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BOOLEAN RINGS THAT ARE BAIRE SPACES

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Abstract. Weak completeness properties of Boolean rings are related to the property of being a Baire space (when suitably topologised) and to renorming properties of the Banach spaces of continuous functions on the corresponding Stone spaces.

1. Introduction. This paper is loosely based on a talk entitled "Banach spaces without a strictly convex equivalent norm", delivered to the Murcia Functional Analysis Seminar in April 1989. In that talk, I exhibited a Banach space X with no subspace isomorphic to ℓ_∞ which admits no strictly convex equivalent norm. The space X was of the type $\mathcal{C}(K)$ for a certain compact, totally disconnected space K. The construction involved the notion of tree-completeness

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for a Boolean ring, something which, together with the related notion of subsequential completeness, had previously appeared in the construction of examples of non-reflexive Grothendieck spaces [8] and of spaces without locally uniformly convex renormings [1, 10]. In each of these cases the aim was to build a space that would behave rather like ℓ_∞ without containing a copy of that space. Shortly after my stay in Murcia, I realised that the example I had spoken about there was also an example of a compact space K which does not have the Namioka property but is such that $\mathcal{C}(K)$ has no subspace isomorphic to ℓ_∞; it was also the case that $\mathcal{C}(K)$ under the pointwise topology was not σ-fragmentable by the norm (*c.f.* [11]). I subsequently learned that Namioka had shown that the same things were true of the example of [10]. In planning this article, my original idea was to give a survey of weak completeness properties of Boolean rings and of their applications to renorming theory, σ-fragmentability and the Grothendieck property. However, as time elapsed, it began to seem to me that the properties of subsequential completeness and tree-completeness were not in themselves of such fundamental importance and that most of the results obtained using them could be deduced from the validity of the Baire Category Theorem for certain Boolean rings equipped with certain topologies. The article in its present form is intended to show why this is so. The perceptive reader will see that I have benefited greatly from the thoughts of Isaac Namioka, and his co-authors John Jayne and Ambrose Rogers. I should like to thank them for some useful and stimulating conversations both in Murcia and in London. I should also like to thank members of the Murcia Functional Analysis Group, and José Orihuela in particular, for their hospitality during my all too brief visit.

2. Weak completeness properties and Baire properties. On the whole, Boolean rings are more straightforward objects to construct and to manipulate than are Banach spaces. On the other hand, given a Boolean ring $\mathfrak{A}$, one can associate the Stone space $S(\mathfrak{A})$, a locally compact space such that the ring of compact open subsets of $S(\mathfrak{A})$ is isomorphic to $\mathfrak{A}$. The Banach space $\mathcal{C}_0(S(\mathfrak{A}))$ of continuous real-valued function vanishing at infinity on $S(\mathfrak{A})$ is then a Banach space whose properties are determined by the properties of $\mathfrak{A}$. A good account of these ideas can be found in [13] (One can alternatively go straight from $\mathfrak{A}$ to the Banach space, missing out the construction of the Stone space [6].) This can be a useful way to construct Banach space counterexamples, and the relationship between algebraic properties of a Boolean ring $\mathfrak{A}$ (or topological properties of a locally compact space S (and Banach space properties of the corresponding space of continuous functions can be a topic of interest in itself. As stated in the Introduction, we shall be concerned here with certain weak versions of completeness for Boolean rings, and with the Baire Category Theorem for certain topologies on them. We start by recalling the definition of the Subsequential Completeness Property.

We say that a Boolean ring $\mathfrak{A}$ is *subsequentially complete* if, for every disjoint sequence $(A_n)_{n\in\omega}$ in $\mathfrak{A}$, there is an infinite subset M of the set ω of natural numbers such that $\{A_m : m \in M\}$ has a least upper bound in $\mathfrak{A}$.

This property (in the special case of a Boolean algebra, that is to say a Boolean ring with maximal element (was introduced in [8], where it was shown to be sufficiently strong to guarantee that $\mathcal{C}_0(S(\mathfrak{A}))$ is a Grothendieck space; it was also shown that there exists a subsequentially complete subalgebra of $\mathfrak{P}(\omega)$, the power set of the set of natural numbers such that $\mathcal{C}(S(\mathfrak{A}))$ does not contain ℓ_∞. At the time, the SCP was probably the weakest algebraic property of $\mathfrak{A}$ known to force $\mathcal{C}_0(S(\mathfrak{A}))$ to be a Grothendieck space, though the intention had been to find a simple and convenient property of this type rather than the weakest one. In fact, weaker such properties now known, for instance the Subsequential Interpolation Property of Freniche [7]. A still weaker property (new, as far as I am aware), the Subsequential Separation Property, will be mentioned later on in this paper.

The tree-completeness property, introduced in [10], is a stronger property than the SCP; the version considered in [10] was specially tailored for convenience in proving that $\mathcal{C}_0(S(\mathfrak{A}))$ has no equivalent l.u.r. norm. Alexandrov and Babev, working independently of the authors of [10], gave an ingenious proof that the non-existence of such norms already follows from the SCP. In Section 4 of this paper I shall give an alternative proof of a version of their result using arguments of Baire Category type. It is time to clarify what these arguments will be concerned with.

Given a Boolean ring $\mathfrak{A}$ and an ideal $\mathfrak{I}$ of $\mathfrak{A}$ we define a topology (not necessarily Hausdorff($\tau(\mathfrak{I})$ on $\mathfrak{A}$ by taking basic neighbourhoods of A to be of the form $\mathcal{N}(A, I) = \{B \in \mathfrak{A} : B \cap I = A \cap I\}$ with $I \in \mathfrak{I}$. An alternative way of looking at this involves the Stone space. An ideal $\mathfrak{I}$ of $\mathfrak{A}$ corresponds to the set of all compact open subsets of some open subset U of $S(\mathfrak{A})$: the topology $\tau(\mathfrak{I})$ corresponds to uniform convergence of indicator functions on the compact subsets of U; if we are thinking of in this way we shall write $\tau_{\mathrm{C}}(U)$. It is perhaps remarking right away that if $\mathfrak{I} = \mathfrak{A}$ and $\mathfrak{A}$ has a maximal element (equivalently, $U = S(\mathfrak{A})$ and $S(\mathfrak{A})$ is compact (then $\mathfrak{A}, \tau(\mathfrak{I})$ is discrete; in all other cases, however, $\mathfrak{A}, \tau(\mathfrak{I})$ has no isolated points. The connection between weak completeness properties and the topologies $\tau(\mathfrak{I})$ is the following theorem. Recall that we say that a topological space X is a *Baire space* if X satisfies the conclusion of the Baire Category Theorem, that is to say, if the intersection of any countable collection of dense open sets is again dense in X.

Theorem 2.1. *If $\mathfrak{A}$ is a Boolean ring with the subsequential completeness property and $\mathfrak{I}$ is an ideal of $\mathfrak{A}$ then $\mathfrak{A}, \tau(\mathfrak{I})$ is a Baire space.*

Proof. For each $n \in \omega$ let $\mathcal{G}_n$ be a $\tau(\mathfrak{I})$-dense open subset of $\mathfrak{A}$. To show that $\bigcap_{n\in\omega} \mathcal{G}_n$ is dense in $\mathfrak{A}$, I shall show that this intersection meets an arbitrary basic open set $\mathcal{N}(A, I)$. Since $\mathcal{N}(A, I) = \mathcal{N}(A \cap I, I)$ we may assume that $A \in \mathfrak{I}$. The proof will make repeated use of the following simple lemma.

Lemma 2.2. *Let $\mathfrak{I}$ be an ideal of a Boolean ring $\mathfrak{A}$, let J be a member of $\mathfrak{I}$, let $B_1, \dots, B_m$ be elements of $\mathfrak{A}$ with $B_i \subseteq J$ for all i and let $\mathcal{G}$ be a $\tau(\mathfrak{I})$-dense open subset of $\mathfrak{A}$. Then there exist $J' \in \mathfrak{I}$ with $J' \supseteq J$ and $D \in \mathfrak{A}$, with $D \subseteq J'$, $D \cap B_i = \emptyset \quad (i \leq m)$, such that $\mathcal{N}(N_i \cup D, J') \subseteq \mathcal{G}$ for all $i \leq n$.*

Proof. Set $I_0 = J$ and note that $\mathcal{N}(B_1, I_0)$ is a non-empty $\tau(\mathfrak{I})$-open set, so that the intersection $\mathcal{G} \cap \mathcal{N}(B_1, I_0)$ is nonempty and open too. This intersection must contain a basic neighbourhood of B_1, necessarily of the form $\mathcal{N}(B_1 \cup C_1, I_1)$, with $I_0 \subseteq I_1 \in \mathfrak{I}$, $I_1 \supseteq C_1 \in \mathfrak{I}$, $C_1 \cap I_0 = \emptyset$. Next we consider the nonempty open set $\mathcal{N}(B_2 \cup C_1, I_1)$ and find, by the same method, I_2 and C_2 with $I_1 \subseteq I_2 \in \mathfrak{I}$, $I_1 \subseteq I_2 \in \mathfrak{I}$, $I_2 \supseteq C_2 \in \mathfrak{I}$, $C_2 \cap I_1 = \emptyset$ and $\mathcal{N}(B_2 \cup C_1 \cup C_2, I_2) \subseteq \mathcal{G}$. After several repetitions of the same procedure, we set $D = C_1 \cup C_2 \cup \cdots \cup C_n$, $J' = I_n$ and have the result we need. $\square$

We now resume the proof of the theorem. We shall construct by recursion an increasing sequence $I = J_0 \subseteq J_1 \subseteq \cdots$ in $\mathfrak{I}$ and a disjoint sequence $D_1, D_2 \dots$ in $\mathfrak{I}$ such that $D_j \subseteq J_j \setminus J_{j-1}$ for all j, in such a way that

$$\mathcal{N}(A \cup \bigcup_{j\in F} D_j \cup D_{n+1}, J_n) \subseteq \mathcal{G}_n$$

for all $n \in \omega$ and all subsets F of $\{1, 2, \dots, n\}$. The inductive step from n to $n+1$ (and the initial stage, which is a trivial degenerate case(follow easily from the lemma. If J_j and D_j have been defined for $j \leq n$ we set $\mathcal{G} = \mathcal{G}_n$, $J = J_n$ and let $B_1 \dots B_{2^n}$ be the sets $A \cup \bigcup_{j\in F} D_j$ with F a subset of $\{1, 2, \dots, n\}$. We take D_{n+1} and J_{n+1} to be the objects D, J' the existence of which is guaranteed by 2.2. $\square$

While we are thinking about such things it will be worth making a remark about the topologies $\tau(\mathfrak{I})$ for a special class of ideals $\mathfrak{I}$. This will show that in some circumstances we can assert the validity of a very strong version of the Baire Category Theorem.

Remark 2.3. Let $\mathfrak{A}$ be a Boolean ring and let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$ having the property that every countable subset of $\mathfrak{I}$ is bounded above in $\mathfrak{I}$. Then for the topology $\tau(\mathfrak{I})$ on $\mathfrak{A}$ every $\mathcal{G}_\delta$ subset is open.

Proof. Let $\mathcal{G}_n$ be a sequence of $\tau(\mathfrak{I})$-open sets and let A be in the intersection $\mathfrak{H} = \bigcap_{n\in\omega} \mathcal{G}_n$. For each n there exists $I_n \in \mathfrak{I}$ such that $\mathcal{N}(A, I_n) \subseteq \mathcal{G}_n$. By hypothesis there exists $J \in \mathfrak{I}$ such that $I_n \subseteq J$ for all n, and the neighbourhood $\mathcal{N}(A, J)$ is contained in $\mathfrak{H}$. □

Corollary 2.4. *Let $\mathfrak{A}$ be a Boolean ring with the SCP and let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$ with the property that every countable subset of $\mathfrak{I}$ is bounded above in $\mathfrak{I}$. Then the intersection of any countable collection of $\tau(\mathfrak{I})$-dense open subsets of $\mathfrak{A}$ is again a $\tau(\mathfrak{I})$-dense open subset.*

3. Construction of Boolean rings with the SCP. As I have already mentioned, the paper [8] contained a construction of a Boolean subalgebra of $\mathfrak{P}(\omega)$ which has the SCP but is such that $\mathcal{C}(S(\mathfrak{A}))$ has no subspace isomorphic to ℓ_∞. Alexandrov and Babev showed that this $\mathcal{C}(S(\mathfrak{A}))$ admits no locally uniformly convex renorming. In order to get a stronger example, in which there is no strictly convex renorming, we shall need to carry out a related construction working with the countable subsets of an uncountable set rather than with the subsets of ω. The resulting space $\mathcal{C}_0(S(\mathfrak{A}))$ will resemble the space $\ell_\infty^{\mathrm{c}}(\omega_1)$ of bounded functions of countable support on ω_1 in the same sort of way that the original space of [8] resembled ℓ_∞. The role of the condition (iii) in the theorem that follows is to ensure that ℓ_∞ does not embed in $\mathcal{C}_0(S(\mathfrak{A}))$. The proof of this is exactly as given in [8] so I shall not include it. In fact, the proof of the theorem is almost the same as in [8], but since we have shifted to an uncountable base set it may be best to spell it out.

Theorem 3.1. *There exists a subring $\mathfrak{A}$ of the Boolean ring $\mathfrak{P}_c(\omega_1)$ of countable subsets of ω_1 with the following properties:*

(i) $[0, \alpha(\in \mathfrak{A}$ *for each countable ordinal α;*

(ii) $\mathfrak{A}$ *has the SCP;*

(iii) *for every countably infinite subset M of ω_1 there is a subset N of M such that for no $A \in \mathfrak{A}$ is N equal to $M \cap A$.*

Proof. We start by establishing some notation: we write $\mathfrak{c}$ for the smallest ordinal of cardinality the continuum and fix a surjection $\phi : \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$ having the property that $\beta < \alpha$ when $\phi(\alpha) = (\beta, \gamma)$. We fix an enumeration $(M_\alpha)_{\alpha\in\mathfrak{c}}$ of the countably infinite subsets of ω_1. We let $\mathfrak{A}_0$ be the subring of $\mathfrak{P}(\omega_1)$ generated by the initial segments $[0, \alpha)$ $(\alpha < \omega_1)$. The cardinality $\#\mathfrak{A}_0$ is at most $\mathfrak{c}$ so that there are a total of $\mathfrak{c}$ disjoint sequences in $\mathfrak{A}$. We fix an enumeration of these, denoting the ξ^{th} such sequence by $(A_n(0, \xi))_{n\in\omega}$. We note that $\{A \cap M : A \in \mathfrak{A}_0\}$ is countable whenever M is a countable subset of ω_1, so that we may choose a subset N_0 of M_0 which is not of the form $A \cap M_0$ with $A \in \mathfrak{A}_0$.

Now let $\delta < \mathfrak{c}$ be an ordinal and assume inductively that we have constructed the following:

- an increasing family of subrings $\mathfrak{A}_\alpha$ $(\alpha < \delta)$ of $\mathfrak{P}_C(\omega_1)$;
- for each α, an enumeration $(A_n(\alpha, \xi))$ $(\xi \in \mathfrak{c})$ of the disjoint sequences in $\mathfrak{A}_\alpha$;
- for each α a subset N_α of M_α.

Assume also that the following are true for all $\alpha, \gamma < \delta$:

(a) for each countable $M \subset \omega_1$, the cardinality $\#\{A \cap M : A \in \mathfrak{A}_\alpha\}$ is at most $\max\{\omega, \#\alpha\}$;

(b) if $\phi(\alpha) = (\beta, \xi($ and $\alpha+1 < \delta$, then, for some infinite $L \subset \omega$, $\bigcup_{n\in L} A_n(\beta, \xi) \in \mathfrak{A}_{\alpha+1}$;

(c) for no $A \in \mathfrak{A}_\alpha$ does $M_\gamma \cap A$ equal N_γ.

If δ is a limit ordinal, we just take $\mathfrak{A}_\delta = \bigcup_{\alpha<\delta} \mathfrak{A}_\alpha$. Condition (a) still holds by cardinal arithmetic. Conditions (b) and (c) still hold trivially.

If δ is of the form $\alpha + 1$ we have to show how to choose an infinite subset L of ω so that, when we define $\mathfrak{A}_{\alpha+1}$ to be the ring generated by $\mathfrak{A}_\alpha$ together with the extra element $\bigcup_{n\in L} A_n(\phi(\alpha))$, the condition (c) remains true for all $\gamma \le \alpha$. The following lemma shows us how to do it.

Lemma 3.2. *Let $\mathcal{B}$ be a Boolean subring of $\mathfrak{P}_C(\omega_1)$ and assume that there is a cardinal $\mathfrak{d} < \mathfrak{c}$ such that $\#\{A \cap M : A \in \mathcal{B}\} \le \mathfrak{d}$ for all countable subsets M of ω_1. Assume further that M_γ, N_γ $(\gamma < \mathfrak{d})$ are countably infinite subsets of ω_1 with the property that $N_\gamma \in \mathfrak{P} \setminus \{A \cap M_\gamma : A \in \mathcal{B}\}$ for all γ. Let $(A_n)_{n\in\omega}$ be any disjoint sequence in $\mathcal{B}$. Then there exists an infinite subset L of ω such that $M_\gamma \cap A \ne N_\gamma$ for all $\gamma < \mathfrak{d}$ and all A in the ring generated by $\mathcal{B} \cup \{\bigcup_{n\in L} A_n\}$.*

P r o o f. Let $L(\eta)$ $(\eta \in \mathfrak{c})$ be a family of infinite subsets of ω with the property that $L(\eta) \cap L(\zeta)$ is finite whenever $\eta \neq \zeta$. For each η, let $\mathcal{B}(\eta)$ be the ring generated by $\mathcal{B}$ together with the additional element $A(\eta) = \bigcup_{n \in L(\eta)} A_n$. Note that the elements of $\mathcal{B}(\eta)$ have the form $(B \cap A(\eta)) \cup (C \setminus A(\eta))$ with B, C disjoint elements of $\mathcal{B}$. We suppose that none of the $L(\eta)$ will do as the L we are looking for. This means that for each η there exist $B(\eta), C(\eta) \in \mathcal{B}$ and $\gamma(\eta(< \mathfrak{d}$ such that the intersection $M_{\gamma(\eta)} \cap ((B(\eta) \cap A(\eta)) \cup (C(\eta) \setminus A(\eta)))$ is exactly the forbidden subset $N_{\gamma(\eta)}$. By our cardinality assumptions there must exist distinct η, ζ such that $\gamma(\eta) = \gamma(\zeta)$ $(= \gamma$, say(and such that $B(\eta) \cap M_\gamma = B(\zeta) \cap M_\gamma$, $C(\eta) \cap M_\gamma = C(\zeta) \cap M_\gamma$. Now $L(\eta) \cap L(\zeta)$ is some finite set F and, since the sets A_n are disjoint, we have $A(\eta) \cap A(\zeta) = \bigcup_{n \in F} A_n$. Writing A_F for this last set, we see that $N_\gamma = M_\gamma \cap ((B(\eta) \cap A_F) \cup (C(\eta) \setminus A_F))$. This is a contradiction since $(B(\eta) \cap A_F) \cup (C(\eta) \setminus A_F)$ is in $\mathcal{B}$. □

In the proof of 3.1, we can now see how to obtain $\mathfrak{A}_{\alpha+1}$ as the algebra generated by $\mathfrak{A}$ and an additional element $A = \bigcup_{n \in L} A_n(\phi(\alpha))$. Recalling from the proof of 3.2 the formula $(B \cap A) \cup (C \setminus A)$ for the general element of $\mathfrak{A}_{\alpha+1}$, we see that for a countable subset M of ω_1, the number of new sets of the form $M \cap ((B \cap A) \cup (C \setminus A))$ that we have created is at most the number of ways of choosing $B \cap M$ and $C \cap M$ with B and C in $\mathfrak{A}_\alpha$. By our inductive hypothesis, this is at most $\max\{\omega, \#\alpha\}$. Thus (a) remains true.

In either the limit ordinal case or the successor ordinal case we now fix an enumeration $(A_n(\delta, \xi))$ $(\xi \in \mathfrak{c})$ of the disjoint sequences in $\mathfrak{A}_\delta$, and (using (a)) choose any subset N_δ of M_δ which is not of the form $D \cap M_\delta$ with $D \in \mathfrak{A}_\delta$.

When the recursion is complete, the algebra $\mathfrak{A} = \bigcup_{\alpha < \mathfrak{c}} \mathfrak{A}_\alpha$ has the properties we wanted. □

4. Baire properties and convex renormings. This section is devoted to showing that suitable properties of Baire Category type for the ring $\mathfrak{A}$ imply that $\mathcal{C}_0(S(\mathfrak{A}))$ admits no locally uniformly convex renorming, or even no strictly convex renorming.

Theorem 4.1. *Let $\mathfrak{A}$ be a Boolean ring without maximal element. If $\mathfrak{A}$ is a Baire space for the topology $\tau(\mathfrak{A})$ then $\mathcal{C}_0(S(\mathfrak{A}))$ admits no locally uniformly convex equivalent norm. If, in addition, every countable subset of $\mathfrak{A}$ is bounded above, then $\mathcal{C}_0(S(\mathfrak{A}))$ admits no strictly convex equivalent norm.*

P r o o f. Let $\|\cdot\|$ be any equivalent norm on $\mathcal{C}_0(S(\mathfrak{A}))$. We consider the function $\nu : \mathfrak{A} \to \mathfrak{R}$ given by $\nu A = \|\mathbb{1}_A\|$. (We are identifying an element of $\mathfrak{A}$ with the corresponding compact open subset of $S(\mathfrak{A})$.) Since the measures of compact support are norm-dense in the dual space of $\mathcal{C}_0(S(\mathfrak{A}))$, any equivalent norm on $\mathcal{C}_0(S(\mathfrak{A}))$ is lower semicontinuous for the topology $\tau_{\mathrm{C}}(S(\mathfrak{A}))$ of uniform convergence on the compact subsets of $S(\mathfrak{A})$. By the way the topology $\tau(\mathfrak{A})$ is defined, the map $A \mapsto \mathbb{1}_A$ is a homeomorphic embedding of $\mathfrak{A}, \tau(\mathfrak{A})$ into $\mathcal{C}_0(S(\mathfrak{A})), \tau_{\mathrm{C}}$, so that the function ν is l.s.c. on $\mathfrak{A}, \tau(\mathfrak{A})$. Now an l.s.c. function on a Baire space is continuous at all points of some dense $\mathcal{G}_\delta$; let us choose one continuity point A of ν. By continuity, there exists, for each $n \in \omega$, an element I_n of $\mathfrak{A}$ with $I_n \supseteq A$ such that $|\nu(A \cup C) - \nu(A)| \leq 2^{-n}$ whenever $C \in \mathfrak{A}$ and $C \cap I_n = \emptyset$. We may define recursively a sequence (C_n) of elements of $\mathfrak{A}$ in such a way that, for each n, $C_n \cap [I_n \cup \bigcup_{j<n} C_j] = \emptyset$. Indeed, to do this we only have to note at stage n that $I_n \cup \bigcup_{j<n} C_j$ is not a maximal element of $\mathfrak{A}$ (since we are assuming none exists). We now have $\|\mathbb{1}_{A\cup C_n}\| \to \|\mathbb{1}_A\|$ as $n \to \infty$. Since the norm $\|\cdot\|$ is equivalent to the supremum norm, we do not have $\|\mathbb{1}_{A\cup C_n} - \mathbb{1}_A\| \to 0$. Thus if $\|\cdot\|$ were l.u.r. we should have $\limsup_{n\to\infty} \|\mathbb{1}_A + \frac{1}{2}\mathbb{1}_{C_n}\| < \|\mathbb{1}_A\|$. This is a contradiction, since $\mathbb{1}_A$ is the weak limit of the sequence $(\mathbb{1}_A + \frac{1}{2}\mathbb{1}_{C_n})$.

We now make the additional assumption that every countable subset of $\mathfrak{A}$ is bounded above in $\mathfrak{A}$. We obtain a continuity point of ν and the sequence (I_n) as before but are now able to construct a transfinite sequence $(C_\alpha)_{\alpha\in\omega_1}$ such that, for each $\beta \in \omega_1$, $C_\beta \cap [\bigcup_{\alpha<\beta} C_\alpha \cup \bigcup_{n\in\omega} I_n] = \emptyset$. The point is that at stage β we know that the countable subset $\{C_\alpha : \alpha < \beta\} \cup \{I_n : n \in \omega\}$ is bounded above by some $J \in \mathfrak{A}$ and that this J is not a maximal element of $\mathfrak{A}$. By the choice of the sets I_n we have $\|\mathbb{1}_{A\cup C_\alpha}\| = \|\mathbb{1}_A\|$ for all α. If the norm were strictly convex we should have $\|\mathbb{1}_A + \frac{1}{2}\mathbb{1}_{C_\alpha}\| < \|\mathbb{1}_A\|$ for all α. So for some $\epsilon > 0$) there would exist uncountably many α with $\|\mathbb{1}_A + \frac{1}{2}\mathbb{1}_{C_\alpha}\| < \|\mathbb{1}_A\| - \epsilon$. This leads to the same contradiction that we had in the first part. □

Corollary 4.2. *There exists a Banach space which does not have a subspace isomorphic to ℓ_∞ and which admits no strictly convex equivalent norm.*

P r o o f. We take $\mathfrak{A}$ to be the Boolean ring constructed in 3.1 and consider the Banach space $\mathcal{C}_0(S(\mathfrak{A}))$. □

5. The Namioka Property and σ-fragmentability. A compact space K has the Namioka Property ($\mathcal{N}^*$) if, for every Baire space B, and every separately continuous function $\phi : B \times K \to \mathfrak{R}$, there is a dense $\mathcal{G}_\delta$ subset H of B such that ϕ is (jointly) continuous at every point of $H \times K$. The Namioka Property can be equivalently formulated as follows: a compact space has ($\mathcal{N}^*$)

if and only if, for every Baire space B, every function $\chi : B \to \mathcal{C}(K)$, continuous into the topology $\tau_{\mathrm{p}}(K)$ of pointwise convergence, is continuous into the norm topology at every point of some dense $\mathcal{G}_\delta$ subset of B. It is known that members of several familiar classes of "good" compact spaces (for example, Corson compacta [2] and dyadic compacta [5]) have this property, but that not all compact spaces do [14, 3]. Talagrand's example was essentially based on the space $\ell_\infty^{\mathrm{c}}(\omega_1)$ and Deville's on ℓ_∞. Namioka and the present author observed independently that examples of non-Namioka compact spaces K such that $\mathcal{C}(K)$ does not contain ℓ_∞ could be constructed using the tree-completeness property of [10]. In fact, the crucial issue is exactly the kind of Baire property that we have been looking at, and indeed it was the role of Baire ideas in this context that made it seem worth examining them in greater detail.

As well as dealing with the Namioka Property, both Namioka and I also proved a similar result about σ-fragmentability. We recall from [11] that a topological space X is said to be *σ-fragmentable* by a metric d if, for each $\epsilon > 0$, there exists a countable covering $(X_n)_{n\in\omega}$ of X such that, for each n and each nonempty subset Y of X_n, some non-empty, relatively open subset of Y has d-diameter at most ϵ. We say that a Banach space X is σ-fragmentable if (X, weak) is σ-fragmentable by the norm. It is shown in [11] that many Banach spaces are of this type, but that ℓ_∞ is not, while $\ell_\infty^{\mathrm{c}}(\omega_1)$, under the weak topology, is σ-fragmentable by no l.s.c. norm. We shall now look at what σ-fragmentability of $\mathcal{C}_0(S(\mathfrak{A}))$ says about the Boolean ring $\mathfrak{A}$.

We start by defining a property of Namioka type for a general topological space. We shall say that a topological space X has the property (DN) if, for every non-empty Baire space B and every continuous function $\phi : B \to X$, there is a non-empty open subset U of B on which ϕ is constant. We start with an easy lemma of general topology. Recall that a topological space Y is said to be *scattered* if every non-empty subset Z of Y has a (relatively) isolated point. We shall say that X is σ-scattered if X is the union of countably many scattered subsets.

Proposition 5.1. *If the topological space is σ-scattered, then X has Property* (DN).

P r o o f. Let ϕ be a continuous mapping from a Baire space B into X and let $(X_n)_{n\in\omega}$ be a covering of X by scattered subsets. By the Baire property of B, there is some n such that $\operatorname{cl}\phi^{-1}[X_n]$ has nonempty interior. Set $V = \operatorname{int}\operatorname{cl}\phi^{-1}[X_n]$ and $W = V \cap \phi^{-1}[X_n]$, noting that $V \subseteq \operatorname{cl} W$. The nonempty subset $\phi[W]$ of X_n has an isolated point x, which means that there exists an open subset U of X

with $U \cap \phi[W] = \{x\}$. So $\phi^{-1}[U \setminus \{x\}] \cap W$ is empty. But $\phi^{-1}[U \setminus \{x\}]$ is open in B and W is dense in V, so that $\phi^{-1}[U \setminus \{x\}] \cap V$ is also empty. This shows that ϕ takes the value x everywhere on the non-empty open set $\phi^{-1}[U] \cap V$. □

Proposition 5.2. *Let $\mathfrak{A}$ be a Boolean ring and let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$.*

- *If $\mathcal{C}_0(S(\mathfrak{A}))$ is σ-fragmentable then $\mathfrak{I}, \tau(\mathfrak{I})$ is σ-scattered.*
- *If $S(\mathfrak{A})$ has the Namioka Property then $\mathfrak{I}, \tau(\mathfrak{I})$ has Property* (DN).

Proof. (1) If $\mathcal{C}(S(\mathfrak{A}))$ is σ-fragmentable by the norm then so is its closed subspace $\mathcal{C}_0(S(\mathfrak{I}))$. We apply σ-fragmentability, with $\epsilon = \frac{1}{2}$. Let X_n $(n \in \omega)$ be the subsets whose existence is thus guaranteed. We may assume that each X_n is bounded. Since the measures of compact support form a norm-dense subspace of $\mathcal{C}_0(S(\mathfrak{I}))^*$, the weak topology on each each X_n is now coarser than τ_{C}. It follows that if $\mathfrak{X}_n = \{I \in \mathfrak{I} : \mathbb{1}_I \in X_n\}$ then each map $I \mapsto \mathbb{1}_I : \mathfrak{X}_n \to X_n$ is continuous from $\tau(\mathfrak{I})$ into the weak topology. If $\mathfrak{Y}$ is a non-empty subset of $\mathfrak{X}_n$ there thus exists $I \in \mathfrak{Y}$ and a $\tau(\mathfrak{I})$ neighbourhood $\mathfrak{N}$ of I such that $\|\mathbb{1}_I - \mathbb{1}_J\|_\infty < \frac{1}{2}$ whenever $J \in \mathfrak{N} \cap \mathfrak{Y}$. Of course, this implies that $\mathfrak{N} \cap \mathfrak{Y} = \{I\}$, so that I is an isolated point of $\mathfrak{Y}$.

(2) Again we identify $\mathcal{C}_0(S(\mathfrak{I}))$ with a closed subspace of $\mathcal{C}_0(S(\mathfrak{A}))$ and note that the topologies $\tau_{\mathrm{p}}(S(\mathfrak{A}))$, $\tau_{\mathrm{p}}(S(\mathfrak{I}))$ coincide on it. Thus if $\phi : B \to \mathfrak{I}, \tau(\mathfrak{I})$ is a continuous map, the map $x \mapsto \mathbb{1}_{\phi(x)} : B \to \mathcal{C}_0(S(\mathfrak{A})), \tau_{\mathrm{p}}$ is continuous too. Provided B is a Baire space, this map has points of continuity into the norm topology. Since the set of indicator functions is norm-discrete, this gives what we want. □

Proposition 5.3. *Let $\mathfrak{A}$ be a Boolean algebra and assume either that $S(\mathfrak{A})$ has the Namioka Property or that $\mathcal{C}(S(\mathfrak{A}))$ is σ-fragmentable. If $\mathfrak{I}$ is an ideal of $\mathfrak{A}$ and $\mathfrak{I}, \tau(\mathfrak{I})$ is a Baire space then $\mathfrak{I}$ has a maximal element.*

Proof. It follows from 5.1 and 5.2 that, subject to either of our hypotheses, $\mathfrak{I}, \tau(I)$ must have the (DN) property. If this space is also Baire then the identity map must be constant on some nonempty open set. This is a way of saying that $\mathfrak{I}$ must have an isolated point, A say. By the way $\tau(\mathfrak{I})$ is defined, there exists $I \in \mathfrak{I}$ such that if $B \in \mathfrak{I}$ and $B \cap I = A \cap I$ then $B = A$. A moment's thought allows us to deduce that I is a maximal element of $\mathfrak{I}$. □

The above proposition gives us a way to construct many examples of non-Namioka compacta with non-σ-fragmentable spaces of continuous functions. In particular, we have the following corollary.

Corollary 5.4. *Let $\mathfrak{A}$ be an infinite Boolean ring with the SCP. Then $S(\mathfrak{A})$ does not have the Namioka Property and $\mathcal{C}_0(S(\mathfrak{A}))$ is not σ-fragmentable.*

Proof. If $\mathfrak{A}$ is not a Boolean algebra we may take $\mathfrak{I} = \mathfrak{A}$ in the theorem. Otherwise, let $\mathfrak{I}$ be a non-principal maximal ideal of $\mathfrak{A}$. It is easy to see that $\mathfrak{I}$ has the SCP. Indeed, given a disjoint sequence (A_n) in $\mathfrak{I}$ there exist infinite subsets M, N of ω such that $\{A_{2m} : m \in M\}$ and $\{A_{2n+1} : n \in N\}$ have least upper bounds A, B respectively in $\mathfrak{A}$. Of the disjoint elements A, B of $\mathfrak{A}$ at least one lies in $\mathfrak{I}$ by maximality of this ideal. Since $\mathfrak{I}$ has the SCP it is a Baire space. On the other hand, saying that $\mathfrak{I}$ is non-principal is the same as saying that it has no maximal element. □

Corollary 5.5. *There exists a non-Namioka compact space K such that $\mathcal{C}(K)$ is not σ-fragmentable and has no subspace isomorphic to ℓ_∞.*

It is shown in [11] that a Banach space that admits an equivalent l.u.r. norm is necessarily σ-fragmentable. Thus the results above give an alternative way to prove the "l.u.r. part" of 4.1. The "strictly convex part", on the other hand, seems to need the technique of Section 4.

Corollary 5.4 is an analogue of the results of [3] and and of [11] about ℓ_∞. Not surprisingly, there is also an analogue of the result of [11] about ℓ_∞^{c}.

Theorem 5.6. *Let $\mathfrak{A}$ be a Boolean ring without maximal element. Assume that every countable subset of $\mathfrak{A}$ is bounded above and that $\mathfrak{A}$ is a Baire space for the topology $\tau(\mathfrak{A})$. Then there is no weakly l.s.c. metric on $\mathcal{C}_0(S(\mathfrak{A}))$ for which this space is σ-fragmentable.*

From this theorem and the construction of Section 3 we immediately get the following corollary.

Corollary 5.7. *There exists a Banach space X which contains no copy of ℓ_∞ but which is σ-fragmentable for no weakly l.s.c. metric.*

We devote the rest of the present section to the proof of Theorem 5.6. Applying arguments similar to those used earlier, we see that it will be enough to show that $\mathfrak{A}, \tau(\mathfrak{A})$ is not σ-fragmentable for a $\tau(\mathfrak{A})$-l.s.c. metric. We shall deduce this from a result about σ-fragmentability in more general Baire spaces. More subtle arguments of the same type appear in [11].

Proposition 5.8. *Let B be a Baire space which is σ-fragmentable by a metric d. Then there is a non-empty $\mathcal{G}_\delta$ subset of B which is d-separable. If d is l.s.c. then B contains a dense set of $\mathcal{G}_\delta$ points.*

Proof. For each $n \in \omega$ let $(X_{n,k})_{k\in\omega}$ be a covering of B such that, for every k and every non-empty $Y \subseteq X_{n,k}$, there exists an open subset U of B with $U \cap Y \neq \emptyset$ and d-diam $Y \cap U < 2^{-n}$. For each n, k we define

$$G_{n,k} = \bigcup \{U : U \text{ is open in } B \text{ and } d\text{-diam}\,(U \cap X_{n,k}) < 2^{-n}\}.$$

Our convention is that d-diam $\emptyset = -\infty$ so that $G_{n,k} \supseteq \operatorname{int}(B \setminus X_{n,k})$. I claim that each $G_{n,k}$ is dense in B; because of what we have just said, it is enough to show that $G_{n,k}$ meets every non-empty open subset V of $\operatorname{int}\operatorname{cl} X_{n,k}$. For such a V, $V \cap X_{n,k}$ is non-empty and so there exists an open subset W of B such that $W \cap V \cap X_{n,k} \neq \emptyset$ and d-diam $W \cap V \cap X_{n,k} < 2^{-n}$. Thus $G_{n,k} \cap V$ contains the non-empty set $W \cap V$.

By the Baire property of B, the intersection $\bigcap_{n,k\in\omega} G_{n,k}$ is dense in B. Let x be any point of this intersection and choose, for each n, k an open set $U_{n,k}$ containing x, with d-diam $(U_{n,k} \cap X_{n,k}) < 2^{-n}$. Let T be the intersection of all the $U_{n,k}$, a non-empty $\mathcal{G}_\delta$ subset of B. I claim that T is d-separable. Indeed, for any $\mathbf{m} = (m(n))_{n\in\omega} \in \omega^\omega$ and any $l \in \omega$, the d-diameter of $T \cap \bigcap_{n\le l} X_{n,m(n)}$ is at most 2^{-l}. Thus, for the d-topology, T is a continuous image of the subset $\{\mathbf{m} : T \cap \bigcap_{n\in\omega} X_{n,m(n)} \neq \emptyset\}$ of ω^ω, the mapping ϕ being such that $\{\phi(\mathbf{m})\} = T \cap \bigcap_{n\in\omega} X_{n,m(n)}$.

Finally, if d is lower semicontinuous, then every closed d-ball is closed for the topology of B. If x and T are as above and D is a countable d-dense subset of T then

$$\{x\} = T \setminus \bigcup_{z\in D, n\in\omega} \{y \in B : d(y,z) \le d(x,z) - 2^{-n}\},$$

so that x is a $\mathcal{G}_\delta$ point. □

Corollary 5.9. *Let $\mathfrak{A}$ be a Boolean ring in which every countable subset is bounded above and assume that $\mathfrak{A}$ is a Baire space for $\tau(\mathfrak{A})$. If d is a metric for which $\mathfrak{A}, \tau(\mathfrak{A})$ is σ-fragmentable then there is a nonempty $\tau(\mathfrak{A})$-open subset of $\mathfrak{A}$ which is d-separable. If d is $\tau(\mathfrak{A})$-l.s.c. then $\mathfrak{A}$ must have a maximal element.*

Proof. We have already noted that the hypothesis about countable subsets of $\mathfrak{A}$ implies that all $\mathcal{G}_\delta$ subsets of $\mathfrak{A}$ are open. Thus the first assertion follows from 5.8. When we assume lower semicontinuity of d, we see that $\mathfrak{A}$ must contain a point that is $\tau(\mathfrak{A})$-isolated. We have already noted that such a point exists only if $\mathfrak{A}$ has a maximal element (in which case $\tau(\mathfrak{A})$ is the discrete topology). □

We have now completed the proof of 5.6. It is worth noting that the assertion in 5.9 about d-separable $\tau(\mathfrak{A})$-open sets shows that the space $\mathcal{C}_0(S(\mathfrak{A}))$

constructed in Section 3 is σ-fragmentable for no "sensible" metric. For instance, since this space $\mathcal{C}_0(S(\mathfrak{A}))$ can be regarded as a subspace of $\ell_\infty^{\mathrm{c}}(\omega_1)$, one notion of a sensible metric might be one that induces a topology finer than the topology of pointwise convergence on ω_1. It is easy to see that no non-empty $\tau(\mathfrak{A})$-open set is separable for this pointwise topology. On the other hand there do exist metrics (not "sensible") on this space, for which it is σ-fragmentable. Indeed, since the cardinality of the space is equal to the continuum, we may equip it with the structure of a separable metric space.

6. Grothendieck spaces. As we noted in the introduction, the Subsequential Completeness Property was originally introduced to answer a question about Grothendieck spaces. It is natural to ask, after what we have seen in the preceding sections, whether a Baire property for $\mathfrak{A}$ of the kind that we have been looking at will suffice to prove that $\mathcal{C}_0(S(\mathfrak{A}))$ is a Grothendieck space. The answer is yes, provided we take a rather strong Baire assertion, namely the conclusion of Theorem 2.1. The proof goes somewhat against the philosophy of the previous sections, since, rather than using the Baire property itself, we use it to deduce a new weak completeness property, the SSP, which we then show implies the our desired conclusion. On the other hand we shall indicate at the end of this section a way in which the SSP may turn out to be of independent interest.

We shall say that a Boolean ring $\mathfrak{A}$ has the *Subsequential Separation Property* if, whenever $(A_n)_{n\in\omega}$ is a disjoint sequence in $\mathfrak{A}$ there exists an infinite subset $M \subseteq \omega$ and an element D of $\mathfrak{A}$ such that $A_{2n} \subseteq D$ and $A_{2n+1} \cap D = \emptyset$ for all $n \in M$.

Theorem 6.1. *Let $\mathfrak{A}$ be a Boolean algebra and assume that $\mathfrak{A}, \tau(\mathfrak{I})$ is a Baire space for every ideal $\mathfrak{I}$ of $\mathfrak{A}$. Then $\mathfrak{A}$ has the SSP.*

Proof. Let $(A_n)_{n\in\omega}$ be a disjoint sequence in $\mathfrak{A}$ and let $\mathfrak{I}$ be the ideal generated by the A_n. For each positive integer m define

$$\begin{aligned}\mathcal{G}_m &= \{B \in \mathfrak{A} : A_{2n} \subseteq B \text{ and } A_{2n+1} \cap B = \emptyset \text{ for at least } m \text{ values of } n\}\\ &= \textstyle\bigcup_{F\subseteq\omega,\ \#F=m}\{B \in \mathfrak{A} : B \cap \bigcup_{n\in F}(A_{2n} \cup A_{2n+1}) = \bigcup_{n\in F} A_{2n}\}.\end{aligned}$$

The second expression above shows that $\mathcal{G}_m$ is $\tau(\mathfrak{I})$-open. I shall show that it is also $\tau(\mathfrak{I})$-dense. Let A be any element of $\mathfrak{A}$ and let I be in $\mathfrak{I}$. Then $I \subseteq \bigcup_{j<2k}$ for some integer k and the neighbourhood $\mathcal{N}(A, I)$ of A contains the element $(A \cap I) \cup \bigcup_{0\le j<m} A_{2k+2j}$ of $\mathcal{G}_m$. The assumption that $\tau(\mathfrak{I})$ is a Baire topology implies that $\bigcap_{m\ge1} \mathcal{G}_m \ne \emptyset$ and any element D of this intersection has the property required for the SSP. $\square$

We now recall that a Banach space X is said to be a *Grothendieck space* if every weak* convergent sequence in X^* is also weakly convergent. The key to all existing criteria for $\mathcal{C}_0(S(\mathfrak{A}))$ to be a Grothendieck space is a reduction to the case of sequences of measures supported by disjoint compact open sets. The following version of this idea is the one used in [15].

Proposition 6.2. *Let S be a locally compact, totally disconnected space and assume that $c_0(S)$ is not a Grothendieck space. Then there exist a disjoint sequence (B_n) of compact open subsets of S and a sequence of Radon measures (μ_n) on S such that $\|\mu_n\| = |\mu_n|(B_n) = 1$ for all n, and $\mu_n \to 0$ in the weak* topology.*

Theorem 6.3. *Let $\mathfrak{A}$ be a Boolean ring with the SSP. Then $\mathcal{C}_0(S(\mathfrak{A}))$ is a Grothendieck space.*

Proof. Suppose that $C_0(S(\mathfrak{A}))$ is not a Grothendieck space. Let the sequences (B_n) in $\mathfrak{A}$ and (μ_n) in $\mathcal{C}_0(S(\mathfrak{A}))^*$ be as in the conclusion of 6.2. For each n, $|\mu_n|(B_n) = 1$, so there exists an element A_{2n} of $\mathfrak{A}$ with $A_{2n} \subseteq B_n$ and $|\mu_n(A_{2n})| > \frac{1}{3}$. We set $A_{2n+1} = B_n \setminus A_{2n}$. If M and D are as in the definition of the SSP, then $|\mu_n(D)| = |\mu_n(D \cap B_n)| = |\mu_n(A_{2n})| > \frac{1}{3}$ for all $n \in M$, contradicting the weak* convergence of (μ_n) to 0. □

The Subsequential Completeness Property and the weaker Subsequential Interpolation Property introduced by Freniche [7] both imply the so-called Nikodym Boundedness Property as well as the Grothendieck Property. Now Talagrand [16] gave an example, constructed using the Continuum Hypothesis, to show that, in general, Grothendieck does not imply Nikodym. The SSP seems to be a good candidate for an algebraic property of $\mathfrak{A}$ which would imply the Grothendieck Property but not the Nikodym Property. I conjecture that this is indeed the case and that the necessary construction can be carried out without a special axiom such as CH.

Final remarks. The topologies $\tau(\mathfrak{I})$ which we have considered in this article have the advantage that they are quite "algebraic" and lend themselves to nice combinatorial arguments. Also, as we have noted, the map $A \mapsto \mathbb{1}_A$ is continuous from $\tau(\mathfrak{A})$ into the the weak topology of $\mathcal{C}_0(S(\mathfrak{A}))$. However, it is possible to look at the topology τ_{p} of *pointwise* convergence, defined so that $A \mapsto \mathbb{1}_A$ is a homeomorphic embedding into $\mathcal{C}_0(S(\mathfrak{A})), \tau_{\mathrm{p}}$. Certain implications are easy to establish:

- if $S(\mathfrak{A})$ has the Namioka Property, then A, τ_{p} has the property (DN);

- if $\mathcal{C}_0(S(\mathfrak{A})), \tau_{\mathrm{p}}$ is σ-fragmentable by the norm then $\mathfrak{A}, \tau_{\mathrm{p}}$ is σ-scattered;
- if $\mathcal{C}_0(s(\mathfrak{A}))$ admits a τ_{p}-l.s.c., l.u.r. equivalent norm, then $\mathfrak{A}, \tau_{\mathrm{p}}$ is σ-discrete.

It is not clear whether any of the reverse implications hold, that is to say whether properties of the Boolean ring imply the corresponding properties of the Banach space. If this turned out to be the case it might be a useful tool for investigating problems such as whether σ-fragmentability implies l.u.r. renormability.

To finish I must mention a very recent result [9] which provides a counterexample stronger than 4.2 and 5.5 of this paper. It is a compact *scattered* space K such that $\mathcal{C}(K)$ admits no strictly convex equivalent norm and is not σ-fragmentable; K does not have the Namioka Property. The reader will recall that one crucial property of the spaces $\mathcal{C}_0(S(\mathfrak{A}))$ was that they contained no copy of ℓ_∞; when K is a scattered compact the space $\mathcal{C}(K)$ does not contain ℓ_1, a much more restrictive condition.

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