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ON A CLASS OF GENERALIZED ELLIPTIC-TYPE INTEGRALS

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ABSTRACT. The aim of this paper is to study a generalized form of elliptic-type integrals which unify and extend various families of elliptic-type integrals studied recently by several authors. In a recent communication [1] we have obtained recurrence relations and asymptotic formula for this generalized elliptic-type integral. Here we shall obtain some more results which are single and multiple integral formulae, differentiation formula, fractional integral and approximations for this class of generalized elliptic-type integrals.

1. Introduction and definition. Elliptic-integrals occur frequently in some problems of radiation physics. Due to their importance and applications, several authors, notably Epstein and Hubbell [2], Kalla et al. [3], Srivastava and Siddiqi [4], Kalla and Tuan [5], A. Al-Zamel et al. [6] have defined and studied

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various families of elliptic-type integrals, generalizing the well known complete elliptic integrals $K(k)$ and $E(k)$ [7]. In the present paper we shall study the following form of generalized elliptic-type integrals which provides generalization of the results cited above.

$$A^{(\alpha,\beta)}_{(a,b,c)}(k) = A^{(\alpha,\beta)}_{(a_1,\ldots,a_n,b_1,\ldots,b_n,c_1,\ldots,c_n)}(k_1,\ldots,k_n)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^n\left\{{}_2F_1\left(a_j,b_j;c_j;\left(\frac{k_j^2}{k_j^2-1}\right)(1-\cos\theta)\right)\right\}d\theta, \tag{1.1}$$

where $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta) > 0$, $|k_j| < 1$, $j = 1,\ldots,n$.

Special Cases:

1. Taking $b_j = c_j$, $a_j = \gamma_j$ $(j = 1,\ldots,n)$ in (1.1) and simplifying, we get

$$\begin{aligned} A^{(\alpha,\beta)}_{(a,b,c)}(k) &= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^n\left[(1-k_j^2)^{\gamma_j}(1-k_j^2\cos\theta)^{-\gamma_j}\right]d\theta \\ &= \prod_{j=1}^n(1-k_j^2)^{\gamma_j}Z^{(\alpha,\beta)}_{(\gamma)}(k) \end{aligned} \tag{1.2}$$

where $Z^{(\alpha,\beta)}_{(\gamma)}(k)$ is the family of elliptic-type integrals defined by A. Al-Zamel et al. [6].

2. Taking $n = 3$, $b_j = c_j$ $(j = 1,2,3)$, $a_1 = \mu + \frac{1}{2}$, $a_2 = \lambda$, $a_3 = \gamma$, $k_1^2 = k^2$, $k_2^2 = \frac{\rho}{\rho-2}$, $k_3^2 = \frac{\delta}{2+\delta}$ in (1.1) and simplifying, we get

$$\begin{aligned} A^{(\alpha,\beta)}_{(\mu+\frac{1}{2},\lambda,\gamma)}(k) &= (1-k^2)^{\mu+\frac{1}{2}}(1+\delta)^\gamma\int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\times \\ &\quad\times(1-k^2\cos\theta)^{-\mu-\frac{1}{2}}\left(1-\rho\sin^2\left(\frac{\theta}{2}\right)\right)^{-\lambda}\left(1+\delta\cos^2\left(\frac{\theta}{2}\right)\right)^{-\gamma}d\theta \\ &= (1-k^2)^{\mu+\frac{1}{2}}(1+\delta)^\gamma\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k) \end{aligned} \tag{1.3}$$

where $\Lambda^{(\alpha,\beta)}_{(\mu,\lambda,\gamma)}(\rho,\delta;k)$ is a generalized form of the elliptic-type integrals defined by Kalla and Tuan [5].

Further, set $\delta = 0$ and $\gamma = 0$, then the above elliptic-type integrals reduce to elliptic-type integral defined by Srivastava and Siddiqi [4].

3. Taking $n = 1$, $b = c$, $a = \mu + \frac{1}{2}$, $\beta = \gamma - \alpha$ in equation (1.1), we get

$$\begin{aligned} A^{(\alpha,\gamma-\alpha)}_{(\mu+\frac{1}{2})}(k) &= (1-k^2)^{\mu+\frac{1}{2}} \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2(\gamma-\alpha)-1}\left(\frac{\theta}{2}\right) \times \\ &\times (1-k^2\cos\theta)^{-\mu-\frac{1}{2}} d\theta = (1-k^2)^{\mu+\frac{1}{2}} R_\mu(k,\alpha,\gamma) \end{aligned} \tag{1.4}$$

where $R_\mu(k,\alpha,\gamma)$ is the family of functions defined by Kalla et al. [3].

4. Let $n = 1$, $b = c$, $a = j + \frac{1}{2}$, $\alpha = \beta = \frac{1}{2}$ in (1.1), we get

$$\begin{aligned} A^{(\frac{1}{2},\frac{1}{2})}_{(j+\frac{1}{2})}(k) &= (1-k^2)^{j+\frac{1}{2}} \int_0^\pi (1-k^2\cos\theta)^{-j-\frac{1}{2}} d\theta \\ &= (1-k^2)^{j+\frac{1}{2}} \Omega_j(k), \quad j = 0,1,2,\ldots \end{aligned} \tag{1.5}$$

where $\Omega_j(k)$ is the Epstein and Hubbell family of elliptic-type integrals [2].

Further, for $j = 0$ equation (1.5), reduces to

$$A^{(\frac{1}{2},\frac{1}{2})}_{(\frac{1}{2})}(k) = 2\sqrt{\frac{1-k^2}{1+k^2}} K\left(\sqrt{\frac{2k^2}{1+k^2}}\right) \tag{1.6}$$

where $K(k)$ is the complete elliptic-type integral of first kind [7].

For $j = 1$ in equation (1.5), we get

$$A^{(\frac{1}{2},\frac{1}{2})}_{(\frac{3}{2})}(k) = 2\sqrt{\frac{1-k^2}{1+k^2}} E\left(\sqrt{\frac{2k^2}{1+k^2}}\right) \tag{1.7}$$

where $E(k)$ is the complete elliptic-type integral of second kind [7].

Explicit form. Substituting $\sin^2\left(\frac{\theta}{2}\right) = u$ in (1.1), it assumes the following form

$$\int_0^1 (1-u)^{\alpha-1} u^{\beta-1} \prod_{j=1}^n {}_2F_1\left[a_j, b_j; c_j; \frac{2k_j^2 u}{k_j^2 - 1}\right] du. \tag{1.8}$$

Now comparing the above integral with the result [8, p. 184, Eq. (A.1.3.1)], we can express our generalized elliptic-type integral in terms of multivariable extension of the Kampé Fériet function [9, p. 65, Eq. (2.1)] as follows

$$\text{(1.9)}\quad A_{(a,b,c)}^{(\alpha,\beta)}(k) = B(\alpha,\beta) F_{1:1;\ldots;1}^{1:2;\ldots;2}\left[\begin{matrix} \beta & : a_1, b_1\,; \ldots ; a_n, b_n\,; \\ \alpha+\beta : & c_1 \quad ; \ldots ; \quad c_n \quad ; \end{matrix} \frac{2k_1^2}{k_1^2-1}, \ldots, \frac{2k_n^2}{k_n^2-1}\right]$$

2. Integral formulae. We note here some integral formulae involving our generalized elliptic-type integrals:

$$\text{(i)}\quad \int_0^d \frac{x^{\gamma-1}}{(x^r+z^r)^\rho} A_{(a,b,c)}^{(\alpha,\beta)}\left(\sqrt{\frac{(1-\frac{x}{d})h_1}{h_1(1-\frac{x}{d})-2}}, \ldots, \sqrt{\frac{(1-\frac{x}{d})h_n}{h_n(1-\frac{x}{d})-2}}\right) dx =$$

$$\text{(2.1)}\quad = B(\alpha,\beta)\frac{d^\gamma}{z^{r\rho}} \sum_{l=0}^{\infty} \frac{(-1)^l(\rho)_l}{l!}\left(\frac{d}{z}\right)^{rl} \frac{\Gamma(\gamma+rl)}{\Gamma(\gamma+rl+1)} \times$$

$$\times F_{2:1;\ldots;1}^{2:2;\ldots;2}\left[\begin{matrix} \beta \quad , \quad 1 \quad : a_1, b_1\,; \ldots ; a_n, b_n; \\ \alpha+\beta\,, \gamma+rl+1 : \quad c_1 \quad ; \ldots ; \quad c_n \quad ; \end{matrix} h_1, \ldots, h_n\right]$$

$\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\gamma) > 0$, $d, r > 0$, $|z| > d$, $r|\arg\ z| < \pi$, $|h_j| <$ for $(j = 1, \ldots, n)$.

$$\text{(ii)}\quad \int_0^{d_1} \cdots \int_0^{d_n} \prod_{j=1}^{n}\left[\frac{x_j^{\gamma_j-1}}{(x_j^{r_j}+z_j^{r_j})^{\rho_j}}\right] \times$$

$$\times A_{(a,b,c)}^{(\alpha,\beta)}\left(\sqrt{\frac{h_1(1-\frac{x_1}{d_1})}{h_1(1-\frac{x_1}{d_1})-2}}, \ldots, \sqrt{\frac{h_n(1-\frac{x_n}{d_n})}{h_n(1-\frac{x_n}{d_n})-2}}\right) dx_1 \ldots dx_n =$$

$$\text{(2.2)}\quad = B(\alpha,\beta)\prod_{j=1}^{n}\left[\frac{d_j^{\gamma_j}}{z_j^{r_j\rho_j}} \sum_{l_j=0}^{\infty} \frac{(-1)^{l_j}(\rho_j)_{l_j}}{l_j!}\left(\frac{d_j}{z_j}\right)^{r_j l_j} \frac{\Gamma(\gamma_j+r_j l_j)}{\Gamma(\gamma_j+r_j l_j+1)}\right] \times$$

$$\times F_{1:2;\ldots;2}^{1:3;\ldots;3}\left[\begin{matrix} \beta \quad : \quad a_1, b_1, 1 \quad ; \ldots ; \quad a_n, b_n, 1 \quad ; \\ \alpha+\beta : c_1, \gamma_1+r_1 l_1+1\,; \ldots ; c_n, \gamma_n+r_n l_n+1; \end{matrix} h_1, \ldots, h_n\right]$$

$\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0$, $d_j, r_j > 0$, $|z_j| > d_j$, $r|\arg\ z_j| < \pi$, $|h_j| < 1$ for $(j = 1, \ldots, n)$.

$$\text{(iii)}\quad \int_0^\infty \frac{x^{\gamma-1}}{(x+z)} A_{(a,b,c)}^{(\alpha,\beta)}\left[h_1\sqrt{\frac{2xz}{2h_1^2xz-(x+z)^2}}, \ldots, h_n\sqrt{\frac{2xz}{2h_n^2xz-(x+z)^2}}\right] dx =$$

$$\text{(2.3)}\quad = \pi B(\alpha,\beta)\frac{z^{\gamma-1}}{\sin(\pi\gamma)} F_{3:1;\ldots;1}^{3:2;\ldots;2}\left[\begin{matrix} \beta, \gamma, 1-\gamma : a_1, b_1\,; \ldots ; a_n, b_n; \\ \alpha+\beta, \frac{1}{2}, 1 : \quad c_1 \quad ; \ldots ; \quad c_n \quad ; \end{matrix} h_1^2, \ldots, h_n^2\right]$$

$\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, $0 < \mathrm{Re}(\gamma) < 1$, $\mathrm{Re}(z) > 0$, $|h_j| < 1$ for $(j = 1, \dots, n)$.

$$(iv) \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n} \left[\frac{x_j^{\gamma_j - 1}}{(x_j + z_j)} \right] \times$$

$$\times A_{(a,b,c)}^{(\alpha,\beta)} \left[h_1 \sqrt{\frac{2x_1 z_1}{2h_1^2 x_1 z_1 - (x_1 + z_1)^2}}, \dots, \right.$$

$$(2.4) \qquad \left. h_n \sqrt{\frac{2x_n z_n}{2h_n^2 x_n z_n - (x_n + z_n)^2}} \right] dx_1 \dots dx_n =$$

$$= B(\alpha, \beta) \prod_{i=1}^{n} \left[\frac{\pi}{\sin(\pi\gamma_j)} z_j^{\gamma_j - 1} \right] \times$$

$$\times F_{1:3;\dots;3}^{1:4;\dots;4} \left[\begin{array}{lllll} \beta & : a_1, b_1, \gamma_1, 1-\gamma_1 & ; \dots ; & a_n, b_n \gamma_n, 1-\gamma_n ; & \\ \alpha+\beta : & c_1, \frac{1}{2}, 1 & ; \dots ; & c_n, \frac{1}{2}, 1 \quad ; & \end{array} h_1^2, \dots, h_n^2 \right]$$

$\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, $0 < \mathrm{Re}(\gamma) < 1$, $\mathrm{Re}(z_j) > 0$, $|h_j| < 1$ for $(j = 1, \dots, n)$.

Here $B(\alpha, \beta)$ stands for the beta function and the function occurring in the right hand side of each of equations (2.1) to (2.4) is the well-known Kampé de Fériet function [9, p. 65, Eq. (21)].

$$(v) \int_0^d x^{2-4\delta} (d^2 - x^2)^{\delta - 1} A_{(a,b,c)}^{(\alpha,\beta)} \left(\sqrt{\frac{h_1 x}{h_1 x - 2d}}, \dots, \sqrt{\frac{h_n x}{h_n x - 2d}} \right) dx =$$

$$= \frac{B(\alpha, \beta)}{2} \frac{\Gamma(\frac{3}{2} - 2\delta) \Gamma(\delta)}{\Gamma(\frac{3}{2} - \delta)} d^{-2\delta + 1} \times$$

$$(2.5) \qquad \times F_{2:1;\dots;1}^{2:2;\dots;2} \left[\begin{array}{l} (\beta : 1, \dots, 1), (\frac{3}{2} - 2\delta; \frac{1}{2}, \dots, \frac{1}{2}) \ : \\ (\alpha+\beta : 1, \dots, 1), (\frac{3}{2} - \delta; \frac{1}{2}, \dots, \frac{1}{2}) : \end{array} \right.$$

$$\left. \begin{array}{lll} (a_1, 1)(b_1, 1) ; \dots ; & (a_n, 1)(b_n, 1) ; & \\ (c_1, 1) \quad ; \dots ; & (c_n, 1) \quad ; & \end{array} h_1, \dots, h_n \right]$$

$\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$, $0 < \mathrm{Re}(\delta) < \frac{3}{4}$, $d > 0$, $|h_j| < 1$ for $(j = 1, \dots, n)$.

$$(vi)\ \int_0^{d_1}\cdots\int_0^{d_n}\prod_{j=1}^{n}\left[x_j^{2-4\delta_j}(d_j^2-x_j^2)^{\delta_j-1}\right]\times$$

$$\times A_{(a,b,c)}^{(\alpha,\beta)}\left(\sqrt{\frac{h_1x_1}{h_1x_1-2d_1}},\ldots,\sqrt{\frac{h_nx_n}{h_nx_n-2d_n}}\right)dx_1\ldots dx_n=$$

(2.6)

$$=\frac{B(\alpha,\beta)}{2}\prod_{j=1}^{n}\left[\frac{\Gamma(\frac{3}{2}-2\delta_j)\Gamma(\delta_j)}{\Gamma(\frac{3}{2}-\delta_j)}d_j^{-2\delta_j+1}\right]F^{1:3;\ldots;3}_{1:2;\ldots;2}\left[\begin{array}{c}(\beta:1,\ldots,1):\\(\alpha+\beta:1,\ldots,1):\end{array}\right.$$

$$\left.\begin{array}{c}(a_1,1)(b_1,1)(\frac{3}{2}-2\delta_1,\frac{1}{2});\ldots;(a_n,1)(b_n,1)(\frac{3}{2}-2\delta_n,\frac{1}{2});\\(c_1,1)(\frac{3}{2}-\delta_1,\frac{1}{2})\quad;\ldots;\quad(c_n,1)(\frac{3}{2}-\delta_n,\frac{1}{2})\quad;\end{array}h_1,\ldots,h_n\right]$$

$\mathrm{Re}(\alpha),\mathrm{Re}(\beta)>0$, $0<\mathrm{Re}(\delta_j)<\frac{3}{4}$, $d_j>0$, $|h_j|<1$ for $(j=1,\ldots,n)$.
The function occurring in the right hand side of equations (2.5) and (2.6) is the generalized Lauricella function of several variables [9, p. 64, Eq. (18)].

Proof. To evaluate the integral (2.1), first we express the generalized elliptic-type integral $A_{(a,b,c)}^{(\alpha,\beta)}(k)$ occurring in its left hand side in the explicit form given by (1.8) and write the Kampé de Fériet function occurring therein in series form. Next, we interchange the order of summation and integration and substitute $\frac{x}{d}=t$ and express the binomial expression thus obtained in the form of a series. Finally, evaluating the integral thus obtained as a beta integral and expressing the result in terms of Kampé de Fériet function, we arrive at the desired result.

The remaining integrals given by the equations (2.2) to (2.6) can be derived by following the same procedure as given in the proof of first integral (2.1) and making appropriate modifications and changes.

Let $\alpha=\beta=\frac{1}{2}$, $n=1$, $b=c$, $a=\frac{1}{2}$ and $h=1$ in the integral formulas (2.1), (2.3) and (2.5), then we get the known result given in [10, p. 180, Eq. (2.16.1.3); p. 188, Eq. (2.16.7.1); p. 184, Eq. (2.16.4.12)] respectively.

If we take $\alpha=\beta=\frac{1}{2}$, $n=1$, $b=c$, $a=-\frac{1}{2}$ and $h=1$ in the integral formula (2.1), we get another known result given in [10, p. 180, Eq. (2.16.1.3)].

3. Differentiation formula. In this section we derive a differentiation formula for $A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\dfrac{k}{\sqrt{2-k^2}}\right)$, which may be considered as an extension of the result of Kalla and Al-Saqabi [11].

$$\frac{d}{dk}\left[k^{\lambda}(2-k^2)^{\sigma}(1-k^2)^{\nu}A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right)\right]=$$

$$\begin{aligned}&=4a_nk^{\lambda-1}(2-k^2)^{\sigma}(1-k^2)^{\nu-1}A^{(\alpha,\beta)}_{(a_1,\ldots,a_n+1,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right)-\\&-(2-k^2)^{\sigma-1}(1-k^2)^{\nu}\left[2\left(\frac{2-k^2}{1-k^2}\right)k^{\lambda-1}(a_n+k^2\nu)+2\sigma k^{\lambda+1}-\lambda k^{\lambda-1}(2-k^2)\right]\times\\&\times A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right).\end{aligned}\tag{3.1}$$

P r o o f. To prove the result (3.1), first we express $A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\dfrac{k}{\sqrt{2-k^2}}\right)$ in the integral form given by (1.1) and interchange the orders of differentiation and integration. Next using the differential properties and recurrence relations for the Gauss hypergeometric function [12], we get (3.1) after a little simplifications.

For $\lambda=0$, (3.1) becomes

$$\frac{d}{dk}\left[(2-k^2)^{\sigma}(1-k^2)^{\nu}A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right)\right]=$$

$$\begin{aligned}&=4a_nk^{-1}(2-k^2)^{\sigma}(1-k^2)^{\nu-1}A^{(\alpha,\beta)}_{(a_1,\ldots,a_n+1,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right)-\\&-(2-k^2)^{\sigma-1}(1-k^2)^{\nu}\left[2\left(\frac{2-k^2}{1-k^2}\right)k^{-1}(a_n+k^2\nu)+2k\sigma\right]\times\\&\times A^{(\alpha,\beta)}_{(a,b,c)}\left(k_1,\ldots,k_{n-1},\frac{k}{\sqrt{2-k^2}}\right)\end{aligned}\tag{3.2}$$

For $n=1$, $b=c$, $a=\mu+\frac{1}{2}$, $\alpha=\beta=\frac{1}{2}$, $\sigma=\rho+\mu$ and $\nu=-\mu-\frac{1}{2}$, (3.1) and (3.2) reduce to the differentiation formulas obtained earlier by Kalla and Saqabi [11, p. 510, Eqs. (10), (9)].

4. Fractional integral. Fractional calculus deals with the differentiation and integration of arbitrary order (Ross [13], Miller and Ross [14], Kiryakova [15], Samko et al. [16]). An integral of the form

$$D_x^{-\mu}[f(x)] = \begin{cases} \frac{1}{\sqrt{\mu}}\int_0^x (x-t)^{\mu-1} f(t)dt, & \mathrm{Re}(\mu)>0 \\ \frac{d^m}{dx^m} D_x^{-\mu-m}\{f(x)\}, & -m<\mathrm{Re}(\mu)\le 0, m\in N \end{cases} \tag{4.1}$$

is known as the Riemann-Liouville operator of fractional integration of order μ.

Here we shall express the elliptic-type integral $A_{(a,b,c)}^{(\alpha,\beta)}(k_1,\dots,k_n)$ as fractional integral of Gauss hypegeometric functions as follows

$$A_{(a,b,c)}^{(\alpha,\beta)}\left(\sqrt{\frac{z_1k}{z_1k-2}},\dots,\sqrt{\frac{z_nk}{z_nk-2}}\right) = \\ = \frac{\Gamma(\alpha)}{k^{\alpha+\beta-1}} D_k^{-\alpha}\left[k^{\beta-1}\prod_{j=1}^{n}\{{}_2F_1(a_i,b_i;c_i;z_ik)\}\right]. \tag{4.2}$$

To establish the above result we shall require the definition (1.1) and the following known result [17, p. 569, Eq. (3.5)]

(4.3)

$$D_k^{\lambda-\mu}\left[k^{\lambda-1}F_{q:q_1;\dots;q_n}^{p:p_1;\dots;p_n}\begin{pmatrix} z_1k \\ \vdots \\ z_nk\end{pmatrix}\right] = k^{\mu-1}\frac{\Gamma(\lambda)}{\Gamma(\mu)}F_{q:q_1;\dots;q_n}^{p:p_1;\dots;p_n}\left[\begin{matrix}(\lambda:1,\dots,1), \\ (\mu:1,\dots,1),\end{matrix}\right.$$

$$\left.\begin{matrix}(a_j:\alpha_j^{(1)},\dots,\alpha_j^{(n)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\dots;(c_j^{(n)},\gamma_j^{(n)})_{1,p_n}; \\ (b_j:\beta_j^{(1)},\dots,\beta_j^{(n)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\dots;(d_j^{(n)},\delta_j^{(n)})_{1,q_n};\end{matrix}\; z_1^k,\dots,z_n^k\right],$$

$$\mathrm{Re}(\lambda)>0.$$

Let $p=q=0$, $p_i=2$, $q_i=1$, $a_j^{(i)}=\beta_j^{(i)}=\gamma_j^{(i)}=\delta_j^{(i)}=1$, $c_1^{(i)}=a_i$, $c_2^{(i)}=b_i$,

$d_1^{(i)} = c_i$ for $(i = 1, \dots, n)$, $\lambda = \beta$ and $\mu = \alpha + \beta$ in (4.3) and set $k_i = \sqrt{\dfrac{z_i^k}{z_i^k - 2}}$ $(i = 1, \dots, n)$ in equation (1.1). Now comparing the results thus obtained, we easily arrive at (4.2).

If in the main result (4.2), we take $n = 1$, $b = c$, $a = \mu + \frac{1}{2}$, $\beta = \gamma - \alpha$ and $n = 1$, $b = c$, $a = \mu + \frac{1}{2}$, $\alpha = \beta = \frac{1}{2}$ respectively, we obtain the fractional integral representations for the elliptic type integrals $R_\mu(k, \alpha, \gamma)$ and $\Omega_\mu(k)$. The results thus obtained are essentially similar to the known results given in [18, p. 31, Eqs. (72), (71)]. They can be established on using the transformation formulae [12, p. 60, Eq. (38.4) and p. 65, Eq. (41.1)] respectively.

5. Approximations. In this section we shall derive three single term approximations for elliptic-type integral $z_\gamma^{(\alpha,\beta)}(k)$.

First we establish the following single term approximation for the Lauricella function $F_D^{(n)}$:

$$F_D^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) = \frac{\Gamma(\gamma)}{\Gamma(\beta_n)\Gamma(\gamma - \beta_1 - \dots - \beta_n)} \prod_{i=1}^{n-1} \left[\frac{(1 - x_i)^{-\alpha}}{\Gamma(\beta_i)}\right] \times$$

$$\times \prod_{i=1}^{n-2} \left\{ B\left(\beta_i + \sum_{l=1}^{i} \frac{\alpha x_l}{1 - x_l}, \alpha + \beta - \sum_{r=i}^{n} \beta_r\right)\right\} B\left(\beta_{n-1} + \frac{\alpha x_{n-1}}{1 - x_{n-1}}, \gamma + \frac{\alpha x_1}{1 - x_1} + \dots + \right. \tag{5.1}$$

$$\left. + \frac{\alpha x_{n-2}}{1 - x_{n-2}} - \beta_{n-1} - \beta_n\right) B\left(\beta_n, \gamma + \frac{\alpha x_1}{1 - x_1} + \dots + \frac{\alpha x_{n-1}}{1 - x_{n-1}} - \alpha x_n - \beta_n\right)$$

$$\mathrm{Re}(\beta_i) > 0, \ \mathrm{Re}(\gamma - \beta_1 - \ldots - \beta_n) > 0, \ 0 < |x_i| \ll 1 \ (i = 1, \dots, n).$$

Proof. The n-integral representation of $F_D^{(n)}$ [19, p. 49, Eq. (2.3.5)] is given by

(5.2)

$$F_D^{(n)}(\alpha, \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n) = \frac{\Gamma(\gamma)}{\Gamma(\beta_1) \dots \Gamma(\beta_n)\Gamma(\gamma - \beta_1 - \dots - \beta_n)} \times$$

$$\times \underbrace{\int \dots \int}_{(n)} u_1^{\beta_1 - 1} \dots u_n^{\beta_n - 1} (1 - u_1 - \dots - u_n)^{\gamma - \beta_1 - \dots - \beta_n - 1} \times$$

$$\times (1 - u_1 x_1 - \dots - u_n x_n)^{-\alpha} du_1 \dots du_n$$

where the region of integration of (5.2) is $u_i \geq 0$ $(i = 1, \ldots, n)$; $\sum_{i=1}^{n} u_i \leq 1$.

If we use the following approximation,

$$(1 - u_1x_1 - \cdots - u_nx_n) \cong (1 - u_1x_1)(1 - u_2x_2)\ldots(1 - u_nx_n) \quad \text{for } |x_i| \ll 1, \quad (i = 1, \ldots, n) \tag{5.3}$$

and make the substitutions

$$w_i = 1 - \frac{u_i}{\left(1 - \sum\limits_{r=i+1}^{n} u_r\right)} \quad (i = 1, \ldots, n-1), \quad w_n = u_n$$

successively in the right hand side of (5.2), it leads to the following form,

$$\begin{aligned}
&\frac{\Gamma(\gamma)}{\Gamma(\beta_1)\ldots\Gamma(\beta_n)\Gamma(\gamma - \beta_1 - \cdots - \beta_n)} \times \\
&\times \underbrace{\int_0^1 \cdots \int_0^1}_{(n)} \prod_{i=1}^{n-1} \left[(1 - w_i)^{\beta_i - 1} w_i^{\gamma} - \sum_{r=i}^{n} \beta_r - 1\right] \times \\
&\times (1 - w_n)^{\gamma - \beta_n - 1} w_n^{\beta_n - 1} \prod_{i=1}^{n-2} \left[1 - \sum_{r=i}^{n-2} (1 - w_r) w_{n-1} (1 - w_n) x_i\right]^{-\alpha} \times \\
&\times \left[1 - (1 - w_{n-1})(1 - w_n) x_{n-1}\right]^{-\alpha} \left[1 - w_n x_n\right]^{-\alpha} dw_1 \ldots dw_n.
\end{aligned} \tag{5.4}$$

Now we use the approximations,

$$[1-(1-w_1)\ldots(1-w_n)x_i] \cong (1-x_1)\left(1+\frac{x_i}{1-x_i}w_1\right)\ldots\left(1+\frac{x_i}{1-x_i}w_n\right), \quad 0 < |x_i| \ll 1 \tag{5.5}$$

and

$$\left(1 - \frac{x}{r}\right)^{\alpha - 1} \cong (1 + x)^{-\frac{(\alpha - 1)}{r}} \quad \text{in the neibourhood of } x = 0$$

or, equivalently

$$\left(1 + \frac{x}{1 - x} w\right)^{-\alpha} \cong (1 - w)^{\frac{\alpha x}{1 - x}} \quad \text{in the neibourhood of } w = 0 \tag{5.6}$$

in the term $[1-(1-w_{n-1})(1-w_n)x_{n-1}]^{-\alpha}$ of the integrand of (5.3) and make some algebraic manipulations and simplifications to arrive at the result (5.1).

Now on using the approximation (5.1) for $F_D^{(n)}$, we obtain the following single term approximation for $z_{(a)}^{(\alpha,\beta)}(k)$ after a little simplification

$$\begin{aligned} z_{(a)}^{(\alpha,\beta)}(k) &= \frac{B(\alpha,\beta)\Gamma(\alpha+\beta)}{\Gamma(a_n)\Gamma(\alpha+\beta-a_1-\cdots-a_n)}(1-k_n^2)^{-a_n}\times \\ &\times \prod_{i=1}^{n-1}\left[\frac{(1-k_i)^{\beta-a_i}(1+k_i^2)^{-\beta}}{\Gamma(a_i)}\right]\prod_{i=1}^{n-2}\left\{B\left(a_i-\sum_{l=1}^{i}\frac{2\beta k_i^2}{1+k_i^2},\alpha+\beta-\sum_{r=i}^{n}a_r\right)\right\}\times \\ &\times B\left(a_{n-1}-\frac{2k_{n-1}^2\beta}{1+k_{n-1}^2},\alpha+\beta-\frac{2k_1^2\beta}{1+k_1^2}-\cdots-\frac{2k_{n-1}^2\beta}{1+k_{n-1}^2}-a_{n-1}-a_n\right)\times \\ &\times B\left(a_n,\alpha+\beta-a_n-\frac{2k_1^2\beta}{1+k_1^2}-\cdots-\frac{2k_{n-1}^2\beta}{1+k_{n-1}^2}-\frac{2k_n^2\beta}{k_n^2-1}\right) \end{aligned} \tag{5.7}$$

$\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(a_i)>0$, $\mathrm{Re}(\alpha+\beta-a_1-\cdots-a_n)>0$ for small k_i $(i=1,\dots,n)$.

Other single term approximation for $z_{(a)}^{(\alpha,\beta)}(k)$ can be established in the following form on using the integral representation of $F_D^{(n)}$ [19, p. 49, Eq. (2.3.6)], making slight change in the variable of integration and applying the approximation (5.6)

$$\begin{aligned} &z_{(a)}^{(\alpha,\beta)}(k) = \\ &=\Gamma(\alpha)(1-k_n^2)^{-a_n}\prod_{i=1}^{n-1}\left[(1+k_i^2)^{-a_i}\right]\frac{\Gamma\left[\beta-\frac{2a_1k_1^2}{1+k_1^2}-\cdots-\frac{2a_nk_n^2}{1+k_n^2}\right]}{\Gamma\left[\alpha+\beta-\frac{2a_1k_1^2}{1+k_1^2}-\cdots-\frac{2a_nk_n^2}{1+k_n^2}\right]} \end{aligned} \tag{5.8}$$

$\mathrm{Re}(\beta)>\mathrm{Re}(\alpha+\beta)>0$, $\mathrm{Re}\left(\beta-\frac{2a_1k_1^2}{1+k_1^2}-\cdots-\frac{2a_nk_n^2}{1+k_n^2}\right)>0$, for small k_i $(i=1,\dots,n)$. Appealing to the integral representation of $F_D^{(n)}$ [19, p. 49, Eq. (2.3.6)] and the approximation (5.6), we get yet another approximation for

$z_{(a)}^{(\alpha,\beta)}(k)$ as follows

$$(5.9)\qquad z_{(a)}^{(\alpha,\beta)}(k) = \Gamma(\alpha)\prod_{i=1}^{n}\left[(1-k_i^2)^{-a_i}\right]\frac{\Gamma\left[\beta - \dfrac{2k_1^2a_1}{k_1^2-1} - \cdots - \dfrac{2k_n^2a_n}{k_n^2-1}\right]}{\Gamma\left[\alpha+\beta - \dfrac{2k_1^2a_1}{k_1^2-1} - \cdots - \dfrac{2k_n^2a_n}{k_n^2-1}\right]}$$

$\mathrm{Re}(\beta) > \mathrm{Re}(\alpha+\beta) > 0$, for small k_i $(i = 1, \ldots, n)$.

A suitable choice of parameters in the result (5.7), say $n = 1$, $b = c$, $a = \mu + \frac{1}{2}$, $\beta = \gamma - \alpha$ and $n = 1$, $b = c$, $a = \mu + \frac{1}{2}$, $\alpha = \beta = \frac{1}{2}$ leads to the approximation formulae for the families of elliptic-type integrals $R_\mu(k, \alpha, \gamma)$ and $\Omega_\mu(k)$. The results thus obtained are essentially similar to the known results given in [18, p. 30, Eq. 67; p. 29, Eq. 65] They can be established on using the transformation formulae [12, p. 60, Eq. (3.8.4) and p. 65, Eq. (41.1)] respectively.

The results obtained in section 2 to 5 would yield the corresponding results for other known families of elliptic-type integrals on specializing the parameters of the generalized elliptic-type integral $A_{(a,b,c)}^{(\alpha,\beta)}(k)$ as mentioned in section 1. We however omit the details.

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