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FURTHER GENERALIZATION OF KOBAYASHI'S GAMMA FUNCTION

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Abstract. In this paper, we introduce a further generalization of the gamma function involving Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ by means of:

$$(1)\qquad D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = v^{-a}\int_0^\infty t^{u-1}\left(1-\frac{t}{v}\right)^{\delta-1} e^{-pt}{}_2F_1\left(a,b;c;-\frac{t}{v}\right)dt$$

where $Re\ u > 0, Re\ p > 0, |\arg v| < \pi$. This reduces to Kobayashi's [7] generalized gamma function when $\delta = 1$, $p = 1$ and $b = c$. Also, it reduces to a function defined by Al-Musallam and Kalla [2, 3] when $\delta = 1$. The generalized incomplete and the complementary incomplete functions associated with $D\left(\begin{smallmatrix} a,b,c,p \\ u,v,\delta \end{smallmatrix}\right)$ are also introduced. For these functions we obtain some properties and recurrence relations satisfied by them and we establish asymptotic series expansions for each of them.

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1. Introduction. Special functions in advanced mathematics are frequently defined by means of improper integrals.An important example is the gamma function. The gamma function $\Gamma(u)$ for a complex variable u is defined by

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t} dt, \qquad Re\ u > 0 \tag{1.1}$$

This function is one of the simplest but very important special functions occuring in many branches of mathematical physics and a knowledge of this function is required to study other special functions [4, 5, 9]. Kobayashi [7, 8] has considered and studied the function:

$$\Gamma_m(u, v) = \int_0^\infty \frac{t^{u-1} e^{-t}}{(t+v)^m} dt \tag{1.2}$$

where m is a positive integer, $Re\ u > 0, |v| > 0$ and $|\arg v| < \pi$; which he called it a generalized gamma function. The function, defined by (1.2) is essentially a confluent hypergeometric function of the second kind [4, 5, 9].

This function is of great importance in the wave scattering and diffraction theory as related to the Wiener-Hopf technique [8], since the multiple edge diffraction process can be described explicity in terms of this special function.

Although the Wiener-Hopf technique [10] is a powerful tool for studying wave scattering and diffraction problems related to strips and slits [8], it is however restricted to the case where obstacles have a semi-infinite boundary. Some geometries (obstacles) with finite boundaries can still be formally treated by this technique to obtain some form of exact solutions. Applying asymptotic to the formal solution, one obtains explicit approximate solutions.

Anticipating that some diffraction problems with obstacles of different geometries (other than slits and strips), one may require the study of other forms of generalized gamma functions, Al-Musallam and Kalla [1] introduced a furthest generalization of the gamma function in the following form:

$$D\begin{pmatrix} a, b, c, p \\ u, v \end{pmatrix} = v^{-a} \int_0^\infty t^{u-1} e^{-pt} {}_2F_1\left(a, b; c; -\frac{t}{v}\right) dt \tag{1.3}$$

$Re\ u > 0, |\arg v| < \pi$ where a, b and c are complex parameters with $c \neq$

$0, -1, -2, \ldots,$ $Re\ p > 0$, and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function [6]. These functions have been used to define and study several gamma-type distributions [1, 6]. In this paper, we consider another generalization of the Kobayashi's function (1.2) in the following form:

$$D\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix} = v^{-a} \int_0^\infty t^{u-1} \left(1 - \frac{t}{v}\right)^{\delta-1} e^{-pt} {}_2F_1\left(a, b; c; -\frac{t}{v}\right) dt \tag{1.4}$$

where $Re\ u > 0, Re\ p > 0, |\arg v| < \pi, b, a$ and c are complex parameters with $c \neq 0, -1, -2, \ldots$ and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function.

We observe that for $\delta = 1$, (1.4) reduces to the function (1.3) defined and studied in [1, 2].

The rest of the paper is organized as follows. In section 2 we describe the generalized incomplete and complementary incomplete functions of $D\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix}$. In section 3 we give some properties including recurrence formulas associated with these functions. Section 4 contains asymptotic series expansions associated with the functions. Section 5 contains a summary and a brief discussion.

2. The generalized incomplete and complementary incomplete functions of $D\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix}$.

In relation with $D\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix}$ we introduce, for $w > 0$, the generalized incomplete gamma function defined by

$$D_0^w\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix} = v^{-a} \int_0^w t^{u-1} \left(1 - \frac{t}{v}\right)^{\delta-1} e^{-pt} {}_2F_1\left(a, b; c; -\frac{t}{v}\right) dt \tag{2.1}$$

where $Re\ u > 0, Re\ p > 0, |\arg v| < \pi, a, b$ and c are complex parameters with $c \neq 0, -1, -2, \ldots,$ and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function, and the generalized complementary incomplete gamma function given by

$$D_w^\infty\begin{pmatrix} a, b, c, p \\ u, v, \delta \end{pmatrix} = v^{-a} \int_w^\infty t^{u-1} \left(1 - \frac{t}{v}\right)^{\delta-1} e^{-pt} {}_2F_1\left(a, b; c; -\frac{t}{v}\right) dt \tag{2.2}$$

with the same conditions established in (2.1).

Thus, it follows that

$$D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} + D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}.$$

For $\delta = 1, p = 1, b = c$, and $a = m$ (m a positive integer) and using the relation for Gauss hypergeometric function [9, p. 258]

$$F(\alpha,\beta;\beta;z) = (1-z)^{-\alpha}, \quad \arg|1-z| < \pi$$

equation (2.2) reduces to the generalized incomplete gamma function $\Gamma_m(u,v,w)$ considered by Kobayashi [7]

$$\Gamma_m(u,v,w) = \int_w^\infty \frac{t^{u-1}e^{-t}}{(t+v)^m}dt \tag{2.3}$$

If m=0 this result reduces to the well known complementary incomplete gamma function

$$\Gamma(u,w) = \int_w^\infty t^{u-1}e^{-t}dt \tag{2.4}$$

On the other hand, if $\delta = 1$, $p = 1$ and $a = 0$ in (2.1) we obtain the incomplete gamma function

$$\gamma(u,w) = \int_0^w t^{u-1}e^{-t}dt. \tag{2.5}$$

This function most commonly arises in probability theory especially in those applications involving the chi-square distribution.

Now,we list some properties of $D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$ and $D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$ which can be easily deduced by using the fundamental theorem of Calculus and from the corresponding definition.

$$\frac{d}{dw}\left[w^{-u}D_0^w\right] = -w^{-u-1}\left[\frac{(\delta-1)}{v}D_0^w\begin{pmatrix}a,b,c,p\\u+1,v,\delta-1\end{pmatrix} + pD_0^w\begin{pmatrix}a,b,c,p\\u+1,v,\delta\end{pmatrix}\right.$$

$$\left.+\frac{ab}{c}D_0^w\begin{pmatrix}a+1,b+1,c+1,p\\u+1,v,\delta\end{pmatrix}\right] \tag{2.6}$$

$$\frac{d}{dw}\left[\left(1-\frac{w}{v}\right)^{1-\delta}D_0^w\right] = w^{-1}\left(1-\frac{w}{v}\right)^{-\delta+1}\left[w\left(1-\frac{w}{v}\right)^{-1}\frac{(\delta-1)}{v}D_0^w + uD_0^w\right.$$

$$-\frac{(\delta-1)}{v}D_0^w\begin{pmatrix}a,b,c,p\\u+1,v,\delta-1\end{pmatrix} - pD_0^w\begin{pmatrix}a,b,c,p\\u+1,v,\delta\end{pmatrix} \tag{2.7}$$

$$\left.-\frac{ab}{c}D_0^w\begin{pmatrix}a+1,b+1,c+1,p\\u+1,v,\delta\end{pmatrix}\right]$$

$$\frac{d}{dw}\left[e^{pw}D_0^w\right] = pe^{pw}D_0^w\begin{pmatrix}a,b,c,p\\u,v,\delta\end{pmatrix}$$

$$+v^{-a}w^{u-1}\left(1-\frac{w}{v}\right)^{\delta-1}{}_2F_1\left(a,b;c;-\frac{w}{v}\right). \tag{2.8}$$

$$\frac{d}{dw}\left[w^{-u}D_w^\infty\right] = -w^{-u-1}\left[\frac{(\delta-1)}{v}D_w^\infty\begin{pmatrix}a,b,c,p\\u+1,v,\delta-1\end{pmatrix} + pD_w^\infty\begin{pmatrix}a,b,c,p\\u+1,v,\delta\end{pmatrix}\right.$$

$$\left.+\frac{ab}{c}D_w^\infty\begin{pmatrix}a+1,b+1,c+1,p\\u+1,v,\delta\end{pmatrix}\right] \tag{2.9}$$

$$\frac{d}{dw}\left[\left(1-\frac{w}{v}\right)^{1-\delta}D_w^\infty\right] = w^{-1}\left(1-\frac{w}{v}\right)^{-\delta+1}\left[\frac{(\delta-1}{v}w\left(1-\frac{w}{v}\right)^{-1}D_w^\infty + uD_w^\infty\right.$$

$$-\frac{(\delta-1)}{v}D_w^{\infty}\begin{pmatrix} a,b,c,p \\ u+1,v,\delta-1\end{pmatrix} - pD_w^{\infty}\begin{pmatrix} a,b,c,p \\ u+1,v,\delta\end{pmatrix}$$

$$-\frac{ab}{c}D_w^{\infty}\begin{pmatrix} a+1,b+1,c+1,p \\ u+1,v,\delta\end{pmatrix}\Bigg] \tag{2.10}$$

$$\frac{d}{dw}\left[e^{pw}D_w^{\infty}\right] = pe^{pw}D_w^{\infty} - v^{-a}w^{u-1}\left(1-\frac{w}{v}\right)^{\delta-1} {}_2F_1\left(a,b;c;-\frac{w}{v}\right) \tag{2.11}$$

For $p=1$, $a=0$ and $\delta=1$ then the above formulas become:

$$\begin{aligned}
&\frac{d}{dw}\left[w^{-u}\gamma(u,w)\right] = -w^{-u-1}\gamma(u+1,w),\\
&\frac{d}{dw}\left[\gamma(u,w)\right] = w^{-1}[u\gamma(u,w)-\gamma(u+1,w)],\\
&\frac{d}{dw}\left[e^{w}\gamma(u,w)\right] = e^{w}\gamma(u,w)+w^{u-1},\\
&\frac{d}{dw}\left[w^{-u}\Gamma(u,w)\right] = -w^{-u-1}\Gamma(u+1,w),\\
&\frac{d}{dw}\left[\Gamma(u,w)\right] = w^{-1}[u\Gamma(u,w)-\Gamma(u+1,w)],\\
&\frac{d}{dw}\left[e^{w}\Gamma(u,w)\right] = e^{w}\Gamma(u,w)-w^{u-1}.
\end{aligned}$$

These formulas indicate relationship between the generalized incomplete and complementary incomplete functions with the familiar complementary gamma function and its companion.

For $p=1, b=c$ and $\delta=1$ then (2.9), (2.10), (2.11) give the new relations for $\Gamma_a(u,v,w)$

$$\begin{aligned}
&\frac{d}{dw}[w^{-u}\Gamma_a(u,v,w)] = -w^{-u-1}[\Gamma_a(u+1,v,w)+a\Gamma_{a+1}(u+1,v,w)],\\
&\frac{d}{dw}[\Gamma_a(u,v,w)] = w^{-1}[u\Gamma_a(u,v,w)-\Gamma_a(u+1,v,w)-a\Gamma_{a+1}(u+1,v,w)],\\
&\frac{d}{dw}[e^{w}\Gamma_a(u,v,w)] = e^{w}\Gamma_a(u,v,w)-w^{u-1}(v+w)^{-a}.
\end{aligned}$$

3. Recurrence relations. Using the relations of Gauss between contiguous functions $\ _2F_1(a,b;c;z)$ [5, pgs. 103–104, No.(2.8.31)–(2.8.45)] and the definition of the generalization of the gamma function (1.4), the following relations hold.

For Convenience $D = D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$.

$$(b-a)D = bD\begin{pmatrix} a,b+1,c,p \\ u,v,\delta \end{pmatrix} - avD\begin{pmatrix} a+1,b,c,p \\ u,v,\delta \end{pmatrix} \tag{3.1}$$

$$(c-2a)vD = (a-b)D\begin{pmatrix} a,b,c,p \\ u+1,v,\delta \end{pmatrix} + (c-a)D\begin{pmatrix} a-1,b,c,p \\ u,v,\delta \end{pmatrix}$$

$$-avD\begin{pmatrix} a+1,b,c,p \\ u+1,v,\delta \end{pmatrix} \tag{3.2}$$

$$(c-a-b)D = (c-b)D\begin{pmatrix} a,b-1,c,p \\ u,v,\delta \end{pmatrix} - avD\begin{pmatrix} a+1,b,c,p \\ u,v,\delta \end{pmatrix}$$

$$-aD\begin{pmatrix} a+1,b,c,p \\ u+1,v,\delta \end{pmatrix} \tag{3.3}$$

$$acvD = -c(c-b)D\begin{pmatrix} a,b,c,p \\ u+1,v,\delta \end{pmatrix} + acv^2D\begin{pmatrix} a+1,b,c,p \\ u,v,\delta \end{pmatrix} + acvD\begin{pmatrix} a+1,b,c,p \\ u+1,v,\delta \end{pmatrix}$$

$$+(c-a)(c-b)D\begin{pmatrix} a,b,c+1,p \\ u+1,v,\delta \end{pmatrix} \tag{3.4}$$

$$(c-a-1)D = (c-1)D\begin{pmatrix} a,b,c-1,p \\ u,v,\delta \end{pmatrix} - avD\begin{pmatrix} a+1,b,c,p \\ u,v,\delta \end{pmatrix} \tag{3.5}$$

$$(c-a-b)vD = (c-a)D\begin{pmatrix} a-1, b, c, p \\ u, v, \delta \end{pmatrix} - bvD\begin{pmatrix} a, b+1, c, p \\ u, v, \delta \end{pmatrix}$$

$$-bD\begin{pmatrix} a, b+1, c, p \\ u+1, v, \delta \end{pmatrix} \tag{3.6}$$

$$(b-a)vD = (c-a)D\begin{pmatrix} a-1, b, c, p \\ u, v, \delta \end{pmatrix} - (b-a)D\begin{pmatrix} a, b, c, p \\ u+1, v, \delta \end{pmatrix}$$

$$-(c-b)vD\begin{pmatrix} a, b-1, c, p \\ u, v, \delta \end{pmatrix} \tag{3.7}$$

$$cvD = cD\begin{pmatrix} a-1, b, c, p \\ u, v, \delta \end{pmatrix} - cD\begin{pmatrix} a, b, c, p \\ u+1, v, \delta \end{pmatrix} + (c-b)D\begin{pmatrix} a, b, c+1, p \\ u+1, v, \delta \end{pmatrix} \tag{3.8}$$

$$(a-1)vD = (c-1)vD\begin{pmatrix} a, b, c-1, p \\ u, v, \delta \end{pmatrix} + (c-1)D\begin{pmatrix} a, b, c-1, p \\ u+1, v, \delta \end{pmatrix}$$

$$-(c-b-1)D\begin{pmatrix} a, b, c, p \\ u+1, v, \delta \end{pmatrix} - (c-a)D\begin{pmatrix} a-1, b, c, p \\ u, v, \delta \end{pmatrix} \tag{3.9}$$

$$(c-2b)vD = (c-b)vD\begin{pmatrix} a, b-1, c, p \\ u, v, \delta \end{pmatrix} + (b-a)D\begin{pmatrix} a, b, c, p \\ u+1, v, \delta \end{pmatrix} - bvD\begin{pmatrix} a, b+1, c, p \\ u, v, \delta \end{pmatrix}$$

$$-bD\begin{pmatrix} a, b+1, c, p \\ u+1, v, \delta \end{pmatrix} \tag{3.10}$$

$$cbvD = -c(c-a)D\begin{pmatrix} a, b, c, p \\ u+1, v, \delta \end{pmatrix} + bcvD\begin{pmatrix} a, b+1, c, p \\ u, v, \delta \end{pmatrix} + bcD\begin{pmatrix} a, b+1, c, p \\ u+1, v, \delta \end{pmatrix}$$

$$+(c-a)(c-b)D\begin{pmatrix} a,b,c+1,p \\ u+1,v,\delta \end{pmatrix} \tag{3.11}$$

$$(c-b-1)D = (c-1)D\begin{pmatrix} a,b,c-1,p \\ u,v,\delta \end{pmatrix} - bD\begin{pmatrix} a,b+1,c,p \\ u,v,\delta \end{pmatrix} \tag{3.12}$$

$$cvD\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = cvD\begin{pmatrix} a,b-1,c,p \\ u,v,\delta \end{pmatrix} - cD\begin{pmatrix} a,b,c,p \\ u+1,v,\delta \end{pmatrix}$$

$$-(c-a)D\begin{pmatrix} a,b,c+1,p \\ u,v,\delta \end{pmatrix} \tag{3.13}$$

$$(b-1)vD = (c-1)\left[vD\begin{pmatrix} a,b,c-1,p \\ u,v,\delta \end{pmatrix} + D\begin{pmatrix} a,b,c-1,p \\ u+1,v,\delta \end{pmatrix}\right]$$

$$-(c-a-1)D\begin{pmatrix} a,b,c,p \\ u+1,v,\delta \end{pmatrix} - (c-b)vD\begin{pmatrix} a,b-1,c,p \\ u,v,\delta \end{pmatrix} \tag{3.14}$$

$$c(c-1)vD = -c(2c-a-b-1)D\begin{pmatrix} a,b,c,p \\ u+1,v,\delta \end{pmatrix} + (c-a)(c-b)D\begin{pmatrix} a,b,c+1,p \\ u+1,v,\delta \end{pmatrix}$$

$$+c(c-1)\left[vD\begin{pmatrix} a,b,c-1,p \\ u,v,\delta \end{pmatrix} + D\begin{pmatrix} a,b,c-1,p \\ u+1,v,\delta \end{pmatrix}\right] \tag{3.15}$$

To show the first recurrence relations we use the formula and definition (1)

$$(b-a)F(a,b;c;z) + aF(a+1,b;c;z) - bF(a,b+1;c;z) = 0 \text{ where } z = -\frac{t}{v}$$

$$(b-a)D = (b-a)v^{-a}\int_0^\infty t^{u-1}e^{-pt}\left(1-\frac{t}{v}\right)^{\delta-1} e^{-pt}{}_2F_1\left(a,b;c;-\frac{t}{v}\right)dt$$

$$= v^{-a}\int_0^\infty t^{u-1}e^{-pt}\left(1-\frac{t}{v}\right)^{\delta-1} e^{-pt}\times$$

$$\times\left[-aF\left(a+1,b;c;-\frac{t}{v}\right)+bF\left(a,b+1;c;-\frac{t}{v}\right)\right]dt$$

$$= -av\left[v^{-(a+1)}\int_0^\infty t^{u-1}e^{-pt}\left(1-\frac{t}{v}\right)^{\delta-1}\left[F\left(a+1,b;c;-\frac{t}{v}\right)\right]dt\right]$$

$$+bv^{-a}\int_0^\infty t^{u-1}e^{-pt}\left(1-\frac{t}{v}\right)^{\delta-1}F\left(a,b+1;c;-\frac{t}{v}\right)dt$$

$$= -avD\begin{pmatrix} a+1,b;c;p \\ u,v \end{pmatrix} + bD\begin{pmatrix} a,b+1;c;p \\ u,v \end{pmatrix},$$

and this is the required recurrence relation. Applying the same method we can show the remaining relations.

Similarly, we can obtained the relations of $D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$ and $D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$.

4. Asymptotic expansions for $D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$, $D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$ and $D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$. In this section we assume that $v\to\infty$. Using the expansion in series of the Gauss hypergeometric function [6, p. 238, No (9.1.2)]

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^\infty \frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k, \quad |z|<1 \tag{4.1}$$

After substitution in the definition (1) we get

$$D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = v^{-a}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k! v^k}\int_0^{\infty} t^{k+u-1}\left(1-\frac{t}{v}\right)^{\delta-1} e^{-pt}dt \tag{4.2}$$

Where we have interchanged the order of integral and sum. From the binomial expansion [9, p. 275]

$$(1-z)^{-\alpha} = \sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!}z^m, \quad |z|<1$$

then the result (4.2) is equivalent to

$$D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = v^{-a}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k! v^k}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m! v^m}\int_0^{\infty} t^{m+k+u-1}e^{-pt} \tag{4.3}$$

To facilitate our analysis we will make the following change of variables:

$$y = pt$$

$$D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k!}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(vp)^{m+k}}$$

$$\int_0^{\infty} y^{m+k+u-1}e^{-y}dy. \tag{4.4}$$

from the definition of gamma function (1.1) equation (4.4) can be written as

$$D\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k!(pv)^k}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(vp)^m}\Gamma(m+k+u) \tag{4.5}$$

Following the same procedure we get from the definition (2.1),

$$D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k!}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(vp)^{m+k}}$$

$$\int_0^{pw} y^{m+k+u-1}e^{-y}dy.$$

and by using the result (2.5) this can be written as

$$D_0^w\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k!}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(vp)^{m+k}}\gamma(m+k+u,pw) \tag{4.6}$$

Similarly, we can find the asymptotic expansion of $D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix}$. From equation (2.2) we have

$$D_w^\infty\begin{pmatrix} a,b,c,p \\ u,v,\delta \end{pmatrix} = \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}\frac{(-1)^k(a)_k(b)_k}{(c)_k k!(pv)^k}$$

$$\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(pv)^m}\int_{pw}^{\infty} y^{m+k+u-1}e^{-y}dy$$

$$= \frac{v^{-a}}{p^u}\sum_{k=0}^{\infty}(-1)^k\frac{(a)_k(b)_k}{(c)_k k!(pv)^k}\sum_{m=0}^{\infty}\frac{(1-\delta)_m}{m!(pv)^m}\Gamma(m+k+u,pw) \tag{4.7}$$

where we have used the result (2.4).

If in results of this paper we put $\delta = 1$ we obtain the expressions given by Al-Musallam and Kalla [2]. This concludes our extension of the generalized complementary and incomplete gamma function.

5. Summary and conclusion. In the previous sections, we have presented an extension of Kobayashi [7] function that include a further generalization of incomplete and complementary incomplete gamma function.

The aim of this paper was to consider further generalization of the gamma function. Also, to obtain some properties and recurrence relations and establishing the asymptotic series expansions for each of them.

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