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WEAK POLYNOMIAL IDENTITIES FOR $M_{1,1}(E)$

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Abstract. We compute the cocharacter sequence and generators of the ideal of the weak polynomial identities of the superalgebra $M_{1,1}(E)$.

1. Introduction. Since the 80s, together with the usual matrix algebras over a field, a fundamental role in the theory of PI-algebras has been played by the superalgebras $M_{k,l}(E)$ of matrices with entries in the Grassmann algebra E [Kem]. A main purpose in the study of $M_{k,l}(E)$ is the description of bases for ideals of polynomial identities satisfied by such algebras. In particular, Razmyslov [Raz2, Raz3] introduced the notion of "weak polynomial identity" for both the algebras $M_n(F)$ and $M_{k,l}(E)$, and explained how these identities are correlated with the central polynomials and the identities in the traces. The concept of weak polynomial identity has been further generalized in the context of Jordan and Lie

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algebras (see [Kos]). In the present paper, for a characteristic zero base field, we show that the ideal T of the weak polynomial identities for the superalgebra $M_{1,1}(E)$ is generated by $[[x_1, x_2], x_3]$ and a suitable proper multilinear polynomial of degree 6. The ideals of weak polynomial identities containing $[[x_1, x_2], x_3]$ have been studied by Volichenko [Vo] as an important step to his result that a variety of Lie algebras $\mathfrak{AN}_2$ defined by the identity $[[x_1, x_2, x_3], [x_4, x_5, x_6]] = 0$ satisfies the Specht property. The results of [Vo] may be used to obtain alternative proofs of some of our results. Our approach consists in determining the structure of the representation of the symmetric group over the vector space of the proper multilinear polynomials of the ideal T. Such description allows us to compute the cocharacter sequence together with the generating function of the codimension sequence of T. Note that Razmyslov used the weak identities of the pair $(M_2(F), sl_2(F))$ to describe the polynomial identities of the algebra $M_2(F)$. We expect that our results may be used also to give a new proof of the description of the polynomial identities of $M_{1,1}(E)$ given by Popov [Po].

2. Preliminaries. Let F be a field and denote by $F\langle X\rangle$ the free associative algebra generated by a countable set $X = \{x_1, x_2, \ldots\}$. If R is an associative algebra and $S \subset R$ is a vector space, then the polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is called *weak polynomial identity* for the pair (R, S) if $f(s_1, \ldots, s_n) = 0$, for all the elements $s_1, \ldots, s_n \in S$. The set of all weak polynomial identities is an ideal $T = T(R, S)$ of $F\langle X\rangle$. It is well known that, for an adequate description of T, it is convenient to determine endomorphisms of $F\langle X\rangle$ which stabilizes T, that is to establish rules that allows to take consequences from any set of weak polynomial identities. More precisely, let Ω be a non-empty subset of $F\langle X\rangle$ such that $\omega(s_1, \ldots, s_n) \in S$, for all $\omega \in \Omega$ and $s_1, \ldots, s_n \in S$. If the polynomial $f(x_1, \ldots, x_n)$ is a weak polynomial identity for R, then clearly $f(\omega_1, \ldots, \omega_n) \in T$, for every choice of elements $\omega_1, \ldots, \omega_n \in \Omega$. In other words, the ideal T is stable under the endomorphisms of the algebra $F\langle X\rangle$ corresponding to polynomials in Ω. In general, each ideal $I \subset F\langle X\rangle$ which verifies such property is called Ω-*stable*. Let now B be a non-empty subset of $F\langle X\rangle$. A polynomial $g(x_1, \ldots, x_m)$ is an Ω-*consequence* of B if g belongs to the minimal Ω-stable ideal $I \subset F\langle X\rangle$ containing B. Moreover, we say that B is an Ω-*generating set* of the ideal I.

Let now F be a field of characteristic zero. We denote by E the Grassmann algebra generated by a vector space on F of countable dimension, and by E_0, E_1 the homogeneous components of the $\mathbb{Z}_2$-graduation of E. We define R the following matrix superalgebra:

$$R = M_{1,1}(E) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in E_0,\, b, c \in E_1 \right\}$$

It is well known that a *supertrace* is defined for $M_{1,1}(E)$ in the following way:

$$\mathrm{str}(A) = a - d$$

This supertrace verifies the usual properties of traces. We denote by S the vector space given by all the matrices of R with supertrace equal to zero. We define then $T = T(R, S)$, the ideal of $F\langle X\rangle$ of the weak polynomial identities for the pair (R, S). Let now Ω be the subspace of $F\langle X\rangle$ spanned by the set $X \cup \{1\}$. The main purpose of this paper is to compute an Ω-generating set for the ideal T.

Since $\mathrm{char}(F) = 0$, by the linearization process we have that each Ω-stable ideal $I \subset F\langle X\rangle$ is determined by its multilinear components. Moreover, from our definition of Ω it follows that I is completely determined also by its proper multilinear components. More precisely, let V_n be the vector space of the multilinear polynomials of $F\langle X\rangle$ in the indeterminates $x_1, \ldots, x_n$, and put $V_n(I) = V_n/(V_n \cap I)$. Note that the symmetric group $\mathbb{S}_n$ acts on $V_n(I)$ in the natural way. Denote by $\chi_n(I)$ the character of this representation. Since F is a characteristic zero field, we have:

$$\chi_n(I) = \sum_{\lambda} m_\lambda \chi_\lambda$$

where λ ranges over the partitions of n, χ_λ is the character of the irreducible representation corresponding to λ and $m_\lambda \geq 0$ is an integer. We denote also by Γ_n the $\mathbb{S}_n$-submodule of V_n given by the proper multilinear polynomials of $F\langle X\rangle$ in $x_1, \ldots, x_n$ (see [Dre2]). We put $\Gamma_n(I) = \Gamma_n/(\Gamma_n \cap I)$ and let $\xi_n(I)$ denote the character of $\Gamma_n(I)$. Finally, we define $c_n(I), \gamma_n(I)$ the dimensions of the vector spaces $V_n(I), \Gamma_n(I)$ respectively, for any $n \geq 0$, and we denote by $c(I, z), \gamma(I, z)$ the generating functions of such sequences.

Proposition 2.1. *For any integer $n \geq 0$, it holds:*

$$\chi_n(I) = \sum_{i=0}^{n} \chi_{(n-i)} \hat{\otimes} \xi_i(I) \tag{1}$$

$$c_n(I) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i(I) \tag{2}$$

$$c(I, z) = \frac{1}{1-z} \gamma\left(I, \frac{z}{1-z}\right) \tag{3}$$

where $\chi_{(n-i)}$ is the character of the irreducible representation of $\mathbb{S}_{n-i}$ corresponding to the partition $\lambda = (n-i)$, and $\hat{\otimes}$ denotes the outer product between characters of the symmetric group.

Proof. The argument follows verbatim the proof of the similar result for the ordinary case of polynomial identities. The starting point is the following remark. Let $L(X) \subset F\langle X\rangle$ be the free Lie algebra generated by $X = \{x_1, x_2, \ldots\}$. By the Poincaré-Birkoff-Witt theorem, the algebra $F\langle X\rangle$ as the basis:

$$x_1^{p_1} \cdots x_n^{p_n} [x_{i_1}, x_{i_2}]^{q_1} \cdots [x_{l_1}, \ldots, x_{l_p}]^{q_m}$$

where $p_1, \ldots, p_n, q_1, \ldots, q_m \geq 0$ are integers and $[x_{i_1}, x_{i_2}] < \ldots < [x_{l_1}, \ldots, x_{l_p}]$ are in the ordered basis of $L(X)$. Hence, we may write each polynomial of $F\langle X\rangle$ in the following way:

$$f(x_1, \ldots, x_n) = \sum_{p,q} \alpha_{p,q} x_1^{p_1} \cdots x_n^{p_n} u_1^{q_1} \cdots u_m^{q_m}$$

where $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_m)$, $\alpha_{p,q}$ belongs to F and $u_1, \ldots, u_m$ are commutators. Since the substitution $x \mapsto x + 1$ is allowed by the definition of Ω, we have that if $f(x_1, \ldots, x_n) \in I$ then, for any fixed p, the proper polynomial $f_p = \sum_q \alpha_{p,q} u_1^{q_1} \cdots u_m^{q_m}$ belongs to I. Now, the argument proceeds as in [Dre2], Proposition 4.3.3. □

3. Computing the identities. Note that $[[x_1, x_2], x_3]$ is a weak polynomial identity for $R = M_{1,1}(E)$. Actually, for each $s_1, s_2 \in S$, the commutator $[s_1, s_2]$ is in the center of R. It follows immediately that $\Gamma_{2m+1}(T) = 0$ $(m \geq 0)$ and the space $\Gamma_{2m}(T)$ is generated as $F(\mathbb{S}_{2m})$-module by the element:

$$[x_1, x_2] \cdots [x_{2m-1}, x_{2m}] + (\Gamma_{2m} \cap T)$$

We will compute explicitely a basis for the vector space $\Gamma_{2m}(T)$ and prove that its dimension is $\gamma_{2m}(T) = \binom{2m-1}{m}$. For this purpose, let us introduce the following notation. For any subset $A = \{x_{i_1}, \ldots, x_{i_n}\} \subset X$, we denote by V_A the subspace of $F\langle X\rangle$ of all multilinear polynomials of degree n in the indeterminates $x_{i_1}, \ldots, x_{i_n}$. As usual, we put $V_A(T) = V_A/(V_A \cap T)$. In the same way, we define Γ_A and $\Gamma_A(T)$.

Lemma 3.1. *The polynomial*

$$p(x_1, \ldots, x_6) = \sum_{\sigma \in \mathbb{S}(\{4,5,6\})} [x_1, x_{\sigma(4)}][x_2, x_{\sigma(5)}][x_3, x_{\sigma(6)}]$$

is a weak polynomial identity for R.

Proof. Recall that R is a $\mathbb{Z}_2$-graded algebra with homogeneous components:

$$R_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in E_0 \right\} \quad \text{and} \quad R_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid b, c \in E_1 \right\}$$

Note that $S = (S \cap R_0) \oplus (S \cap R_1)$. Since $S \cap R_0$ is in the center of R, a proper multilinear polynomial $f(x_1, \dots, x_n)$ is a weak identity for R if and only if it vanishes for all substitutions of indeterminates with elements in $S \cap R_1 = R_1$, that is if f is a polynomial identity for the odd $\mathbb{Z}_2$-component of R.

In particular, the polynomial $[[x_1, x_2], x_3]$ vanishes on R_1. Moreover, it is proved in [DiVi] that $g := x_1x_2x_3 + x_3x_2x_1$ is also a polynomial identity for the odd component of R. By commuting progressively g with the variables x_4, x_5, x_6, we obtain a polynomial congruent to $2p$ modulo the consequences of $[[x_1, x_2], x_3]$. Since p is a proper polynomial, we have so proved that p is a weak polynomial identity for R. $\square$

Let I denote the ideal of $F\langle X\rangle$ which is Ω-generated by the polynomials $[[x_1, x_2], x_3]$ and $p(x_1, \dots, x_6)$. Define the vector space $\Gamma_{2m}(I)$ as the quotient $\Gamma_{2m}/(\Gamma_{2m} \cap I)$ and denote $\gamma_{2m}(I) = \dim_F \Gamma_{2m}(I)$. In the same way, we may define $\Gamma_{2m+1}(I)$ and $\gamma_{2m+1}(I)$, and we get $\gamma_{2m+1}(I) = 0$. We want to prove that $T = I$.

Let $A = \{x_{i_1}, \dots, x_{i_{2n}}\}$ be a subset of X and consider for the polynomials of type $[x_{a_1}, x_{b_1}] \cdots [x_{a_n}, x_{b_n}]$ ($\{x_{a_1}, x_{b_1}, \dots, x_{a_n}, x_{b_n}\} = A$) the following conditions:

1. $a_i < b_i$ $(i = 1, 2, \dots, n)$
2. $a_1 < a_2 < \dots < a_n$
3. there exists no integers $p < q < r$ such that $b_p < b_q < b_r$

Denote by C_A the set of such polynomials which verify conditions 1 and 2. Let $B_A \subset C_A$ be the subset of polynomials which verify also condition 3. Moreover, for $A = \{x_1, \dots, x_{2m}\}$ we will write simply $C_{2m} = C_A, B_{2m} = B_A$. Each set C_A can be totally ordered in the following way:

$$[x_{a_1}, x_{b_1}] \cdots [x_{a_n}, x_{b_n}] < [x_{c_1}, x_{d_1}] \cdots [x_{c_n}, x_{d_n}] \iff$$

$$(a_1, b_1, \dots, a_n, b_n) < (c_1, d_1, \dots, c_n, d_n) \text{ in the lexicographic order}$$

With such notation, we prove:

Lemma 3.2. *A generating set for the vector space $\Gamma_{2m}(I)$ is given by the polynomials of B_{2m}, for any integer $m \geq 0$.*

Proof. Since $[x_i, x_j] = -[x_j, x_i]$ and $[[x_1, x_2], x_3] \in I$, we have clearly that a generating set of $\Gamma_{2m}(I)$ is given by the polynomials of C_{2m} (note that $\# C_{2m} = \prod_{i=1}^{m}(2i-1)$). Let us introduce the following notation:

$$[x_{a_1}, x_{b_1}][x_{a_2}, x_{b_2}] \cdots [x_{a_m}, x_{b_m}] = \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{bmatrix}$$

Note immediately that $C_{2m} = B_{2m}$, for any $m < 3$. For $m = 3$, among the 15 elements of C_6, five of them do not satisfy the condition 3. Precisely, they are:

$$\begin{bmatrix} 1\,3\,5 \\ 2\,4\,6 \end{bmatrix}, \begin{bmatrix} 1\,3\,4 \\ 2\,5\,6 \end{bmatrix}, \begin{bmatrix} 1\,2\,5 \\ 3\,4\,6 \end{bmatrix}, \begin{bmatrix} 1\,2\,4 \\ 3\,5\,6 \end{bmatrix}, \begin{bmatrix} 1\,2\,3 \\ 4\,5\,6 \end{bmatrix} \tag{4}$$

Moreover, the polynomial p determines in $\Gamma_6(I)$ the following identities:

$$\begin{bmatrix} 1\,3\,5 \\ 2\,4\,6 \end{bmatrix} + \begin{bmatrix} 1\,3\,5 \\ 2\,6\,4 \end{bmatrix} + \begin{bmatrix} 1\,3\,5 \\ 4\,2\,6 \end{bmatrix} + \begin{bmatrix} 1\,3\,5 \\ 6\,2\,4 \end{bmatrix} + \begin{bmatrix} 1\,3\,5 \\ 4\,6\,2 \end{bmatrix} + \begin{bmatrix} 1\,3\,5 \\ 6\,4\,2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1\,3\,4 \\ 2\,5\,6 \end{bmatrix} + \begin{bmatrix} 1\,3\,4 \\ 2\,6\,5 \end{bmatrix} + \begin{bmatrix} 1\,3\,4 \\ 5\,2\,6 \end{bmatrix} + \begin{bmatrix} 1\,3\,4 \\ 6\,2\,5 \end{bmatrix} + \begin{bmatrix} 1\,3\,4 \\ 5\,6\,2 \end{bmatrix} + \begin{bmatrix} 1\,3\,4 \\ 6\,5\,2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1\,2\,5 \\ 3\,4\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,5 \\ 3\,6\,4 \end{bmatrix} + \begin{bmatrix} 1\,2\,5 \\ 4\,3\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,5 \\ 6\,3\,4 \end{bmatrix} + \begin{bmatrix} 1\,2\,5 \\ 4\,6\,3 \end{bmatrix} + \begin{bmatrix} 1\,2\,5 \\ 6\,4\,3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1\,2\,4 \\ 3\,5\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,4 \\ 3\,6\,5 \end{bmatrix} + \begin{bmatrix} 1\,2\,4 \\ 5\,3\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,4 \\ 6\,3\,5 \end{bmatrix} + \begin{bmatrix} 1\,2\,4 \\ 5\,6\,3 \end{bmatrix} + \begin{bmatrix} 1\,2\,4 \\ 6\,5\,6 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1\,2\,3 \\ 4\,5\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,3 \\ 4\,6\,5 \end{bmatrix} + \begin{bmatrix} 1\,2\,3 \\ 5\,4\,6 \end{bmatrix} + \begin{bmatrix} 1\,2\,3 \\ 6\,4\,5 \end{bmatrix} + \begin{bmatrix} 1\,2\,3 \\ 5\,6\,4 \end{bmatrix} + \begin{bmatrix} 1\,2\,3 \\ 6\,5\,4 \end{bmatrix} = 0$$

Owing to $[x_i, x_j] = -[x_j, x_i]$ and $[[x_1, x_2], x_3] \in I$, the above five identities make us able to rewrite in $\Gamma_6(I)$ the five polynomials (4) as a linear combination of elements of B_6 which are greater of them in the defined ordering.

We prove now for any $m > 3$. Let $w \in C_{2m}$ be the maximum in the subset of elements of C_{2m} which can't be rewritten in $\Gamma_{2m}(I)$ as a linear combination of elements of B_{2m}, say $w = \begin{bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{bmatrix}$. In particular, there exist indexes $p < q < r$ such that $b_p < b_q < b_r$ and

$$w = w_0 \begin{bmatrix} a_p \\ b_p \end{bmatrix} w_1 \begin{bmatrix} a_q \\ b_q \end{bmatrix} w_2 \begin{bmatrix} a_r \\ b_r \end{bmatrix} w_3$$

with $w_i \in C_{A_i}$ for suitable subsets $A_i \subset \{x_1, \ldots, x_{2m}\}$. We put therefore $A = \{x_{a_p}, x_{b_p}, x_{a_q}, x_{b_q}, x_{a_r}, x_{b_r}\}$ and let $w' = \begin{bmatrix} a_p & a_q & a_r \\ b_p & b_q & b_r \end{bmatrix}$. Since the polynomial $[[x_1, x_2], x_3] \in I$, we have that $w = w_0 w_1 w_2 w_3 w'$ in $\Gamma_{2m}(I)$. Clearly w' belongs to C_A and hence, using the first part of the proof, it holds that w' is a linear combination of elements $v_i \in B_A$ that are greater than w' in the fixed ordering. Then, $w = \sum_i \alpha_i w_0 w_1 w_2 w_3 v_i$ ($\alpha_i \in \mathbb{Z}$) and again each term of the sum can be written in $\Gamma_{2m}(I)$ as an element of B_{2m} greater than w. $\square$

We want now to compute the number of elements of the generating set B_{2m}. For doing this, let us introduce the following notation. For any $i = 2, 3, \ldots, 2m$, let $B_{2m,i}$ be the subset of B_{2m} of the generators of type $[1, i]\, w$, with w a product of commutators. Denote moreover $\beta_m = \#B_{2m}$ and $\beta_{m,i} = \#B_{2m,i}$ (hence $\beta_m = \beta_{m,2} + \beta_{m,3} + \cdots + \beta_{m,2m}$).

Proposition 3.3. *For any integer $m \geq 1$, it holds:*

i) $\beta_{m,2} = 1$

ii) $\beta_{m+1,i} = \sum_{j=2}^{\min(i-1,2m)} \beta_{m,j} \qquad \forall i = 3, 4, \ldots, 2(m+1)$

P r o o f. We prove (i). Let w be any element of $B_{2m,i}$. With the notation of Lemma 3.2, we put:

$$w = \begin{bmatrix} 1 & a_2 & \ldots & a_m \\ 2 & b_2 & \ldots & b_m \end{bmatrix}$$

with $\{a_2, b_2, \ldots, a_m, b_m\} = \{3, 4, \ldots, 2m\}$. Since $2 < b_k$ for any k, from the condition 3 of B_{2m} it follows that $b_k > b_{k+1}$. Owing to the conditions 1,2, we conclude that $(a_2, \ldots, a_m, b_m, \ldots, b_2) = (3, 4, \ldots, 2m)$ and hence $\beta_{m,2} = 1$.

We prove now (ii). For simplify the notation, assume $\beta_{m,j} = 0$ for $j > 2m$. Let w be any element of $B_{2(m+1),i}$ and put $A_{m+1} = \{1, 2, \ldots, 2m+2\}$. Since 2 is the minimum of the set $A_{m+1} \setminus \{1, i\}$, we have:

$$w = \begin{bmatrix} 1 & 2 & a_2 & \ldots & a_{m+1} \\ i & j & b_2 & \ldots & b_{m+1} \end{bmatrix}$$

with $j \in \{3, 4, \ldots, 2m+2\}, j \neq i$. Moreover, we define:

$$B'_{2(m+1),i} = \{w \in B_{2(m+1),i} \,|\, i < j\} \quad \text{and} \quad B''_{2(m+1),i} = \{w \in B_{2(m+1),i} \,|\, i > j\}$$

If $i < j$ then by the condition 3, we have that $j = \max\{3, 4, \ldots, 2m+2\} = 2m+2$. Then, we put $A'_{m+1} = A_{m+1} \setminus \{2, 2m+2\}$ and we order this set in the natural way. Since $i > 2$ and $2m+2 = \max A_{m+1}$, from the condition 3 it follows that the sets $B'_{2(m+1),i}$, $B_{\{x_k | k \in A'_{m+1}\}}$ and $B_{2m,i-1}$ have all the same number of elements which is $\beta_{m,i-1}$.

We assume now $i > j$. We put $A''_{m+1,j} = A_{m+1} \setminus \{1, i\}$ with the natural ordering. Then, by the condition 3 we have that the sets:

$$B''_{2(m+1),i}, \quad \bigcup_{j=3}^{i-1} B_{\{x_k | k \in A''_{m+1,j}\}} \quad \text{and} \quad \bigcup_{j=3}^{i-1} B_{2m,j-1}$$

have all the same number of elements that is $\sum_{j=3}^{i-1} \beta_{m,j-1}$. Owing to $B_{2(m+1),i} = B'_{2(m+1),i} \cup B''_{2(m+1),i}$, we conclude that $\beta_{m+1,i} = \beta_{m,i-1} + \sum_{j=3}^{i-1} \beta_{m,j-1} = \sum_{j=2}^{i-1} \beta_{m,j}$. □

4. Bounding the codimensions. In this section, we finally prove:

Proposition 4.1. *For any integer $m \geq 1$, it holds $\beta_m = \binom{2m-1}{m}$ and hence*

$$\gamma_{2m}(I) \leq \binom{2m-1}{m}$$

For proving this proposition, some new notation and preparatory results are needed. Proposition 3.3 proves that we have the equations:

$$\epsilon_{n+1,j} : \quad \beta_{n+1,j} = \sum_{k=2}^{\min(j-1,2n)} \beta_{n,k} \quad (j \geq 3)$$

with $\beta_{n,2} = 1$, for any integer $n \geq 1$. Since $\beta_m = \beta_{m,2} + \beta_{m,3} + \cdots + \beta_{m,2m}$, to prove Proposition 4.1 it is sufficient to show that:

$$\beta_{m+1,2(m+1)} = \binom{2m-1}{m}$$

Example 4.2. $\beta_{m,i}$, *for* $m = 1, \ldots, 5$.

m/i	2	3	4	5	6	7	8	9	10
1	1								
2	1	1	1						
3	1	1	2	3	3				
4	1	1	2	4	7	10	10		
5	1	1	2	4	8	15	25	35	35

By eliminating $\beta_{n+1,j}$ from the equations $\epsilon_{n+1,j}$, for all the integers $n = 1, 2, \ldots, m$ and $j = 3, 4, \ldots, 2(n+1)$, we get:

$$\beta_{m+1,2(m+1)} = \eta_1\beta_{1,2} + \eta_2\beta_{2,2} + \ldots + \eta_m\beta_{m,2} = \eta_1 + \eta_2 + \ldots + \eta_m$$

for some integers $\eta_n \geq 1$ $(n = 1, 2 \ldots, m)$. Clearly we have $\eta_m = 1$. In the elimination process, note that the equation $\epsilon_{m+1,2(m+1)}$ is just used once. Then, we say that the multiplicity of the equation $\epsilon_{m+1,2(m+1)}$ is equal to 1. In general, for $n = 2, 3, \ldots, m$ and $k = 3, 4, \ldots, 2n$, we may define recursively the *multiplicity* of the equation $\epsilon_{n,k}$ as the number of times that $\beta_{n,k}$ occurs in the equations $\epsilon_{n+1,j}$, the latter computed with their multiplicity. We will denote by $\eta_{n-1,k}$ the multiplicity of $\epsilon_{n,k}$. From the shape of these equations it follows therefore:

$$\eta_{n-1} = \eta_{n-1,3} + \eta_{n-1,4} + \ldots + \eta_{n-1,2n}$$

Moreover, the integers $\eta_{n,j}$ $(n = 1, 2, \ldots, m-1$ and $j = 3, 4, \ldots, 2(n+1))$ satisfy the equations:

$$\delta_{n,j}: \quad \eta_{n,j} = \sum_{k=j+1}^{2(n+2)} \eta_{n+1,k}$$

with $\eta_{m-1,j} = 1$, for all j. The claim of Proposition 4.1 may be rewritten now as:

$$\sum_{n=1}^{m-1} \sum_{j=3}^{2(n+1)} \eta_{n,j} = \binom{2m-1}{m} - 1 \tag{5}$$

Example 4.3 $\eta_{n,j}$, *for* $n = 1, \ldots, 3$.

n/j	3	4	5	6	7	8
1	9	5				
2	5	4	3	2		
3	1	1	1	1	1	1

To prove the equality (5), it is easier to use the normalized sequence $\nu_{n,j} = \eta_{m-n-1,2m-j}$. The claim now is:

$$\sum_{n=0}^{m-2} \sum_{j=2n}^{2m-3} \nu_{n,j} = \binom{2m-1}{m} - 1 \tag{6}$$

For $n = 0, 1, \ldots, m-2$ and $j = 2n, 2n+1, \ldots, 2m-3$, the integers $\nu_{n,j}$ verify hence the recursion:

$$\kappa_{n,j}: \quad \nu_{n,j} = \sum_{k=2(n-1)}^{j-1} \nu_{n-1,k}$$

with $\nu_{0,j} = 1$, for all j.

Example 4.4. *$\nu_{n,j}$, for $n = 0, \ldots, 2$.*

n/j	0	1	2	3	4	5
0	1	1	1	1	1	1
1	0	0	2	3	4	5
2	0	0	0	0	5	9

Lemma 4.5. *For all $n = 0, 1, \ldots, m-2$ and $j = 2n, 2n+1, \ldots, 2m-3$, it holds:*

$$\nu_{n,j} = \binom{j}{n} - \binom{j}{n-2}$$

where we assume $\binom{j}{k} = 0$ if $k < 0$.

Proof. We argument by induction on n, j. The case $n = j = 0$ is obvious. Moreover, by the equations $\kappa_{n,j}$ we get the equalities:

i) $\nu_{n+1,j+1} = \nu_{n,j} + \nu_{n+1,j}$ per ogni $j \geq 2(n+1)$;

ii) $\nu_{n+1,2(n+1)} = \nu_{n,2n} + \nu_{n,2n+1}$.

For $\nu_{n+1,j+1}$ with $j \geq 2(n+1)$, the claim follows immediately from the equality (i) and the Stifel formula. For $\nu_{n+1,2(n+1)}$ we use instead the equality (ii), and by induction we get:

$$\begin{aligned}
&\nu_{n+1,2(n+1)} = \\
&\quad = \binom{2n}{n} - \binom{2n}{n-2} + \binom{2n+1}{n} - \binom{2n+1}{n-2} = \\
&\quad = \binom{2n}{n+1} + \binom{2n}{n} + \binom{2n+1}{n} - \binom{2n}{n-1} - \binom{2n}{n-2} - \binom{2n+1}{n-2} = \\
&\quad = \binom{2n+1}{n+1} + \binom{2n+1}{n} - \binom{2n+1}{n-1} - \binom{2n+1}{n-2} = \\
&\quad = \binom{2n+2}{n+1} - \binom{2n+2}{n-1}
\end{aligned}$$

□

We have finally:

P r o o f o f P r o p o s i t i o n 4.1. By induction on $m \geq 1$, we prove the equality (6). For $m = 1$, the left hand side vanishes and so the difference in the right hand side. For $m > 1$, we denote briefly $\alpha_m = \binom{2m-1}{m}$. We get:

$$\sum_{n=0}^{m-1} \sum_{j=2n}^{2m-1} \nu_{n,j} = \sum_{n=0}^{m-2} \sum_{j=2n}^{2m-3} \nu_{n,j} + \sum_{n=0}^{m-1} \nu_{n,2m-2} + \sum_{n=0}^{m-1} \nu_{n,2m-1}$$

By induction, the first term of the sum in the right hand side is equal to $\alpha_m - 1$. Moreover, still by induction and using the equations $\kappa_{n,j}$, we obtain that the second term of the sum is α_m and the third $2\alpha_m - \nu_{m-1,2m-2}$. Therefore, the sum in the left hand side equals:

$$-1 + 4\alpha_m - \nu_{m-1,2m-2} = -1 + 4\binom{2m-1}{m} - \binom{2m-2}{m-1} + \binom{2m-2}{m-3} =$$

$$-1 + \left[\frac{4(m+1)}{2(2m+1)} - \frac{m(m+1)}{2(2m+1)(2m-1)} + \frac{(m-1)(m-2)}{2(2m+1)(2m-1)}\right] \alpha_{m+1} = -1 + \alpha_{m+1}$$

□

5. Computing the cocharacters.

Lemma 5.1. *Let $2m$ be an even integer and $\lambda = (2^{2p}, 1^{2q})$ be a partition of $2m$ with $p, q \geq 0$ integers such that $m = 2p+q$. The character of the irreducible representation associated to λ is a component of the proper cocharacter $\xi_{2m}(T)$.*

P r o o f. Let us consider the following standard tableau:

$$T_\lambda = \left.\begin{array}{|c|c|} \hline 1 & 2r+1 \\ \vdots & \vdots \\ 2p & 2m \\ \hline \end{array}\right\} 2p \qquad \left.\begin{array}{|c|} \hline 2p+1 \\ \vdots \\ 2r \\ \hline \end{array}\right\} 2q$$

with $r = p + q$, and denote by e_{T_λ} the essential idempotent of the group algebra $F(\mathbb{S}_{2m})$ associated to such tableau. It is sufficient to prove that the polynomial

$$e_{T_\lambda} \cdot [x_1, x_2] \cdots [x_{2m-1}, x_{2m}]$$

is not a weak polynomial identity for R. By identifying the variables whose indexes occur in the same row of T_λ, the previous polynomial becomes a scalar multiple of $g \cdot h$, where:

$$g = \sum_{\sigma \in \mathbb{S}_{2r}} (-1)^\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(2r-1)}, x_{\sigma(2r)}]$$
$$h = \sum_{\sigma \in \mathbb{S}_{2p}} (-1)^\sigma [x_{\sigma(1)}, x_{\sigma(2)}] \cdots [x_{\sigma(2p-1)}, x_{\sigma(2p)}]$$

In the ordered basis of the vector space W which generate the Grassmann algebra E, consider the elements $a_1 < b_1 < \ldots < a_{2r} < b_{2r}$. For $i = 1, 2, \ldots, 2r$, we define the following matrices of the superalgebra $R = M_{1,1}(E)$

$$M_i = A_i + B_i = \begin{pmatrix} 0 & a_i \\ a_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}$$

Note that $[M_{\sigma(i)}, M_{\sigma(i+1)}] = 2(a_{\sigma(i)} a_{\sigma(i+1)} - b_{\sigma(i)} b_{\sigma(i+1)})I$, where I is the identity matrix, and hence $g(M_1, \ldots, M_{2r}) = \alpha I$ and $h(M_1, \ldots, M_{2p}) = \beta I$, with $\alpha, \beta \in E_0$. In the expansion of α in terms of the canonical basis of E, there occur monomials v of lenght $2r$ with an even number of vectors a_i and an even number of vectors b_j, in such a way that the set of indexes of all vectors a_i, b_j is equal to $\{1, 2, \ldots, 2r\}$. Denoted by $2s$ the number of vectors b_j occurring in v, it is easy to verify that the coefficient of v in α is:

$$(-1)^s 2^r \binom{r}{s} (2s)!(2r - 2s)!$$

In the same way, the coefficient of a monomial w in the expansion of β which has $2t$ occurrences of the vectors b_j equals:

$$(-1)^t 2^p \binom{p}{t} (2t)!(2p - 2t)!$$

Denote by γ the monomial $b_{2p+1} \cdots b_{2r}$ ($\gamma = 1$ if $p = r$) and let us verify that the product $\alpha\beta\gamma$ is different from zero in E. Let v, w be monomials of α, β respectively, and suppose that $vw\gamma \neq 0$. Let $B = \{j_1, \ldots, j_{2t}\}$ be the subset of $S = \{1, 2, \ldots, 2p\}$ given by the indexes of the vectors b_j which occurs in w. The subset of the indexes of the vectors a_i which occur in w is hence $A = S \setminus B$. Since $vw\gamma \neq 0$, then the set of indices of the vectors b_j occurring in v is precisely A. In the same way, the set of indices of the vectors a_i occurring in v is equal to $B \cup \{2p + 1, \ldots, 2r\}$. Therefore, by operating $2t$ transpositions between v and

w, we have that the product $vw\gamma$ equals the element $z = a_1 \cdots a_{2r} b_1 \cdots b_{2r}$. The product of the coefficients of v, w is then:

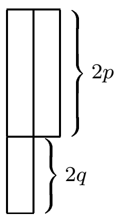
$$(-1)^{(p-t)} 2^r \binom{r}{p-t} (2p-2t)!(2r-2p+2t)! \cdot (-1)^t 2^p \binom{p}{t} (2t)!(2p-2t)!$$

which has sign $(-1)^p$ for any t. We conclude that $\alpha\beta\gamma = cz$, for some integers $c \neq 0$. $\square$

Lemma 5.2. *The sequence of proper codimensions of the ideal T satisfies, for any $m \geq 0$, the following equalities:*

1. $\gamma_{2m+1}(T) = 0$
2. $\gamma_{2m}(T) \geq \binom{2m-1}{m}$

P r o o f. Owing to Lemma 5.1, the module $\Gamma_{2m}(T)$ contains the sum of the irreducible modules of $F(\mathbb{S}_{2m})$ associated to partitions of type $\lambda = (2^{2p}, 1^{2q})$

$\lambda =$ 

with $p, q \geq 0$ integers and $m = 2p + q$. Denoted by d_λ the dimension of the irreducible representation of shape λ, by means of the hook length formula we compute easily:

$$d_\lambda = \frac{(2m)!(2q+1)}{(2p)!(2p+2q+1)!} = \frac{(2m)!(2q+1)}{(m-q)!(m+q+1)!} = \binom{2m}{m-q} - \binom{2m}{m-q-1}$$

By putting $d = \sum_\lambda d_\lambda$, we have then $\gamma_{2m}(T) \geq d$. If m is odd, from $m - q = 2p$ it follows:

$$d = \sum_{\substack{q \in \{0,\dots,m\} \\ q \text{ odd}}} \left[\binom{2m}{m-q} - \binom{2m}{m-q-1} \right] = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} = \frac{1}{2}\binom{2m}{m}$$

owing to $d - \binom{2m}{m} + d = (-1+1)^{2m} = 0$. For m an even integer, in a similar way we are still able to prove that $d = \frac{1}{2}\binom{2m}{m} = \binom{2m-1}{m}$. $\square$

Theorem 5.3. *The ideal T of the weak polynomial identities satisfied by the superalgebra $M_{1,1}(E)$ is Ω-generated by the polynomials:*

$$[[x_1, x_2], x_3] \quad \text{and} \quad \sum_{\sigma\in\mathbb{S}(\{4,5,6\})} [x_1, x_{\sigma(4)}][x_2, x_{\sigma(5)}][x_3, x_{\sigma(6)}] \tag{7}$$

Moreover, it holds $\chi_n(T) = \sum_\lambda \chi_\lambda$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ runs over all the partitions of the integer n that verify the condition:

$$r = 1 \quad \text{or} \quad \lambda_2 \leq 2 \tag{8}$$

Proof. Lemma 3.1 states that the polynomials (7) belong to the ideal T. Recall that we have defined I as the ideal of $F\langle X\rangle$ which is Ω-generated just by these polynomials. We have then $I \subset T$ and hence $\gamma_n(T) \leq \gamma_n(I)$. By using the identity $[[x_1, x_2], x_3] = 0$, we get immediately that $\gamma_{2m+1}(T) = \gamma_{2m+1}(I) = 0$, for any integer $m \geq 0$. Moreover, by means of Lemma 3.2 and Proposition 4.1 we have shown that $\gamma_{2m}(I) \leq \binom{2m-1}{m}$. Finally, Lemma 5.2 states that $\binom{2m-1}{m} \leq \gamma_{2m}(T)$ and we conclude that $I = T$, where $\gamma_{2m}(T) = \binom{2m-1}{m}$ for any integer $m \geq 0$.

We prove now the result about the cocharacter sequence $\chi_n(T)$. We make use of the formula (1) contained in Proposition 2.1, that is:

$$\chi_n(T) = \sum_{i=0}^{n} \chi_{(n-i)} \hat{\otimes}\, \xi_i(T)$$

If $i = 2m + 1$ we get clearly $\xi_{2m+1}(T) = 0$. If instead $i = 2m$, since $\gamma_{2m}(T) = \binom{2m-1}{m}$, from Lemma 5.1 and 5.2 it follows that:

$$\xi_{2m}(T) = \sum_{q=0}^{m} \chi_{(2^{m-q},1^{2q})}$$

Now the claim follows by the Young's rule. $\square$

Theorem 5.4. *It holds:*

$$c_n(T) = \sum_{i=0}^{\lfloor n/2\rfloor} \binom{n}{2i}\binom{2i-1}{i} \quad \text{and} \quad c(T, z) = \frac{1}{2}\Big(\frac{1}{1-z} + \frac{i}{\sqrt{(z+1)(3z-1)}}\Big)$$

P r o o f. The computation of the sequence of the codimensions $c_n(T)$ follows immediately from the formula (2) of Proposition 2.1. Moreover, such proposition contains also the relation (3) between the generating functions of the sequences $c_n(T)$ and $\gamma_n(T)$, that is:

$$c(T,z) = \frac{1}{1-z}\gamma(T, \frac{z}{1-z})$$

It is sufficient therefore to prove that the generating function $\gamma(T,z)$ equals:

$$\gamma(T,z) = \frac{1}{2}(1 + \frac{i}{\sqrt{(2z+1)(2z-1)}})$$

Denote by $g(z)$ the function defined in the right hand side. By induction on $m \geq 1$, it is sufficient to verify:

$$\begin{aligned} g^{(2m)}(z) &= \frac{i\, p_m(z^2)}{\sqrt{(4z^2-1)^{4m+1}}} \\ g^{(2m+1)}(z) &= \frac{-i\, z\, q_m(z^2)}{\sqrt{(4z^2-1)^{4m+3}}} \end{aligned}$$

where $p_m(z), q_m(z)$ are polynomials with integer coefficients both of degree at most m such that:

$$\begin{aligned} p_m(0) &= \frac{1}{2}\frac{(2m)!^2}{(m)!^2} \\ p'_m(0) &= \frac{1}{4}\frac{(2(m+1))!^2}{(m+1)!^2} - 2(4m+1)\, p_m(0) \end{aligned}$$

In fact, such relations are needed to correlate the derivative $g^{(2m)}(0)$ with $g^{(2(m+1))}(0)$ and allow hence the induction. $\square$

Finally, note that radius of convergence of the series $c(T,z)$ is $1/3$ and therefore $\limsup_{n\to\infty} \sqrt[n]{c_n(T)} = 3$, in analogy with the results of Giambruno and Zaicev [GiZa] about the exponent of ideals of ordinary polynomial identities.

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