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A SURVEY OF COUNTEREXAMPLES TO HILBERT'S FOURTEENTH PROBLEM

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Abstract. We survey counterexamples to Hilbert's Fourteenth Problem, beginning with those of Nagata in the late 1950s, and including recent counterexamples in low dimension constructed with locally nilpotent derivations. Historical framework and pertinent references are provided. We also include 8 important open questions.

1. Introduction. One of the main ideas of classical invariant theory is to study and classify groups of linear transformations $\rho : G \hookrightarrow GL_n(\mathbb{C})$ via their rings of invariants R^G. Here, R denotes the polynomial ring in n variables over the complex field $\mathbb{C}$, and R^G is the subring of G-invariant polynomials, i.e., polynomials $f(x)$ such that $f(gx) = f(x)$ for all $g \in G$ and $x \in \mathbb{C}^n$.

A fundamental question about R^G is the following.

Is R^G finitely generated as a $\mathbb{C}$-algebra?

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This is a special case of a problem formulated by Hilbert, namely, Problem 14 in his famous list of 23 problems delivered by him to the International Congress in 1900. At the time, it was thought that this special case had been settled positively by Mauerer, but his proof was incomplete. So in fact, except for certain groups for which the solution was known to be positive, the solution to the Fourteenth Problem, even for this special case, was not known at the time.

Standing on the other side of the Twentieth Century, we see two main results which have emerged from work on this problem.

(1) If k is a field, and G is a finite or reductive algebraic k-group acting by algebraic automorphisms on an affine k-variety X, then the algebra of invariants $k[X]^G$ is finitely generated over k.

(2) If k is a field, there exists a unipotent k-group G which acts by linear transformations on affine space $X = \mathbb{A}_k^n$ (for some positive integer n) in such a way that the algebra of invariants $k[X]^G$ is not finitely generated over k.

Each result has now been proved, with the exception of a single special case of (2). Here, the *locally finite* case remains open, i.e., the case k is an algebraic extension of a finite field (see Section 2 below).

Result (1) has come to be called Hilbert's Finiteness Theorem, though it represents the culmination of the efforts of many mathematicians over the past century. Since several excellent descriptions of its development are already present in the literature, it will be the purpose of this article to focus on (2), the counterexamples. The reader is referred to the article of Humphreys [18], which provides an introductory survey of reductive group actions, and to the monograph of Popov [31], which gives a more extended treatment of the subject; each contains insightful historical background and a good list of pertinent references. The article of Mumford [22] is also required reading for anyone interested in the subject. Other standard references include [9], [12], [23], and [27]. In view of (1), the question of finite generation for reductive groups has been replaced by other questions about invariant rings. For example, in what has come to be called *constructive invariant theory*, the idea is to determine degree bounds for a system of generators, or to find algorithms which produce minimal generating sets, for invariant rings of reductive group actions; see [7].

Results (1) and (2) correspond to the convenient division of algebraic groups into two basic "generating" types, namely, reductive groups and unipotent groups. Recall that $G \hookrightarrow GL_n(k)$ is *unipotent* if it can be conjugated into the group of upper triangular matrices with ones on the diagonal, or equivalently, the only eigenvalue for elements of G is 1. G is *reductive* if G has no non-trivial connected normal unipotent subgroup. For example, the two irreducible algebraic groups of dimension 1 are $\mathbb{G}_m$ (the unit group of the field k) and $\mathbb{G}_a$ (the additive group of the field k) which are respectively reductive and unipotent. Given $r \geq 1$, $(\mathbb{G}_m)^r$ is a reductive group, called an algebraic *torus*, which is represented in $GL_n(k)$ by diagonal matrices. The unipotent counterpart of the torus, $(\mathbb{G}_a)^r$, is often represented by sub-diagonal matrices, i.e., matrices of the form $\begin{pmatrix} I & 0 \\ D & I \end{pmatrix}$, where D is diagonal. As we shall see, the groups $(\mathbb{G}_m)^r$ and $(\mathbb{G}_a)^r$ appear in the first counterexamples to Hilbert's Fourteenth Problem.

Before discussing counterexamples, we discuss two important cases in which the solution to Hilbert's Fourteenth Problem is positive. As mentioned, Hilbert's original statement of the problem was more general than the statement above:

> *For a field k, let $k^{[n]}$ denote the polynomial ring in n variables over k, and let $k^{(n)}$ denote its field of fractions. If K is a subfield of $k^{(n)}$ containing k, is $K \cap k^{[n]}$ finitely generated over k?*

In 1954, Zariski [45] showed:

(3) If tr $\deg_k K \leq 2$, then $K \cap k^{[n]}$ is finitely generated over k.

It follows from Zariski's Theorem that, if $X = \mathbb{A}^n_k$ for $n \leq 3$, then $k[X]^G$ is finitely generated for any algebraic group G acting algebraically on X: If tr $\deg_k k[X]^G \leq 2$, apply Zariski's Theorem; and if tr $\deg_k k[X]^G = 3$, then $k[X]$ is algebraic over $k[X]^G$, and G is necessarily finite. In this case, result (1) above applies. (For finite groups, result (1) is due to E. Noether; see [18].)

Another important positive result is the following.

(4) If k is a field of characteristic 0, and if $\mathbb{G}_a$ acts by linear transformations on $X = \mathbb{A}^n_k$, then $k[X]^{\mathbb{G}_a}$ is finitely generated.

This is known as Weitzenböck's Theorem. Weitzenböck proved this for the case $k = \mathbb{C}$ in 1932 [43], and Seshadri gave a proof over any field of characteristic zero in 1962 [36]. A modified version of Seshadri's proof appears in [40] for $k = \mathbb{C}$. The main idea in these proofs is (roughly) to embed the given $\mathbb{G}_a$-action in a linear $SL_2(k)$-action, and show that the $\mathbb{G}_a$-invariant ring is finitely generated over the ring of $SL_2(k)$-invariants, and thus over k. This method also works for certain kinds of linear $\mathbb{G}_a$-actions in positive characteristic, which are included in the paper of Seshadri. In [39], Tan gives an algorithm for calculating explicit generators for invariant rings of linear $\mathbb{G}_a$-actions in cases where finite generation is known. Additional positive results for certain kinds of unipotent actions are discussed in Section 4 below.

The speech delivered by Hilbert in 1900 includes the 23 problems, and was later published in [15]. In contrast to its influence on mathematics in the following century, this speech bears the unassuming title "Mathematical Problems". In 1903, the speech appeared in English translation in [16]. In 1974, the American Mathematical Society sponsored a special Symposium on the mathematical consequences of Hilbert's problems. The volume [22] contains the proceedings of that symposium, as well as the English translation of Hilbert's speech. The purpose of the Symposium was "to focus upon those areas of importance in contemporary mathematical research which can be seen as descended in some way from the ideas and tendencies put forward by Hilbert in his speech" (from the Introduction). In particular, the volume contains one paper discussing each of the 23 problems, written by 23 of the most influential mathematicians of the day.

2 Linear counterexamples

2.1. Nagata's examples. The first counterexamples to Hilbert's Fourteenth Problem were presented by Nagata in 1958. Prior to the appearance of Nagata's examples, Rees [32] constructed a counterexample to Zariski's generalization of the Fourteenth Problem, which asks:

> Let R be a normal affine ring over a field k. If L is a field with $k \leq L \leq \text{frac}(R)$, is $(R \cap L)$ an affine ring?

In Rees' example, $\text{frac}(R \cap L)$ contains the function field of a non-singular cubic projective plane curve, and cannot therefore be a counterexample to Hilbert's problem. But Rees' example was very important in its own right, and indicated that counterexamples to Hilbert's problem might be found in a similar fashion.

Shortly thereafter, Nagata discovered two counterexamples to Hilbert's problem. In [24], he describes the situation as follows. (By "original 14-th problem", he means the specific case using fixed rings for linear actions of algebraic groups.)

> In 1958, the writer found at first a counter-example to the 14-th problem and then another example which is a counter-example to the

> original 14-th problem. This second example was announced at the International Congress in Edinburgh (1958) (see [25]). Though the first example is in the case where dim $K = 4$, in the second example dim K is equal to 13 [should read 19]. Then the writer noticed that the first example is also a counter-example to the original 14-th problem.

Here is the construction of the first example, as presented in [24].

Let π be a prime field, and let $k = \pi(a_{ij})$, $1 \leq i \leq 3$ and $1 \leq j \leq 16$, where a_{ij} are elements algebraically independent over π. Let V denote the vector space k^{16}, and define vectors $v_i \in V$ by $v_i = (a_{i1}, \ldots, a_{i16})$, $1 \leq i \leq 3$. As an additive group, $V \cong \mathbb{G}_a^{16}$, and we identify V with the set of diagonal matrices of order 16.

Now define an algebraic group G in the following way.

(i) Let $U \subset V$ be the subspace of vectors perpendicular to each v_i; then $U \cong \mathbb{G}_a^{13}$.

(ii) Let $T \subset GL(V)$ be the subgroup of all diagonal matrices C such that the product of diagonal entries $c_1 c_2 \cdots c_{16} = 1$; then $T \cong \mathbb{G}_m^{15}$ (a torus).

(iii) Let $G = (U \times T) \cong (\mathbb{G}_a^{13} \times \mathbb{G}_m^{15})$.

Let W denote the vector space $V \oplus V = k^{32}$, and define a function $\rho : G \to GL(W)$ by

$$\rho(B, C) = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \quad (B \in U, C \in T) .$$

Then ρ is an injective homomorphism, and in this way we obtain an action of G on W. The coordinate ring $k[W]$ is a polynomial ring in 32 variables, and the fixed ring of the action $k[W]^G$ has dimension 4 over k. In his paper, Nagata proves:

$k[W]^G$ *is not finitely generated over* k.

The second (Edinburgh) example presented by Nagata in [25] is simply the restriction to the subgroup $U \cong \mathbb{G}_a^{13}$ of the above action. In this case, the action itself is simpler than the first, but the dimension of the fixed ring has increased: $\dim_k k[W]^U = 19$.

How did Nagata find these examples? As Steinberg [38] points out, the heart of Nagata's method is to relate the structure of $k[W]^G$ to an interpolation problem in the projective plane, namely, that for each $m \geq 1$, there does not exist a curve of degree $4m$ having multiplicity at least m at each of 16 general points of the projective plane. Steinberg writes: "Nagata's ingenious proof of this is a tour de force but the results from algebraic geometry that he uses are by no means elementary" (p. 377).

The foundation of this geometric approach to the problem was laid by Zariski in the early 1950s. His idea was to look at rings of the form $R(D)$, where D is a positive divisor on some non-singular projective variety X, and $R(D)$ is the ring of rational functions on X with poles only on D. Mumford writes:

> In his penetrating article [45], Zariski showed that Hilbert's rings $K \cap k[x_1, \ldots, x_n]$ were isomorphic to rings of the form $R(D)$ for a suitable X and D; asked more generally whether all the rings $R(D)$ might not be finitely generated; and proved $R(D)$ finitely generated if dim $X = 1$ or 2... Unfortunately, it was precisely by focusing so clearly the divisor-theoretic content of Hilbert's 14th problem that Zariski cleared the path to counter-examples. [22]

In the example constructed by Rees, X is birational to $\mathbb{P}^2 \times C$ for an elliptic curve C; and in Nagata's examples, X is the surface obtained by blowing up $\mathbb{P}^2$ at 16 general points. For further detail, the reader is referred to Mumford's article, *ob. cit.*, as well as Nagata's 1965 lectures on the subject, found in [26].

2.2. The example of A'Campo-Neuen. It is surprising that (to the writer's knowledge) no new linear counterexamples to Hilbert's problem appeared until almost 40 years after those of Nagata. By *linear* we mean that an algebraic group acts on a finite-dimensional vector space by linear transformations. To be sure, the methods described above were much-studied and discussed in that time. But it is natural to ask: For a linear action of an algebraic group G on $\mathbb{A}^n$ yielding a counterexample to the Fourteenth Problem, what are the smallest dimensions of G and $\mathbb{A}^n$ which can occur?

The first linear counterexample to appear after Nagata's is due to A'Campo-Neuen [1] in 1994. This example arises as the fixed ring of a linear action of $G = \mathbb{G}_a^{12}$ on $\mathbb{A}^{19}$ in the case k is a characteristic 0 field. In particular, given $(t_1, \ldots, t_{12}) \in G$, the G-action is defined explicitly by the lower triangular matrix

$$\begin{pmatrix} I & 0 \\ M^T & I \end{pmatrix}$$

of order 19, where the identities are of order 4 and 15 respectively, and M is the

(4×15) matrix

$$M = \begin{pmatrix} t_1 & t_2 & 0 & t_3 & t_4 & 0 & t_5 & t_6 & 0 & t_7 & t_8 & t_9 & t_{10} & t_{11} & 0 \\ t_{12} & t_1 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{12} & t_3 & t_4 & 0 & 0 & 0 & 0 & 0 & t_8 & t_9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_{12} & t_5 & t_6 & 0 & 0 & 0 & 0 & t_{10} & t_{11} \end{pmatrix}.$$

This example is based on a (non-linear) counterexample to Hilbert's problem published by Roberts [35] in 1990. The methods used by Roberts in his construction are somewhat different from those employed by Nagata; the details of Roberts' example are discussed in Section 3 below.

A'Campo-Neuen's proof is quite elegant. She first shows that Roberts' example can be realized as the fixed ring R of a non-linear algebraic $\mathbb{G}_a$-action on $\mathbb{A}^7$. She then proceeds to unravel this action by defining a sequence of 11 linear $\mathbb{G}_a$-actions for which the successive fixed rings are polynomial rings. More precisely, if G_i denotes the ith copy of $\mathbb{G}_a$, and if $k^{[n]}$ denotes a polynomial ring in n variables, then

$$(k^{[19]})^{(G_1\times\cdots\times G_{11})} = \left((k^{[19]})^{G_1}\right)^{(G_2\times\cdots\times G_{11})} \cong (k^{[18]})^{(G_2\times\cdots\times G_{11})}.$$

Note that the subring $k^{[18]} \subset k^{[19]}$ is abstractly a polynomial ring, but is not a coordinate subring. Continuing in this way, we eventually obtain

$$(k^{[19]})^{(G_1\times\cdots\times G_{11})} \cong \cdots \cong (k^{[9]})^{G_{11}} \cong k^{[8]}.$$

For the total action, $(k^{[19]})^G \cong (k^{[8]})^{G_{12}} \cong ((k^{[7]})^{\mathbb{G}_a}))^{[1]}$, and the $\mathbb{G}_a$-action on the polynomial ring $k^{[7]}$ is exactly that defined by Roberts' example. In other words, $(k^{[19]})^G \cong R^{[1]}$, and since R is not finitely generated, neither is $R^{[1]}$.

2.3. Steinberg's examples. In 1997, Steinberg [38] published a lucid exposition of Nagata's original constructions, and modified Nagata's approach to obtain further linear counterexamples of reduced dimension. The smallest counterexample obtained in Steinberg's paper is the fixed ring of a linear action of $G = \mathbb{G}_a^6$ on $\mathbb{A}^{18}$, where k is any infinite field which is not locally finite (i.e., not an algebraic extension of a finite field). It is constructed as follows.

Represent the group $\mathbb{G}_a^9$ in $GL_{18}(k)$ by matrices of the form

$$\begin{pmatrix} I & 0 \\ D & I \end{pmatrix},$$

where, given $(t_1, \ldots, t_9) \in \mathbb{G}_a^9$, D is the (9×9) diagonal matrix whose i^{th} diagonal entry is t_i. In the case char $k = 0$, choose $a_1, \ldots, a_9 \in k$ such that $a_i \neq a_j$ for $i \neq j$, and $\sum a_i \neq 0$. Let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum a_i^3 t_i = 0$. In the case char $k > 0$, choose distinct $a_1, \ldots, a_9 \in k$ so that Πa_i is neither 0 nor any root of 1, and let $G \subset \mathbb{G}_a^9$ be the subgroup for which $\sum t_i = \sum a_i t_i = \sum (a_i^2 - a_i^{-1}) t_i = 0$. Then in both cases, $G \cong \mathbb{G}_a^6$ and the fixed ring $(k^{[18]})^G$ is not finitely generated.

Steinberg's paper is self-contained and very well-written. In particular, two main lemmas (2.1 and 2.2) are presented and proved, which provide the crucial link between an interpolation problem in the projective plane and the structure of certain fixed rings. The group associated with a cubic curve plays an important role in this approach to the problem. Steinberg goes on to discuss the status of the classical geometric problem lying at the heart of this approach, which is of interest in its own right, described by the author as follows:

> Find the dimension of the space of all polynomials (or curves) of a given degree with prescribed multiplicities at the points of a given finite set in general position in the plane, thus also determine if there is a curve, i.e., a nonzero polynomial, in the space and if the multiplicity conditions are independent. (p. 383)

2.4. Open questions I. The following questions of finite generation are open. They concern linear actions of unipotent groups on $\mathbb{A}^n$.

- *Question 1.* Given a field k which is not locally finite, what is the smallest n such that, for some r, there exists a linear action of $\mathbb{G}_a^r$ on $V = k^n$ with non-finitely generated ring of invariants?

- *Question 2.* Given a field k which is not locally finite, what is the smallest r such that, for some n, there exists a linear action of $\mathbb{G}_a^r$ on $V = k^n$ with non-finitely generated ring of invariants?

So far, Steinberg's example gives the smallest value for both n and r, namely, $n = 18$ and $r = 6$. In general we must have $n \geq 4$ due to Zariski's Theorem (c.f. Section 1). In addition, if char $k = 0$, then $r \geq 2$ by Weitzenböck's Theorem.

Naturally, we would also like to know whether Weitzenböck's theorem generalizes to all fields:

- *Question 3.* If k is a field of positive characteristic, does there exist any linear action of $\mathbb{G}_a$ on $\mathbb{A}^n$ whose ring of invariants is not finitely generated as a k-algebra? (See [11])

Finally:

- *Question 4.* If k is a finite or locally finite field, does there exist a counterexample to Hilbert's Fourteenth Problem for this field (linear or otherwise)?

Section 4 below contains a couple of recent candidates for counterexamples in the case of a finite field.

3. Non-Linear counterexamples

3.1. Roberts' example. In the mid-1980s, Paul Roberts was studying the examples of Rees and Nagata from a point of view somewhat different than that presented above. The main idea of this approach is to consider a ring R which is the symbolic blow-up of a prime ideal P in a commutative Noetherian ring A. What this means is that R is isomorphic to a graded ring of the form $\oplus_{n\geq 0}P^{(n)}$, where $P^{(n)}$ denotes the nth symbolic power of P, defined as

$$P^{(n)} = \{x \in A \mid xy \in P^n \text{ for some } y \notin P\}.$$

Rees used symbolic blow-ups in constructing his counterexample to Zariski's Problem. In his 1985 paper [34], Roberts writes:

> In a few nice cases the symbolic blow-up of P is a Noetherian ring or, equivalently, a finitely generated A-algebra. In general, however, $\oplus P^{(n)}$ is not Noetherian. The first example of this is due to Rees.

It was in this paper that Roberts constructed new a counterexample to Zariski's problem similar to Rees' example, but having somewhat nicer properties. Subsequently, in a 1990 paper [35] he constructed an important new counterexample to Hilbert's Fourteenth Problem along similar lines. In the latter paper, Roberts gives the following description of these developments.

> In his example, Rees takes R to be the coordinate ring of the cone over an elliptic curve and shows that if P is the prime ideal corresponding to a point of infinite order then the ring $\oplus_{n\geq 0}P^{(n)}$ is not finitely generated and is a counterexample to Zariski's problem. Shortly thereafter Nagata gave a counterexample to Hilbert's original problem, and, in fact gave a counterexample which was a ring of invariants of a linear group acting on a polynomial ring, which is the special case which motivated the original problem. In his example a similar construction to that of Rees was used in which P was not prime, but was the

> ideal defining sixteen generic lines through the origin in affine space of three dimensions. The proof was based on the existence of points of infinite order on elliptic curves.
>
> But this did not totally end the story. Rees's example uses a ring which is not regular, and Nagata's uses an ideal which is not prime; Cowsick then asked whether there were examples in which the ring was regular and the ideal prime. Such an example was given in Roberts [34]. However, this still did not totally finish the problem, since this example was based on that of Nagata and made crucial use of the fact that when the ring was completed the ideal broke up into pieces and did not remain prime.

Roberts proceeds to construct an example of a prime ideal in a complete regular local ring (a power series ring in seven variables) whose symbolic blow-up is not finitely generated.

Explicitly (and in his notation) he takes F to be any field of characteristic 0 and $R = F^{[7]} = F[X, Y, Z, S, T, U, V]$, and defines a graded $F[X, Y, Z]$-module homomorphism $\phi : R \to R$. He proves that the kernel of ϕ is not finitely generated over F. This construction is then "completed" to give the example in terms of symbolic blow-ups.

In the paper, ϕ is defined explicitly by its effect on monomials in S, T, U, V. Though Roberts does not use the language of derivations, one easily recognizes from his description of these images that ϕ is equivalent to the F-derivation $\mathcal{D}$ of R defined by

$$\mathcal{D} = (X^{t+1})\frac{\partial}{\partial S} + (Y^{t+1})\frac{\partial}{\partial T} + (Z^{t+1})\frac{\partial}{\partial U} + (X^tY^tZ^t)\frac{\partial}{\partial V},$$

where $t \geq 2$. According to Roberts, this example originated in his study of Hochster's Monomial Conjecture, which had been proved for any field of characteristic 0. The conjecture asserted that for any local ring of dimension 3 with system of parameters X, Y, Z, and for any non-negative integer t, the monomial $X^tY^tZ^t$ is not in the ideal generated by the monomials X^{t+1}, Y^{t+1}, and Z^{t+1}.

As mentioned above, A'Campo-Neuen recognized that Roberts' example can be realized as the invariant ring of an algebraic (but non-linear) $\mathbb{G}_a$-action on $\mathbb{A}^7$. This was recognized independently by Deveney and Finston at about the same time. In [8], they give a different proof that the kernel of the derivation $\mathcal{D}$ above is not finitely generated in the case $t = 2$. We next examine the precise connection between derivations and $\mathbb{G}_a$-actions.

3.2. Locally nilpotent derivations. When k is a field of characteristic 0, the study of algebraic $\mathbb{G}_a$-actions on an affine variety X is equivalent to the study of locally nilpotent k-derivations of the coordinate ring $k[X]$. So from now on, we assume that char $k = 0$, and we consider derivations in place of $\mathbb{G}_a$-actions, since these are generally easier to work with than the associated actions.

A derivation $D : B \to B$ of a commutative k-algebra B is *locally nilpotent* if, given $f \in B$, $D^n f = 0$ for all sufficiently large n. If $B = k[x_1, \dots, x_n]$ (a polynomial ring), we are especially interested in *triangular* derivations, namely, those for which $Dx_i \in k[x_1, \dots, x_{i-1}]$ for each i; note that any triangular derivation of B is locally nilpotent. Likewise, we say D on B is *linear* if each image Dx_i is a linear polynomial in $x_1, \dots, x_n$; in this case, D is locally nilpotent iff the induced linear map of $(k \cdot x_1 \oplus \cdots \oplus k \cdot x_n)$ is nilpotent.

The correspondence between locally nilpotent derivations and $\mathbb{G}_a$-actions is as follows. Given the locally nilpotent derivation D on B, and given $t \in \mathbb{G}_a$, the exponential $\exp(tD) = \sum\limits_{i=0}^{\infty} \frac{t^i}{i!} D^i$ defines an algebraic action of $\mathbb{G}_a$ on B (and on $\operatorname{Spec} B$). Conversely, given a $\mathbb{G}_a$-action $\rho(t) : B \to B$ ($t \in \mathbb{G}_a$), the function $D = \rho'(0)$ defines a locally nilpotent derivation of B. The reader is referred to [37] for further details regarding this correspondence[1].

In this setting, the fixed ring of the $\mathbb{G}_a$-action is precisely the kernel of the associated derivation. So we are led to the following special case of Hilbert's Fourteenth Problem.

> *Given a locally nilpotent k-derivation D of the polynomial ring $B = k^{[n]}$, is* $\ker D$ *finitely generated?*

We know the answer to be positive when D is linear (Weitzenböck's Theorem) or if $n \le 3$ (Zariski's Theorem), but negative in general due to Roberts' example.

In fact, it is easy to see that the answer is generally negative for all $n \ge 7$. In [42], van den Essen and Janssen observe that if $\mathcal{D}$ is the derivation on $k[x_1, \dots, x_7]$ associated with Roberts' example, and if A is its kernel, then the extension of $\mathcal{D}$ to $k[x_1, \dots, x_n]$ ($n \ge 8$) obtained by setting $\mathcal{D}x_i = 0$ for $8 \le i \le n$ has kernel equal to $A[x_8, \dots, x_n]$, which is also not finitely generated. Another family of counterexamples in higher (odd) dimensions was given by Kojima and Miyanishi in [20]. They consider the triangular derivations λ on polynomial rings $k[x_1, \dots, x_n, y_1, \dots, y_n, z]$ defined by $\lambda(x_i) = 0$, $\lambda(y_i) = x_i^{t+1}$, and $\lambda(z) =$

[1]This correspondence is lost for fields of characterstic $p > 0$. For example, $t(x, y) = (x, y + t^p f(x))$ is a triangular $\mathbb{G}_a$-action on the plane $\mathbb{A}^2_k$ for any polynomial $f(x)$, but it is not the exponential of a derivation.

$(x_1 \cdots x_n)^t$. They prove that, for each $n \geq 3$ and $t \geq 2$, the kernel of λ is not finitely generated. Since Roberts' examples are included in this family as the case $n = 3$, their paper provides a new proof for Roberts' example as well.

Actually, it was Derksen [6] who first recognized a connection between counterexamples to Hilbert's problem and derivations, but the derivations he uses are in general not locally nilpotent. In particular, he constructs a derivation of the polynomial ring in 32 variables whose kernel coincides with the fixed ring $k[W]^G$ of Nagata's example. Earlier, Nagata and Nowicki [29] investigated kernels of derivations, and succeeded in giving several positive results, including the finite generation of kernels in dimension 3.

In general, the subject of locally nilpotent derivations is one of growing importance in algebra. The recent books of van den Essen [41] and Nowicki [28] are very good references on the subject.

3.3. The Dixmier map. Beginning in 1997, the author began looking for counterexamples in lower dimension of the type Roberts constructed, focusing on locally nilpotent derivations rather than symbolic blow-ups. The first goal was to find a more constructive proof that the kernel of the derivation $\mathcal{D}$ above was not finitely generated. The key fact in Roberts' paper is *Lemma 3*, which asserts the existence of a sequence of homogeneous kernel elements of the form $f_n = XV^n +$ (lower-degree V-terms); using homogeneity, it follows that no finitely-generated subring of $\ker \mathcal{D}$ can contain every f_n. In [8], the authors list f_n explicitly for $n = 1, 2, 3$. But the general construction of the f_n was not at all apparent in any of the existing literature.

One idea was to consider the *Dixmier map* of $\mathcal{D}$ relative to V. For $\mathbb{G}_a$-actions, this takes the place of the *Reynolds operator*: For a reductive group action, the Reynolds operator is a natural map from the coordinate ring $k[X]$ to the invariant ring $k[X]^G$, and it is an essential tool, for example, in the proof of Hilbert's Finiteness Theorem (see [7]). For $\mathbb{G}_a$-actions, we consider the corresponding locally nilpotent derivation $D : B \to B$. The Dixmier map is a ring homomorphism which transforms elements of B into elements in some localization of $\ker D$.

Specifically, if $D \neq 0$, choose $\sigma \in B$ for which $D^2\sigma = 0$ but $D\sigma \neq 0$; such an element always exists by local nilpotency. Then the Dixmier map is the homomorphism $\pi_\sigma : B \to (\ker D)_{D\sigma}$ defined by

$$\pi_\sigma(b) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} D^i b \frac{\sigma^i}{(D\sigma)^i}.$$

(See [10], 4.7.5).

Returning to the derivation $\mathcal{D}$ in dimension 7, one then asks whether there exists a sequence $g_n \in k[X,Y,Z,S,T,U,V]$ such that $\pi_V(g_n) = f_n$ for every n. The main point is that the elements g_n might be easier to construct and work with than the f_n. In short, I eventually did manage to construct such a sequence g_n, and noticed that it could also be used to give a counterexample to Hilbert's problem in dimension six.

3.4. Dimensions six and five. In an effort to generalize Roberts' method, we give in [3] the following criterion for non-finite generation of kernels.

Lemma 1. *Let $K = \oplus_{i\in\mathbb{N}} K_i$ be a graded k-domain such that $K_0 = k$, and let δ be a homogeneous locally nilpotent k-derivation of K. Given $\alpha \in \ker\delta$ which is not in the image of δ, let $\tilde{\delta}$ be the extension of δ to $K[T]$ defined by $\tilde{\delta}T = \alpha$, where T a variable over K. Suppose ϕ_n is a sequence of non-zero elements of $\ker\tilde{\delta}$ having leading T-coefficients $b_n \in K$. If $\deg b_n$ is bounded, but $\deg_T \phi_n$ is not bounded, then $\ker\tilde{\delta}$ is not finitely generated over k.*

This criterion can be used to show the existence of counterexamples in dimensions five and six, as follows.

Let $R = k[a,b,s,t,u]$, a polynomial ring in 5 variables over k, and define a triangular derivation

$$\Delta = a\frac{\partial}{\partial s} + bs\frac{\partial}{\partial t} + bt\frac{\partial}{\partial u}.$$

Define a sequence $t_n \in R$ by

$$t_1 = a\ , \quad t_2 = b\ , \quad t_3 = ab\ ; \quad \text{and} \quad t_n = t_{n-3} \quad \text{for } n \geq 4.$$

The central result of [13] is:

Theorem 1. *There exist $w_n \in R$ $(n \geq 0)$ such that $w_0 = 1$, $w_1 = s$, and $\Delta w_n = t_n \cdot w_{n-1}$ for all $n \geq 1$.*

The main ingredients in the proof of this theorem are homogeneity and linear algebra.

To obtain a counterexample in dimension 6, let x and y be integral elements over R such that $x^3 = a$ and $y^3 = b$, and let v be transcendental over R. Then $B := R[x,y,v] = k[x,y,s,t,u,v]$ is a polynomial ring in 6 variables over k. If D is the triangular derivation on B defined by

$$D = (x^3)\frac{\partial}{\partial s} + (y^3 s)\frac{\partial}{\partial t} + (y^3 t)\frac{\partial}{\partial u} + (x^2y^2)\frac{\partial}{\partial v},$$

then $D|_R = \Delta$. Therefore, $Dw_n = t_n \cdot w_{n-1}$. The Dixmier map π_v defines a homomorphism from B to $(\ker D)_{Dv}$. It can be shown using *Theorem 2* that $\pi(xw_{3n})$ is a polynomial for all n. Direct calculation shows that $\pi(xw_{3n}) = c_n xv^{3n}+$ (lower-degree v-terms), where $c_n \in k^*$.

Now the lemma above may be applied: If $K = k[x, y, s, t, u]$, then $D|_K$ is homogeneous with respect to the grading on $K = \oplus_{i \geq 0} K_i$ defined by $k = K_0$, $x, y \in K_1$, $s \in K_3$, $t \in K_6$, and $u \in K_9$. Moreover, $Dv = x^2y^2$ does not lie in the image of $D|_K$. Using the sequence $\phi_n := \pi(xw_{3n})$, it follows from the lemma that $\ker D$ is not finitely generated.

To obtain a counterexample in dimension 5, we now simply set $y = 1$ in the above example. More precisely, if $\overline{B} = B \bmod (y-1)$ and $\overline{D} = D \bmod (y-1)$, then $\overline{B} = k[x, s, t, u, v]$, a polynomial ring in 5 variables, and $\overline{D}$ is the triangular derivation

$$\overline{D} = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v}.$$

We see that, if $\overline{K} = k[x, s, t, u]$, then $\overline{D}|_{\overline{K}}$ is homogeneous with respect to the grading on $\overline{K}$ for which $\deg x = 1$ and $\deg s = \deg t = \deg u = 3$. The elements $\overline{\phi_n}$ of $\ker \overline{D}$ are of the form $c_n xv^{3n}+$ (lower-degree v-terms). Since $\overline{D}(v) = x^2$ is not in the image of $\overline{D}|_{\overline{K}}$, we conclude by the lemma above that $\ker \overline{D}$ is not finitely generated.

Note that this last example can be simplified by changing coordinates in $k[x, s, t, u, v]$. If σ fixes x, t, u, and v, and maps s to $(s + xv)$, then

$$\sigma \overline{D} \sigma^{-1} = x^2 \frac{\partial}{\partial v} + (xv + s) \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} \ .$$

The dimension 6 and dimension 5 counterexamples may thus be summarized as follows.

Theorem 2 (see [13]). *Let $B = k[x, y, s, t, u, v]$ be the polynomial ring in 6 variables over k, and let D be the triangular derivation on B defined by*

$$D = (x^3) \frac{\partial}{\partial s} + (y^3 s) \frac{\partial}{\partial t} + (y^3 t) \frac{\partial}{\partial u} + (x^2 y^2) \frac{\partial}{\partial v}.$$

Then the kernel of D is not finitely generated as a k-algebra.

Theorem 3 (see [3]). *Let $A = k[a, b, x, y, z]$ be the polynomial ring in 5 variables over k, and let d be the triangular derivation on A defined by*

$$d = a^2 \frac{\partial}{\partial x} + (ax + b) \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.$$

Then the kernel of d is not finitely generated as a k-algebra.

In [13], the proof of the main result (*Theorem 1* above) gives an algorithm for constructing the polynomials w_n. So we also get an algorithm for constructing the sequence of kernel elements $f_n = xv^n +$ (lower v-terms) for the six-dimensional example D; and by setting $y = 1$ in f_n, we get the corresponding sequence for the five-dimensional example $\overline{D}$. In addition, at the end of [13] it is shown that there exists a homomorphism $\rho : k[x, y, s, t, u, v] \to k[X, Y, Z, S, T, U, V]$ for which $\rho(\ker \mathrm{D}) \subset \ker \mathcal{D}$, and which transforms the kernel elements $f_n = xv^n +$ (lower v-terms) into those of the form $XV^n +$ (lower V-terms). We thus obtain a new proof for Roberts' example, as well as an algorithm for constructing the kernel elements $XV^n +$ (lower V-terms).

The description of recent developments given here is chronological. We could also start in dimension four and see that these counterexamples to Hilbert's Fourteenth Problem in dimensions five, six, and seven all stem from the simple Weitzenböck (linear) derivation in dimension four, namely

$$\delta = a\frac{\partial}{\partial s} + s\frac{\partial}{\partial t} + t\frac{\partial}{\partial u}$$

on $k[a, s, t, u]$. This is just the quotient of the derivation Δ above gotten by setting $b = 1$. The crucial fact about δ is the existence of the sequence $\overline{w_n}$ for which $\delta\overline{w_n} = a^{\tau(n)}\overline{w}_{n-1}$, where $\tau(n) = 0$ if $n \equiv 2$ modulo 3, and $\tau(n) = 1$ otherwise. The sequence $\overline{w_n}$ is gotten by setting $b = 1$ in the sequence w_n, and we thus have an algorithm for constructing $\overline{w_n}$.

3.5. A new example. Here is another example of a non-finitely generated kernel in dimension six related to Roberts' example $\mathcal{D}$. Observe that the polynomial $P := (X^{t+1}T - Y^{t+1}S)$ is in the kernel of $\mathcal{D}$. Also, Z is in the kernel, and the polynomial $(Z - P)$ is a triangular variable of $B = k[X, Y, Z, S, T, U, V]$ not involving V. Therefore $B/(Z - P)$ is a polynomial ring in 6 variables with quotient derivation $\overline{\mathcal{D}}$. In particular, if $B/(Z - P) = k[X, Y, S, T, U, V]$, then

$$\overline{\mathcal{D}} = (X^{t+1})\frac{\partial}{\partial S} + (Y^{t+1})\frac{\partial}{\partial T} + (P^{t+1})\frac{\partial}{\partial U} + (X^tY^tP^t)\frac{\partial}{\partial V}.$$

Taking quotients of kernel elements of $\mathcal{D}$, we see that the kernel of $\overline{\mathcal{D}}$ contains elements of the form $XV^n +$ (lower V-terms) for each $n \geq 0$, and the same reasoning as above leads to the conclusion that the kernel of $\overline{\mathcal{D}}$ is not finitely generated.

Note that $\overline{\mathcal{D}}$ is triangular, and has the additional nice property that $\overline{\mathcal{D}}^2$ is 0 when applied to each of the generators X, Y, S, T, U, and V. Such phenomena are discussed in Section 4.2 below.

4. Positive results

4.1. Dimension four. Naturally, one would like to know whether a counterexample to Hilbert's Fourteenth Problem can be found in dimension four, and to date this question is open. *Every* counterexample above is triangular, i.e., the invariant ring of a group acting on $\mathbb{A}^n$ by triangular automorphisms. So in dimension four we first examine triangular derivations, and here we have the following positive result, due to the author and Daigle.

Theorem 4 [5]**.** *Let k be an algebraically closed field of characteristic zero, and let R be a k-affine Dedekind domain or a localization of such a ring. The kernel of any triangular R-derivation of $R[x,y,z]$ is finitely generated as an R-algebra.*

This result easily implies a positive answer to our question when k is algebraically closed.

Corollary 1. *If T is a triangular derivation of $k[w,x,y,z]$, then the kernel of T is finitely generated.*

In this case, we may assume with no loss of generality that $Tw = 0$. Thus, T is a triangular R-derivation of $R[x,y,z]$, where $R = k[w]$, and the theorem implies that $\ker T$ is finitely generated. We remark that a special case of the corollary was earlier proved in [21].

Finite generation notwithstanding, $\ker T$ may be very complicated. In [4] we prove the following.

Theorem 5. *For each integer $n \geq 3$, there exists a triangular derivation of $k[w,x,y,z]$ whose kernel, though finitely generated, cannot be generated by fewer than n elements.*

The actual construction of such derivations is a bit complicated, and the reader should see the article for details.

Finally, it should be noted that *Theorem 4* fails for more general rings R. For example, if $R = k[a,b]$, a polynomial ring in two variables over k, then the derivation d of *Theorem 3* is a triangular R-derivation of $R[x,y,z]$ with non-finitely generated kernel.

4.2. Additional results. The foregoing examples suggest three categories of derivations to study. Let D be a locally nilpotent derivation of the polynomial ring $B = k[x_1,\ldots,x_n]$.

(i) D is *monomial* if each image Dx_i is a monomial in $x_1,\ldots,x_n$;

(ii) D is *elementary* if $k[x_1,\ldots,x_i] \subset \ker D$ and $Dx_{i+1},\ldots,Dx_n \in k[x_1,\ldots,x_i]$;

(iii) D is *nice* if $D^2 x_i = 0$ for each i.

Note that an elementary derivation is both triangular and nice. Of the counterexamples above, we have:

dim 7: $\mathcal{D}$ is elementary monomial

dim 6: $\overline{\mathcal{D}}$ is nice, non-elementary, non-monomial

dim 6: D is triangular monomial, non-nice

dim 5: $\overline{D}$ is triangular, non-nice, non-monomial

Two recent positive results concern elementary derivations.

Theorem 6 (van den Essen, Janssen [42]). *Let D be an elementary derivation of $B = k[x_1, \ldots, x_n]$ for which $Dx_1 = \cdots = Dx_i = 0$ and $Dx_j \in k[x_1, \ldots, x_i]$ for $j > i$.*

(*a*) *If either $i \le 2$ or $(n - i) \le 2$ then* $\ker D$ *is finitely generated.*

(*b*) *If* 1 *is in the ideal generated by the image of D, then* $\ker D$ *is a polynomial ring.*

Theorem 7 (Khoury [19]). *For $n \le 6$, the kernel of every elementary monomial derivation of $k^{[6]}$ is generated by at most 6 elements.*

From a geometric point of view, counterexamples to Hilbert's problem show that, in general, the ring of invariants of an algebraic group acting on an affine variety need not be the coordinate ring of an affine variety. However, the recent result of Winkelmann [44] asserts that these rings are always at least quasi-affine, that is, isomorphic to the coordinate ring of a Zariski-open subset of an affine variety.

For other positive results in a more general setting, the reader is referred to the very nice recent book of Grosshans [14]. In this book, the author studies invariant rings for non-reductive groups, and gives an up-to-date account of the subject. The book does a good job of bringing together the classical (Nineteenth Century) and modern points of view. In addition to the positive results, he also considers the counterexamples of Nagata and Roberts (Chapter 2.8), and points out that these extend to give counterexamples for any non-reductive group (p. 47).

Along similar lines, the reader is also referred to the article of Hochschild and Mostow [17], where the finite generation of the ring of invariants is established for a special class of unipotent groups.

4.3. Open questions II. As mentioned, the following question is open.

- *Question 5.* If D is a locally nilpotent derivation of $k[x,y,z,w]$, is the kernel of D always finitely generated?

Note that every locally nilpotent derivation of $k[x,y]$ is triangular in some coordinate system (see [33]), but that for any $n \geq 3$, there are locally nilpotent derivations of $k[x_1,\dots,x_n]$ which are not triangular in any coordinate system; see for example [2] and [30]. We are a long way from being able to classify the locally nilpotent derivations of polynomial rings in any usable sense, even in dimension three, so this question may be very difficult.

Our next question appears in Nagata's 1959 paper [24] as *Problem 2.* In spite of being very important, the question seems to have been largely ignored.

- *Question 6.* (NAGATA'S SECOND PROBLEM) Let K be a subfield of $k(x_1,\dots,x_n)$ such that $\dim_k K = 3$. Is $K \cap k[x_1,\dots,x_n]$ always finitely generated?

Recall that the counterexample he presents in this paper has $\dim_k K = 4$. Also, the above counterexample in dimension 5 (the kernel of the derivation d) has $\dim_k(\ker d) = 4$. On the other hand, Rees' counterexample to the Zariski problem has dimension three.

Finally, note that we now have some candidates for counterexamples over finite fields. In particular, consider the following (non-linear) $\mathbb{G}_a$-actions. They are obtained as exponentials of locally nilpotent derivations in dimensions 7 and 5 as above, but their invariant rings no longer equal the kernel of the derivation.

- *Question 7.* Let k be a finite field, and define a $\mathbb{G}_a$-action on $\mathbb{A}_k^7$ by: $t(X,Y,Z,S,T,U,V)$ equals

  $$(X,Y,Z,S+tX^3,T+tY^3,U+tZ^3,V+t(XYZ)^2)\ .$$

  Is the invariant ring of this action finitely generated?

- *Question 8.* Let k be a finite field of characteristic $p \neq 2,3$, and define a $\mathbb{G}_a$-action on $\mathbb{A}_k^5$ by: $t(a,b,x,y,z)$ equals

  $$\left(a,b,x+ta^2,y+t(ax+b)+\frac{t^2}{2}a^3,z+ty+\frac{t^2}{2}(ax+b)+\frac{t^3}{6}a^3\right).$$

  Is the invariant ring of this action finitely generated?

Note that in order to answer these last 2 questions, we can replace the finite field k with any overfield K, for example, K equal to the algebraic closure of k; see [18], (6.3).

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REFERENCES

[1] A. A'Campo-Neuen. Note on a counterexample to Hilbert's fourteenth problem given by P. Roberts. *Indag. Math., N. S.* **5** (1994), 253–257.

[2] H. Bass. A non-triangular action of G_a on A^3. *J. Pure Appl. Algebra* **33**, (1984), 1–5.

[3] D. Daigle, G. Freudenburg. A counterexample to Hilbert's Fourteenth Problem in dimension five. *J. Algebra*, **221** (1999), 528–535.

[4] D. Daigle, G. Freudenburg. A note on triangular derivations of $k[X_1, X_2, X_3, X_4]$, *Proc. Amer. Math. Soc.* **129** (2001), 657–662.

[5] D. Daigle, G. Freudenburg. Triangular derivations of $k[X_1, X_2, X_3, X_4]$. *J. Algebra* **241** (2001), 328–339.

[6] H. Derksen. The kernel of a derivation. *J. Pure Appl. Algebra* **84**, (1993), 13–16.

[7] H. Derksen. Computation of invariants for reductive groups. *Adv. Math.* **141** (1999), 366–384.

[8] J. K. Deveney, D. R. Finston. G_a-actions on $\mathbb{C}^3$ and $\mathbb{C}^7$. *Comm. Algebra* **22** (1994), 6295–6302.

[9] J. Dieudonne, J. Carrell. Invariant theory, old and new. *Adv. Math.* **4** (1970), 1–80.

[10] J. Dixmier. Enveloping Algebras. North-Holland Publishing, Amsterdam, New York, Oxford, 1977.

[11] A. Fauntleroy. On Weitzenböck's theorem in positive characteristic. *Proc. Amer. Math. Soc.* **64** (1977), 209–213.

[12] J. Fogarty. Invariant Theory. Benjamin, New York, 1969.

[13] G. Freudenburg. A counterexample to Hilbert's Fourteenth Problem in dimension six. *Transformation Groups* **5** (2000), 61–71.

[14] F. D. Grosshans. Algebraic Homogeneous Spaces and Invariant Theory. Lect. Notes in Math. vol. **1673**, Springer Verlag, 1997.

[15] D. Hilbert. Mathematische Probleme. *Archiv der Math. und Physik* **1** (1901), 44–63, 213–237.

[16] D. Hilbert. Mathematical Problems. Bull. Amer. Math. Soc. **8** (1902), 437–479.

[17] G. Hochschild, G. D. Mostow. Unipotent groups in invariant theory. *Proc. Nat. Acad. Sci. USA* **70** (1973), 646–648.

[18] J. E. Humpreys. Hilbert's Fourteenth Problem. *Amer. Math. Monthly* **70** (1978), 341–353.

[19] J. Khoury. On some properties of elementary derivations in dimension six. *J. Pure Appl. Algebra* **156** (2001), 69-79.

[20] H. Kojima, M. Miyanishi. On P. Roberts' counterexample to the fourteenth problem of Hilbert. *J. Pure Appl. Algebra* **122** (1997), 247–268.

[21] S. Maubach. Triangular monomial derivations on $k[X_1, X_2, X_3, X_4]$ have kernel generated by at most four elements. *J. Pure Appl. Algebra* **153** (2000), 165–170.

[22] D. Mumford. Hilbert's fourteenth problem–the finite generation of subrings such as rings of invariants. Proc. Symp. Pure Math. vol. **28**, Amer. Math. Soc., Providence, 1976, 431–444.

[23] D. Mumford, J. Fogarty. Geometric Invariant Theory (Third enlarged edition). Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **34**, Springer-Verlag, 1994.

[24] M. Nagata. On the 14-th Problem of Hilbert. *Amer. J. Math.* **81**, 1959, 766–772.

[25] M. Nagata. On the Fourteenth Problem of Hilbert. Proc. I.C.M. 1958, Cambridge University Press, 1960, 459–462.

[26] M. Nagata. Lectures on the Fourteenth Problem of Hilbert. Lecture Notes vol. **31**, Tata Inst., Bombay, 1965.

[27] P. E. Newstead. Introduction to Moduli Problems and Orbit Spaces. Tata Institute, Bombay, 1978.

[28] A. Nowicki. Polynomial Derivations and their Rings of Constants. Uniwersytet Mikolaja Kopernika, Toruń, 1994.

[29] A. Nowicki, M. Nagata. Rings of constants for k-derivations in $k[x_1, \ldots, x_n]$. *J. Math. Kyoto Univ.* **28** (1988), 111–118.

[30] V. L. Popov. On actions of G_a on A^n. Algebraic groups, Utrecht 1986, Lectures Notes in Math., vol. **1271**, Springer-Verlag, 1987, 237–242.

[31] V. L. Popov, Groups, Generators, Syzygies, and Orbits in Invariant Theory. Translations of Math. Monographs, vol. **100**, Amer. Math. Soc., Providence, 1992.

[32] D. Rees. On a problem of Zariski. *Illinois J. Math.* **2** (1958), 145–149.

[33] R. Rentschler. Opérations du groupe additif sur le plan affine. *C. R. Acad. Sci. Paris* **267**, (1968), 384–387.

[34] P. Roberts. A prime ideal in a polynomial ring whose symbolic blow-up is not noetherian. *Proc. Amer. Math. Soc.*, **94**, (1985), 589–592.

[35] P. Roberts. An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem. *J. Algebra* **132** (1990), 461–473.

[36] C. S. Seshadri. On a theorem of Weitzenböck in invariant theory. *J. Math. Kyoto Univ.* **1** (1962), 403–409.

[37] D. M. Snow. Unipotent actions on affine space. Topological Methods in Algebraic Transformation Groups, Progress in Mathematics, vol. **80**, Birkhäuser, 1989, 165–176.

[38] R. Steinberg. Nagata's example. Algebraic Groups and Lie Groups, Cambridge University Press, 1997, 375–384.

[39] L. Tan. An algorithm for explicit generators of the invariants of the basic G_a-actions. *Comm. Algebra* **17** (1989), 565–572.

[40] A. Tyc. An elementary proof of the Weitzenböck theorem. *Colloq. Math.* **78** (1998), 123–132.

[41] A. van den Essen. Polynomial Automorphisms and the Jacobian Conjecture. Birkhauser, Boston, 2000.

[42] A. van den Essen, T. Janssen. Kernels of elementary derivations, Dept. of Mathematics, Univ. Nijmegen, No. 9548, 1995.

[43] R. Weitzenböck. Über die Invarianten von linearen Gruppen. *Acta Math.* **58** (1932), 231–293.

[44] J. Winkelmann. Hilbert's 14th Problem, (to appear).

[45] O. Zariski. Interpretations algebrico-geometriques du quatorzieme problem de Hilbert. *Bull. Sci. Math.* **78** (1954), 155–168.

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