \mathbb{R}$ so that $f|_{\{t\}\times Y}$ is relatively continuous for all $t \in X$ and the space of all functions $F : X \to C(Y)$, that is determined by the rule $f(t,y) = F(t)(y)$ for $t \in X$ and $y \in Y$. We identify these spaces via the above correspondence and notice that pointwise to pointwise topology on $C(Y)^X$ is the same with the pointwise topology on the space $\{f : X \times Y \to \mathbb{R}$ so that f is relatively continuous on $\{t\}\times Y$ for all $t \in X\}$. Sometimes we will use $\hat{f}$ instead of $F \in C(Y)^X$, where $\hat{f}(t,y) = F(t)(y)$, $t \in X$, $y \in Y$.

Proposition 1.2. (a). *Let $t \in X$; f is continuous at every point of $\{t\} \times Y$ if and only if the corresponding function $F : X \to (C(Y), \|\cdot\|)$ is continuous at t.*

(b). *There is a sequence of continuous functions $f_n : X \times Y \to \mathbb{R}$, $n \in \mathbb{N}$ so that for every $t \in X$ we have $\lim_{n\to\infty} \|f_n - f\|_{\{t\}\times Y} = 0$, if and only if there is a sequence of continuous functions $F_n : X \to (C(Y), \|\cdot\|)$, $n \in \mathbb{N}$ so that for every $t \in X$ we have $\lim_{n\to\infty} \|F_n(t) - F(t)\| = 0$ (that is, F is a Baire-1 function).*

Remarks 1.3. (i). From Proposition 1.2 (a) we see that $f : X \times Y \to \mathbb{R}$ is continuous if and only if the corresponding function $F : X \to (C(Y), \|\cdot\|)$ is continuous.

(ii) It is clear that every Baire-1 is a weak-Baire-1 function.

(iii) The class of bounded weak-Baire-1 functions $F : X \to C(Y)$ is in one-to-one correspodence with the class of bounded Baire-1 functions $f : X \times Y \to \mathbb{R}$ with the property: $\forall\ t \in X$, $f|_{\{t\}\times Y} : Y \to \mathbb{R}$ is (relatively) continuous. We simply set $f(t,y) = F(t)(y)$ for $t \in X$ and $y \in Y$. The same analogy exists between bounded Baire-1 functions $F : X \to C(X)$ and bounded Baire-1 functions $f : X \times Y \to \mathbb{R}$ with the property: There is a (uniformly bounded) sequence of continuous functions $f_n : X \times Y \to \mathbb{R}$ so that $\lim_{n\to\infty} \|f_n - f\|_{\{t\}\times Y} = 0$, for all $t \in X$ (cf. Proposition 1.2 (b)).

(iv) Analogous remarks hold for Baire-1 (resp. weak-Baire-1) functions $F : X \to E$ where E is a Banach space. We associate to F the function $f :$

$X \times (B_{E^*}, w^*) \to \mathbb{R}$ by $f(t, x^*) = x^*(F(t)),\ t \in X,\ \|x^*\| \leq 1$.
We notice here that the pointwise-weak topology for the space $B_1(X, E)$ (resp. $B_{1,w}(X, E)$) becomes the pw to pw topology for the space $B_1(X, C(B_{E^*}))$ (resp. $B_{1,w}(X, C(B_{E^*}))$), this is because the space (E, w) is naturally identified with a pointwise closed subspace of $C(B_{E^*})$ (see Example 1.4 (2)).

(v) By 1.3 (iv) above every Rosenthal-Banach compact K is homeomorphic to a compact subset of a space of the form $B_1(X, C(Y))$ endowed with pw-to-pw topology (if $K \subseteq B_1(X, E)$ take as $Y = (B_{E^*}, w^*)$). On the converse direction if K is a pointwise- to-pointwise compact set in $B_1(X, C(Y))$ (X polish Y compact) then K is a Rosenthal-Banach compact. Indeed, it is easy to see that K is homeomorphic to a (pointwise to pointwise) uniformly bounded subset Ω of $B_1(X, C(Y))$ ($\|f(t)\| \leq 1$ for every $t \in X$ and $f \in \Omega$), therefore by Grothendieck's theorem Ω is pointwise weak-compact.

(vi) It is easily proved that the function $f : X \times Y \to \mathbb{R}$ is separately continuous exactly when the corresponding function $F : X \to C(Y)$ is continuous when $C(Y)$ has the pointwise topology τ_p. If X is a metric space (Y compact) and F bounded, then f is separately continuous exactly when the corresponding function $F : X \to C(Y)$ is continuous when $C(Y)$ has the weak topology.

Examples of Baire-1 functions 1.4.

1. Let $f : M \times Y \to \mathbb{R}$ be a separately continuous function where M is a complete metric space and Y compact. It then follows from Namioka's theorem and Baire's characterization theorem (see the introduction) that the corresponding $F : M \to C(Y)$ is Baire-1. (If Y is in addition metrizable, then Baire's characterization theorem suffices, because then condition (v) of this theorem is easily satisfied).

2. Let M be a complete metric space, E a Banach space and $F : M \to (E, w)$ continuous. Then F is a Baire-1. Indeed, the operator $T : E \to C(B_{E^*})$ with $T(x) = x|_{(B_{E^*}, w^*)}$ is an isometry and an homeomorphic embedding when E has the weak topology and $C(B_{E^*})$ the pointwise one. It follows that the function $\hat{f} : M \times B_{E^*} \to \mathbb{R}$ with $\hat{f}(t, x^*) = x^*(F(t)),\ t \in M,\ \|x^*\| \leq 1$ is separately continuous and from example 4.1 above $F : M \to E \subseteq C(B_{E^*})$ is Baire-1.

3. Let M be a complete metric space, E an Asplund space and $f : M \to (E^*, w^*)$ continuous and bounded. Then f is Baire-1. We use here the fact that (B_{E^*}, w^*) is norm-fragmented ([4], Theorem 5.2, p. 26) and that in this case the identity $I : (A, w^*) \to (A, \| \cdot \|)$ has a point of continuity for every $A \subseteq B_{E^*}$ weak*-closed. ([15], Lemma 1.1). See also Theorem 4 of [23]. (A Banach space

E is said to be Asplund if every separable subspace of E has separable dual).

4. There is a bounded function $F : [0,1] \to c_0(\mathbb{N})$ that is weak- Baire-1 but not Baire-1. We put $x_n = 1/2^n$, $n \in \mathbb{N}$ and $K = \{x_n,\ n \in \mathbb{N}\} \cup \{0\}$ so that K is homeomorphic to the one point compactification $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ ($n \in \mathbb{N} \to x_n$, $\infty \to 0$) of the discrete set $\mathbb{N}$. Let (d_n) be a countable dense subset of $(0,1]$. The set $D = \{e_n \equiv (d_n, x_n) :\ n \in \mathbb{N}\}$ is a discrete (hence) F_σ and G_δ subset of $\mathbb{R}^2$.

Let $f : [0,1] \times K \to \mathbb{R}$ be the characteristic function $\mathcal{X}_D$ of D as a subset of the space $[0,1] \times K$. It is clear that f is Baire-1 with the further property that for every $t \in [0,1]$ $f|_{\{t\}\times K}$ is relatively continuous. We define now the function $F : [0,1] \to c_0(\mathbb{N})$ by $F(t)(m) = f(t, x_m)$ for $t \in [0,1]$ and $m \in \mathbb{N}$. Using Remark 1.3 (iii) we have that F is a weak-Baire-1 function. But F is not Baire-1 because for every open non empty subset $G \subseteq [0,1]$ we can find two points t, x of G so that $\|F(t) - F(x)\| = 1$ which means that F has no point of continuity in $[0,1]$. We notice that (given any one-to-one dense sequence (d_n) in $[0,1]$) the function $F : [0,1] \to c_0(\mathbb{N})$ could be defined directly by the rule $F(t)(n) = 0$ if $t \neq d_m$ for all $n \in \mathbb{N}$ and $F(t)(n) = e_n$ if $t = d_n$ for some $n \in \mathbb{N}$, where (e_n) is the usual unit vector basis of $c_0(\mathbb{N})$.

Proposition 1.5. *Let X and Y be Banach spaces with X polish. If $T : X \to Y$ is a bounded linear operator then the function $T|_{B_X} : (B_X, w) \to (Y, \|\cdot\|)$ is Baire-1.*

We recall that a Banach space X is called polish if (B_X, w) is a polish space. It is clear that every polish Banach space is separable and every separable reflexive Banach space is polish. On the other hand there exist examples of non-reflexive polish Banach spaces (see [5] Example 3.6 and [22]).

Proof. The operator T is weak-weak continuous so $T|_{B_X} : (B_X, w) \to (Y, w)$ is continuous and because (B_X, w) is polish the conclusion follows from Remark 1.3 (iv) and example 1.4 (2).

Definition 1.6. *Let X be a Hausdorff topological space.*

(*a*) *The space X is angelic if it is regular and (i) every relatively countably compact set is relatively compact, (ii) the closure of a relatively compact set is precicely the set of limits of its sequences.*

(*b*) *The space X has countable tightness, if for every $A \subseteq X$ and $x \in \overline{A}$ there exists a countable $B \subseteq A$ with $x \in \overline{B}$.*

(*c*) *The space X is said to be K-analytic (resp. countably determined) if it is a continuous image of a closed subset of a space of the form $M \times \Omega$, where M is polish (resp. M is separable metric) and Ω compact and Hausdorff.*

We note that every Banach space endowed with the weak topology has countable tightness (Kaplansky, for a proof see [22], Lemma 4.5) and it is an angelic space (Eberlein-Smulyan, see [19]). We also note that Banach spaces which are K-analytic or countably determined in their weak topology are studied in [25] and [12] (see also [16] and [4]).

The foundamental results of Rosenthal ([20] and [21]) and Bourgain, Fremlin and Talagrand ([2]) concerning the properties of compact sets in $B_1(M)$, where M is a polish space, can be summarized in the following,

Theorem 1.7. *The space $B_1(M)$ is angelic in its pointwise topology.*

The central result of this section is the following analogous of the above result.

Theorem 1.8. *The space $B_1(M,E)$ endowed with pointwise weak topology is angelic, for every Banach space E.*

By using some elementary properties of angelic spaces (see [19] and [2], 3A) Theorem 1.8 can be stated as follows:

Theorem 1.9. *Let M be a polish space, E a Banach space and $\Omega \subseteq B_1(M,E)$. The following are equivalent:*

1. *Ω is relatively compact in $B_1(M,E)$ in the pointwise weak topology.*
2. *Every countable infinite subset of Ω has a cluster point in $B_1(M,E)$ in the pointwise weak topology.*
3. *Every sequence of elements of Ω has a convergent subsequence in the pointwise-weak topology, in $B_1(M,E)$.*

Suppose that Ω satisfies one and hence all of these conditions. Then,

(a) every function in the closure of Ω is the limit of a sequence of elements of Ω.

We note that if $E=\mathbb{R}$ (so $B_1(M,\mathbb{R})=B_1(M)$) the above theorem has been shown by Rosenthal with (a) replaced by the weaker,

(b) every function in the closure of Ω is in the closure of a countable subset of Ω (see MAIN THEOREM in [20]). Bourgain, Fremlin and Talagrand relied on the work of Rosenthal and completed the proof that $B_1(M)$ is angelic by showing the stronger statement (a) (see [2], Theorem 3F and [21], Lemmas 3.11–3.15).

Our goal now is to prove Theorem 1.8 (or its equivalent Theorem 1.9). To achieve this goal we will make strong use of the work of the above authors. We start with:

Definition 1.10. *Let M be a metric space, E a Banach space and $f : M \to E$ a function. We say that f satisfies the Discontinuity Criterion (D.C.) provided there exists a non-empty subset $L \subseteq M$ and $\varepsilon > 0$ so that for every non-empty relatively open subset U of L, there are t_1, t_2 in U with $\|f(t_1) - f(t_2)\| > \varepsilon$. If there is such an L we say that f satisfies the D.C. on L.*

Remark 1.11.

(i) If $E = \mathbb{R}$, then our definition is equivalent to the well known definition of Discontinuity Criterion given in [17], (see also [20] and [21]).

(ii) It is easy to see that if M is a separable metric space and $f : M \to E$ satisfies the D.C. on $L \subseteq M$ then there exists a countable subset L' of L so that f satisfies the D.C. on L' (see [20], p. 368 and [21], p. 819).

The following result is proved as in case $E = \mathbb{R}$ (see [21, p. 819]).

Proposition 1.12. *Let M be a complete metric space, E a Banach space and $f : M \to E$ a function. The following are equivalent:*

(i) *f fails to belong to $B_1(M, E)$.*

(ii) *f satisfies the Discontinuity Criterion.*

Proof of Theorem 1.8. We deduce from Remarks 1.3 (iv) that it is enough to prove the theorem in case $E = C(K)$ for K compact and Hausdorff and for the pointwise-to-pointwise topology of the space $B_1(M, C(K))$ (that is, the pointwise topology if it is regarded as a subset of $\mathbb{R}^{M \times K}$) (see Definition 1.0 (iii)).

We begin with countable tightness.

Claim (I). *Suppose that $\Omega \subseteq B_1(M, C(K))$ is relatively compact. Let Ω' be the union of the closures (in $B_1(M, C(K))$ of all the countable subsets of Ω. Then Ω' is the closure of Ω (cf. [20], Lemma 4).*

Proof of Claim (I). We note first that if $A \subseteq \Omega'$ is countable, then its closure in $B_1(M, C(K))$ is included in Ω'. Let $f \in B_1(M, C(K))$ belong to the closure of Ω and $\delta > 0$. For each countable set $H \subseteq \Omega'$, write

$$D(H) = \{t \in M : \inf_{h \in H} \sup_{x \in K} |f(t, x) - h(t, x)| \geq \delta\}.$$

Since M has a countable base, there must exist a countable set $H_0 \subseteq \Omega$ such that $\overline{D(H_0)} = \overline{D(H_0 \cup \{h\})}$ for every $h \in \Omega'$. Suppose, if possible, that $D(H_0) \neq \emptyset$.

Let $D_0 \subseteq D(H_0)$ be a countable dense set. Then there is $h_0 \in \Omega'$ such that h_0 agrees with f on D_0. But now D_0 and $D(H_0 \cup \{h_0\})$ are both dense in $D(H_0)$, so $f - h_0$ satisfies the Discontinuity Criterion on $D(H_0)$ and cannot belong to $B_1(M, C(K))$ (see Proposition 1.12). This shows that $D(H_0) = \emptyset$. So we have proved that given $f \in B_1(M, C(K))$ belonging to the closure of Ω then for every $\delta > 0$ there exists a countable subset H of Ω so that

$$\inf_{h \in H} \sup_{x \in K} |f(t,x) - h(t,x)| < \delta, \quad \forall t \in M.$$

Now following the proof of the implication $2 \Rightarrow (a)$ of the MAIN THEOREM in [20] (and without essential change) we get the derived conclusion.

Claim (II). *If $A \subseteq B_1(M, C(K))$ is countable and relatively countably compact in $B_1(M, C(K))$ then the closure of A in $\mathbb{R}^{M \times K}$ is an angelic space.*

P r o o f o f C l a i m (II). The space $M \times K$ is K-analytic and of course $B_1(M, C(K)) \subseteq B_1(M \times K, \mathbb{R})$, which is a subset of the space $B(M \times K) =$ the space of Borel functions on $M \times K$. It then follows from Theorem 4D of [2] that the closure of A in $\mathbb{R}^{M \times K}$ is an angelic space.

Now let Ω be a relatively countably compact set in $B_1(M, C(K))$. For any countable set D, $C(K)^D$ can be identified with $C(D \times K)$, where D is given its discrete topology. Since $D \times K$ is σ-compact, $C(D \times K)$ is angelic in its pointwise topology (see [19], Theorem 2.5 and [2], 3A). Let f belong to the closure of Ω in $\mathbb{R}^{M \times K}$ and D a countable subset of M; then $\Omega' = \{h|_{D \times K} : h \in \Omega\}$ is relatively countably compact in $C(K)^D$ and f belongs to the closure of Ω' in $\mathbb{R}^{D \times K}$. But because $C(D \times K)$ is angelic, Ω' is a relatively compact set in $C(D \times K)$ and $f|_{D \times K}$ is the pointwise limit of a sequence in Ω'. So there is a sequence (h_n) in Ω such that $\lim_{n \to \infty} h_n(t,x) = f(t,x) \forall\ (t,x) \in D \times K$. Since (h_n) has a cluster point $h \in B_1(M, C(K))$, f agrees on $D \times K$ with h. As D is arbitrary we conclude that $f \in B_1(M, C(K))$ by the Discontinuity Criterion (see Remark 1.11 (ii)).

This shows that any relatively countably compact set in $B_1(M, C(K))$ is relatively compact. So if $f \in \overline{\Omega}$ then by Claim (I) there exists a countable subset A of Ω so that $f \in \overline{A}$. Now by Claim (II) f must be the limit of a sequence in A.

It follows immediately from the above that $B_1(M, C(K))$ is an angelic space.

Corollary 1.13. *The space $B_1(M, E)$ is angelic in the pointwise-norm topology.*

P r o o f. The property of being "angelic" is preserved by (taking subspaces and) finer regular topologies (see [19] and [2], 3A). So Theorem 1.8 implies that $B_1(M, E)$ is still angelic when given the pointwise-norm topology.

Remark 1.14. (i) The above result has already been obtained by Stegall ([24], Corollary 7).

(ii) A function f from the topological space M into the Banach space E will be called ***weak-weak Baire-1*** if there is a sequence of continuous functions $f_n : M \to (E, w)$, $n \in \mathbb{N}$, so that $\forall\ t \in M$, $w - \lim\limits_{n\to\infty} f_n(t) = f(t)$. The set of these functions will be denoted by $B^w_{1,w}(M, E)$. If M is polish, E has separable dual and $\Omega \subseteq B^w_{1,w}(M, E)$ is uniformly bounded then Theorem 1.9 still stands in the pointwise-weak topology. To see this assume (without loss of generality) that $\|f(t)\| \le 1$, $\forall\ t \in M$, $\forall\ f \in \Omega$; since the space (B_E, w) is metrizable and separable it is embedded in $(\ell^2(\mathbb{N}), \|\cdot\|)$. Now the result follows from Corollary 1.13, since Ω is embedded in $B_1(M, \ell^2(\mathbb{N}))$ endowed with pointwise-norm topology (cf. Proposition 2.9).

The following result asserts that Rosenthal-Banach and Rosenthal compact cannot be separated at the level of separable compact sets.

Proposition 1.15. *Let M be a polish space, E a Banach space and $\Omega \subseteq B_1(M, E)$ a pointwise-weak compact set. If either E or Ω is separable then Ω is a Rosenthal compact, i.e. there is a polish space P so that Ω is embedded in $B_1(P)$.*

Proof. Assume that E is separable. We define a linear one-to-one operator $\Psi : B_1(M, E) \to B_1(M \times B_{E^*})$ by $\Psi(f)(t, x^*) = x^*(f(t))$, $t \in M$, $\|x^*\| \le 1$ (see Remark 1.3 (iv)). It is easily seen that Ψ is pointwise-weak-to-pointwise-weak continuous; since E is separable, the space $M \times B_{E^*}$ is polish and thus Ω is homeomorphic to the Rosenthal compact set $\Psi(\Omega)$. Now let Ω be separable and let A be a countable dense subset of Ω. Since each $g \in B_1(M, E)$ has separable range, the set $U\{g(M) : g \in A\}$ is a separable subset of E; this allows us to accept E as a separable space and then to get the conclusion from the first case.

Now we are able to show that another deep result of Bourgain, Fremlin and Talagrand (see [2], Theorem 5E) also stands in the space $B_1(M, E)$.

Theorem 1.16 (Krein's theorem for the space $B_1(M, E)$)**.** *Let $\Omega \subseteq B_1(M, E)$ be a uniformly bounded and relatively compact set for the pointwise-weak topology. Then the convex hull of Ω is relatively compact in $B_1(M, E)$.*

Proof. It is enough by Theorem 1.9 to show that every countable infinite subset of the convex hull $\mathrm{conv}(\Omega)$ of Ω has a cluster point in $B_1(M, E)$. So assume without loss of generality that Ω itself is countable. But if Ω is countable then there exists a separable closed linear subspace F of E such that $\Omega \subseteq B_1(M, F) \subseteq B_1(M, E)$ (cf. the proof of Proposition 1.15). For the some reason as in proof of Theorem 1.8 we can assume that $F = C(K)$, where K is some compact (necessarily) metrizable space and that $B_1(M, C(K))$ has the pointwise-to-pointwise topology. So Ω is a uniformly bounded relatively compact set in $B_1(M \times K, \mathbb{R})$ in the pointwise topology, with the further property that for each $t \in M$ the set $\{f|_{\{t\}\times K} : f \in \Omega\}$ is a (uniformly bounded and) relatively compact subset of the space $C(\{t\} \times K)$. Since the space $M \times K$ is polish it follows from Theorem 5E of [2] that the convex hull $\mathrm{conv}(\Omega)$ of Ω is relatively compact in $B_1(M \times K, \mathbb{R})$ and from Krein's theorem (see [2], 5K (a)) that the set $\{g|_{\{t\}\times K} : g \in \mathrm{conv}(\Omega)\}$ is a relatively compact set in $C(\{t\} \times K)$ for every $t \in M$.

It clearly follows that $\mathrm{conv}(\Omega)$ is a relatively compact set in $B_1(M, C(K))$, so we are done.

Remark 1.17. Let M be an analytic Hausdorff space, that is, a continuous image of the Baire space of irrationals $\mathbb{N}^{\mathbb{N}}$. Then for every Banach space E, there exists a one-to-one linear oparator $T : B_1(M, E) \to B_1(\mathbb{N}^{\mathbb{N}}, E)$ which is continuous while $B_1(M, E)$ and $B_1(\mathbb{N}^{\mathbb{N}}, E)$ have the pointwise weak topology; if $\varphi : \mathbb{N}^{\mathbb{N}} \to M$ is any continuous surjection, we just put $T(f) = f \circ \varphi$, for every $f \in B_1(M, E)$. So every pointwise- weak compact set in $B_1(M, E)$ is homeomorphic to a pointwise-weak compact set in $B_1(\mathbb{N}^{\mathbb{N}}, E)$. We note in this connection the well known fact that every polish space is analytic.

As Godefroy has proved, if K is a Rosenthal compact then the space $P(K)$ (= the set of Radon probability measures on K with the weak* topology) is also a Rosenthal compact ([7]). The same property enjoys the class of Rosenthal-Banach compact (cf. the corresponding proof in [7]).

Theorem 1.18. *Let K be a Rosenthal-Banach compact. Then the space $P(K)$ of Radon probability measures on K with the weak* topology is a Rosenthal-Banach compact.*

Proof. It is clear that (for some M polish and Ω compact and Hausdorff) K can be considered as a pointwise compact and uniformly bounded set in the space of all (Baire-1) functions $f : M \times \Omega \to \mathbb{R}$, so that there exists a sequence of continuous functions $f_n : M \times \Omega \to \mathbb{R}$, $n \geq 1$, with $\lim_{n\to\infty} \|f_n - f\|_{\{t\}\times\Omega} = 0$ for all $t \in M$ (see Proposition 1.2 (b) and Remarks 1.3 (iii)–(v)). We denote by Y the disjoint union of $M^n \times \Omega^n$, $n \in \mathbb{N}$ and Y_1 the disjoint union of M^n, $n \in \mathbb{N}$, i.e.:

$$Y = \sum_{n=1}^{\infty} (M^n \times \Omega^n), \quad Y_1 = \sum_{n=1}^{\infty} M^n.$$

Then it is easy to see that the map $\Psi : Y_1 \times \Omega^N \to Y$ defined by,

$$\Psi\left((t_1, t_2, \ldots, t_n), (x_1, x_2, \ldots, x_n, \ldots)\right) = ((t_1, x_1), \ldots, (t_n, x_n))$$

is continuous and onto. We notice also that the space Y_1 is a polish space. We define now for every $(t, x) \in M \times \Omega$ the map

$$\pi_{(t,x)} : K \to \mathbb{R} \quad \text{with} \quad \pi_{(t,x)}(f) = f(t, x) \ \forall\ f \in K$$

and set

$$L = \{\pi_{(t,x)} : (t, x) \in M \times \Omega\}.$$

It is easy to see that $L \subseteq C(K)$ and L separates the points of K. ¿From the Stone-Weierstrass theorem we have that the set $A = \{1\} \cup \bigcup_{n=1}^{\infty} L^n$ where,

$$L^n = \{\ell_1 \cdot \ell_2 \ldots \ell_n : \ell_i \in L \text{ for } i = 1, 2, \ldots, n\}$$

is a total subset of $C(K)$ (its linear span is a norm dense subalgebra of $C(K)$).

From now on we will use the following easy claim:

Claim. *If K is a compact Hausdorff space and $A \subseteq C(K)$ is a total set, then $(P(K), w^*)$ is affinely homeomorphic with the closed convex hull of the set $\{\delta_t : t \in K\}$ in the locally convex space $\mathbb{R}^A$ through the correspondence $\mu \to (\mu(f))_{f \in A}$ $(: P(K) \to \mathbb{R}^A)$.*

Because, in our case, $1 \in A$ and $\mu(1) = 1$, $\forall\ \mu \in P(K)$, we have that the correspondence $\mu \to (\mu(f))_{f \in A \setminus \{1\}}$ $\left(: P(K) \to \mathbb{R}^{A \setminus \{1\}}\right)$ is an homeomorphism.

We can easily see that for every $f \in K$, the function $\widetilde{f} : Y_1 \times \Omega^{\mathbb{N}} \to \mathbb{R}$ defined by: $\widetilde{f}\left((t_1, \cdots, t_n), (x_1, \ldots, x_n, \ldots)\right) = f(t_1, x_1) \cdot f(t_2, x_2) \cdots f(t_n, x_n)$ is a Baire-1 function of the type that we need. It is enough to notice here that if $f \in K$ and (f_m) is a sequence of continuous functions on $M \times \Omega$ so that for all $t \in M$ $(f_m|_{\{t\} \times \Omega})$ converges uniformly (on $\{t\} \times \Omega$) to $f|_{\{t\} \times \Omega}$, then for all $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in M$ the sequence of functions $(f_m \times f_m \times \cdots \times f_m)$ $(-n$ times) $m \in \mathbb{N}$, converges uniformly on $(\{t_1\} \times \Omega) \times \cdots \times (\{t_n\} \times \Omega) \equiv \{t_1, \ldots, t_n\} \times \Omega^n$, to the function $(f \times f \times \cdots \times f)$ (n-times). (We notice here that the function $f \times f \times \cdots \times f$ (n-times) is defined in the obvious way). Since Ψ is continuous, the space $\bigcup_{n=1}^{\infty} L^n$ with the pointwise-topology is a continuous image of $Y_1 \times \Omega^{\mathbb{N}}$ via the map

$$((t_1, t_2, \ldots t_n), (x_1, \ldots, x_n, \ldots)) \in Y_1 \times \Omega^{\mathbb{N}} \to \pi_{(t_1,x_1)} \cdot \pi_{(t_2,x_2)} \cdot \ldots \cdot \pi_{(t_n,x_n)} \in \bigcup_{n=1}^{\infty} L^n.$$

So the last claim, Theorem 1.16 (K is uniformly bounded) and the above observations conclude the proof of the Theorem (see also Remark 1.17).

Now we shall prove another analogue of a result of Godefroy: every Radon probability measure on a Rosenthal compact has separable support ([7]).

Corollary 1.19. *If K is a Rosenthal-Banach compact then every Radon probability measure on K has separable support.*

Proof. Let $\mu \in P(K)$ and also let S be the support of μ. Then μ is in the weak*-closure of the set of finitely supported probability measures on S. Since by Theorems 1.8 and 1.18 the space $P(K)$ is angelic, there exists a sequence (μ_n) of convex combinations of Dirac measures supported on S so that $w^* - \lim \mu_n = \mu$. We set $F_n = \operatorname{supp} \mu_n$ for $n \in \mathbb{N}$. Then it is clear that $\bigcup_{n=1}^{\infty} F_n$ is dense in S.

Proposition 1.20. *If K is a Rosenthal-Banach compact then for every Radon probability measure μ on K the Banach space $L^1(K, \mu)$ is separable.*

Proof. This result was established by Bourgain if K is a (classical) Rosenthal compact (see [26] and [27]). The general case is an obvious consequence of this result combined with Corollary 1.19 and Proposition 1.15.

Note. Todorcevic has recently proved the deep result that every Rosenthal compact contains a dense metrizable subset (see [26], Theorem 1). As Todorcevic kindly informed us, the method of his proof can also be adapted in the wider class of Rosenthal-Banach compact to give the same result. Since the support of a Radon measure on a compact space has the ccc (countable chain condition) his result obviously implies Corollary 1.19.

From the definition of Rosenthal-Banach compact we have immediately that every Eberlein compact is such one. We will prove that the new class includes also Gulko (and so Talagrand) compact. We have to notice here that the reason to insert the concept of Rosenthal-Banach compact was the following conclusion from [12] (Theorem 3.5): Every Gulko compact with topological weight at most 2^ω (= the cardinality of the continuum) is (classical) Rosenthal compact. The proof we give here is analogous to that in [12].

We first recall the definition of spaces of the form $c_1(M \times \Gamma)$, where M is a separable metric space and Γ a non-empty set, due to Mercourakis (see [12] Definition 1.1 and [4], Definition 6.1, p. 248). A bounded function $f : M \times \Gamma \to \mathbb{R}$ is a member of $c_1(M \times \Gamma)$ iff for every compact non-empty subset K of M the function $f|_{K\times\Gamma}$ belongs to the space $c_0(K \times \Gamma)$. We notice that: (i) The space $c_1(M \times \Gamma)$ is a closed linear subspace of $\ell^\infty(M \times \Gamma)$ and so a Banach space and

(ii) every member f of $c_1(M \times \Gamma)$ has countable support (i.e. the set $\{(t,\gamma) : f(t,\gamma) \neq 0\}$ is at most countable, see, [12], Definition 1.1 (d) and [4] Remark 6.3, p. 249).

A compact space K is said to be Gulko (resp. Talagrand) compact if the space $C(K)$ is countably determined (resp. $\mathcal{K}$-analytic) in its pointwise topology (see [25], [12], [1], [13]).

Let $\Gamma^* = \Gamma \cup \{\infty\}$ be the one point compactification of the discrete set Γ and $\widehat{M}$ a completion of M (so $\widehat{M}$ is Polish).

Let $f \in c_1(M \times \Gamma)$. We define a **mapping** $\hat{f} : \widehat{M} \times \Gamma^* \to \mathbb{R}$ by

$$\hat{f}(t,\gamma) = \begin{cases} f(t,\gamma), & \text{if} \quad t \in M,\ \gamma \in \Gamma \\ 0 & \quad t \in \widehat{M},\ \gamma = +\infty \\ 0 & \quad t \in \widehat{M}\backslash M,\ \gamma \in \Gamma. \end{cases}$$

It is obvious that $\forall\, t \in \widehat{M},\ \hat{f}|_{\{t\}\times\Gamma^*} \in c_0(\{t\} \times \Gamma^*)$. We define now the mapping $\widehat{F} : \widehat{M} \to c_0(\Gamma)$ by

$$\widehat{F}(t)(\gamma) = \hat{f}(t,\gamma) \quad t \in \widehat{M}, \quad \gamma \in \Gamma.$$

Lemma 1.21. *If $f \in c_1(M \times \Gamma)$ then the map $\widehat{F} : \widehat{M} \to c_0(\Gamma)$ is Baire-1 (cf. Remarks* 1.3 *(iii) and (iv)).*

P r o o f. First of all we notice that (since f has countable support) we can take $\Gamma = \mathbb{N}$.

Claim 1. *For every $\varepsilon > 0$ the set $\sigma_\varepsilon(\widehat{F}) = \{t \in \widehat{M} : \|\widehat{F}(t)\| \geq \varepsilon\|$ is a discrete subset of $\widehat{M}$ (and so a countable G_δ subset of $\widehat{M}$).*

P r o o f o f C l a i m 1. It is obvious that $\sigma_\varepsilon(\widehat{F}) = \{t \in \widehat{M} : \|\widehat{F}(t)\| \geq \varepsilon\} = \{t \in M :\ \exists\, n \in \mathbb{N}$ with $|f(t,n)| \geq \varepsilon\}$. So the set $\sigma_\varepsilon(\widehat{F})$ is a closed and discrete subset of M and thus a discrete subset of the Polish space $\widehat{M}$.

Claim 2. *If $g \in c_0(\mathbb{N})$ with $\|g\| > \varepsilon > 0$ then the set $\widehat{F}^{-1}(S(g,\varepsilon))$ is a countable G_δ subset of $\widehat{M}$, where $S(g,\varepsilon)$ is the open ball of center g and radious ε in $c_0(\mathbb{N})$.*

P r o o f o f C l a i m 2. Let $\delta > 0$ so that $S(0,\delta) \cap S(g,\varepsilon) = \emptyset$. Then we have $\widehat{F}^{-1}(S(g,\varepsilon)) \subseteq \widehat{M}\backslash\widehat{F}^{-1}(S(0,\delta)) = \sigma_\delta(\widehat{F})$.

The conclusion comes now from Claim 1. Now let V be a non empty norm-open subset of $c_0(\mathbb{N})$. Then V is a countable union of open balls, let

$V = \bigcup_{n=0}^{\infty} S(g_n, \varepsilon_n)$ where $(g_n) \subseteq c_0(\mathbb{N})$ and $\varepsilon_n > 0$ for $n = 0, 1, 2, \ldots$.

We distinguish two cases:

a) $0 \notin V$. Then $0 \notin S(g_n, \varepsilon_n)\ \forall\ n \in \mathbb{N}$, and from Claim 2 we conclude that the set $\widehat{F}^{-1}(V)$ is an F_σ subset of $\widehat{M}$ (as a matter of fact it is a countable subset of $\widehat{M}$).

b) $0 \in V$. We can suppose without loss of generality that $g_0 = 0$ and $0 \notin S(g_n, \varepsilon_n)$ for all $n \in \mathbb{N}$. Then we have that $\widehat{F}^{-1}(S(0, \varepsilon_0)) = \widehat{M} \backslash \sigma_{\varepsilon_0}(\widehat{F})$ is an F_σ subset of $\widehat{M}$ and each of the sets $\widehat{F}^{-1}(S(g_n, \varepsilon_n))$ is a countable subset of $\widehat{M}$. It is obvious then that $\widehat{F}^{-1}(V) = \bigcup_{n=0}^{\infty} \widehat{F}^{-1}(S(g_n, \varepsilon_n))$ is an F_σ subset of $\widehat{M}$.

From both cases we conclude that for every V non empty norm-open subset of $c_0(\mathbb{N})$, $\widehat{F}^{-1}(V)$ is an F_σ subset of $\widehat{M}$ and so $\widehat{F}$ is a Baire-1 function (cf. Baire's characterization theorem in the introduction).

Theorem 1.22. *Every Gulko compact is a Rosenthal-Banach compact.*

Proof. The previous Lemma allows us to define a linear one-to-one operator

$$T : c_1(M \times \Gamma) \to B_1(M, c_0(\Gamma)), \text{ by } T(f) = \widehat{F}.$$

It is easy to see that T is continuous whenever $c_1(M \times \Gamma)$ has the pointwise and $B_1(M, c_0(\Gamma))$ the pointwise to pointwise topology. Since by a result in [12] (Theorem 3.1) a compact space K is Gulko compact iff it is homeomorphically embedded into some $c_1(M \times \Gamma)$, with the pointwise topology we have the conclusion (see also remark 1.3 (v)).

We also notice that if E is a Banach space so that there is a bounded, linear one-to-one operator $T : E \to c_0(\Gamma)$ for some set Γ, and M a Polish space, then there is a linear one-to-one operator $\Psi : B_1(M, E) \to B_1(M, c_0(\Gamma))$ which is pointwise-weak to pointwise weak continuous. Indeed, we just define for every $f : M \to E$ which is Baire-1, $\tilde{f} : M \to c_0(\Gamma)$ with $\tilde{f} = T \circ f$. The new function $\tilde{f}$ is Baire-1 and $\Psi : B_1(M, E) \to B_1(M, c_0(\Gamma))$ defined by $\Psi(f) = T \circ f$ is the desired operator. So we proved the following:

Theorem 1.23. *If E is a Banach space such that there is a bounded linear, one-to-one operator $T : E \to c_0(\Gamma)$ for the some set Γ, and Ω a pointwise-weak compact subset of $B_1(M, E)$ then Ω is affinely homeomorphic with some pointwise-weak compact subset of $B_1(M, c_0(\Gamma))$.*

Remark 1.24. We recall that a Banach space is weakly compactly generated (WCG) if it contains a weakly compact total subset. It is well known that if

E is WCG then there exists a bounded linear one to one operator $T : E \to c_0(\Gamma)$ for some set Γ (Amir-Lindenstrauss theorem, see [11], [16], [13] and [4] p. 246).

Corollary 1.25. *Let M be a Polish space and E a WCG or more generaly weakly Lindelöf determined (WLD) Banach space (see [1] Definition 1.1). If Ω is a pointwise-weak compact subset of $B_1(M, E)$. Then Ω is homeomorphic with some pointwise-weak compact subset of $B_1(M, c_0(\Gamma))$, for some set Γ.*

Proof. Since E is WLD there exists a bounded linear one to one operator $T : E \to c_0(\Gamma)$ for some set Γ (see [1] Theorem 13 and [13]). The conclusion comes now from Theorem 1.23.

It is well known that each of the classes of Eberlein and Rosenthal compact is closed under countable products. The same is true for the class of Rosenthal-Banach compact. The proof which we are going to give is analogous of that given in [11] (Proposition 3.3) for Eberlein compact.

Proposition 1.26. *Let (K_n) be a sequence of Rosenthal-Banach compact, then the set $K = \prod_{i=1}^{\infty} K_i$ is a Rosenthal-Banach compact.*

Proof. It follows from Remarks 1.17 and the fact that every Banach space is isometrically embedded into some $\ell^\infty(\Gamma)$ (= the Banach space of all bounded functions on Γ), that we can assume that each K_n is embedded as a pointwise-to-pointwise compact set in the space $B_1(\mathbb{N}^{\mathbb{N}}, C(\Omega))$, for some compact Hausdorff space Ω. (cf also Remarks 1.3 (iv) and (v)).

For every $n \in \mathbb{N}$, we define a map $\Phi_n : \prod_{i=1}^{\infty} K_i \to B_1\left(\left(\mathbb{N}^{\mathbb{N}}\right)^n, C(\Omega)\right)$ by putting $\Phi_n(f)(\sigma_1, \ldots, \sigma_n) = \prod_{i=1}^{n} f_i(\sigma_i)$, if $f = (f_1, \ldots, f_i, \ldots)$. Let $M = \sum_{n=1}^{\infty} \left(\mathbb{N}^{\mathbb{N}}\right)^n$ (= the disjoint union of $\left(\mathbb{N}^{\mathbb{N}}\right)^n$, $n \in \mathbb{N}$). Note that M is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$. The map $\Phi : \prod_{i=1}^{\infty} K_i \to B_1(M, C(\Omega))$ defined by the rule, $\Phi(f)(\sigma_1, \ldots, \sigma_n) = \Phi_n(f)(\sigma_1, \ldots, \sigma_n)$ is one-to-one continuous map (taking in $B_1(M, C(\Omega))$ the pointwise-to- pointwise topology). This finishes the proof.

Note. It follows in particular that if K_1 is an Eberlein and K_2 a Rosenthal compact, then $K_1 \times K_2$ is a Rosenthal-Banach and hence an angelic space (cf. Theorem 2.6 in [5].

Elementary considerations conclude the following result:

Proposition 1.27. *Let M be a polish space, E a Banach space and Ω a pointwise-weak compact subset of E^M. If every member f of Ω is continuous* *on M while E has the weak topology then Ω is a Rosenthal-Banach compact. Furthermore Ω is an Eberlein compact.*

P r o o f. By Example 1.4. (2) and Remark 1.3 (vi), every member of Ω is a Baire-1 function on M. As a result Ω is a pointwise-weak compact subset of $B_1(M, E)$ and thus a Rosenthal-Banach compact. Now let D be a dense countable subset of M. The restriction operator $T : B_1(M, E) \to B_1(D, E) : T(f) = f|_D$ is of course continuous for the pointwise-weak topology; since D is dense in M, T is also one-to-one on the subspace of $B_1(M, E)$ that consists of continuous functions $f : M \to (E, w)$. It clearly follows that Ω is homeomorphic to a pointwise weak compact subset of $B_1(D, E) \subseteq E^D$. Since D is countable and a countable product of Eberlein compact is again an Eberlein compact we get the conclusion. (see the remarks before Proposition 1.26).

Questions 1.28. **1)** Let X, Y be compact Hausdorff spaces and $\Omega \subseteq \mathbb{R}^{X\times Y}$ a pointwise compact set so that every $f \in \Omega$ is a (bounded) separately continuous function. Is then Ω an Eberlein compact?

2) Let Ω be a Rosenthal-Banach compact with topological weight at most 2^ω, is then Ω a (classical) Rosenthal compact?

Clearly such a set is embedded as pointwise-weak compact set into a space of the form $B_1(M, E)$ with M polish and $\dim E \leq 2^\omega$.

3) Can a Rosenthal-Banach compact be embedded into $B_1(M, E)$ where E is reflexive or at least WCG Banach space?

It is clear that if the answer is yes, then every Rosenthal-Banach compact embeds into $B_1(M, c_0(\Gamma))$, for some set Γ. (see Theorems 1.22, 1.23). Clearly it would be a strong extension (of the consequence) of the classical Amir-Lindenstrauss theorem: every weakly compact subset of a Banach space is homeomorphic to a weakly compact subset of some $c_0(\Gamma)$ (see Remark 1.24).

4) Let E be an Asplund Banach space (every separable subspace of E has separable dual), so that $(B_{E^{**}}, w^*)$ is a Rosenthal-Banach compact. Is then E^* weakly countably determined? Equivalently, is $(B_{E^{**}}, w^*)$ a Gulko compact?

2. Baire-1 Operators.

In this paragraph we study those bounded, linear operators $T : X^* \to Y$ where X, Y are (separable) Banach spaces so that the function $T|_{B_{X^*}} : (B_{X^*}, w^*) \to Y$ is Baire-1 or weak Baire-1. We call them Baire-1 or weak Baire-1 operators.

The leading question that concerns us is up to what point a Baire-1 (resp. weak Baire-1) operator $T : X^* \to Y$, is pointwise-norm (resp. pointwise-weak) limit of a sequence of linear operators $T_n : X^* \to Y$ so that the function

$T_n|_{B_{X^*}} : (B_{X^*}, w^*) \to (Y, \|\cdot\|)$ is continuous for all $n \in \mathbb{N}$. We answer partially this question with Theorems 2.12 and 2.20 supposing the B.A.P. (Bounded Aproximation Property) for Y or X^*. Also, with Examples 2.29 and 2.30 we separate the class of Baire-1 from the class of weak Baire-1 operators.

Definition 2.1. *Let X and Y Banach spaces and $T : X^* \to Y$ a bounded linear operator. The operator T is called* ***Baire-1*** *(resp.* ***weak Baire-1****) if $T|_{B_{X^*}} : (B_{X^*}, w^*) \to Y$ is Baire-1 (resp. weak Baire-1) function i.e., if there is a sequence of functions (f_n) so that:*

(*i*) $f_n : (B_{X^*}, w^*) \to (Y, \|\cdot\|)$ *is continuous for all $n \in \mathbb{N}$ and*

(*ii*) $\|\cdot\| - \lim\limits_{n\to\infty} f_n(x^*) = T(x^*)$ *(resp.* $w - \lim\limits_{n\to\infty} f_n(x^*) = T(x^*)$*) for all $x^* \in B_{X^*}$*

Remark 2.2. We can define in an analogous way Baire-1 (resp. weak-Baire-1) bounded linear operators $T : X \to Y$ like this: T is called Baire-1 operator if there is a sequence of continuous functions $f_n : (B_X, w) \to (Y, \|\cdot\|)$, $n \in \mathbb{N}$, so that $\|\cdot\| - \lim\limits_{n\to\infty} f_n(x) = T(x)$, (resp. $w - \lim f_n(x) = T(x)$) for all $x \in B_X$. However, if X is a dual Banach space ($X = Z^*$) we shall always mean that T is Baire-1 with respect of the weak* topology of $X = Z^*$.

Examples 2.3. (1) Let X be a separable Banach space. It is well known that X has a separable dual if and only if for every non-empty weak* compact subset of X^* the identity map $I : (K, w^*) \to (K, \|\cdot\|)$ has a point of continuity (see [4] Proposition 5.5, p. 28). So by Baire's theorem the identity map $I : (B_{X^*}, w^*) \to (B_{X^*}, \|\cdot\|)$ is a Baire-1 function which means that $I : X^* \to X^*$, the identity operator, is a Baire-1 operator.

(2) If $T : X \to Y$ is a bounded linear operator and X is a separable reflexive or a polish Banach space then from the Proposition 1.5 we have that T is a Baire-1 operator. (If X is polish then T is Baire-1 in the sense of Remark 2.2).

(3) Let X be a separable Banach space and $\{x_n, n \in \mathbb{N}\}$ be a total subset of X with $\|x_n\| = 1$ for all $n \in \mathbb{N}$. We set

$$T(x^*) = (x^*(x_n)/n)_{n\in\mathbb{N}} \text{ for every } x^* \in X^*.$$

We easily see that in this way a bounded linear operator ($\|T\| \leq 1$) $T : X^* \to c_0(\mathbb{N})$ is defined, which is 1-1 and weak* to weak continuous. It follows from 1.4 (2) that T is a Baire-1 operator (observe that (B_{X^*}, w^*) is a compact metric space).

We begin with a simple but useful consequence of Mazur's classical theorem.

Proposition 2.4. *Let M be a compact space, E a Banach space, and $F : M \to E$ a Baire-1 function. Let also (F_n) be a sequence of continuous functions $F_n : M \to (E, \|\cdot\|)$, $n \in \mathbb{N}$, so that*

(*i*) *For every $t \in M$, weak-$\lim_{n\to\infty} F_n(t) = F(t)$,*

(*ii*) *there is $C > 0$ with $\|F_n(t)\| \le C$, for all $t \in M$ and $n \in \mathbb{N}$.*

Then there is a sequence $(G_n)_{n\in\mathbb{N}}$ of convex combinations of the sequence (F_n) so that $\|\cdot\| - \lim_{n\to\infty} G_n(t) = F(t)$, for all $t \in M$.

Proof. Since F is a bounded Baire-1 function, there is a sequence of continuous functions $g_n : M \to (E, \|\cdot\|)$, $n \in \mathbb{N}$ so that for every $t \in M$, $\|\cdot\| - \lim_{n\to\infty} g_n(t) = F(t)$ and $\|g_n\| \le \|F\|$, for $n \in \mathbb{N}$. We set $f_n = F_n - g_n$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the function $\hat{f}_n$ is continuous on $M \times K$, $K = (B_{X^*}, w^*)$ (see Remarks 1.3 (i), (iv) and the comments before Proposition 1.2), so that

(i) $(\hat{f}_n)$ is uniformly bounded and

(ii) $\lim_{n\to\infty} \hat{f}_n(t, x^*) = 0$ for $t \in M$ and $\|x^*\| \le 1$.

From Mazur's Theorem there is a sequence (C_{f_n}) of convex combinations of (f_n) so that $(\widehat{C}_{f_n})$ converges uniformly to 0. We can write, $\widehat{C}_{f_n} = \widehat{C}_{F_n} - \widehat{C}_{g_n}$, where (C_{F_n}) and (C_{g_n}) are the sequences of convex combinations corresponding to the sequences (F_n) and (g_n). Then it is easy to verify that the sequence $G_n = C_{F_n}$, $n \in \mathbb{N}$, is the one we were asking for.

Note. If D is a subset of a linear space then by conv(D) we denote the convex hull of D.

Lemma 2.5. *Let $X = C(K)$, where K is a compact Hausdorff space and $A = \{f_{i,n} : i, n \in \mathbb{N}\} \subseteq X$. We suppose that $f : K \to \mathbb{R}$ and $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, are Baire-1 functions with:*

(*i*) *$\|f_{i,n}\| \le C < +\infty$ for all $i, n \in \mathbb{N}$,*

(*ii*) *$\tau_p - \lim_{i\to\infty} f_{i,n} = f_n$, for all $n \in \mathbb{N}$ and*

(*iii*) *$\tau_p - \lim_{n\to\infty} f_n = f$.*

Then $f \in \overline{A}^{w^*}$ *and so there is a sequence* (g_n) *of convex combinations of members of* A *so that* $w^* - \lim g_n = f$. *(By* $\tau_p - \lim_{n\to\infty} h_n = h$ *we mean that the sequence* (h_n) *converges pointwise on* K *to* h*).*

For the proof of this Lemma we need both of the following facts that can be found in [17].

Fact 2.6. Let K be a Hausdorff compact space and $X = C(K)$. The set of bounded Baire-1 functions on K is identified with $B_1(X)$ i.e. the set of Baire-1 elements of X^{**}.
(We recall that an element of the second dual X^{**} of a Banach space X is said to be a Baire-1 element if it is the weak* limit of a sequence of elements of X).

Fact 2.7. Let X be a Banach space and D a convex subset of X. If $f \in B_1(X)$ and $f \in \overline{D}^{w^*} \subseteq X^{**}$ then there is a sequence $(f_n) \subseteq D \subseteq X$ which converges in the weak* topology to f.

Proof of Lemma 2.5. From Fact 2.6 $A \cup \{f_n, n \in \mathbb{N}\} \cup \{f\} \subseteq B_1(X) \subseteq X^{**}$. From (i), (ii) and the theorem of dominated convergence of Lebesque we have that $w^* - \lim_i f_{i,n} = f_n$, so $f_n \in \overline{A}^{w^*}$ for all $n \in \mathbb{N}$. For the same reason $f \in \overline{A}^{w^*}$, $(w^* - \lim f_n = f)$.

The set $D = \text{conv}(A)$ is convex and $f \in \overline{D}^{w^*}$. So, from Fact 2.7 we have the conclusion.

Proposition 2.8. *Let* M *be a compact space,* Y *a Banach space and* $F_{n,m}$, F_n, F, $(m, n \in \mathbb{N})$, *functions from* M *to* Y *with the following properties*

1. $F_{n,m} : M \to (Y, \|\cdot\|)$ *is continuous for every* $n, m \in \mathbb{N}$.
2. *weak-* $\lim_{m\to\infty} F_{n,m}(t) = F_n(t)$, *for every* $t \in M$, $n \in \mathbb{N}$,
3. *weak-* $\lim_{n\to\infty} F_n(t) = F(t)$, *for all* $t \in M$,
4. F *is a weak-Baire-1 (resp. Baire-1) function and* $\|F_{n,m}\|$ $(= \sup\{\|F_{n,m}(t)\| : t \in M\|\}) \leq C < +\infty$, *for all* $m, n \in \mathbb{N}$.

Then there is $(G_n) \subseteq$ *conv*$(\{F_{n,m} : n, m \in \mathbb{N}\})$ *so that weak*$-\lim_n G_n(t) = F(t)$ *(resp. norm*$-\lim_n G_n(t) = F(t)$*) for every* $t \in M$.

Proof. Assume that F is a weak-Baire-1 function. We use here the previous Lemma 2.5 for $X = C(K)$ where $K = M \times (B_{Y^*}, w^*)$, through the correspondence of Remark 1.3. (iv). If F is furthermore Baire-1 then by Prop. 2.4 there exists a sequence (G'_n) of convex combinations of (G_n) satisfying our requirement.

Proposition 2.9. *Let (M,τ) be a compact metric space, Y a Banach space with a Schauder basis and separable dual, and $f : M \to Y$ a bounded function. Then the following are equivalent:*

1. *There is a uniformly bounded sequence of continuous functions $G_n : M \to (Y, \|\cdot\|)$, $n \in \mathbb{N}$ so that for every $t \in M$ weak-$\lim_{n\to\infty} G_n(t) = f(t)$ (i.e., f is weak-Baire-1 function).*

2. *There is a uniformly bounded sequence of continuous functions $f_n : M \to (Y, w)$, $n \in \mathbb{N}$ so that for every $t \in M$ weak-$\lim_n f_n(t) = f(t)$ (i.e. f is a weak-weak-Baire-1 function, see Remark* 1.14 *(ii)).*

3. *For every $F \subseteq M, F$ closed, the function $f|_F : F \to (Y, w)$ has a point of continuity (i.e. $f : M \to (f(M), w)$ is a Baire-1 function).*

Proof. $1 \Rightarrow 2$. It is obvious.

$2 \Rightarrow 1$. We can take $f(M) \subseteq B_Y$ (f is bounded). Because Y^* is separable, (B_Y, w) is metrizable and separable. For this reason it can be homeomorphically embedded in $(\ell_2(\mathbb{N}), \|\cdot\|)$ as a bounded subset, i.e. there is a homeomorphism $I : (B_Y, w) \to I(B_Y) \subseteq (\ell_2(\mathbb{N}), \|\cdot\|)$ (*).

Then $f'_n = I \circ f_n : (M, \tau) \to (\ell^2(\mathbb{N}), \|\cdot\|)$ is a continuous function for all $n \in \mathbb{N}$ and $\|\cdot\| - \lim f'_n(t) = f'(t)$, for all $t \in M$, where $f' = I \circ f$; so f' is a Baire-1 function. (1).

Let (P_n) be the sequence of projections associated to the basis in Y. We set $f_{n,m} = P_n \circ f_m$, for $n, m \in \mathbb{N}$. The set $\{f_{n,m}, n, m \in \mathbb{N}\}$ is uniformly bounded and every $f_{n,m}$ is τ-weak continuous and so τ-norm continuous, as a function with a finite dimensional range. For the same reason $f'_{n,m} = I \circ f_{n,m} : M \to \ell_2(\mathbb{N})$, will be τ-norm continuous for every $n, m \in \mathbb{N}$. (2).

We have

$$\text{weak} - \lim_{m\to\infty} f_{n,m}(t) = \text{weak} - \lim_{m\to\infty} P_n(f_m(t)) = P_n(f(t)), \text{ for } t \in M \text{ and } n \in \mathbb{N}$$

therefore, $\|\cdot\| - \lim_{m\to\infty} f_{n,m}(t) = P_n(f(t))$ for $t \in M$ and $n \in \mathbb{N}$. (3).

So for every $n \in \mathbb{N}$ $g_n \equiv P_n \circ f$ is a Baire-1 function and

$$\|\cdot\| - \lim_n g_n(t) = f(t), \text{ for } t \in M. \text{ (4)}$$

If we set $g'_n = I \circ g_n$ we conclude that:

(i) $f'_{n,m} : M \to (\ell_2(\mathbb{N}), \|\cdot\|)$ is continuous for all $m, n \in \mathbb{N}$ (from (2))

(ii) $\|\cdot\| - \lim_{m\to\infty} f'_{n,m}(t) = g'_n(t)$, for $t \in M$ and $n \in \mathbb{N}$ (from (3) and (*))

(iii) $\|\cdot\| - \lim_{n\to\infty} g'_n(t) = f'(t)$, for $t \in M$ (from (4) and (*))

(iv) $f' : M \to (\ell_2(\mathbb{N}), \|\cdot\|)$ is a Baire-1 function (from (1)).

Using Proposition 2.8, there is $(G_n) \subseteq \text{conv}(\{f_{n,m} : n, m \in \mathbb{N}\})$ so that if $G'_n = I \circ G_n$ $n \in \mathbb{N}$, then

$$\|\cdot\| - \lim_{n\to\infty} G'_n(t) = f'(t), \text{ for all } t \in M.$$

Consequently, weak-$\lim_{n\to\infty} G_n(t) = f(t)$, for $t \in M$ and G_n is a τ-norm continuous for every $n \in \mathbb{N}$.

$2 \Leftrightarrow 3$ Because (B_Y, w) is metrizable and (we can assume that) $f(M) \subseteq B_Y$ the conclusion comes from characterization theorem of Baire (see the introduction).

Corollary 2.10. *Under the circumstances of Proposition* 2.9. *we have that the subspace of bounded members of $B_{1,w}(M, Y)$ is equal to the subspace of bounded members of $B^w_{1,w}(M, Y)$ (see Definition* 1.0 *(ii) and Remark* 1.14 *(ii)).*

Remark 2.11. 1) We recall at this point the definition of the approximation property (A.P.) and bounded approximation property (B.A.P.) for a Banach E (see [10] pp. 30-38).

(i) We say that E has the A.P. if the identity operator $id : E \to E$ is the limit of a net of finite rank operators $T_i : E \to E$, $i \in I$ in the topology of uniform convergence on compact subsets of E.

(ii) We say that E has the B.A.P. if there exists $\lambda \geq 1$ and a net $(T_i)_{i\in I}$ of finite rank operators with $\|T_i\| \leq \lambda$ for $i \in I$, so that $\|\cdot\| - \lim_{i\in I} T_i(x) = x$ $(= id(x))$, for all $x \in E$.

2) Pelczynski and independently Johnson, Rosenthal and Zippin have proved the following characterization of separable Banach spaces with B.A.P.: A separable Banach space E has the B.A.P. iff E is isomorphic to a complemented subspace of a space with a basis. ([18], [8], [10], Theorem 1.e.13). From the method of the proof of this result in [8] (Cor. 4.12) we see that in the case E is a Banach space with B.A.P. and E^* separable, then there exists a Banach space X with basis and X^* separable, where E is embedded as a complemented subspace. So we conclude that Proposition 2.9 still stands if Y is a Banach space with B.A.P. and separable dual.

Let X, Y be Banach spaces and $T : X^* \to Y$ a bounded linear Baire-1 (weak-Baire-1) operator. If Y is finite dimensional, then from the theorem of Choquet, mentioned in the introduction or from an analogous result of Odell and Rosenthal (see [17], Sublemma p. 378), there exists a sequence of weak*-norm continuous linear operators $T_n : X^* \to Y$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \|T_n(x^*) - T(x^*)\| = 0$ for every $x^* \in X^*$. In fact if Y is of dimension one then we may have that $\|T_n\| \leq \|T\|$ for all $n \in \mathbb{N}$ (see [17], p. 378).

We will see now what happens if Y is infinite dimensional.

Theorem 2.12. *Let $T : X^* \to Y$ be a bounded linear Baire-1 (resp. weak-Baire-1) operator, where Y is a separable Banach space with the B.A.P. Then there exists a sequence of bounded linear operators $G_n : X^* \to Y$, $n \in \mathbb{N}$ such that:*

(*i*) $G_n|_{B_{X^*}} : (B_{X^*}, w^*) \to (Y, \|\cdot\|)$ *is continuous, for all $n \in \mathbb{N}$, (in fact, G_n is weak* to norm continuous and hence of finite rank).*

(*ii*) $\|\cdot\| \lim_{n\to\infty} G_n(x^*) = T(x^*)$ *(resp. weak–$\lim G_n(x^*) = T(x^*)$), for every $x^* \in X^*$.*

Definition 2.13. *We say that a bounded linear operator $T : X^* \to Y$ is an **affine Baire-1** (resp. **affine weak-Baire-1**) operator, if it satisfies the conclusion of Theorem* 2.12. *We define, in an analogous way, the concept of an affine Baire-1 (resp. affine weak-Baire-1) operator $T : X \to Y$.*

Proof of Theorem 2.12. We will prove first the case Y has a Schauder basis. Let (e_n) be such a basis for Y. We suppose without loss of generality that $\sum_{n=1}^{\infty} \|e_n\| \leq 1$. Let K be the basis constant, i.e. $K = \sup_n \|P_n\|$, where (P_n) is the sequence of projections associated to that basis.

We define,

$$T_n = P_n \circ T,\ P_0 = 0 \text{ and } R_n = P_n \circ T - P_{n-1} \circ T, \text{ for } n \in \mathbb{N},$$

which have the following properties:

(i) $\|T_n\| \leq K\|T\|$ and $\|R_n\| \leq 2K\|T\|$, for $n \in \mathbb{N}$,

(ii) each T_n and R_n is a Baire-1 operator.

We can consider each R_n as an operator from X^* to $(\mathbb{R}, \|\cdot\|)$ where $\|\cdot\|$ in $\mathbb{R}$ is the norm which comes from the norm in Y. From the remarks above there is a family of linear operators $R_{n,m} : X^* \to (\mathbb{R}, \|\cdot\|)$, $m, n \in \mathbb{N}$ with the following properties:

a) $R_{n,m} : (X^*, w^*) \to \mathbb{R}$ is continuous for all $m, n \in \mathbb{N}$,

b) $\lim_m R_{n,m}(x^*) = R_n(x^*)$, for every $x^* \in X^*$ and $n \in \mathbb{N}$,

c) $\|R_{n,m}\| \leq \|R_n\| \leq 2K\|T\|$, for all $n, m \in \mathbb{N}$.

We define now an operator $T_{n,m} : X^* \to Y$ by $T_{n,m}(x^*) = \sum_{k=1}^{n} R_{k,m}(x^*)e_k$ for $x^* \in X^*$ and $m, n \in \mathbb{N}$ and notice that $\|T_{n,m}\| \leq \sup_{\|x^*\|\leq 1} \sum_{k=1}^{n} \|R_{k,m}(x^*)\| \cdot \|e_k\| \leq 2K\|T\| \cdot \sum_{k=1}^{n} \|e_k\| \leq 2K\|T\|$, for every $m, n \in \mathbb{N}$.

Finally we get from the above the following:

1) $\|T_{n,m}\| \leq 2K\|T\|$ for $m, n \in \mathbb{N}$ and each $T_{n,m}$ is weak* to norm continuous.

2) pointwise-$\|\cdot\|$ $\lim_{m\to\infty} T_{n,m} = T_n$, for every $n \in \mathbb{N}$

3) pointwise-$\|\cdot\|$ $\lim_{n\to\infty} T_n = T$ and T is a Baire-1 (resp. weak-Baire-1) operator.

Using Proposition 2.8 we conclude that there exists a sequence (G_n) of convex combinations of $(T_{n,m})$ so that pointwise- $\|\cdot\|$ $\lim_{n\to\infty} G_n = T$ (resp. pointwise-weak $\lim_{n\to\infty} G_n = T$); therefore T is an affine Baire-1 (resp. weak Baire-1) operator.

For the case Y has the B.A.P. we use the characterization of separable Banach spaces having the B.A.P. stated in Remark 2.11. (2). Let Z be a Banach space with a basis and $I : Y \to Z$ an embedding of Y into Z so that $I(Z)$ is a complemented subspace of Z; also let $P : Z \to Z$ be a projection of Z onto $I(Y)$. Set $T' = I \circ T$, then $T' : X^* \to Z$ is a Baire-1 (resp. weak-Baire 1) operator and Z has a basis. By the first case of our theorem there exists a sequence of operators $G'_n : X^* \to Z$, $n \in \mathbb{N}$ satisfying requirements (i) and (ii) for T'. We set $G_n = I^{-1} \circ P \circ G'_n$ for $n \in \mathbb{N}$; then it is easily verified that (G_n) is the desired sequence of operators for T.

Remarks 2.14. 1) The operators (G_n) in Theorem 2.12 are in fact weak* to norm continuous and (hence) of finite rank. We also notice that $\|G_n\| \leq 2K\|T\|$ for $n \in \mathbb{N}$.

2) If X, Y are Banach spaces and $T : X^* \to Y$ a linear operator so that $T : (X^*, w^*) \to (Y, \|\cdot\|)$ is continuous then it is easily seen that T is of finite rank (especially T is compact). On the other hand every compact operator from

a reflexive space X to a Banach space Y is weak-to norm continuous whenever it is restricted on the closed unit ball of X. Such an operator T is not necessarily of finite rank, thus T is not weak-to norm continuous on the whole space X. For example take $T : \ell^2 \to \ell^2 : T((a_n)) = \left(\frac{a_n}{n}\right)$.

3) As an obvious consequence of Theorem 2.12 (or by a direct argument) the operator $T : X^* \to c_0(\mathbb{N})$ given in example 2.3 (3) is an affine Baire-1 operator.

Question 2.15. Does the previous Theorem still stand if Y (is separable and) has the A.P.?

It is well known that a separable Banach space has the B.A.P. if and only if there is a sequence of bounded linear operators $T_n : X \to X$, $n \in \mathbb{N}$, of finite rank so that

$$\lim_{n\to\infty} \|T_n(x) - x\| = 0, \text{ for all } x \in X$$

(see [18] or the proof of Theorem 1.e.13 in [10]).

In an analogous way we give the following:

Definition 2.16. *Let X be a (separable) Banach space*

*(i) We say that X has the **Baire-1 Approximation Property** (B-1. A.P) if there is a sequence of bounded linear operators $T_n : X \to X$, $n \in \mathbb{N}$ so that*

a) $T_n|_{B_X} : (B_X, w) \to (X, \|\cdot\|)$ is continuous, for all $n \in \mathbb{N}$,

b) $\lim_{n\to\infty} \|T_n(x) - x\| = 0$ for all $x \in X^*$.

(i.e. if the identity operator $I : X \to X$ is affine Baire-1, see Definition 2.13).

*(ii) We say that X^* has the w^*-**Baire-1 Approximation Property** (w^*-B-1 A.P.) if there is a sequence of bounded linear operators $T_n : X^* \to X^*$, $n \in \mathbb{N}$, so that*

a) $T_n|_{B_{X^*}} : (B_{X^*}, w^*) \to (X^*, \|\cdot\|)$ is continuous, for all $n \in \mathbb{N}$,

b) $\lim_{n\to\infty} \|T_n(x^*) - x^*\| = 0$ for all $x^* \in X^*$,

(i.e. if the identity operator $I : X^* \to X^*$ is affine Baire-1).

We continue now with some applications of Theorem 2.12.

Proposition 2.17. *Let X be a Banach space so that X^* is separable with A.P. Then X^* has the w^*-B-1 A.P.*

(In fact, there exists a sequence of weak to norm continuous linear operators $I_n : X^* \to X^*$ so that $\lim_{n\to\infty} \|I_n(x^*) - x^*\| = 0$ for every $x^* \in X^*$).*

Proof. We know that in this case $Y = X^*$ has the B.A.P. ([10], Theorem 1.e.15). From example 2.3 (1) the identity operator $I : X^* \to Y$ is a Baire-1 operator, so the conclusion comes from Theorem 2.12.

More generally we have the next,

Proposition 2.18. *Let X be a Banach space so that X^* is separable and has the A.P. (so it has the B.A.P.) and Y a Banach space. Then every bounded linear operator $T : X^* \to Y$ is an affine Baire-1 operator.*

Proof. From the previous proposition, the identity operator $I : X^* \to X^*$ has the w^*-B-1 A.P., therefore there is a sequence of operators $I_n : X^* \to X^*$, $n \in \mathbb{N}$ so that $I_n|_{B_{X^*}} : (B_{X^*}, w^*) \to (X^*, \|\cdot\|)$ is continuous for every $n \in \mathbb{N}$ and moreover $\lim_{n\to\infty} \|I_n(x^*) - x^*\| = 0$, for all $x^* \in X^*$. It follows immediately that the sequence $T \circ I_n : X^* \to Y$, $n \in \mathbb{N}$ makes T an affine Baire-1 operator.

Remarks 2.18.1. a) Let X be a separable Banach space not containing ℓ^1 and Y a finite dimensional Banach space. If $T : X^* \to Y$ is a bounded linear operator then T is affine Baire-1. This is an easy consequence of Odell-Rosenthal's characterization of separable Banach spaces containing ℓ^1 ([17]). (This result is almost obvious if we make the further assumption that X^* is separable, because then $(B_{X^{**}}, w^*)$ is metrizable). In any case we can see that there is a sequence of continuous linear operators $T_n : (X^*, w^*) \to Y$, $n \in \mathbb{N}$ so that,

(i) $\lim_{n\to\infty} \|T_n(x^*) - T(x^*)\| = 0$, for $x^* \in X^*$ and

(ii) $\|T_n\| \leq 2K_1K_2\|T\|$, with $K_1 = \sum_{i=1}^{n} \|e_i\|$, where $\{e_1, \ldots, e_n\}$ is a basis of Y and K_2 is the basis constant (cf. the proof of Theorem 2.12).

b) If $Y = \mathbb{R}^n$ with maximum-norm then it is easy to see that $\|T_n\| \leq \|T\|$ for all $n \in \mathbb{N}$.

We are going to prove a generalization of (the case X^* separable of) the above result with Y infinite dimensional.

Theorem 2.19. *Let X be a Banach space with separable dual X^* and Y a separable Banach space with the B.A.P. Then every bounded linear operator $T : X^* \to Y$ is affine Baire-1.*

P r o o f. As in Theorem 2.12 it is enough to prove the result assuming that Y has a basis, for which we have that $\sum_{n=1}^{\infty} \|e_n\| \leq 1$.

Let (P_n) be the sequence of projections associated to the basis, $K > 0$ be the basis constant and $\{x_n^* : n \in \mathbb{N}\}$ a norm dense sequence in X^*. Let $T_n = P_n \circ T$, $n \in \mathbb{N}$. From our assumption (X^* separable and $\sum_{n=1}^{\infty} \|e_n\| \leq 1$) and since each T_n is of finite rank there exists, according to Remark 2.18.1, for every $n \in \mathbb{N}$ a sequence of continuous linear operators $T_{n,m} : (X^*, w^*) \to P_n(Y) \subseteq Y$, $m \in \mathbb{N}$ so that:

a) $\lim_{m\to\infty} \|T_{n,m}(x^*) - T_n(x^*)\| = 0$, for every $x^* \in X^*$,

b) $\|T_{n,m}\| \leq 2K\|T_n\| \leq 2K^2\|T\|$, for every $m \in \mathbb{N}$.

Claim. *There is a sequence of operators (T'_k) in $\{T_{n,m} : n, m \in \mathbb{N}\}$ so that for every $n \in \mathbb{N}$, $w^* - \|\cdot\| \lim_{k\to\infty} T'_k(x_n^*) = T(x_n^*)$.*

P r o o f o f t h e C l a i m. We can suppose from the beginning (by passing to a subsequence of (T_n) if it is necessary) that for every $n \in \mathbb{N}$ and $i = 1, 2, \dots, n$

$$\|T_n(x_i^*) - T(x_i^*)\| < 1/2^n.$$

Since $(T_{1,m})_m$ converges pointwise-norm to T_1 there is $m_1 \in \mathbb{N}$ so that

$$\|T_{1,m}(x_1^*) - T_1(x_1^*)\| < 1/2 \text{ for every } m \geq m_1$$

Since for every $k \in \mathbb{N}$ the sequence $(T_{k,m})_m$ converges pointwise-norm to T_k, we can choose by induction a sequence of positive integers $m_1 < m_2 < \ldots < m_k < \ldots$ so that for every $k \in \mathbb{N}$ we have that,

$$\|T_{k,m}(x_i^*) - T_k(x_i^*)\| < 1/2^k \text{ for every } m \geq m_k \text{ and } i = 1, 2, \dots, k.$$

The sequence $(T_{k,m_k})_k$ has the desired property. For every $i \in \mathbb{N}$ and $\varepsilon > 0$, there is $n_0 > i$ so that $1/2^{n_0} < \varepsilon$. Then for every $k > n_0$ we have,

$$\begin{aligned} \|T_{k,m_k}(x_i^*) - T(x_i^*)\| &< \|T_{k,m_k}(x_i^*) - T_k(x_i^*)\| + \|T_k(x_i^*) - T(x_i^*)\| \\ &< 1/2^k + 1/2^k = 1/2^{k-1} \leq 1/2^{n_0} < \varepsilon. \end{aligned}$$

The proof of the claim is complete.

From the norm density of $\{x_n^* : n \in \mathbb{N}\}$ we have the pointwise-norm convergence on X^* of the sequence $T_k' = T_{k,m_k}$, $k \in \mathbb{N}$ to the operator T.

As an immediate consequence of Theorems 2.18 and 2.19 we get the following,

Theorem 2.20. *Let X, Y be separable Banach spaces with X^* separable and $T : X^* \to Y$ a bounded linear operator. If X^* or Y has the B.A.P. then T is affine-Baire-1.*

We recall that a Banach space X has the compact approximation property (C.A.P.) if the identity operator $I : X \to X$ is in the closure of the set of compact operators from X into itself with respect to the topology τ of uniform convergence on compact subsets of X (see [10] p. 94).

Proposition 2.21. *Let X be a Banach space with separable dual so that X^* has the w^*-B-1 A.P. Then X^* has the C.A.P.*

P r o o f. Let $T_n : X^* \to X^*$, $n \in \mathbb{N}$ be the sequence of operators which satisfies the requirements (a) and (b) of Definition 2.16 (ii). From (a) follows immediately that each T_n is compact; from (b) we get easily that (T_n) converges to the identity operator I in the τ-topology.

Example 2.22. There is a separable, reflexive Banach space X so that the identity operator $I : X \to X$ is (necessarily) Baire-1 and it is not an affine-Baire-1 operator: We set $X \equiv Y_P$ where Y_P is the subspace of ℓ^p, $p > 2$ which has not the C.A.P. (see [10] p. 94). In this case $I : X \to X$ does not have (by Proposition 2.21) the w^*-B-1 A.P.; equivalently, I is not an affine Baire-1 operator (cf. Definition 2.16).

Questions 2.23. Let X be a Banach space with a separable dual:

1. If X^* has the w^*-B-1 A.P. does it have the B.A.P.?

2. If X^* has the C.A.P. does it have the w^*-B-1 A.P.?

We notice here that as it follows from previous results, for X^* the following implications are true:

$$\text{B.A.P.} \Rightarrow w^* - \text{B-1 A.P.} \Rightarrow \text{C.A. P.}$$

From now on we are going to examine the place where the space of Baire-1 operators belong to, in comparison with the other spaces of operators we already know; for example the space of compact operators.

We will denote by $L(X^*, Y)$ the Banach space of bounded operators from X^* to Y with the usual operator norm, by $L_{i,w}(X^*, Y)$ the space of weak-B-1 operators, by $L_1(X^*, Y)$ the space of Baire-1 operators and by $K(X, Y)$ the space of compact operators from X to Y.

The following proposition has a routine proof. The main point for its proof is that a uniform limit of a sequence of bounded Baire-1 functions on a compact space is again Baire-1 (see [17]). So its proof is ommited.

Proposition 2.24. *Let X, Y be Banach spaces and $T_n : X^* \to Y$, $n \in \mathbb{N}$ a sequence of Baire-1 (resp. weak-Baire-1) operators so that $\lim_{n\to\infty} \|T_n - T\| = 0$, where $T : X^* \to Y$ is an operator. Then T is B-1 (resp. w-B-1) operator.*

Remark 2.25. From Proposition 2.24 we have that $L_1(X^*, Y) \subseteq L_{1,w}(X^*, Y)$ and both are closed linear subspaces of $L(X^*, Y)$. We shall see later that there are separable Banach spaces X and Y with Y having a basis such that, $L_1(X^*, Y) \subsetneq L_{1,w}(X^*, Y)$, i.e. the inclusion is strict (see Theorems 2.29 and 2.30). As we have already seen, in some instances we have equality (see Theorem 2.20 and Example 2.3 (2)).

Corollary 2.26. *Let X be a Banach space, Y a Banach space with A.P. and $T : X^* \to Y$ a compact operator. Then T is a weak-Baire-1 operator if and only if T is a Baire-1 operator.*

Proof. Since Y has the A.P. and T is compact we can find a sequence of finite rank operators $T'_n : Y \to Y$, $n \geq 1$ so that, $\|T'_n(T(x^*)) - T(x^*)\| \leq \frac{1}{n}$ for every $\|x^*\| \leq 1$ and $n \geq 1$ (see Theorem 1.e. 14, p. 32 in [10]). Set $T_n = T'_n \circ T$, for $n \in \mathbb{N}$, then clearly $\lim_{h\to\infty} \|T_n - T\| = 0$. The operator T being weak Baire-1, we have that T_n is weak-Baire-1 for every $n \in \mathbb{N}$. As an operator of finite rank T_n is also a Baire-1 operator. What we need now is Proposition 2.24.

The following result should be compared with Theorems 2.20 and 2.30.

Proposition 2.27. *Let X be a separable Banach space not containing ℓ^1 and Y a Banach space with A.P. Then every compact operator from X^* to Y is an affine Baire-1 operator.*

Proof. Since Y has the A.P. and T is compact there exists a sequence of finite rank operators $T_n : X^* \to Y$, $n \in \mathbb{N}$ so that $\|T_n - T\| \to 0$, for $n \to \infty$ (see the proof of Prop. 2.26). From Odel-Rosenthal's theorem ([17]) each T_n is Baire-1 so the conclusion comes from Proposition 2.24 (cf. Remark 2.18.1).

Remark 2.28. The identity operator $I : c_0^* \to \ell^1$ is B-1 but not a compact operator. Therefore, we have the following relation: $K(X^*, Y) \nsubseteq L_1(X^*, Y)$.

We close this section with two examples of operators $T : X^* \to c_0(\mathbb{N})$, that separate the classes of weak-Baire-1 and Baire-1 operators. In the first example

$X = C[0,1]$; in the second X is separable, X^* non-separable but X does not contain ℓ^1. Notice that if X^* is separable then every bounded linear operator $T : X^* \to c_0(\mathbb{N})$ is affine Baire-1, according to Theorem 2.20. We also mention here that both examples are constructed using an example of a bounded function defined on a compact metric space and with values in $c_0(\mathbb{N})$ which is weak-Baire-1 but not Baire-1 (see 1.4 (4)).

Theorem 2.29. *There is a bounded linear operator $T : X^* \to c_0(\mathbb{N})$, where $X = C[0,1]$, which is affine weak-Baire-1, but not Baire-1.*

Proof. Let $f : [0,1] \to c_0(\mathbb{N})$ be a weak-Baire-1 but not Baire-1 function with $\|f(t)\| \leq 1$, for all $t \in [0,1]$ (see, 1.4 (4)). If $k \in \mathbb{N}$ then the function $\varphi_k : [0,1] \to \mathbb{R}$, $t \to f(t)(k)$ is bounded and Baire-1 and so the integral $\int_0^1 \varphi_k d\mu$ exists for every positive regular Borel measure μ on $[0,1]$. Because $\lim\limits_{k\to\infty} \varphi_k(t) = 0$ for all $t \in [0,1]$ and $|\varphi_k(t)| = |f(t)(k)| \leq 1$ for all $t \in [0,1]$ and $k \in \mathbb{N}$, using the theorem of dominated convergence of Lebesgue we have, $\lim\limits_{k\to\infty} \int_0^1 \varphi_k d\mu = 0$ for every positive regular Borel measure μ. From Riesz's representation theorem we know that the dual of the Banach space $C[0,1]$ is identified with the space $M[0,1]$ of the regular Borel measures on $[0,1]$. We also know that for every $\mu \in M[0,1]$ there are two positive measures μ^+, μ^- so that $\mu = \mu^+ - \mu^-$. For these reasons the operator $T : M[0,1] \to c_0(\mathbb{N})$ defined with,

$$T(\mu) = \left(\int_0^1 f(t)(k) d\mu(t) \right)_{k\in\mathbb{N}}$$

is well defined, linear and bounded.

We notice that if $t \in [0,1]$ and δ_t is the corresponding Dirac measure, then $T(\delta_t) = f(t)$. Because the set $\{\delta_t : t \in [0,1]\} \subseteq B_{X^*}$ with the weak* topology is homeomorphic to $[0,1]$, the function $T|_{(B_{X^*}}, w^*)$ is just an extension of f on the compact space (B_{X^*}, w^*), which means that $T|_{(B_{X^*}}, w^*)$ can not be Baire-1, so T is not a Baire-1 operator.

Let now $f_n : [0,1] \to (c_0(\mathbb{N}), \|\cdot\|)$, $n \in \mathbb{N}$, be a sequence of continuous functions with $\|f_n(t)\| \leq 1$, for $n \in \mathbb{N}$ and $t \in [0,1]$ such that weak-$\lim\limits_{n\to\infty} f_n(t)$, for every $t \in [0,1]$. Thus $\lim\limits_{n\to\infty} f_n(t)(k) = f(t)(k)$, for all $t \in [0,1]$ and $k \in \mathbb{N}$. We define a sequence of operators $T_n : X^* \to c_0(\mathbb{N})$, $n \in \mathbb{N}$, by the rule

$$T_n(\mu) = \left(\int_0^1 f_n(t)(k) d\mu(t) \right)_{k\in\mathbb{N}}.$$

It is easy to see that (T_n) is uniformly bounded.

Claim 1. *For every $m \in \mathbb{N}$ the function $T_m|_{B_{X^*}} : B_{X^*} \to c_0(\mathbb{N})$ is weak*-weak continuous.*

Proof. It follows easily from the definition of the operators T_m, $m \in \mathbb{N}$ and the fact that the functions $f_m(\cdot)(k) : [0,1] \to \mathbb{R}$ are continuous for all $m, k \in \mathbb{N}$.

Claim 2. *For every $\mu \in B_{X^*}$ we have, weak-$\lim_{n\to\infty} T_n(\mu) = T(\mu)$.*

Proof. Let $\mu \in B_{X^*}$, $\mu \geq 0$. We have that, $\lim_{n\to\infty} f_n(t)(k) = f(t)(k)$ for all $t \in [0,1]$ and $k \in \mathbb{N}$. So from Lebesgue's dominated convergence theorem we get that, $\lim_{n\to\infty} T_n(\mu)(k) = \lim_{n\to\infty} \mu(f_n(\cdot)(k)) = \mu(f(\cdot)(k) = T(\mu)(k)$ for all $k \in \mathbb{N}$. Since the weak and pointwise topology on bounded subsets of $c_0(\mathbb{N})$ coincide, the proof of the claim is complete.

The conclusion of the theorem comes now from the previous claims and (the method of the proof of the implication (2) $\Rightarrow$ (1) of) Proposition 2.9.

Theorem 2.30. *There exist a separable Banach space X with non-separable dual, not containing ℓ^1 and a bounded linear operator $T : X^* \to c_0(\mathbb{N})$, which is affine weak-Baire-1 but not Baire-1.*

Proof. The space X will be the James tree space JT. We shall follow (the definition and) notation for JT given in [22]. So, let D be the set of finite zero-one sequences (including the empty sequence). For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_\ell)$ members of D, define $\alpha < \beta$ iff $k < \ell$ and $\alpha_i = \beta_i$ for all i, $1 \leq i \leq k$. JT may then be defined as the space of all families of real numbers $x = (x_\alpha)_{\alpha\in D}$ so that

$$\|x\|_{JT} = \sup \left(\sum_{i=1}^{n} |\sum_{\alpha\in S_i} x_\alpha|^2 \right)^{\frac{1}{2}} < +\infty,$$

where the supremum is taken over all $k \in \mathbb{N}$ and k-tuples $s_1, \ldots, s_n$ of finite disjoint segments, If s is any (non-empty) segment of D and $x \in JT$ then $s^*(x) = \sum_{\alpha\in s} x_\alpha$ exists; it is also easy to see that s^* is a linear functional on JT with $\|s^*\| = 1$. If s has only one element, say $s = \{\alpha\}$, then we set $s = e_\alpha$; clearly each $e_\alpha \in JT$.

We denote by B the set of branches of the tree D (i.e., the set of all maximal segments of D). It is a well known fact that JT^*, the dual of JT, is equal to the closed linear span of the set $\{b^* : b \in B\} \cup \{e^*_\alpha : \alpha \in D\}$ (see [22]).

Set $S = B \cup \{e_\alpha : \alpha \in D\}$.

Claim 1. *The set $S^* = \{s^* : s \in S\}$ is linear independent.*

Proof. Let $s_1,\dots,s_n$ be distinct members of S. Assume without loss of generality that $s_1,\dots,s_k$ $(1\le k\le n)$ are the only members of B between of $s_1,\dots,s_n$. Let us assume that $\sum\limits_{i=1}^{n}\lambda_i s_i^*=0$; then there exists some $\alpha\in s_1\setminus\bigcup\limits_{i=2}^{n}s_i$. Then we have, $\sum\limits_{i=1}^{n}\lambda_i s_i^*(e_\alpha)=\lambda_1 s_1^*(e_\alpha)=\lambda_1 e_\alpha(\alpha)=\lambda_1=0$. In the same way we get that $\lambda_1=\lambda_2=\ldots=\lambda_k=0$. So, $\sum\limits_{i=k+1}^{n}\lambda_i s_i^*=0$ and $s_i=e_{\alpha_i}$, $k+1\le i\le n$. It clearly follows that $\lambda_{k+1}=\ldots=\lambda_n=0$.

Claim 2. *Let $x^*=\sum\limits_{i=1}^{n}\lambda_i b_i^*+\sum\limits_{j=1}^{m}\mu_j e_{\alpha j}^*$, where $b_1,\dots,b_n$ be members of B and $\alpha_1,\dots,\alpha_m$ members of D. Then we have $|\lambda_i|\le\|x^*\|$ for all $i=1,2,\dots,n$.*

Proof. Let i_0, $1\le i_0\le n$. There exists, $\alpha\in b_{i_0}\setminus\bigcup\left(\bigcup\limits_{i=1}^{n}b_i\cup\{\alpha_1,\dots,\alpha_m\}\right)$. Therefore, $|\lambda_{i_0}|=|x^*(e_\alpha)|\le\|x^*\|$.

Let's turn back to our example. We notice that the space $B^*=\{s^*:s\in B\}$ is a weak* compact subset of B_{X^*} and (as it is easily proved) homeomorphic to the Cantor set. Let $\{d_n:n\in\mathbb{N}\}$ be a one-to-one dense sequence in (B^*,w^*). We define a map $F:S^*\to c_0(\mathbb{N})$ as follows:

$$F(s^*)=\begin{cases} e_n, & \text{if } s=d_n \text{ for some } n\in\mathbb{N}\\ 0, & \text{for } s\in S\setminus\{d_n:n\in\mathbb{N}\}.\end{cases}$$

Then it is proved as in Example 1.4.4 that F is weak Baire-1 but not Baire-1 on S^* (notice that the set S^* together with zero, is weak* compact). The function F is naturally extended (by using Claim 1)) to a linear operator T from the linear span $[S^*]$ of S^* to $c_0(\mathbb{N})$. If we prove that T is bounded on the normed space $[S^*]$ then we can extend it (uniquely) to bounded operator $\widehat{T}$ defined on the closed linear span $\overline{[S^*]}$ which is the space JT^*.

So let $x^*=\sum\limits_{i=1}^{n}\lambda_i b_i^*+\sum\limits_{j=1}^{m}\mu_j e_{\alpha j}^*$ be a member of $[S^*]$. Set $I=\{i\le n:b_i=d_{k_i}$ for some $k_i\in\mathbb{N}\}$; then we have (by using Claim 2)) that $T(x^*)=\sum\limits_{i\in I}\lambda_i e_{k_i}$, hence $\|T(x^*)\|\le\max\{|\lambda_1|,\dots,|\lambda_n|\}\le\|x\|$. Therefore ($\|T\|\le 1$ and) $\|\widehat{T}\|\le 1$.

The operator $\widehat{T}:JT^*\to c_0(\mathbb{N})$ is not Baire-1, because $\widehat{T}|_{S^*}=F$ is not Baire-1. But as we shall prove, it is an affine weak Baire-1 operator. So, let $\widehat{T}_n=P_n\circ\widehat{T}$, where (P_n) is the sequence of projections associated to the usual basis of $c_0(\mathbb{N})$. Then we have:

(i) $\|\widehat{T}_n\|\le 1$ for all $n\in\mathbb{N}$,

(ii) each $\widehat{T}_n$ is of finite rank and (since X does not contain ℓ^1) a Baire-1 operator (see Remark 2.18.1 (a)).

(iii) $\| \cdot \| - \lim \widehat{T}_n(x^*) = \widehat{T}(x^*)$, for all $x^* \in X^*$.

It clearly follows from (ii) and remark 2.18.1 (b) that for every $n \in \mathbb{N}$ there exists a sequence of weak*-norm continuous linear operators $T_m^n : X^* \to c_0(\mathbb{N})$, $m \in \mathbb{N}$, so that $\| \cdot \| - \lim\limits_{m\to\infty} T_m^n(x^*) = \widehat{T}_n(x^*)$ for every $x^* \in X^*$ and $\|T_m^n\| \leq 1$ for all $m \in \mathbb{N}$.

By Proposition 2.8 there exists a sequence $(G_n) \subseteq \mathrm{conv}(\{T_m^n : m, n \in \mathbb{N}\})$ such that weak- $\lim\limits_{n\to\infty} G_n(s^*) = \widehat{T}(s^*) = F(s^*)$ for every $s \in S$ (F is weak Baire-1 on S^*). Therefore,

$$\text{weak-}\lim_{n\to\infty} G_n(x^*) = \widehat{T}(x^*) \text{ for all } x^* \in [S^*].$$

The conclusion of the theorem now follows from the following easily proved,

Lemma 2.31. *Let X, Y be Banach spaces, $T, T_n : X \to Y$, $n \in \mathbb{N}$, bounded linear operators with $\|T_n\| \leq M < +\infty$ for all $n \in \mathbb{N}$. Assume that weak-$\lim\limits_{n\to\infty} T_n(x) = T(x)$ for every element of a norn-dense subset of X. Then weak-$\lim\limits_{n\to\infty} T_n(x) = T(x)$, for all $x \in X$.*

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