Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **25** (1999), 283-296

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

REPRESENTATIONS AND POSITIVE DEFINITE FUNCTIONS ON HYPERGROUPS

A. Nasr-Isfahani

Communicated by S. L. Troyanski

Abstract. Some relationships between representations of a hypergroup X, its algebras, and positive definite functions on X are studied. Also, various types of convergence of positive definite functions on X are discussed.

1. Introduction. The theory of locally compact hypergroups in harmonic analysis was initiated with a slight difference independently by Dunkl [5], Jewett [8], and Spector [19] in the early 1970's. In 1968, Pym [14] also considered convolution structures which are close to this theory. Ross [16] gives a nice survey of the subject; see also [17] and [18].

Nevertheless, the term "hypergroup" is used long before by a number of mathematicians to describe some different and important mathematical structures. For examples see Bruck [2] and Delsarte [4].

Furthermore, the ideas of hypergroups in analysis appear in works of Delsarte [3] in 1938 and Levitan [11] in 1940 on generalized translations operators. In the early 1950's, using these ideas, Berezansky and Krein studied some structures similar to hypergroups in harmonic analysis and called them hypercomplex

1991 *Mathematics Subject Classification:* Primary 43A65, 43A35. Secondary 43A62, 43A10.
Key words: Hypergroup, representation, positive definite function.

systems. The theory of generalized translation operators and hypercomplex systems and their connections with harmonic analysis are explained in Berezansky and Kalyuzhnyi [1].

While the hypergroup structure of Jewett has become the standard in many papers of harmonic analysis, here we follow that of Dunkl [5], Definition 1.1 without the commutativity assumption, that is more general than that of Jewett. Suppose that X is a locally compact Hausdorff space, $M(X)$ is the Banach space of all complex regular Borel measures on X, and $M_p(X)$ is the set of all probability Borel measures on X. As usual, we reserve the symbols $C_b(X), C_0(X)$, and $C_{00}(X)$ for the spaces of bounded continuous complex-valued functions on X, those that vanish at infinity, and those that have compact support.

The space X is called a *hypergroup* if there is a map λ from $X \times X$ into $M_p(X)$ with the following properties:

(i) for each $x, y \in X$, the measure $\lambda_{(x,y)}$ has compact support.

(ii) for each $f \in C_{00}(X)$, the mapping $(x,y) \mapsto \int_X f(t)\, d\lambda_{(x,y)}(t)$ is continuous, and the mappings $x \mapsto \int_X f(t)\, d\lambda_{(x,y)}(t)$ and $x \mapsto \int_X f(t)\, d\lambda_{(y,x)}(t)$ are in $C_{00}(X)$ for all $y \in X$.

(iii) the convolution $*$ on $M(X)$ defined implicitly by

$$\int_X f(t)\, d(\mu * \nu)(t) = \int_X \int_X \int_X f(t)\, d\lambda_{(x,y)}(t)\, d\mu(x)\, d\nu(y)$$

$(\mu, \nu \in M(X), f \in C_0(X))$, is associative.

(iv) there is a point $e \in X$ (the *identity*) such that $\lambda_{(x,e)} = \delta_x = \lambda_{(e,x)}$ for all $x \in X$, where δ_x denotes the Dirac measure at x.

For a hypergroup X, define $L(X)$ to be the subalgebra of $M(X)$, consisting of all measures μ for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from X into $M(X)$ are norm continuous. Then $L(X)$ is a closed ideal in $M(X)$ (see Medghalchi [12] or [13]).

Throughout this paper, X will denote a hypergroup with

$$X = \textit{ The closure of the set } \bigcup \{\mathrm{supp}(\mu) : \mu \in L(X)\ \}.$$

Finally, we assume that the reader is familiar with the representation theory of normed algebras. However, we recall that a representation T of an algebra A by bounded operators on a normed space E is said to be *cyclic* if there exists a vector $\zeta \in E$, *cyclic vector*, such that the linear subspace $\{T_x\zeta \ : \ x \in A\}$ is dense in E. The representation T is said to be *nondegenerate* if for every $0 \neq \xi \in E$, there exists an element $x \in A$ such that $T_x\xi \neq 0$.

2. Some relations between representations of X and representations of $L(X)$. In [9], Lashkarizadeh-Bami generalized to a large family of topological semigroups parts of well-defined theory of representations of topological groups. In this section, we develop these results for hypergroups. We begin with the following definition.

Definition 2.1. *Let E be a reflexive Banach space. A* representation *V of X by bounded operators on E^*, the dual space of E, is a mapping $x \longmapsto V_x$ of X into $B(E^*)$, the space of all bounded operators on E^*, such that the following hold:*

(i) the function $t \longmapsto \langle V_t\xi, \eta\rangle$ is bounded and continuous on X for all $\xi \in E^*$ and $\eta \in E$.

(ii) $\int_X \langle V_t\xi, \eta\rangle \; d\lambda_{(x,y)}(t) = \langle V_x V_y \xi, \eta\rangle$ for all $x, y \in X$, $\xi \in E^*$, and $\eta \in E$.

The representation V is said to be bounded *if there is a positive real number k such that $\| V_x \| \leq k$ for all $x \in X$; the infimum of all such k will be denoted by $\| V \|$.*

Theorem 2.2. *Let E be a reflexive Banach space and V be a bounded representation of X by bounded operators on E^*. Then for every subalgebra A of $M(X)$, the formula*

$$\langle T_\mu \xi, \eta\rangle = \int_X \langle V_t\xi, \eta\rangle \; d\mu(t) \quad (\mu \in A, \xi \in E^*, \eta \in E)$$

defines a bounded representation T of A by bounded operators on E^* with $\|T\| \leq \|V\|$.

P r o o f. The proof is similar to that given for Theorem 22.3 of [6], in the group case, so we omit it. □

The following two lemmas are needed for the proof of the main result of this section. We omit the proof of the first lemma, since it is straightforward.

Lemma 2.3. *If $f \in C_b(X)$ is such that $\int_X f(x) \; d\mu(x) = 0$ for every $\mu \in L(X)$, then $f(x) = 0$ for all $x \in X$.*

Lemma 2.4. *Let $\mu \in L(X)$, $\nu \in M(X)$, and let τ be a continuous linear functional on $L(X)$. Then*

$$\tau(\mu * \nu) = \int_X \tau(\mu * \delta_t) \; d\nu(t) \quad \text{and} \quad \tau(\nu * \mu) = \int_X \tau(\delta_t * \mu) \; d\nu(t).$$

P r o o f. We first show that $\mu * \delta_x \in L^1(X, |\mu| * |\nu|)$ for all $x \in \text{supp}(\nu)$. To this end, let K be a compact subset of X and $|\mu| * |\nu| (K) = 0$. Then

$$\int_X |\mu| *\delta_x(K)\, d\,|\nu|\,(x) = \int_X \int_K d\,|\mu| *\delta_x(y)\, d\,|\nu|\,(x) = |\mu| * |\nu|\,(K) = 0.$$

Thus, since the function $x \longmapsto |\mu| *\delta_x(K)$ is in $C_b(X)$, it follows from Lemma 2.3 that $|\mu| *\delta_x(K) = 0$ for all $x \in \text{supp}(\nu)$.

Now, since τ is a bounded linear functional on $L^1(X, |\mu| * |\nu|)$, by the Radon-Nikodym theorem, there exists a bounded Borel measurable function g on X such that $\tau(\sigma) = \int_X g(t)\, d\sigma(t)$ for all $\sigma \in L^1(X, |\mu| * |\nu|)$. Hence

$$\begin{aligned} \tau(\mu * \nu) &= \int_X g(t)\, d(\mu * \nu)(t) = \int_X \int_X \int_X g(t)\, d\lambda_{(y,x)}(t)\, d\mu(y)\, d\nu(x) \\ &= \int_X \int_X g(y)\, d(\mu * \delta_x)(y)\, d\nu(x) = \int_X \tau(\mu * \delta_x)\, d\nu(x). \end{aligned}$$

The proof of the other formula is similar. □

Let V be a representation of X by bounded operators on E^*, where E is a reflexive Banach space. Then V is said to be *faithful* if for each $x, y \in X$ with $x \neq y$, we have $V_x \neq V_y$. A subspace M of E^* is said to be *invariant under* V if $V_x(M) \subseteq M$ for all $x \in X$.

We are now ready to state and prove the main theorem of this section.

Theorem 2.5. *Suppose that T is a bounded cyclic representation of $L(X)$ by bounded operators on E^*, where E is a reflexive Banach space. Then there exists a unique bounded representation V of X by bounded operators on E^* with $V_e = I$ such that $\| V \| = \| T \|$ and*

$$\langle T_\mu \xi, \eta \rangle = \int_X \langle V_t \xi, \eta \rangle\, d\mu(t) \quad (\mu \in L(X), \xi \in E^*, \eta \in E). \tag{1}$$

Furthermore, $V_x T_\mu = T_{\delta_x * \mu}$ and $T_\mu V_x = T_{\mu * \delta_x}$ for all $x \in X$ and $\mu \in L(X)$. If T is faithful, then V is faithful, and in this case $V_x \neq 0$ for every $x \in X$. Moreover, T and V have the same closed invariant subspaces.

P r o o f. By Proposition 1 of [12], there exists a bounded approximate identity $(\mu_\alpha)_{\alpha \in J}$ for $L(X)$ with $\| \mu_\alpha \| = 1$ for all $\alpha \in J$. Let $\zeta \in E^*$ be a fixed cyclic vector for T; thus the linear subspace

$$S = \{T_\mu \zeta : \mu \in L(X)\}$$

of E^* is dense in E^*. For every $x \in X$ and $\mu \in L(X)$, we have

$$\| T_{\delta_x * \mu} - T_{\delta_x * \mu_\alpha * \mu} \| \leq \| T \| \, \| \mu - \mu_\alpha * \mu \| \, .$$

It follows that

$$T_{\delta_x * \mu}(\zeta) = \lim_\alpha T_{\delta_x * \mu_\alpha * \mu}(\zeta).$$

We denote this limit by $U_x T_\mu \zeta$; so

$$U_x T_\mu \zeta = \lim_\alpha T_{\delta_x * \mu_\alpha}(T_\mu \zeta) = \lim_\alpha T_{\delta_x * \mu_\alpha * \mu}(\zeta) = T_{\delta_x * \mu}(\zeta). \tag{2}$$

To prove that U_x is well defined, we suppose $T_{\mu_1}\zeta = T_{\mu_2}\zeta$, where $\mu_1, \mu_2 \in L(X)$. Then for each $\alpha \in J$, we have

$$T_{\delta_x * \mu_\alpha * \mu_1}(\zeta) = T_{\delta_x * \mu_\alpha}(T_{\mu_1}\zeta) = T_{\delta_x * \mu_\alpha}(T_{\mu_2}\zeta) = T_{\delta_x * \mu_\alpha * \mu_2}(\zeta).$$

Now, by (2), we get

$$U_x(T_{\mu_1}\zeta) = \lim_\alpha T_{\delta_x * \mu_\alpha * \mu_1}(\zeta) = \lim_\alpha T_{\delta_x * \mu_\alpha * \mu_2}(\zeta) = U_x(T_{\mu_2}\zeta).$$

This shows that U is well defined.

On the other hand, for each $\alpha \in J$, we have

$$\| T_{\delta_x * \mu_\alpha}(T_\mu \zeta) \| \leq \| T \| \, \| T_\mu \zeta \| \qquad (x \in X, \mu \in L(X)).$$

Therefore $\| U_x \| \leq \| T \|$ for all $x \in X$ by (2). Thus for each $x \in X$, U_x is a bounded operator on S and hence we can extend U_x uniquely to a bounded operator V_x on E^* with $\| V_x \| = \| U_x \|$.

Now let $\xi \in E^*$ and $\eta \in E$. Then for every $\mu \in L(X)$ and every $x, y \in X$, we infer that

$$\begin{aligned} \| V_x \xi - V_y \xi \| &\leq \| V_x \xi - V_x T_\mu \zeta \| + \| V_x T_\mu \zeta - V_y T_\mu \zeta \| \\ &+ \| V_y T_\mu \zeta - V_y \xi \| \\ &\leq \| T \| \; (\, \| \xi - T_\mu \zeta \| + \| \delta_x * \mu - \delta_y * \mu \| \, \| \zeta \| \,). \end{aligned}$$

Since the mapping $t \longmapsto \delta_t * \mu$ is norm continuous on X and S is dense in E^*, the mapping $t \longmapsto \langle V_t \xi, \eta \rangle$ is bounded and continuous on X.

For each fixed $\eta \in E$, the mapping

$$\mu \longmapsto \langle T_\mu \zeta, \eta \rangle \qquad (\mu \in L(X)),$$

defines a bounded linear functional on $L(X)$. So by Lemma 2.4, for each $\mu \in L(X)$ and $\nu \in M(X)$, we have

$$\langle T_{\nu*\mu}\zeta, \eta\rangle = \int_X \langle T_{\delta_t*\mu}\zeta, \eta\rangle \ d\nu(t). \tag{3}$$

Now, let $x, y \in X$. Then for every $\mu \in L(X)$, we obtain that

$$\begin{aligned} \langle V_x V_y T_\mu \zeta, \eta\rangle &= \left\langle T_{\lambda_{(x,y)}*\mu}(\zeta), \eta \right\rangle \\ &= \int_X \langle T_{\delta_t*\mu}(\zeta), \eta\rangle \ d\lambda_{(x,y)}(t) \\ &= \int_X \langle V_t T_\mu \zeta, \eta\rangle \ d\lambda_{(x,y)}(t) \end{aligned}$$

Thus, since for each $\eta \in E$ both functions

$$\xi \longmapsto \langle V_x V_y \xi, \eta\rangle \quad \text{and} \quad \xi \longmapsto \int_X \langle V_t \xi, \eta\rangle \ d\lambda_{(x,y)}(t)$$

are linear and bounded on E^*, and S is dense in E^*, it follows that

$$\langle V_x V_y \xi, \eta\rangle = \int_X \langle V_t \xi, \eta\rangle \ d\lambda_{(x,y)}(t) \quad (x, y \in X, \xi \in E^*, \eta \in E).$$

Hence V defines a bounded representation of X by bounded operators on E^* with $V_e = I$ and $\| V \| \leq \| T \|$.

Now, fix $\mu \in L(X)$ and $\eta \in E$. Then for every $\nu \in L(X)$, it follows from (3) that

$$\langle T_\mu T_\nu \zeta, \eta\rangle = \langle T_{\mu*\nu}(\zeta), \eta\rangle = \int_X \langle V_t T_\nu \zeta, \eta\rangle \ d\mu(t).$$

This establishs the formula (1) since S is dense in E^*. We also have

$$| \langle T_\mu \xi, \eta\rangle | = \left| \int_X \langle V_t \xi, \eta\rangle \ d\mu(t) \right| \leq \| V \| \| \xi \| \| \eta \| \| \mu \|$$

for all $\xi \in E^*, \eta \in E$ and $\mu \in L(X)$ which implies that $\| T \| \leq \| V \|$. Therefore $\| V \| = \| T \|$.

Lemma 2.3 together with the formula (1) imply the uniqueness of V.

By the use of (1), one can easily prove that

$$V_x T_\mu = T_{\delta_x*\mu} \quad \text{and} \quad T_\mu V_x = T_{\mu*\delta_x} \quad (x \in X, \mu \in L(X)).$$

We now suppose that T is faithful. If $x, y \in X$ with $x \neq y$, then there exists a function $f \in C_0(X)$ such that $f(x) \neq f(y)$. Therefore, Lemma 2.3 implies that

$$\int_X f \, d\delta_x * \mu \neq \int_X f \, d\delta_y * \mu \quad \text{for some} \quad \mu \in L(X).$$

Thus $\delta_x * \mu \neq \delta_y * \mu$ and hence $V_x T_\mu = T_{\delta_x * \mu} \neq T_{\delta_y * \mu} = V_y T_\mu$. So $V_x \neq V_y$; that is V is faithful.

To prove the last assertion of the theorem, we consider a closed linear subspace M of E^*. If M is invariant under T, x is in X, and ξ is a vector in M with $V_x \xi \notin M$, then there exists $\eta \in E^{**} = E$ such that $\langle V_x \xi, \eta \rangle = 1$ and $\langle m, \eta \rangle = 0$ for all $m \in M$, and thus Lemma 2.3 together with (1) show that $\langle V_t \xi, \eta \rangle = 0$ for all $t \in X$. In particular $\langle V_x \xi, \eta \rangle = 0$; this contradiction shows that M is also invariant under V. The converse is an easy consequence of (1), and the proof is complete. □

For the rest of this paper, we assume that X has an involution $*$; i.e. a continuous mapping $x \longmapsto x^*$ from X onto X such that $(x^*)^* = x$ and $\lambda^*_{(x,y)} = \lambda_{(y^*, x^*)}$ for all $x, y \in X$, where the adjoint μ^* of a measure $\mu \in M(X)$ is defined by $\mu^*(f) = \int_X f(x^*) \, d\overline{\mu}(x)$ for all $f \in C_0(X)$.

A representation V of X by bounded operators on a Hilbert space H is called $*$-*representation* if $V_{x^*} = V_x^*$ for all $x \in X$, where V_x^* is the adjoint operator of V_x on H. It is obvious that a $*$-representation V of X is bounded if and only if $\| V_x \| \leq 1$ for all $x \in X$.

In the next theorem, we extend the preceding result to nondegenerate (not necessarily cyclic) $*$-representations of the Banach $*$-algebra $L(X)$ by bounded operators on a Hilbert space H and the bounded $*$-representations of X by bounded operators on H.

Theorem 2.6. *A mapping $\mu \longmapsto T_\mu$ from $L(X)$ into the Banach $*$-algebra of bounded operators on a Hilbert space H is a (bounded) nondegenerate $*$-representation of the Banach $*$-algebra $L(X)$ if and only if there exists a unique bounded $*$-representation V of X by bounded operators on H with $V_e = I$ such that $\| V \| = 1$ and*

$$\langle T_\mu \xi, \eta \rangle = \int_X \langle V_t \xi, \eta \rangle \, d\mu(t) \quad (\mu \in L(X), \xi, \eta \in H). \tag{4}$$

In this case, $V_x T_\mu = T_{\delta_x * \mu}$ and $T_\mu V_x = T_{\mu * \delta_x}$ for all $\mu \in L(X)$ and $x \in X$.
If T is faithful, then V is also faithful, and in this case $V_x \neq 0$ for every $x \in X$.
Moreover, T and V have the same closed invariant subspace.

P r o o f. Suppose that T is a nondegenerate $*$-representation of $L(X)$ by bounded operators on H. Then by Theorem 21.13 of [6], H is the direct sum of subspaces $\{H_\gamma\}_{\gamma\in\Gamma}$ which are closed, pairwise orthogonal and invariant under T, such that for each $\gamma \in \Gamma$, $T^{(\gamma)}$ (the restriction of T to H_γ) is a cyclic $*$-representation of $L(X)$ by bounded operators on H_γ. So, by Theorem 2.5, there is a bounded representation $V^{(\gamma)}$ of X by bounded operators on H_γ with $V_e^{(\gamma)} = I$ such that $\| V^{(\gamma)} \| = \| T^{(\gamma)} \|$ and

$$(5) \quad \left\langle T_\mu^{(\gamma)}\xi_\gamma \, , \eta_\gamma \right\rangle = \int_X \left\langle V_t^{(\gamma)}\xi_\gamma \, , \eta_\gamma \right\rangle \, d\mu(t) \quad (\mu \in L(X),\ \xi_\gamma, \eta_\gamma \in H).$$

Every $\xi \in H$ has a unique representation $\xi = \sum_{\gamma\in\Gamma} \xi_\gamma$, where $\xi_\gamma \in H_\gamma$, only a countable number of ξ_γ are nonzero, the series is convergent in the norm of H, and $\| \xi \|^2 = \sum_{\gamma\in\Gamma} \| \xi_\gamma \|^2$; let V_x be the operator on H defined by $V_x\xi = \sum_{\gamma\in\Gamma} V_x^{(\gamma)}\xi_\gamma$. Then for every $x, y \in X$, we have

$$\| V_x\xi - V_y\xi \|^2 = \sum_{\gamma\in\Gamma} \| V_x^{(\gamma)}\xi_\gamma - V_y^{(\gamma)}\xi_\gamma \|^2 \quad (\xi, \eta \in H).$$

Hence the function $x \longmapsto \langle V_x\xi, \eta\rangle$ is bounded and continuous for all $\xi, \eta \in H$. We also have

$$\begin{aligned} \int_X \langle V_t\xi, \eta\rangle \ d\lambda_{(x,y)}(t) &= \sum_{\gamma\in\Gamma} \int_X \left\langle V_t^{(\gamma)}\xi_\gamma \ , \ \eta_\gamma \right\rangle \ d\lambda_{(x,y)}(t) \\ &= \sum_{\gamma\in\Gamma} \left\langle V_x^{(\gamma)} V_y^{(\gamma)}\xi_\gamma \ , \ \eta_\gamma \right\rangle = \langle V_x V_y \xi, \eta\rangle \, . \end{aligned}$$

From (5) follows immediately (4). Lemma 2.3 together with (4) imply that $V_{x^*} = V_x^*$ for all $x \in X$. Thus V is a bounded $*$-representation by bounded operators on H with $V_e = I$ such that $\| V \| = 1$ and (4) holds.

The converse is an easy consequence of Theorem 2.2 with the aid of Lemma 2.3. The rest of the proof is similar to the proof of the corresponding part of Theorem 2.5. □

Corollary 2.7. *Let τ be a linear functional on the Banach $*$-algebra $L(X)$. Then the following are equivalent:*

(i) τ is positive, i.e. $\tau(\mu * \mu^*) \geq 0$ for all $\mu \in L(X)$;

(ii) there exist a bounded $*$-representation V of X by bounded operators on a Hilbert space H with $V_e = I$ and an element ζ of H such that

$$\tau(\mu) = \int_X \langle V_x\zeta, \zeta\rangle \ d\mu(x) \qquad (\mu \in L(X)).$$

Proof. Suppose that τ is positive. Then, since $L(X)$ has a bounded approximate identity [12], by Corollary 32.28 of [7] and Theorem 21.24 of [6], there exists a cyclic $*$-representation T of $L(X)$ by bounded operators on a Hilbert space H with a cyclic vector ζ such that $\tau(\mu) = \langle T_\mu \zeta, \zeta \rangle$ for all $\mu \in L(X)$. Now, (ii) is an immediate consequence of Theorem 2.6. Conversely, suppose that (ii) holds. Then the formula

$$\langle T_\mu \xi, \eta \rangle = \int_X \langle V_x \xi, \eta \rangle \; d\mu(x) \quad (\mu \in L(X), \xi, \eta \in H),$$

defines a $*$-representation T of $L(X)$ by bounded operators on H, by Theorem 2.6. In particular, $\tau(\mu) = \langle T_\mu \zeta, \zeta \rangle$, and so $\tau(\mu * \mu^*) = \langle T_{\mu^*} \zeta, T_{\mu^*} \zeta \rangle \; \geq 0$ for all $\mu \in L(X)$. $\square$

3. Some relations between representations of X and positive definite functions on X. A continuous function $\varphi : X \longrightarrow \mathbf{C}$ is said to be *positive definite* if for every finite subsets $\{x_1, \ldots, x_n\}$ of X and $\{c_1, \ldots, c_n\}$ of $\mathbf{C}$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \int_X \varphi(t) d\lambda_{(x_i, x_j^*)}(t) \geq 0.$$

We denote by $P(X)$ the set of all bounded positive definite functions on X.

Theorem 3.1. *Let $\varphi \in C_b(X)$. Then the following are equivalent:*
(i) $\varphi \in P(X)$;
(ii) the linear functional $\mu \longmapsto \int_X \varphi \, d\mu$ is positive on $L(X)$;
(iii) there exists a bounded $*$-representation V of X by bounded operators on a Hilbert space H with $V_e = I$ such that $\varphi(x) = \langle V_x \zeta, \zeta \rangle$ $(x \in X)$ for some $\zeta \in H$.

Proof. By Lemma 2.3 and Corollary 2.7, (ii) implies (iii). Also, it is trivial that (iii) implies (i). Now, suppose that (i) holds. To prove (ii), we only need to establish $\int_X \varphi \; d(\mu * \mu^*) \geq 0$ for all $\mu \in L(X)$ which are of the form $f\nu$, where f is a continuous function on X with compact support E and ν is a positive measure in $L(X)$. To this end, we note that

$$\int_X \varphi \; d(\mu * \mu^*) = \int_X \int_X g(x, y) \; d\nu(x) \; d\nu(y),$$

where $g(x, y) = f(x) \; \overline{f(y)} \; \int_X \; \varphi(t) \; d\lambda_{(x, y^*)}(t)$ for all $x, y \in X$. But g is continuous on $X \times X$, $\mathrm{supp}(g) \subseteq E \times E$ and $g(x, y) = \overline{g(y, x)}$ for all $x, y \in X$. So, it is easy to see that for every $\varepsilon > 0$, there exists a partition $\{E_1, \ldots, E_n\}$ for E into Borel sets such that each E_i contains a point x_i which satisfies

$$| g(x,y) - g(x_i, x_j) | < \varepsilon / \| \nu \|^2 \quad ((x,y) \in E_i \times E_j,\ 1 \leq i,j \leq n).$$

Therefore, we conclude that

$$\begin{aligned} & | \int_X \varphi \, d(\mu * \mu^*) - \sum_{i=1}^{n} \sum_{j=1}^{n} \nu(E_i)\, f(x_i) \overline{\nu(E_j)\, f(x_j)} \int_X \varphi(t)\, d\lambda_{(x_i, y_j^*)}(t) | \\ \leq & | \int_X \int_X g(x,y)\, d\nu(x)\, d\nu(y) - \int_X \int_X g(x_i, y_j)\, d\nu(x)\, d\nu(y) | \leq \varepsilon. \end{aligned}$$

So, the result follows from the fact that φ is positive definite. □

Corollary 3.2. *Let $\varphi \in P(X)$. Then for every $x, y \in X$,*
(i) $| \varphi(x) | \leq \varphi(e)$,
(ii) $| \int_X \varphi(t)\, d\lambda_{(x,y)}(t) - \varphi(x) |^2 \leq 2\varphi(e)\, [\, \varphi(e) - \mathbf{Re}\, \varphi(y)\,]$.

Proof. By Theorem 3.1, there exists a bounded $*$-representation V of X by bounded operators on a Hilbert space H with $V_e = I$ such that for some $\zeta \in H$, $\varphi(x) = \langle V_x \zeta, \zeta \rangle$ for all $x \in X$. Therefore, by Cauchy-Schwarz inequality for every $x, y \in X$, we have

$$\begin{aligned} | \varphi(x) - \int_X \varphi(t)\, d\lambda_{(x,y)}(t) |^2 &= | \langle V_x \zeta, \zeta \rangle - \langle V_x V_y \zeta, \zeta \rangle |^2 \\ &= | \langle \zeta - V_y \zeta, V_x^* \zeta \rangle |^2 \\ &\leq \| \zeta - V_y \zeta \|^2 \, \| V_x^* \zeta \|^2 \\ &\leq 2\varphi(e)\, [\, \varphi(e) - \mathbf{Re}\, \varphi(y)\,]. \end{aligned}$$

This establishs (ii). The assertion (i) is trivial. □

4. Some types of convergence of positive definite functions on *X*. The main result of this section is the following theorem.

Theorem 4.1. *If (φ_n) is a sequence in $P(X)$ which converges pointwise to a continuous function φ on X, then $\varphi \in P(X)$ and (φ_n) converges to φ in the topology of uniform convergence on compact subsets of X.*

This theorem was first proved by Raikov [15], and independetly by Yoshizawa [20] for locally compact groups. Later, Lashkarizadeh-Bami [10] generalized this result for foundation semigroups. In order to prove Theorem 4.1 we need to the next theorem. We first state the following definition.

Definition 4.2. *For measures $\mu_1, \ldots, \mu_m \in L(X)$, positive numbers α, β, and function $\varphi_0 \in P(X)$, we denote by $\mathcal{F}_{\mu_1,\ldots,\mu_m;\alpha,\beta}(\varphi_0)$ the set of all $\varphi \in P(X)$ such that*

$$| \varphi(e) - \varphi_0(e) | < \alpha \quad \textit{and} \quad | \int_X (\varphi - \varphi_0)\, d\mu_i | < \beta \quad (i = 1, \ldots, m).$$

*Then the family of the sets of the form $\mathcal{F}_{\mu_1,\ldots,\mu_m;\alpha,\beta}(\varphi_0)$ defines an open base for a topology on $P(X)$. We call it the $\mathcal{F}$-*topology *of $P(X)$.*

Theorem 4.3. *The $\mathcal{F}$-topology and the topology of uniform convergence on compact subset of X coincide on $P(X)$.*

Proof. Let K be an arbitrary compact subset of X, $\varphi_0 \in P(X)$, and $\varepsilon > 0$. Choose a real number $0 < \alpha < 1$ such that $\alpha < \varepsilon/6\, [\, 1 + \varphi_0(e)\,]$, and $\alpha < \varepsilon^2/54\, [\, 1 + \varphi_0(e)\,]$. Since φ is continuous, there exists a compact neighbourhood U of e such that

$$| \varphi_0(y) - \varphi_0(e) | < \alpha \qquad (y \in U). \tag{6}$$

Now, there exists $\mu \in L(X)$ such that $U \bigcap \operatorname{supp}(\mu) \neq \emptyset$. Hence, if we put $\mu_0 = |\, \mu\, |\,(U)^{-1} \chi_U\, |\, \mu\, |$, then $\mu_0 \in L(X)$, $\operatorname{supp}(\mu_0) \subseteq U$, and $\mu_0(U) = 1$. By the definition of $L(X)$, the mapping $x \longmapsto \delta_x * \mu_0$ from X into $L(X)$ is norm continuous. Thus from the compactness of K, we infer that there exists $x_1, \ldots, x_m \in K$ such that

$$\{\delta_x * \mu_0 : x \in K\} \subseteq \bigcup_{i=1}^{m} \{\mu \in L(X) : \| \delta_{x_i} * \mu_0 - \mu \| < \alpha \}. \tag{7}$$

We show that

$$\mathcal{F}_{\mu_0,\mu_1,\ldots,\mu_m;\alpha,\alpha}(\varphi_0) \subseteq \{\varphi \in P(X) : | \varphi(x) - \varphi_0(x) | < \varepsilon \text{ for all } x \in K \},$$

where $\mu_i = \delta_{x_i} * \mu_0$ for $1 \leq i \leq m$. To this end, let $\varphi \in \mathcal{F}_{\mu_0,\mu_1,\ldots,\mu_m;\alpha,\alpha}(\varphi_0)$ and $x \in K$. Then, by (6) and (7), we have

$$\begin{aligned}
\int_X [\, \varphi(e) - \mathbf{Re}\, \varphi(y)\,]\, d\mu_0(y) \quad &\leq \quad | \int_X [\, \varphi(e) - \varphi(y)\,]\, d\mu_0(y) | \\
&\leq \quad \int_X | \varphi(e) - \varphi_0(e) |\; d\mu_0(y) \\
&+ \quad \int_U | \varphi_0(e) - \varphi_0(y) |\; d\mu_0(y) \\
&+ \quad | \int_X (\varphi - \varphi_0)\, d\mu_0 | < 3\alpha.
\end{aligned}$$

Now, Corollary 3.2 (ii) and Holder's inequality imply that

$$\begin{aligned} \left| \int_X \varphi \, d\delta_x * \mu_0 - \varphi(x) \right| &\leq \int_X \left| \int_X \varphi(t) \, d\lambda_{(x,y)}(t) - \varphi(x) \right| \, d\mu_0(y) \\ &\leq [\, 2\,\varphi(e)\,]^{1/2} \int_X [\, \varphi(e) - \mathbf{Re}\, \varphi(y)\,]^{1/2} \, d\mu_0(y) \\ &\leq \left\{ 2\,\varphi(e) \int_X [\, \varphi(e) - \mathbf{Re}\, \varphi(y)\,] \, d\mu_0(y) \right\}^{1/2} \\ &\leq [\, 6\,\varphi(e)\,\alpha\,]^{1/2} \\ &\leq \{\, 6\,[\varphi_0(e) + 1\,]\,\alpha\}^{1/2} < \varepsilon/3. \end{aligned}$$

On the other hand, by (7), there is $1 \leq i \leq m$ such that $\| \delta_x * \mu_0 - \mu_i \| < \alpha$, and hence by Corollary 3.2 (i),

$$\begin{aligned} \left| \int_X (\varphi - \varphi_0) \, d\delta_x * \mu_0 \right| &\leq \left| \int_X \varphi \, d(\delta_x * \mu_0 - \mu_i) \right| + \left| \int_X (\varphi - \varphi_0) \, d\mu_i \right| \\ &+ \left| \int_X \varphi_0 \, d(\mu_i - \delta_x * \mu_0) \right| \\ &\leq \| \delta_x * \mu_0 - \mu_i \| \, [\, \varphi(e) + \varphi_0(e)\,] + \alpha \\ &< \alpha\, [\, \varphi(e) + \varphi_0(e) + 1\,] \\ &\leq 2\alpha\, [\, \varphi_0(e) + 1\,] < \varepsilon/3. \end{aligned}$$

We therefore have

$$\begin{aligned} | \varphi(x) - \varphi_0(x) | &\leq \left| \varphi(x) - \int_X \varphi \, d\delta_x * \mu_0 \right| + \left| \int_X (\varphi - \varphi_0) \, d\delta_x * \mu_0 \right| \\ &+ \left| \int_X \varphi_0 \, d\delta_x * \mu_0 - \varphi_0(x) \right| < \varepsilon. \end{aligned}$$

Conversely, let $\mu_1, \ldots, \mu_m \in L(X)$, $\alpha, \beta > 0$ and $\varphi_0 \in P(X)$. Then there exists $\varepsilon > 0$ such that $\varepsilon \leq \alpha$, $\varepsilon \max\{ \| \mu_1 \|, \ldots, \| \mu_m \| \} < \beta/2$, and $\varepsilon\, [\, \varphi_0(e) + \alpha\,] < \beta/2$. Now, choose a compact subset K of X such that $e \in K$ and $| \mu_i | (X \setminus K) < \varepsilon$ for all $i = 1, \ldots, m$. It is easy to check that

$$\{\, \varphi \in P(X) : | \varphi(x) - \varphi_0(x) | < \varepsilon \text{ for all } x \in K \,\} \subseteq \mathcal{F}_{\mu_1, \ldots, \mu_m; \alpha, \beta}(\varphi_0). \qquad \square$$

We conclude this paper with the proof of the main result of this section.

P r o o f o f T h e o r e m 4.1. By Corollary 3.2, we have $| \varphi_n(x) | \leq \varphi_n(e)$ for all $x \in X$ and $n \geq 1$. Hence $| \varphi(x) | \leq \varphi(e)$ for all $x \in X$; thus φ is bounded.

By the Lebesgue Dominated Convergence Theorem, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i\overline{c_j}\int_X \varphi_n(t)\; d\lambda_{(x_i,x_j^*)}(t) \longrightarrow \sum_{i=1}^{m}\sum_{j=1}^{m} c_i\overline{c_j}\int_X \varphi(t)\; d\lambda_{(x_i,x_j^*)}(t)$$

for every subset $\{x_1,\dots,x_m\}$ of X and $\{c_1,\dots,c_m\}$ of $\mathbf{C}$, and also

$$\int_X \varphi_n(x)\; d\mu(x) \longrightarrow \int_X \varphi(x)\; d\mu(x) \qquad (\ \mu\in L(X)\ ).$$

Therefore $\varphi\in P(X)$ and (φ_n) converges to φ in the $\mathcal{F}$-topology of $P(X)$. Now the result follows from Theorem 4.3. $\square$

Acknoledgment. I wish to express my sincere thanks to Professor K. A. Ross for sending some of his papers as well as pointing out some related references. I also wish to thank very much the referee of this paper for his valuable remarks.

REFERENCES

[1] Yu. M. Berezansky, A. A. Kalyushnyi. Hypercomplex systems and hypergroups: connections and distinctions. *Contemp. Math.* **183** (1995), 21-44.

[2] R. H. Bruck. A Survey of Binary Systems. Berlin, Springer, 1966.

[3] J. Delsarte. Sur une extension de la formule de Taylor. *J. Math. Pures et Appl.* **17** (1938), 213-231.

[4] J. Delsarte. Hypergroups and operateurs de permutation et de transmutation. *Colloque de Nancy du CNRS* **LXXI** (1956) 29-45.

[5] C. F. Dunkl. The measure algebra of a locally compact hypergroup. *Trans. Amer. Math. Soc.* **179** (1973), 331-384.

[6] E. Hewitt, K. A. Ross. Abstract Harmonic Analysis I. Berlin, Springer, 1963.

[7] E. Hewitt, K. A. Ross. Abstract Harmonic Analysis II. Berlin, Springer, 1970.

[8] R. I. Jewett. Spaces with an abstract convoluion of measures. *Adv. Math.* **18** (1975), 1-101.

[9] M. Lashkarizadeh-Bami. Representations of foundation semigroups and their algebras. *Canad. J. Math.* **37** (1985), 29-47.

[10] M. Lashkarizadeh-Bami. On various types of convergence of positive definite functions on foundation semigroups. *Math. Proc. Camb. Phil. Soc.* **111** (1992), 325-330.

[11] B. M. Levitan. The generalization of shift operation in connection with almost periodic functions. *Mat. Sbor.* **7** (1940), 449-478.

[12] A. R. Medghalchi. The second dual algebra of a hypergroup. *Math. Z.* **210** (1992), 615-624.

[13] A. R. Medghalchi. Isometric isomorphisms on the dual and second dual of a hypergroup. *Acta Math. Hungar.* **74** (1997), 167-175.

[14] J. S. Pym. Weakly seperately continuous measure algebras. *Math. Ann.* **175** (1968), 208-219.

[15] D. A. Raikov. On various types of convergence of positive definite functions. *Dokl. Akad. Nauk SSSR* **58** (1947), 1279-1281.

[16] K. A. Ross. Hypergroups and centers of measure algebras. *Ist. Naz. di Alta. Mat., Symposia Math.* **22** (1977), 221-231.

[17] K. A. Ross. Signed hypergroups – a survey. *Contemp. Math.* **183** (1995), 319-329.

[18] K. A. Ross. Hypergroups and Signed Hypergroups. In: Harmonic Analysis and Hypergroups. Berlin, Springer, 1998.

[19] R. Spector. Apercu de la theoric des hypergroupes. In: Analse Harmonic sur les Groupes de Lie, Lecture Notes in Mathematics, vol. **497**, Berlin, Springer, 1975.

[20] H. Yoshizawa. On some types of convergence of positive definite functions. *Osaka Math. J.* **1** (1949), 90-94.

Department of Mathematics
University of Isfahan
Isfahan 81745, Iran
`isfahani@math.ui.ac.ir`

Received July 14, 1997
Revised March 16, 1999