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GEOMETRIC STABLE LAWS THROUGH SERIES REPRESENTATIONS

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Abstract. Let (X_i) be a sequence of i.i.d. random variables, and let N be a geometric random variable independent of (X_i). Geometric stable distributions are weak limits of (normalized) geometric compounds, $S_N = X_1 + \cdots + X_N$, when the mean of N converges to infinity. By an appropriate representation of the individual summands in S_N we obtain series representation of the limiting geometric stable distribution. In addition, we study the asymptotic behavior of the partial sum process $S_N(t) = \sum_{i=1}^{[Nt]} X_i$, and derive series representations of the limiting geometric stable process and the corresponding stochastic integral. We also obtain strong invariance principles for stable and geometric stable laws.

1. Introduction. An increasing interest has been seen recently in *geometric stable* (GS) distributions: the class of limiting laws of appropriately normalized random sums of i.i.d. random variables,

$$S_N = X_1 + \cdots + X_N, \tag{1}$$

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where the number of terms is geometrically distributed with mean $1/p$, and $p \to 0$ (see, e.g., [7], [8], [10], [11], [12], [13], [14] and [21]). These heavy-tailed distributions provide useful models in mathematical finance (see, e.g., [1], [20], [13]), as well as in a variety of other fields (see, e.g., [6] for examples, applications, and extensive references for geometric compounds (1)).

Gnedenko and Fakhim [5] in their classical transfer theorem gave the conditions for the *weak* convergence and the form of the limiting distribution of (1). Our approach, which is in the spirit of LePage et al. [17], is fundamentally different from that of Gnedenko and Fakhim [5] and other authors. It utilizes the following representation of a vector of symmetric i.i.d. random variables:

$$(2) \qquad \begin{aligned} (X_1, \ldots, X_n) &\stackrel{d}{=} (X_{1,n}, \ldots, X_{n,n}) \\ &\stackrel{def}{=} \pi((\delta_1 G^{-1}(\Gamma_1/\Gamma_{n+1}), \ldots, \delta_n G^{-1}(\Gamma_n/\Gamma_{n+1})). \end{aligned}$$

In the above equation, (Γ_i) is a sequence of arrival times of a standard Poisson process, (δ_i) is a sequence of independent symmetric signs, and π is a random uniform permutation of coordinates in R^n, all three mutually independent, while

$$(3) \qquad G^{-1}(y) = \inf\{x \geq 0 : G(x) \leq y\},$$

where $G(x) = P(|X_1| \geq x)$.

After preliminary definitions of Section 2, in Section 3 of the paper we generalize (2) to the case of random number of variables and then use it to prove *almost sure* convergence of the obtained version of the random sum S_N. Further, in Theorem 3.1, we establish LePage type series representation of the limiting distribution of (1). We also derive an almost sure version of the well-known stability property of symmetric GS random variables (Proposition 3.1). Then, we extend the representations to processes and consider

$$(4) \qquad S_p(t) = \sum_{i=1}^{[N_p t]} X_i, \quad 0 \leq t \leq 1,$$

where N_p is geometric with mean $1/p$ (independent of an i.i.d. sequence (X_i)). Rachev and Samorodnitsky [21] define a GS *Lévy motion* as the week limit of (4) in $D[0,1]$ (as $p \to 0$). We study *almost sure* asymptotic behavior of a weak representation of (4) using the method of LePage et al. [17]. Our approach leads to a series representation of the limiting process and allows for a natural definition of the stochastic integral with respect to a GS Lévy motion.

We conclude with Section 4, where we use the representation (2) to obtain strong invariance principles for stable and GS laws. In a series of papers concluded by Berkes and Dehling [2], a sum of symmetric, i.i.d. random variables from the normal α-stable domain, $\sum_{i=1}^{n} X_i$, was approximated almost surely by a sum of i.i.d. symmetric α-stable r.v.'s, $\sum_{i=1}^{n} Y_i$. We follow a different approach, and approximate $\sum_{i=1}^{n} X_{i,n}$ strongly by $\sum_{i=1}^{n} Y_{i,n}$, where $(Y_{i,n})_{i=1}^{n} \stackrel{d}{=} (Y_i)_{i=1}^{n}$. While we obtain a weaker form of approximation, the rate of convergence is always at least $o(n^{1/\alpha})$, the rate that may not always hold for other strong invariance principles (see examples [2]). Our proofs, which are essentially different and simpler, allow for an extension to the general stable domain (not necessary normal or symmetric). We also derive analogous strong invariance principle for random sums $\sum_{i=1}^{N_p} X_{i,N_p}$, by approximating them by $\sum_{i=1}^{N_p} Y_{i,N_p}$, where $(Y_{i,N_p})_{i=1}^{N_p}$ is a vector of i.i.d. symmetric GS r.v.'s.

2. Geometric stable laws. Let $(\mathbf{X}_i)$ be a sequence of i.i.d. random vectors in R^d, and let N_p be a r.v. independent of $(\mathbf{X}_i)$ and geometrically distributed with parameter $p \in (0, 1)$,

$$P(N_p = k) = p(1-p)^{k-1},\ k = 1, 2, \ldots. \tag{5}$$

We say that $\mathbf{Y}$ (and its distribution) is *geometric stable* (GS) if and only if

$$a(p) \sum_{i=1}^{N_p} (\mathbf{X}_i - \mathbf{b}(p)) \xrightarrow{d} \mathbf{Y},\ \ p \to 0, \tag{6}$$

for some $a(p) > 0$ and $\mathbf{b}(p) \in R^d$. As shown in Mittnik and Rachev [19], $\mathbf{Y}$ is GS if and only if its characteristic function (ch.f.) has the form

$$\Psi(\mathbf{t}) = (1 - \log \Phi(\mathbf{t}))^{-1},\ \mathbf{t} \in R^d, \tag{7}$$

where Φ is a ch.f. of some multivariate α-stable distribution. A random vector $\mathbf{Y}$ is *strictly* GS if it corresponds to a strictly α-stable r.v. via (7).

In the univariate case, GS laws form a four parameter family given by ch.f.

$$\psi(t) = [1 + \sigma^\alpha |t|^\alpha \omega_{\alpha,\beta}(t) - i\mu t]^{-1}, \tag{8}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sign}(x) \tan(\pi\alpha/2), & \text{if } \alpha \neq 1, \\ 1 + i\beta \frac{2}{\pi} \text{sign}(x) \log|x|, & \text{if } \alpha = 1. \end{cases} \tag{9}$$

The parameter $\alpha \in (0,2]$ is the index of stability, determining the tail of the distribution, $\beta \in [-1,1]$ is the skewness parameter, and $\mu \in R$ and $\sigma \geq 0$ control the location and scale, respectively. A GS distribution given by (8) will be denoted as $GS_\alpha(\sigma,\beta,\mu)$. Similarly, the stable law corresponding to (8) via (7) will be denoted $S_\alpha(\sigma,\beta,\mu)$. Strictly GS laws have ch.f. (8) with $\alpha \neq 1$ and $\mu = 0$, or $\alpha = 1$ and $\beta = 0$. In the symmetric case ($\mu = 0$ and $\beta = 0$), the ch.f. simplifies to $\psi(t) = 1/(1+\sigma^\alpha|t|^\alpha)$, and is known in the literature as the Linnik characteristic function (see, e.g., [9], [3]). Positive and completely skewed GS distributions ($0 < \alpha < 1$ and $\beta = 1$) are Mittag-Leffler distributions, studied in [23] in relation to relaxation phenomena.

Relation (7) leads to the fundamental representation of a GS r.v.'s in terms of stable and exponential variates (see, e.g., [10]).

Proposition 2.1. *Let $Y \sim GS_\alpha(\sigma,\beta,\mu)$. Then,*

$$Y \stackrel{d}{=} \begin{cases} \mu W + W^{1/\alpha}\sigma X, & \text{if } \alpha \neq 1, \\ \mu W + W\sigma X + 2\pi^{-1}\sigma\beta W \log(W\sigma), & \text{if } \alpha = 1, \end{cases} \tag{10}$$

where $X \sim S_\alpha(1,\beta,0)$ and W, independent of X, has the standard exponential distribution.

A GS stochastic process can be defined through its finite-dimensional distributions: $\{X(t),\, t \in T\}$ is geometric stable if all of its finite-dimensional distributions,

$$(X(t_1),\dots,X(t_d)),\; t_1,\dots,t_d \in T,\; d \geq 1, \tag{11}$$

are GS. It is strictly GS (symmetric GS) , if all of its finite-dimensional distributions are strictly GS (symmetric GS), respectively. Note that if the finite-dimensional distributions are GS, then by consistency, they must all have the same index α.

3. Series representations. The series representations of stable and, in general, infinitely divisible random variables, were studied on various occasions (see [18], [4], [15], [16], [22]). It is well-known that a symmetric α-stable distribution has the representation

$$\sum_{i=1}^{\infty} \delta_i \Gamma_i^{-1/\alpha},$$

where (Γ_i) is a sequence of arrival times of a Poisson process with intensity one and (δ_i) is a Rademacher sequence independent of (Γ_i). By Proposition 2.1, the corresponding symmetric GS distribution can be written as

$$W^{1/\alpha}\sum_{i=1}^{\infty}\delta_i\Gamma_i^{-1/\alpha},$$

where W has the standard exponential distribution and is independent of (δ_i) and (Γ_i).

We derive similar representation for the general non-symmetric case through almost sure convergence of an appropriately scaled and centered representation of $\sum_{i=1}^{N_p} X_i$. Our approach is based on the results of LePage et al. [17] and illustrates probabilistic structure of the weak convergence of normalized random sums. Let (X_i) be a sequence of i.i.d. random variables. Define

$$\begin{aligned}
&G_+(x) = P(X_1 \geq x | X_1 \geq 0),\\
&G_-(x) = P(-X_1 > x | -X_1 > 0),\\
&G(x) = P(|X_1| \geq x)\\
&p_0 = P(X_1 \geq 0),\;\; q_0 = P(-X_1 > 0).
\end{aligned}$$

Then,

$$S_n = X_1 + \cdots + X_n \stackrel{d}{=} \delta_1 G_{\delta_1}^{-1}(\Gamma_1/\Gamma_{n+1}) + \cdots + \delta_n G_{\delta_n}^{-1}(\Gamma_n/\Gamma_{n+1}), \tag{12}$$

where (δ_i) is an i.i.d. sequence of random signs: $P(\delta_i = +) = p_0$ and $P(\delta_i = -) = q_0$. If the distribution of X_1 is in an α-stable domain of attraction with $\alpha < 2$, the representation on the right hand side of (12), appropriately normalized, converges almost surely to a stable law:

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}(\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{n+1}) - b_n)}{a_n} \stackrel{a.s.}{=} \sum_{i=1}^{\infty}(z_i\Gamma_i^{-1/\alpha} - C_i), \tag{13}$$

where

$$a_n = G^{-1}(1/n), \quad b_n = p_0 A\int_{1/n}^{1} G_+^{-1}(x)dx - q_0 B\int_{1/n}^{1} G_-^{-1}(x)dx,$$
$$z_i = A\tfrac{\delta_i+1}{2} + B\tfrac{\delta_i-1}{2}, \quad C_i = Ez_i\int_i^{i+1} x^{-1/\alpha}dx,$$

with $A = \lim_{n\to\infty} G_+^{-1}(1/n)/G^{-1}(1/n)$, $B = \lim_{n\to\infty} G_-^{-1}(1/n)/G^{-1}(1/n)$ (see [17]). We extend (13) to the case where n is a random variable. In the following result, N_p is a geometric r.v. (5) independent of (X_i) and

$$G_p(x) = P(N_p \leq x), \;\; G_p^{-1}(y) = \inf\{x : G_p(x) \geq y\}. \tag{14}$$

Theorem 3.1. *The following representation holds*

$$S_{N_p} \stackrel{d}{=} \sum_{i=1}^{N_p^*} \delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{N_p^*+1}), \tag{15}$$

where $N_p^* = G_p^{-1}(U)$ *and* U *is uniformly distributed on* $(0,1)$, *independent of* (δ_i) *and* (Γ_i).

Moreover,

$$\lim_{p\to 0} \frac{\sum_{i=1}^{N_p^*}(\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{N_p^*+1}) - b_p)}{a_p} \stackrel{a.s.}{=} W^{1/\alpha}\sum_{i=1}^{\infty}(z_i\Gamma_i^{-1/\alpha} - C_i) \tag{16}$$

$$+\begin{cases} \frac{\alpha}{\alpha-1}(p_0A^2 - q_0B^2)(W - W^{1/\alpha}), & if \quad \alpha \neq 1, \\ (p_0A^2 - q_0B^2)W(\ln W - 1), & if \quad \alpha = 1, \end{cases}$$

where $b_p = p_0A\int_p^1 G_+^{-1}(x)dx - q_0B\int_p^1 G_-^{-1}(x)dx$, $a_p = G^{-1}(p)$, *and* $W = -\ln U$.

P r o o f. First, by (12), the right hand side of (15), conditionally on $N_p^* = n$, has the same distribution as S_n. Thus, the equality of distributions follows, since N_p is independent of S_n and N_p^* is independent of (δ_i) and (Γ_i).

Next, write

$$\frac{\sum_{i=1}^{N_p^*}(\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{N_p^*+1}) - b_p)}{a_p} =$$

$$= \frac{a_{N_p^*}}{a_p}\left(\frac{\sum_{i=1}^{N_p^*}(\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{N_p^*+1}) - b_{N_p^*})}{a_{N_p^*}} + N_p^*\frac{b_{N_p^*} - b_p}{a_{N_p^*}}\right),$$

and note that $a_{N_p^*}/a_p = G^{-1}(1/N_p^*)/G^{-1}(p)$ converges almost surely to $W^{1/\alpha}$, since pN_p^* converges almost surely to $-\ln U$. Further, since $N_p^* \to \infty$ with probability one, (13) implies

$$\lim_{p\to 0} \frac{\sum_{i=1}^{N_p^*}(\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{N_p^*+1}) - b_{N_p^*})}{a_{N_p^*}} \stackrel{a.s.}{=} \sum_{i=1}^{\infty}(z_i\Gamma_i^{-1/\alpha} - C_i).$$

Finally, note that

$$\lim_{p\to 0}\frac{G_+^{-1}(u/N_p^*)}{G^{-1}(1/N_p^*)} = Au^{-1/\alpha}, \qquad \lim_{p\to 0}\frac{G_-^{-1}(u/N_p^*)}{G^{-1}(1/N_p^*)} = Bu^{-1/\alpha},$$

and apply the dominated convergence theorem to obtain

$$\begin{aligned}
\lim_{p\to 0} N_p^*\frac{b_{N_p^*}-b_p}{a_{N_p^*}} &= p_0AN_p^*\lim_{p\to 0}\frac{\int_{1/N_p^*}^{p} G_+^{-1}(x)dx}{G^{-1}(1/N_p^*)} - q_0BN_p^*\lim_{p\to 0}\frac{\int_{1/N_p^*}^{p} G_-^{-1}(x)dx}{G^{-1}(1/N_p^*)}\\
&= p_0A\lim_{p\to 0}\int_1^{pN_p^*}\frac{G_+^{-1}(u/N_p^*)}{G^{-1}(1/N_p^*)}du - q_0B\lim_{p\to 0}\int_1^{pN_p^*}\frac{G_-^{-1}(u/N_p^*)}{G^{-1}(1/N_p^*)}du\\
&= (p_0A^2 - q_0B^2)\int_1^W u^{-1/\alpha}du,
\end{aligned}$$

which concludes the proof. □

Symmetric GS laws possess the well-known stability property: if X_i are i.i.d. symmetric GS random variables with index $\alpha \in (0,2)$, then for any $p > 0$ we have

$$p^{1/\alpha}(X_1+\cdots+X_{N_p}) \stackrel{d}{=} X_1, \tag{17}$$

where N_p is geometrically distributed and independent of (X_i). Using our series representation we can obtain an almost sure version of (17). Suppose that the arrival times of N_p Poisson processes are combined to define a new process. More precisely, let $\mathcal{N}_1, \mathcal{N}_2, \ldots$ be a sequence of independent standard Poisson processes with corresponding arrival times $(\Gamma_{1,i}), (\Gamma_{2,i}), \ldots$. The observed processes are $\mathcal{N}_1(W_1\cdot), \ldots, \mathcal{N}_{N_p}(W_{N_p}\cdot)$, where (W_i) is a sequence of intensities, and N_p is a geometric r.v. (5) independent of $(\mathcal{N}_i)$. The observed arrival times, $\Gamma_{1,i}/W_1, \ldots, \Gamma_{N_p,i}/W_{N_p}$, $i \in N$, when combined and ordered, define a new counting process,

$$\mathcal{N}((W_1+\cdots+W_{N_p})\cdot).$$

Thus, $\mathcal{N}$ is the combined process with time scaled by the divisor $(W_1+\cdots+W_{N_p})$. Consequently, if Γ_i's are arrival times of $\mathcal{N}$, then the observed arrivals have the form $\Gamma_i/(W_1+\cdots+W_{N_p})$. Assuming that the intensities $W_1, W_2, \ldots$ are standard exponential random variables, mutually independent, and independent of $(\mathcal{N}_i)$, we have the following *almost sure version* of (17).

Proposition 3.1. *The process $\mathcal{N}$ defined above is a standard Poisson process, and with probability one*

$$p^{1/\alpha}\left(\sum_{i=1}^\infty \delta_{1,i}\left(\frac{W_1}{\Gamma_{1,i}}\right)^{1/\alpha} + \cdots + \sum_{i=1}^\infty \delta_{N_p,i}\left(\frac{W_{N_p}}{\Gamma_{N_p,i}}\right)^{1/\alpha}\right) = \sum_{i=1}^\infty \delta_i\left(\frac{W}{\Gamma_i}\right)^{1/\alpha},$$

where $W = p(W_1 + \cdots + W_{N_p})$ *has the standard exponential distribution and* (δ_i) *is an independent of* $((\Gamma_i), W)$ *Rademacher sequence defined by*

$$\delta_i = \delta_{j,k} \text{ if and only if } \frac{\Gamma_i}{W_1 + \cdots + W_{N_p}} = \frac{\Gamma_{j,k}}{W_j}.$$

Proof. Conditionally on $W_1 = \lambda_1, W_2 = \lambda_2, \ldots$, and $N_p = n$, the observed process $\mathcal{N}((W_1 + \cdots + W_{N_p})\cdot)$ is a Poisson process with intensity $\lambda_1 + \cdots + \lambda_n$. Thus, under the same conditioning, $\mathcal{N}$ is a standard Poisson process whose distribution does not depend on the conditioning. Consequently, the unconditional distribution of $\mathcal{N}$ must be the same as that of a standard Poisson process. □

Next let us consider symmetric GS processes. Let (X_i) be a sequence of i.i.d. random variables independent of a geometrically distributed (5) random variable N_p. The partial sum process on $[0, 1]$,

$$S_p(t) = \sum_{i=1}^{[N_p t]} X_i, \tag{18}$$

has càdlàg trajectories with jumps at points k/N_p for $k \in \{1, \ldots, N_p\}$ and is constant between the jumps. The values of jumps are equal to X_k, $k \in \{1, \ldots, N_p\}$. It does not have independent increments. Note that the integral $\int_0^1 v dS_p$ is well defined for any measurable function v on $[0, 1]$.

Rachev and Samorodnitsky [21] introduced a GS Lévy motion $S(t)$ as the weak limit (as $p \to 0$) of an appropriately normalized $S_p(t)$. We study the limits of $S_p(t)$ and $\int_0^1 v dS_p$ through their almost sure representations.

We use a sequence (U_i) of i.i.d. random variables uniformly distributed on $[0, 1]$ to define a random permutation $(J_k^n)_{k=1}^n$ of $(k/n)_{k=1}^n$. Let J_1^n be the right end of the interval $((k-1)/n, k/n]$ which includes U_1. For $l > 1$, define J_l^n as the right end of the shortest interval of the form $(j/n, k/n]$ and containing U_l, with $k/n \neq J_j^n$ for $j < l$. It can be shown that $(J_k^n)_{k=1}^n$ is uniformly distributed over all possible permutations of $(k/n)_{k=1}^n$. Thus, we have the distributional equality

$$\sum_{i=1}^{[nt]} X_i \stackrel{d}{=} \sum_{i=1}^{n} \delta_i |X|_{(i,n)} \mathbf{1}_{\{J_i^n \le t\}},$$

where $(|X|_{(i,n)})$ are order statistics of $|X_i|$ and δ_i's are ± 1's (see [17] for details).

The above equality also produces

$$\sum_{i=1}^{[N_p t]} X_i \stackrel{d}{=} \sum_{i=1}^{N_p} \delta_i |X|_{(i,N_p)} \mathbf{1}_{\{J_i^{N_p} \le t\}}.$$

The random vector $(|X|_{(i,n)})$ has the well-known representation in distribution:

$$(|X|_{(1,n)}, \dots, |X|_{(n,n)}) \stackrel{d}{=} (G^{-1}(\Gamma_1/\Gamma_{n+1}), \dots, G^{-1}(\Gamma_n/\Gamma_{n+1})).$$

Consequently, assuming mutual independence of (Γ_n), (δ_n), (U_n), and U, we can define a process $\tilde{S}_p$ as follows

$$\tilde{S}_p(t) = \sum_{i=1}^{N_p} \delta_i G^{-1}(\Gamma_i/\Gamma_{N_p+1}) \mathbf{1}_{\{J_i^{N_p} \le t\}}.$$

The process $\tilde{S}_p$ has identical distribution in $D[0,1]$ as the randomized partial sum process S_p. We show below that the normalized $\tilde{S}_p$ is almost surely convergent, and derive almost sure series representation for the limiting process. We assume (14), where N_p is geometric with mean $1/p$.

Theorem 3.2. *With the above notation, denoting $c_p = G^{-1}(p)$ and $W = -\ln(U)$, we have*

$$\lim_{p\to 0} S_p/c_p \stackrel{d}{=} \lim_{p\to 0} \tilde{S}_p/c_p \stackrel{a.s}{=} W^{1/\alpha} \sum_{i=1}^{\infty} \delta_i/\Gamma_i^{1/\alpha} \mathbf{1}_{[U_i,1]}$$

in the Skorohod metric in $D[0,1]$. Moreover, if a real function v on $[0,1]$ has bounded variation, then

$$\lim_{p\to 0} \int_0^1 v dS_p/c_p \stackrel{d}{=} \lim_{p\to 0} \int_0^1 v d\tilde{S}_p/c_p \stackrel{a.s}{=} W^{1/\alpha} \sum_{i=1}^{\infty} \delta_i/\Gamma_i^{1/\alpha} v(U_i).$$

Proof. Since $N_p = G_p^{-1}(U)$ and $pG_p^{-1}(u)$ converges to $-\ln u$ as $p \to 0$, N_p diverges almost surely to infinity, and with probability one

$$\lim_{p\to 0} pN_p = W. \tag{19}$$

Moreover,

$$\frac{\tilde{S}_p}{c_p} = \frac{C_{N_p}}{c_p} \frac{\sum_{i=1}^{N_p} \delta_i G^{-1}(\Gamma_i/\Gamma_{N_p+1}) \mathbf{1}_{\{J_i^{N_p} \le t\}}}{C_{N_p}},$$

where $C_{N_p} = G^{-1}(1/N_p)$.

The function G^{-1} is regularly varying at zero with index $-1/\alpha$. Thus, by (19),

$$\lim_{p\to 0} \frac{C_{N_p}}{c_p} = W^{1/\alpha}.$$

Since $N_p \to \infty$, it follows from the corresponding result for the stable case given in [17] that, with probability one,

$$\lim_{p\to 0} \frac{\sum_{i=1}^{N_p} \delta_i G^{-1}(\Gamma_i/\Gamma_{N_p+1})\mathbf{1}_{\{J_i^{N_p}\le t\}}}{C_{N_p}} = \sum_{i=1}^{\infty} \delta_i/\Gamma_i^{1/\alpha}\mathbf{1}_{[U_i,1]}$$

in the Skorohod metric in $D[0,1]$. This establishes the first part of the theorem. The second part follows by almost identical arguments. $\square$

4. Strong invariance principles. Let (X_i) be a sequence of symmetric, i.i.d. random variables from the normal domain of attraction of a stable law. It is well-known that X_1 satisfies the following tail condition

$$G(x) = cx^{-\alpha} + \beta(x)x^{-\alpha}, \tag{20}$$

or, equivalently,

$$G^{-1}(y) = \tilde{c}y^{-1/\alpha} + \gamma(y)y^{-1/\alpha}, \tag{21}$$

where $\beta(x) \to 0$ as $x \to \infty$ and $\gamma(y) \to 0$ as $y \to 0$. Without loss of generality, we may assume that $\tilde{c} = 1$.

We shall exploit the representation $(X_{i,n})_{i=1}^n$ of $(X_i)_{i=1}^n$ given in (2) to derive an almost sure approximation of $\sum_{i=1}^n X_{i,n}$ by $\sum_{i=1}^n Y_{i,n}$, as well as $\sum_{i=1}^{N_p} X_{i,N_p}$ by $\sum_{i=1}^{N_p} Y_{i,N_p}$, where $(Y_{i,n})$ is a sequence of symmetric, i.i.d. α-stable (respectively, GS) random variables.

Using (2) and the definitions of $(\Gamma_{1,j})$, $(\Gamma_{2,j})$, …, and $(\delta_{1,j})$, $(\delta_{2,j})$, … given in Section 3 we have

$$\sum_{i=1}^n X_i \stackrel{d}{=} \sum_{i=1}^n X_{i,n} = \sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1})$$

and

$$X \stackrel{d}{=} \sum_{i=1}^n Y_{i,n} \stackrel{def}{=} \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \delta_{i,j}\Gamma_{i,j}^{-1/\alpha}\right),$$

where the stable r.v. $X = n^{1/\alpha} \sum_{j=1}^{\infty} \delta_j \Gamma_j^{-1/\alpha}$. The following theorem holds in this setting.

Theorem 4.1. *Let M_n be a sequence of integers such that $M_n \to \infty$ and $M_n/n \to 0$ as $n \to \infty$. Then, with probability one,*

$$\sum_{i=1}^{n} X_{i,n} - \sum_{i=1}^{n} Y_{i,n} =$$
$$= O(n^{1/\alpha}) \left[\sum_{i=1}^{M_n} \delta_i \Gamma_i^{-1/\alpha} \gamma(\Gamma_i/\Gamma_{n+1}) + \frac{\sqrt{\log\log M_n}}{M_n^{1/\alpha-1/2}} + \sqrt{\frac{\log\log n}{n}} \right].$$

Remarks.

1. The exact rate of convergence in the above approximation depends on the rate of convergence of γ at the origin: the faster γ tends to zero, the faster M_n can tend to infinity, increasing the rate of convergence of $M_n^{1/2-1/\alpha}\sqrt{\log\log M_n}$ to zero.

2. Regardless of the rate of convergence of γ, we obtain almost sure convergence with the rate at least $o(n^{1/\alpha})$. This rate could not always be obtained without additional assumptions, when another form of strong invariance principle for $\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i$ was considered in [2] (see examples therein).

For the random summation, we have the following representation (see Theorem 3.1),

$$\sum_{i=1}^{N_p} X_i \stackrel{d}{=} \sum_{i=1}^{N_p^*} X_{i,N_p^*} = \sum_{i=1}^{N_p^*} \delta_i G^{-1}(\Gamma_i/\Gamma_{N_p^*+1}).$$

By Proposition 3.1 of Section 3, a GS r.v. $Y = \sum_{i=1}^{\infty} \delta_i (W/\Gamma_i)^{1/\alpha}$ has the representation

$$Y \stackrel{d}{=} p^{1/\alpha} \sum_{i=1}^{N_p} Y_{i,N_p} \stackrel{def}{=} p^{1/\alpha} \sum_{i=1}^{N_p} \left(\sum_{j=1}^{\infty} \delta_{i,j} (W_i/\Gamma_{i,j})^{1/\alpha} \right)$$

(see the definitions in Section 3).

Theorem 4.2. *Let the assumptions of Theorems* 4.1 *and* 3.1 *hold. If* $M_p \to \infty$ *and* $pM_p \to 0$ *as* $p \to 0$, *then, with probability one,*

$$\sum_{i=1}^{N_p^*} X_{i,N_p^*} - \sum_{i=1}^{N_p} Y_{i,N_p} =$$
$$= O(p^{-1/\alpha}) \left[\sum_{k=1}^{M_p} \Gamma_k^{-1/\alpha} \gamma(\Gamma_k/\Gamma_{N_p^*+1}) + \frac{\sqrt{\log\log M_p}}{M_p^{1/\alpha-1/2}} + \frac{\sqrt{\log\log(1/p)}}{(1/p)^{1/2}} \right].$$

We prove Theorem 4.1. Similar proof of Theorem 4.2, based on Theorem 3.1, is omitted.

Proof of Theorem 4.1. It follows directly from the definition of $(X_{i,n})$ and $(Y_{i,n})$ that

$$\left| \sum_{i=1}^{n} X_{i,n} - \sum_{i=1}^{n} Y_{i,n} \right| \le$$
$$\le \left| \sum_{i=1}^{n} \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1}) - n^{1/\alpha} \sum_{i=1}^{n} \delta_i \Gamma_i^{-1/\alpha} \right| + n^{1/\alpha} \left| \sum_{i=n+1}^{\infty} \delta_i \Gamma_i^{-1/\alpha} \right|$$
$$\le I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$I_{1,n} = \left| \textstyle\sum_{i=1}^{n} \delta_i \left[G^{-1}(\Gamma_i/\Gamma_{n+1}) - (\Gamma_i/\Gamma_{n+1})^{-1/\alpha} \right] \right|,$$

$$I_{2,n} = \left| \Gamma_{n+1}^{1/\alpha} - n^{1/\alpha} \right| \left| \textstyle\sum_{i=1}^{n} \delta_i \Gamma_i^{-1/\alpha} \right|, \qquad I_{3,n} = n^{1/\alpha} \left| \textstyle\sum_{i=n+1}^{\infty} \delta_i \Gamma_i^{-1/\alpha} \right|.$$

First, by the Law of Iterated Logarithm (LIL) and integration by parts formula,

$$\begin{aligned} I_{3,n} &\le O(1) \left(\sqrt{n \log\log n} + \int_{\Gamma_{n+1}}^{\infty} x^{-1/\alpha-1/2} \sqrt{\log\log x} dx \right) \\ &= O(n^{1/\alpha}) \frac{\sqrt{\log\log n}}{n^{1/\alpha-1/2}}. \end{aligned}$$

Then, again by LIL,

$$I_{2,n} \le O(n^{1/\alpha}) \sqrt{\frac{\log\log n}{n}}.$$

Equation (21) and the discrete version of integration by parts formula produce

$$\begin{aligned} I_{1,n} &= O(n^{1/\alpha})\Bigg[\sum_{i=1}^{M_n}\delta_i\Gamma_i^{-1/\alpha}\gamma(\Gamma_i/\Gamma_{n+1}) \\ &\quad +\gamma(\Gamma_n/\Gamma_{n+1})\frac{\sum_{i=1}^{n}\delta_i}{n^{1/\alpha}}+\gamma(\Gamma_{M_n}/\Gamma_{n+1})\frac{\sum_{i=1}^{M_n}\delta_i}{M_n^{1/\alpha}} \\ &\quad + n^{-1/\alpha}\left(\sum_{i=M_n+1}^{N}\sum_{k=1}^{i-1}\delta_k\Delta_i+\sum_{i=M_n+1}^{N}\sum_{k=1}^{i-1}\delta_k\tilde{\Delta}_i\right)\Bigg], \end{aligned}$$

where Δ_i and $\tilde{\Delta}_i$ are increments of $x^{-1/\alpha}$ and G^{-1} taken at Γ_i/Γ_{n+1}, $i = 1,\ldots,n+1$.

The approximation (from above) of the last two sums, coupled with the LIL, yields

$$\begin{aligned} I_{1,n} &= O(n^{1/\alpha})\Bigg[\sum_{i=1}^{M_n}\delta_i\Gamma_i^{-1/\alpha}\gamma(\Gamma_i/\Gamma_{n+1})+\frac{\sqrt{\log\log M_n}}{M_n^{1/\alpha-1/2}} \\ &\quad +n^{1/2-1/\alpha}\left(\int_{\Gamma_{M_n}/\Gamma_{n+1}}^{1}\sqrt{x\log\log(2nx)}\,d(x^{-1/\alpha})\right. \\ &\quad \left. + \int_{\Gamma_{M_n}/\Gamma_{n+1}}^{1}\sqrt{x\log\log(2nx)}\,dG^{-1}(x)\right)\Bigg]. \end{aligned}$$

Denote $H(u) = \int_u^1 t^{1/\alpha}dG^{-1}(t)$ and apply integration by parts formula, to obtain

$$\begin{aligned} &\left|\int_{\Gamma_{M_n}/\Gamma_{n+1}}^{1}\sqrt{x\log\log(2nx)}\,dG^{-1}(x)\right| \\ &\leq\ H(1)\sqrt{\log\log(2n)}+H(\Gamma_{M_n}/\Gamma_{n+1})\sqrt{\log\log(\Gamma_{M_n}n/\Gamma_{n+1})} \\ &+\ O(1)\int_{\Gamma_{M_n}/\Gamma_{n+1}}^{1}\frac{1}{\sqrt{\log\log(2nx)}}+\frac{1}{\log(2nx)}+\frac{1}{x}H(x)dx \\ &\leq\ O(1)\left[\sqrt{\log\log n}+n^{-(1/2-1/\alpha)}\frac{\sqrt{\log\log M_n}}{M_n^{1/\alpha-1/2}}+\int_{\Gamma_{M_n}/\Gamma_{n+1}}x^{-1}H(x)dx\right]. \end{aligned}$$

The last integral is of order $O((M_n/n)^{1/2-1/\alpha})$, since the function $u \mapsto \int\limits_u^1 x^{-1}H(x)dx$ is regularly varying at zero with index $1/\alpha - 1/2$. The same arguments produce

$$\int_{\Gamma_{M_n}/\Gamma_{n+1}} \sqrt{x \log\log(2nx)}\, d(x^{-1/\alpha}) \leq$$
$$\leq O(1)\left[\sqrt{\log\log n} + n^{-(1/2-1/\alpha)} \frac{\sqrt{\log\log M_n}}{M_n^{1/\alpha-1/2}}\right].$$

Thus,

$$I_{1,n} = O(n^{1/\alpha})\left[\sum_{i=1}^{M_n} \delta_i \Gamma_i^{-1/\alpha} \gamma(\Gamma_i/\Gamma_{n+1}) + \frac{\sqrt{\log\log M_n}}{M_n^{1/\alpha-1/2}}\right],$$

which concludes the proof. □

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