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LIMIT THEOREMS FOR REGENERATIVE EXCURSION PROCESSES

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Abstract. The regenerative excursion process $Z(t)$, $t = 0, 1, 2, \ldots$ is constructed by two independent sequences $X = \{X_i, i \geq 1\}$ and $Z = \{T_i, (Z_i(t), 0 \leq t < T_i), i \geq 1\}$.

For the embedded alternating renewal process, with interarrival times X_i – the time for the installation and T_i – the time for the work, are proved some limit theorems for the spent worktime and the residual worktime, when at least one of the means of X_i and T_i is infinite.

Limit theorems for the process $Z(t)$ are proved, too.

Finally, some applications to the branching processes with state-dependent immigration are given.

1. Introduction. Let on the probability space $(\Omega, \mathcal{A}, \mathcal{P})$ be given:

i) The set $X = \{X_i, i = 1, 2, \ldots\}$ of independent, identically distributed, positive, integer valued random variables with distribution function (d.f.)

$$A(x) = \sum_{k \leq x} a_k = \mathcal{P}(X_i \leq x),\ x \geq 0, \qquad \sum_{t=1}^{\infty} a_t = 1.$$

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ii) Independent of X set $Z = \{(T_i, (Z_i(t) \neq 0,\ t = 0, 1, \dots, T_{i-1}, Z_i(T_i) = 0)), i = 1, 2, \dots\}$ of independent, identically distributed random processes with discrete-time parameter and common state space $(\mathbf{R}_+^m, \mathcal{B}_+^m)$, where $\mathbf{R}_+ = [0, \infty)$ and $\mathcal{B}_+^m$ is the Borel σ-algebra. We assume that in the life-period $[0, T_i)$ the process $Z_i(t)$ is not zero and $Z_i(t) \equiv 0$ for $t \geq T_i$. So, $Z_i(t)$ are excursion processes which generalize in some sense the branching processes with continuous or discrete state space.

The life-periods T_i are independent, positive, integer valued random variables with d.f.

$$B(x) = \sum_{k \leq x} b_k = \mathcal{P}(T_i \leq x),\ x \geq 0,\ \ \sum_{t=1}^{\infty} b_t = 1.$$

Denote $Y_i = X_i + T_i,\ \ i = 1, 2, \dots$ and

$$c_t = \mathcal{P}(Y_i = t) = \sum_{k=1}^{t-1} a_k b_{t-k},\ t = 1, 2, \dots, C(x) = \mathcal{P}(Y_i \leq x) = (A * B)(x),\ \ x \geq 0.$$

Define the ordinary renewal process

$$S_0 = 0, S_{n+1} = S_n + Y_{n+1},\ n \geq 0,\ \ \nu(t) = \max(n : S_n \leq t < S_{n+1}) \tag{1.1}$$

The sequence $(S_n, S'_{n+1})_{n=0}^{\infty}$ where

$$S'_{n+1} = S_n + X_{n+1}, \quad n = 0, 1, 2, \dots \tag{1.2}$$

is called an alternating renewal process.

Define the processes

$$\sigma(t) = t - S'_{\nu(t)+1} = t - (S_{\nu(t)} + X_{\nu(t)+1}),\ \ \tau(t) = \min(S_{\nu(t)+1} - t, T_{\nu(t)+1}). \tag{1.3}$$

We call $\sigma(t)$ the spent worktime and $\tau(t)$ the residual worktime. In terms of reliability theory we can interpret X_i as the time for installation of the i-th element in some system and T_i as the worktime of this element. Then $\sigma(t)$ and $\tau(t)$ are the spent worktime and the residual worktime of the element working at time t.

Using X and Z we define the regenerative excursion process $Z(t), t \geq 0$, as follows

$$Z(t) = \begin{cases} 0 & \text{for} \quad \sigma(t) < 0, \\ Z_{\nu(t)+1}(\sigma(t)) & \text{for} \quad \sigma(t) \geq 0. \end{cases} \tag{1.4}$$

In terms of reliability theory one can interpret the process $Z_i(t)$ as measuring some characteristic of the i-th element at time t. Then $Z(t)$ is the same characteristic of the element working at time t.

Denote

$$u_t = \mathcal{P}(S_n = t \text{ for some } n = 0, 1, 2, \ldots) = \sum_{n=0}^{\infty} \mathcal{P}(S_n = t),\ U(x) = \sum_{t \le x} u_t,\ x \ge 0,$$

$$v_t = \mathcal{P}(S'_n = t \text{ for some } n = 1, 2, \ldots) = \sum_{n=1}^{\infty} \mathcal{P}(S'_n = t),\ V(x) = \sum_{t \le x} v_t,\ x \ge 0.$$

It is well-known that the sequence u_t satisfies the discrete renewal equation (see [4, Vol.1, Sec.13.3])

$$u_0 = 1, \quad u_t = \sum_{k=0}^{t-1} u_k c_{t-k}, \quad t = 0, 1, 2, \ldots. \tag{1.5}$$

By the definition of v_t and (1.2), one can see that

$$v_t = \sum_{k=1}^{t} a_k u_{t-k}, \quad t = 0, 1, 2, \ldots\ , \tag{1.6}$$

i.e. v_t is the renewal sequence for the discrete time renewal process with delay $X_0 \overset{d}{=} X_i$.

Initial motivation for this work is to prove limit theorems for branching processes with state-dependent immigration when the conditional limit behavior of such processes without immigration is known, using direct probabilistic methods. We want to point out that the branching processes with state-dependent immigration were investigated by many authors using the methods of probability generating functions. (See e.g. [3, 10, 15, 7, 8, 9]).

The regenerative excursion processes considered in the paper include all models of branching processes with state-dependent immigration.

Some applications in this direction are given in Section 6.

Naturally, the limit theorems proved here will be applicable to other models of regenerative processes.

Moreover, it seems that the asymptotic results about alternating renewal processes with infinite means are not published earlier and they are interesting in themselves. One can compare the results of Section 4 with the well-known results for ordinary renewal processes with infinite mean. (See [4, Sec.14.3] and [2]).

We will consider the aperiodic discrete-time processes but all results obtained here can be easily transferred to the continuous time case with slight modifications.

In the sequel we assume some of the following basic conditions:

$$\gcd(t : a_t > 0) = 1, \quad \gcd(t : b_t > 0) = 1. \tag{1.7}$$

$$\mathcal{E}X_i < \infty, \tag{1.8}$$

$$\mathcal{E}X_i = \infty, \quad 1 - A(x) \sim x^{-\alpha} L_A(x), \ x \to \infty, \quad \alpha \in (1/2, 1], \tag{1.9}$$

$$\mathcal{E}T_i < \infty, \tag{1.10}$$

$$\mathcal{E}T_i = \infty, \quad 1 - B(x) \sim x^{-\beta} L_B(x), \ x \to \infty, \quad \beta \in (1/2, 1], \tag{1.11}$$

where $L_A(.)$, $L_B(.)$ are functions slowly varying at infinity.

The lower bound 1/2 for α and β arises because we will use the results of Garsia and Lamperti [5] for the asymptotic behavior of u_t.

In the case when both means are finite the behavior of the processes defined above is well-known. (See e.g. [4, Vol.2, Sec.11.8]). For this reason, we assume that at least one of the means is infinite.

Two different relations between $\mathcal{E}X_i$ and $\mathcal{E}T_i$ will be considered separately.

(C.1) *One of the conditions* (1.10) *or* (1.11), *together with* (1.9), *holds and*

$$1 - B(t) = o(1 - A(t)), \quad t \to \infty.$$

(C.2) *One of the conditions* (1.8) *or* (1.9), *together with* (1.11), *holds and*

$$\lim_{t\to\infty} \frac{1 - A(t)}{1 - B(t)} = c, \ \ 0 \le c < \infty.$$

In other words, the condition (C.1) describes the situation when the installation time is 'longer' than the work time, but the condition (C.2) describes the opposite situation.

Further, we will use the following notations

$$m_A(t) = \int_0^t (1 - A(x))dx, \quad m_B(t) = \int_0^t (1 - B(x))dx,$$
$$m_C(t) = \int_0^t (1 - C(x))dx$$
$$m_A(\infty) = \int_0^\infty (1 - A(x))dx = \mathcal{E}X_i \le \infty,$$
$$m_B(\infty) = \int_0^\infty (1 - B(x))dx = \mathcal{E}T_i \le \infty.$$

2. Preliminary results. Now, we will state some results which are needed in the following sections.

Lemma 2.1. *Let* (1.7) *and one of* (C.1) *or* (C.2) *hold. Then*

$$\gcd(t : c_t > 0) = 1 \tag{2.1}$$

and

$$1 - C(x) \sim x^{-\theta} L_C(x), \quad x \to \infty, \tag{2.2}$$

where $\theta = \alpha$ *and* $L_C(t) = L_A(t)$ *when* (C.1) *holds, and* $\theta = \beta$ *and* $L_C(t) = (1+c)L_B(t)$ *when* (C.2) *holds.*

Proof. (2.1) follows immediately from (1.7), (2.2) follows from the relation [4, Vol.2, Sec.8.8] $1 - C(x) \sim (1 - A(x)) + (1 - B(x)),\ x \to \infty.$ □

The results in the next lemma are due to Garsia and Lamperti [5], and Erickson [2]. (See e.g. [2, (2.3) and (2.4)]).

Lemma 2.2. *Suppose the conditions of Lemma* 2.1*.*

i) If $1/2 < \theta < 1$*, then*

$$u_t \sim \frac{\sin \pi\theta}{\pi} \frac{t^{\theta-1}}{L_C(t)}, \quad t \to \infty. \tag{2.3}$$

ii) If $\theta = 1$*, then*

$$u_t \sim \frac{1}{m_C(t)}, \quad t \to \infty, \tag{2.4}$$

and in this case $m_C(t)$ *is a monotone increasing function which varies slowly at infinity, and* $m_C(t) \uparrow \infty, \quad t \to \infty.$

Lemma 2.3. *Under the conditions of Lemma* 2.1

$$v_t \sim u_t, \quad t \to \infty. \tag{2.5}$$

The proof is easy but long and is omitted.

From the definition of $\sigma(t)$, it is evident that $\sigma(t) \le t$ a.s.

Lemma 2.4. *For* $k = 0, 1, 2, \ldots t$

$$\mathcal{P}(\sigma(t) = k) = v_{t-k}(1 - B(k)). \tag{2.6}$$

For $k > 0$

$$\mathcal{P}(\sigma(t) = -k) = \sum_{j=0}^{t} u_{t-j} a_{j+k}. \tag{2.7}$$

Proof. Let $k \in \{0, 1, \ldots, t\}$ be fixed. From the definition of $\sigma(t)$, it is clear that the events $\{\sigma(t) = k\}$ and $\{S'_{n+1} = t - k, T_{n+1} > k$ for some

$n = 0, 1, 2, \ldots\}$ are equivalent. Thus,

$$\mathcal{P}(\sigma(t) = k) = \sum_{n=0}^{\infty} \mathcal{P}(S'_{n+1} = t - k, T_{n+1} > k).$$

By the independence of S'_{n+1} and T_{n+1} for every $n \geq 0$, we obtain

$$\mathcal{P}(\sigma(t) = k) = \sum_{n=0}^{\infty} \mathcal{P}(S'_{n+1} = t - k)\mathcal{P}(T_{n+1} > k)$$
$$= (1 - B(k)) \sum_{n=0}^{\infty} \mathcal{P}(S'_{n+1} = t - k) = (1 - B(k))v_{t-k},$$

which proves (2.6). The proof of (2.7) is similar. □

Corollary 2.1. *Under the conditions of Lemma* 2.4

$$\mathcal{P}(\sigma(t) < 0) = \sum_{k=0}^{t} u_{t-k}(1 - A(k)), \tag{2.8}$$

If $0 \leq x_1 < x_2 \leq 1$, *then*

$$\mathcal{P}(tx_1 \leq \sigma(t) \leq tx_2) = \sum_{tx_1 \leq k \leq tx_2} v_{t-k}(1 - B(k)). \tag{2.9}$$

Lemma 2.5. *Let* $k \in \{0, 1, \ldots, t\}$ *and* $l \geq 0$. *Then*

$$\mathcal{P}(\sigma(t) = k, \tau(t) = l) = v_{t-k}b_{k+l}. \tag{2.10}$$

The proof is similar to the proof of Lemma 2.4.

Corollary 2.2. *For* $0 \leq x \leq t$ *and* $y \geq 0$

$$\mathcal{P}(0 \leq \sigma(t) \leq x, \tau(t) > y) = \sum_{0 \leq k \leq x} v_{t-k}(1 - B(k + y)), \tag{2.11}$$

$$\mathcal{P}(0 \leq \sigma(t) \leq x, \tau(t) \leq y) = \sum_{0 \leq k \leq x} v_{t-k}(1 - B(k)) - \sum_{0 \leq k \leq x} v_{t-k}(1 - B(k + y)). \tag{2.12}$$

From the definition of $Z(t)$ is clear that

$$\mathcal{P}(Z(t) = 0) = \mathcal{P}(\sigma(t) < 0), \quad \mathcal{P}(Z(t) \neq 0) = \mathcal{P}(\sigma(t) \geq 0). \tag{2.13}$$

Lemma 2.6. *For the distributions of* $Z(t)$ *the following equations hold:*

$$\begin{aligned}&\mathcal{P}(Z(t) \le z) = \mathcal{P}(\sigma(t) < 0) + \\ &\mathcal{P}(\sigma(t) \ge 0) \sum_{k=0}^{t} \mathcal{P}(Z_1(k) \le z | T_1 > k) \mathcal{P}(\sigma(t) = k | \sigma(t) \ge 0),\end{aligned} \tag{2.14}$$

$$\begin{aligned}&\mathcal{P}(Z(t) \le z | Z(t) \ne 0) = \\ &\sum_{k=0}^{t} \mathcal{P}(Z_1(k) \le z | T_1 > k) \mathcal{P}(\sigma(t) = k | \sigma(t) \ge 0).\end{aligned} \tag{2.15}$$

Proof. First of all we see that (2.15) follows immediately from (2.14) when $\mathcal{P}(\sigma(t) \ge 0) > 0$, using also (2.13).

So, we have to prove (2.14). Note that

$$\begin{aligned}&\mathcal{P}(Z(t) \le z) = \mathcal{P}(Z(t) \le z, X_1 > t) \\ &+ \mathcal{P}(Z(t) \le z, X_1 \le t, S_1 > t) + \mathcal{P}(Z(t) \le z, X_1 \le t, S_1 = X_1 + T_1 \le t).\end{aligned} \tag{2.16}$$

For the right-hand side we have

$$\mathcal{P}(Z(t) \le z, X_1 > t) = \mathcal{P}(X_1 > t) = 1 - A(t), \tag{2.17}$$

$$\begin{aligned}\mathcal{P}(Z(t) \le z, X_1 \le t, S_1 > t) &= \sum_{k=0}^{t} \mathcal{P}(Z(t) \le z, T_1 > k, X_1 = t - k) \\ &= \sum_{k=0}^{t} \mathcal{P}(Z_1(k) \le z, T_1 > k) \mathcal{P}(X_1 = t - k) \\ &= \sum_{k=0}^{t} \mathcal{P}(Z_1(t) \le z | T_1 > k)(1 - B(k)) a_{t-k},\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\mathcal{P}(Z(t) \le z, X_1 \le t, S_1 = X_1 + T_1 \le t) = \mathcal{P}(Z(t) \le z, S_1 \le t) \\ &= \sum_{k=0}^{t} \mathcal{P}(Z(k) \le z, S_1 = t - k) = \sum_{k=0}^{t} \mathcal{P}(Z(k) \le z) \mathcal{P}(S_1 = t - k) \\ &= \sum_{k=0}^{t} \mathcal{P}(Z(k) \le z) c_{t-k}.\end{aligned} \tag{2.19}$$

From (2.16)–(2.19) one obtains

$$
\begin{aligned}
\mathcal{P}(Z(t) \le z) &= \sum_{k=0}^{t} \mathcal{P}(Z(k) \le z) c_{t-k} \\
&+1 - A(t) + \sum_{k=0}^{t} \mathcal{P}(Z_1(k) \le z | T_1 > k)(1 - B(k)) a_{t-k},
\end{aligned}
\tag{2.20}
$$

which is a discrete-time equation of renewal type and has the solution (see [4, Vol.1, Sec.13.5])

$$
\begin{aligned}
&\mathcal{P}(Z(t) \le z) \\
&= \sum_{0}^{t} u_{t-k} \Big[1 - A(k) + \sum_{j=0}^{k} \mathcal{P}(Z_1(j) \le z | T_1 > j)(1 - B(j)) a_{k-j}\Big] \\
&= \sum_{k=0}^{t} u_{t-k} [1 - A(k)] \\
&+ \sum_{k=0}^{t} u_{t-k} \Big[\sum_{j=0}^{k} \mathcal{P}(Z_1(j) \le z | T_1 > j)(1 - B(j)) a_{k-j}\Big].
\end{aligned}
\tag{2.21}
$$

Changing the order of summation in the second sum in (2.21), and, using (2.6), (2.8) and (2.9) with $x_1 = 0$ and $x_2 = 1$, we complete the proof. □

Let

$$\Xi = \{\xi_0, \xi_1, \ldots \xi_t, \ldots\}$$

be a sequence of random vectors taking values in $\mathbf{R}_+^m$. Let

$$\mathrm{H} = \{\eta_0, \eta_1, \ldots, \eta_t, \ldots\}$$

be a sequence of non-negative, integer-valued random variables, independent of Ξ.

The next lemma is a version of a result of Dobrushin [1], (see also [13, Th.1, Sec.2.5]).

Lemma 2.7. *Assume $h(x) > 0$ and is regularly varying with exponent $\gamma \ge 0$, ξ is a random vector with a proper m-dimensional distribution $\mathcal{P}(\xi \le z) = P(z)$ and*

$$\frac{\xi_t}{h(t)} \Rightarrow \xi, \quad t \to \infty. \tag{2.22}$$

Assume $N(x) \to \infty$ as $x \to \infty$ and that it is a non-decreasing, positive function, η is an almost surely positive random variable with a proper distribution

$\mathcal{P}(\eta \le x) = Q(x)$, and

$$\frac{\eta_t}{N(t)} \Rightarrow \eta, \quad t \to \infty, \tag{2.23}$$

Then

$$\frac{\xi_{\eta_t}}{h(N(t))} \Rightarrow \xi.\eta^{\gamma}, \quad t \to \infty, \tag{2.24}$$

where ξ and η are independent.

The proof can be found in [1], in the case when ξ_t are random variables, or, in [13], in the case when ξ_t are random vectors.

The next lemma is a generalization of the result of Dobrushin in the case of non-linear normalization of the indexing sequence H.

Lemma 2.8 *Assume $h(x) > 0$ and is regularly varying with exponent $\gamma > 0$ and (2.22) holds. Assume $N(x) \to \infty$ as $x \to \infty$ and that it is non-decreasing function, slowly varying at infinity, η is an almost surely positive random variable with a proper distribution $\mathcal{P}(\eta \le x) = Q(x)$, and*

$$\frac{N(\eta_t)}{N(t)} \Rightarrow \eta, \quad t \to \infty. \tag{2.25}$$

Then

$$\frac{N(h^{-1}(\xi_{\eta_t}))}{N(t)} \Rightarrow \mathbf{1}.\eta, \quad t \to \infty, \tag{2.26}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbf{R}_+^m$ and $h^{-1}(.)$ is the inverse function of $h(.)$.

We omit the proof because it is similar to the proof of Lemma 2.7 and it is based on the principle of embedding established by Skorohod [12], (see also [13, Sec.1.1, Th.1]).

In the same manner one can prove the following lemma.

Lemma 2.9. *Assume that*

$$\eta_t \Rightarrow \eta, \quad t \to \infty,$$

where η is a non-negative, integer valued random variable, independent of Ξ. Then

$$\xi_{\eta_t} \Rightarrow \xi_\eta, \quad t \to \infty. \tag{2.27}$$

Let us note that because of the independence of Ξ and H

$$\mathcal{P}(\xi_{\eta_t} \le z) = \sum_{k=0}^{\infty} \mathcal{P}(\xi_k \le z)\mathcal{P}(\eta_t = k). \tag{2.28}$$

3. Asymptotic behavior of $\mathcal{P}(\sigma(t) \geq 0)$. In this section we establish the asymptotic behavior of $\mathcal{P}(\sigma(t) \geq 0)$ under the conditions (C.1) or (C.2).

Theorem 3.1. *Assume* (1.7) *and* (C.1). *Then*

$$\lim \mathcal{P}(\sigma(t) \geq 0) = 0. \tag{3.1}$$

i) If (1.11) *with* $1/2 < \beta < 1$ *holds, then*

$$\mathcal{P}(\sigma(t) \geq 0) \sim \mathrm{B}(\alpha, 1-\beta) v_t m_B(t), \quad t \to \infty. \tag{3.2}$$

ii) If (1.11) *with* $\beta = 1$ *holds, then*

$$\mathcal{P}(\sigma(t) \geq 0) \sim v_t m_B(t), \quad t \to \infty. \tag{3.3}$$

In this case $m_B(t) \uparrow \infty$, $t \to \infty$ *and* $m_B(t)$ *varies slowly at infinity.*

iii) If (1.10) *holds, then*

$$\mathcal{P}(\sigma(t) \geq 0) \sim v_t m_B(\infty), \quad t \to \infty. \tag{3.4}$$

Theorem 3.2. *Assume* (1.7) *and* (C.2). *Then*

$$\lim_{t\to\infty} \mathcal{P}(\sigma(t) \geq 0) = \frac{1}{1+c}. \tag{3.5}$$

P r o o f o f T h e o r e m 3.1. Under the conditions of the theorem, (2.2) is fulfilled with $\theta = \alpha$ and $L_C(t) = L_A(t)$.

i) Let (1.11) with $1/2 < \beta < 1$ holds. It follows from (C.1) that $1/2 < \alpha \leq \beta < 1$.

Let $\varepsilon \in (0, \frac{1}{2})$ and $\delta > 0$ be fixed. From (2.3), (2.5) and (1.11) we have for every t large enough

$$(1-\delta)\frac{t^{\alpha-1}}{L_A(t)}\frac{\sin \pi\alpha}{\pi} \leq v_t \leq (1+\delta)\frac{t^{\alpha-1}}{L_A(t)}\frac{\sin \pi\alpha}{\pi} \tag{3.6}$$

and

$$(1-\delta)t^{-\beta}L_B(t) \leq 1 - B(t) \leq (1+\delta)L_B(t)t^{-\beta}. \tag{3.7}$$

Setting $x_1 = 0$ and $x_2 = 1$ in (2.9), we can write

$$\mathcal{P}(\sigma(t) \geq 0) = \sum_{0\leq k<[t\varepsilon]} + \sum_{[t\varepsilon]\leq k\leq[t(1-\varepsilon)]} + \sum_{[t(1-\varepsilon)]<k\leq t} = S_1 + S_2 + S_3. \tag{3.8}$$

We estimate S_2 using (3.6) and (3.7)

$$(1-\delta)^2\frac{\sin\pi\alpha}{\pi}\sum_{[t\varepsilon]\le k\le[t(1-\varepsilon)]}\frac{(t-k)^{\alpha-1}}{L_A(t-k)}k^{-\beta}L_B(k)$$

(3.9)

$$\le S_2\le(1+\delta)^2\frac{\sin\pi\alpha}{\pi}\sum_{[t\varepsilon]\le k\le[t(1-\varepsilon)]}\frac{(t-k)^{\alpha-1}}{L_A(t-k)}k^{-\beta}L_B(k)$$

Using the theorem for uniform convergence of slowly varying functions [11, Th. 1.1], we obtain

$$\sum_{[t\varepsilon]\le k\le[t(1-\varepsilon)]}\frac{(t-k)^{\alpha-1}}{L_A(t-k)}k^{-\beta}L_B(k)\sim\frac{L_B(t)}{L_A(t)}\sum_{[t\varepsilon]\le k\le[t(1-\varepsilon)]}(t-k)^{\alpha-1}k^{-\beta}, t\to\infty. \tag{3.10}$$

It is easy to be seen that

$$\sum_{[t\varepsilon]\le k\le[t(1-\varepsilon)]}(t-k)^{\alpha-1}k^{-\beta}\sim t^{\alpha-\beta}\int_{\varepsilon}^{1-\varepsilon}(1-u)^{\alpha-1}u^{-\beta}du,\quad t\to\infty. \tag{3.11}$$

Thus, from (3.9)–(3.11) we obtain

$$S_2\sim t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)}\frac{\sin\pi\alpha}{\pi}\int_{\varepsilon}^{1-\varepsilon}(1-u)^{\alpha-1}u^{-\beta}du,\quad t\to\infty, \tag{3.12}$$

because δ was arbitrary.

For S_1 using (3.6) and (3.7) we have

$$S_1\le(1+\delta)\frac{\sin\pi\alpha}{\pi}\frac{(t-[t\varepsilon])^{\alpha-1}}{\inf_{t-[t\varepsilon]\le x<t}L_A(x)}\sum_{0\le k<[t\varepsilon]}(1-B(k))$$

(3.13)

$$\sim C_1(1+\delta)(1-\varepsilon)^{\alpha-1}\varepsilon^{1-\beta}t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)},\quad t\to\infty,$$

Similarly,

$$S_1\ge(1-\delta)\frac{\sin\pi\alpha}{\pi}\frac{t^{\alpha-1}}{\sup_{t-[t\varepsilon]\le k<t}L_A(k)}\sum_{0\le k<[t\varepsilon]}(1-B(k))$$

(3.14)

$$\sim C_2(1-\delta)\varepsilon^{1-\beta}t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)},\quad t\to\infty.$$

In the same manner we obtain for S_3

$$S_3 \le (1+\delta)[t(1-\varepsilon)]^{-\beta}\Big(\sup_{[t(1-\varepsilon)]\le x\le t} L_B(x)\Big) \sum_{0\le k<t-[t(1-\varepsilon)]} v_k$$
$$\sim C_3(1+\delta)(1-\varepsilon)^{-\beta}\varepsilon^{\alpha} t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)}, \quad t\to\infty \tag{3.15}$$

and

$$S_3 \ge (1-\delta)t^{-\beta}\Big(\inf_{[t(1-\varepsilon)]\le x\le t} L_2(x)\Big) \sum_{0\le k<t-[t(1-\varepsilon)]} v_k$$
$$\sim C_4(1-\delta)\varepsilon^{\alpha} t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)}, \quad t\to\infty. \tag{3.16}$$

In these estimations we use Theorem 5 from [4, Vol.2, Sec.13.5] and the theorem for uniform convergence of slowly varying functions [11, Th. 1.1]. The constants C_i are positive and different in the different cases.

Finally, from (3.12)–(3.16) we obtain

$$\limsup_{t\to\infty}\left((S_1+S_2+S_3)/t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)}\frac{\sin\pi\alpha}{\pi}\right)$$
$$\le \int_{\varepsilon}^{1-\varepsilon}(1-u)^{\alpha-1}u^{-\beta}du + Const.(1+\delta)((1-\varepsilon)^{\alpha-1}\varepsilon^{1-\beta}+\varepsilon^{\alpha}(1-\varepsilon)^{-\beta}),$$

$$\liminf_{t\to\infty}\left((S_1+S_2+S_3)/t^{\alpha-\beta}\frac{L_B(t)}{L_A(t)}\frac{\sin\pi\alpha}{\pi}\right)$$
$$\ge \int_{\varepsilon}^{1-\varepsilon}(1-u)^{\alpha-1}u^{-\beta}du + Const.(1-\delta)(\varepsilon^{1-\beta}+\varepsilon^{\alpha}).$$

The last estimations prove (3.2), because ε and δ were arbitrary.

ii) Let (1.11) with $\beta = 1$ holds. It follows from (C.1) that $1/2 < \alpha \le 1$. On the other hand, $m_B(x)$ is a monotone increasing, slowly varying function (see [4, Vol.2, Sec.8.9, Lemma]), and $m_B(t)\uparrow\infty$ as $t\to\infty$.

Let $\varepsilon\in(0,1)$. Then

$$\mathcal{P}(\sigma(t)\ge 0) = \sum_{0\le j\le[t\varepsilon]} + \sum_{[t\varepsilon]<j\le t} = S_1+S_2. \tag{3.17}$$

Using the properties of regularly varying functions [11], we obtain for S_1 that

$$S_1 \ge \Big(\inf_{t-[t\varepsilon]\le k\le t} v_k\Big)\sum_{0\le j\le[t\varepsilon]}(1-B(j)) \sim v_t m_B(t), \quad t\to\infty, \tag{3.18}$$

(3.19) $$S_1 \le (\sup_{t-[t\varepsilon]\le x\le t} v_k) \sum_{0\le j\le [t\varepsilon]} (1-B(j)) \sim v_t m_B(t), \quad t\to\infty.$$

Using (3.6) and the monotonicity of $1-B(x)$, we have for S_2

(3.20) $$0 \le S_2 \le \sum_{[t\varepsilon]<j\le t} (1-B(j))v_{t-j} \le (1-B([t\varepsilon])) \sum_{0\le j<t-[t\varepsilon]} v_j$$
$$\sim C_5 t(1-B(t))v_t, \quad t\to\infty.$$

In the last relation we apply Theorem 5 from [4, Vol.2, Sec.13.5] to v_t. Moreover, in this case we have (see [4, Vol.2, Sec.8.9, Lemma]) $t(1-B(t)) = o(m_B(t))$ as $t\to\infty$, which together with (3.17)–(3.20) and the fact that $\varepsilon\in(0,1)$ was arbitrary, complete the proof of the theorem.

iii) The proof is quite similar to the previous case and so it is omitted. Finally, (3.1) follows from (3.2)–(3.4).

P r o o f o f T h e o r e m 3.2. The proof is quite similar to that of the cases *i)* and *ii)* of Theorem 3.1 and so it is omitted.

Comment. It is evident, that under the condition (C.1), we can obtain only proper conditional limit distributions for $\sigma(t)$, $\tau(t)$ and $Z(t)$. If (C.2) holds, then we can obtain both conditional and unconditional limit distributions with atom $c/(1+c)$ at zero.

4. Limit theorems for $\boldsymbol{\sigma(t)}$ and $\boldsymbol{\tau(t)}$. In this section we prove the limit theorems for common and marginal distributions of $\sigma(t)$ and $\tau(t)$. The main results are in the following theorems.

Theorem 4.1. *Assume* (1.7) *and* (C.1).

i) If (1.11) *with* $1/2<\beta<1$, *holds then for* $0<x\le 1$ *and* $y>0$

(4.1) $$\lim_{t\to\infty} \mathcal{P}(\frac{\sigma(t)}{t}\le x, \frac{\tau(t)}{t}>y|\sigma(t)\ge 0)=(\int_0^x (1-u)^{\alpha-1}(u+y)^{-\beta}du)/\mathrm{B}(\alpha,1-\beta).$$

ii) If (1.11) *with* $\beta=1$, *then for* $0<x\le 1$ *and* $y>0$

(4.2) $$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\sigma(t))}{m_B(t)}\le x, \frac{m_B(\tau(t))}{m_B(t)}\le y|\sigma(t)\ge 0) = \min(x,y),$$

(4.3) $$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\sigma(t))}{m_B(t)}\le x|\sigma(t)\ge 0) = x,$$

(4.4) $$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\tau(t))}{m_B(t)}\le y|\sigma(t)\ge 0) = \min(y,1).$$

iii) If (1.10) holds, then

$$\lim_{t\to\infty} \mathcal{P}(\sigma(t) \le x, \tau(t) \le y | \sigma(t) \ge 0) = \frac{1}{m_B(\infty)} \sum_{0\le j\le x} (B(j+y) - B(j)), \tag{4.5}$$

$$\lim_{t\to\infty} \mathcal{P}(\sigma(t) \le x, |\sigma(t) \ge 0) = \frac{1}{m_B(\infty)} \sum_{0\le j\le x} (1 - B(j)), \tag{4.6}$$

$$\lim_{t\to\infty} \mathcal{P}(\tau(t) \le y | \sigma(t) \ge 0) = \frac{1}{m_B(\infty)} \sum_{0\le j\le y} (1 - B(j)), \tag{4.7}$$

and for every fixed $k = 0, 1, 2, \dots$

$$\lim_{t\to\infty} \mathcal{P}(\sigma(t) = k, |\sigma(t) \ge 0) = \frac{1}{m_B(\infty)}(1 - B(k)). \tag{4.8}$$

Theorem 4.2. *Assume* (1.7) *and* (C.2).
i) If $1/2 < \beta < 1$, *then for* $0 < x \le 1$ *and* $y > 0$

$$\lim_{t\to\infty} \mathcal{P}(\frac{\sigma(t)}{t} \le x, \frac{\tau(t)}{t} > y) = \frac{c}{1+c} + \frac{\sin \pi\beta}{\pi(1+c)} \int_0^x (1-u)^{\beta-1}(u+y)^{-\beta} du, \tag{4.9}$$

ii) If $\beta = 1$, *then for* $0 < x \le 1$ *and* $y > 0$

$$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\sigma(t))}{m_B(t)} \le x, \frac{m_B(\tau(t))}{m_B(t)} \le y) = \frac{c + \min(x, y)}{1+c}, \tag{4.10}$$

$$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\sigma(t))}{m_B(t)} \le x) = \frac{c + x}{1+c}, \tag{4.11}$$

$$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(\tau(t))}{m_B(t)} \le y) = \frac{c + \min(y, 1)}{1+c}. \tag{4.12}$$

P r o o f o f T h e o r e m 4.1. From (2.11) we have

$$\mathcal{P}(\frac{\sigma(t)}{t} \le x, \quad \frac{\tau(t)}{t} > y) | \sigma(t) \ge 0) = \frac{\sum_{0\le j\le tx} v_{t-j}(1 - B(j + ty))}{\mathcal{P}(\sigma(t) \ge 0)}. \tag{4.13}$$

i) Similarly to the proof of the case *i)* of Theorem 3.1, we obtain for the numerator in the right-hand side of (4.13) that as $t \to \infty$

$$\sum_{0\le j\le tx} v_{t-j}(1 - B(j + ty)) \sim v(t) m_B(t) \int_0^x (1-u)^{\alpha-1}(u+y)^{-\beta} du. \tag{4.14}$$

Now (3.2), (4.13) and (4.14) imply (4.1).

ii) Denote by $m_B^{-1}(.)$ the inverse function of $m_B(.)$. Since $m_B(t) \uparrow \infty$ as $t \to \infty$ and $m_B(.)$ is continuous monotone increasing, so is $m_B^{-1}(.)$. Further, (see [2, Lemma 10]) for every $0 < x < y$

$$m_B^{-1}(xm_B(t)) = o(m_B^{-1}(ym_B(t)), \quad t \to \infty. \tag{4.15}$$

Partiqularly, for $0 \le x < 1$

$$m_B^{-1}(xm_B(t)) = o(t), \quad t \to \infty. \tag{4.16}$$

Denote $q(t,x) = m_B^{-1}(xm_B(t))$ and consider the representation

$$\mathcal{P}(\frac{m_B(\sigma(t))}{m_B(t)} \le x, \frac{m_B(\tau(t))}{m_B(t)} \le y | \sigma(t) \ge 0)$$

$$= \frac{\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j)) - \sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j + q(t,y)))}{\mathcal{P}(\sigma(t) \ge 0)}, \tag{4.17}$$

where, as usual, [.] denotes the greatest integer function.

For the first sum in the numerator of the right hand side of (4.17), we have

$$\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j)) \le (\sup_{t-[q(t,x)] \le j \le t} v_j) \sum_{j=0}^{[q(t,x)]} (1 - B(j))$$
$$\le m_B([q(t,x)])(\sup_{t-[q(t,x)] \le j \le t} v_j).$$

Similarly,

$$\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j)) \ge m_B([q(t,x)])(\inf_{[t-q(t,x)] \le j \le t} v_j).$$

Since v_j is regularly varying and (4.16) holds, we have

$$(\inf_{t-[q(t,x)] \le j \le t} v_j) \sim (\sup_{t-[q(t,x)] \le j \le t} v_j) \sim v_t, \quad t \to \infty.$$

On the other hand, $m_B(x)$ is slowly varying at infinity and $m_B^{-1}(.)$ is monotone increasing. Thus,

$$m_B(q(t,x) - 1) \le m_B([q(t,x)]) \le m_B(q(t,x)) = xm_B(t),$$

which implies

$$m_B([q(t,x)]) \sim x m_B(t), \quad t \to \infty.$$

Hence,

$$\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j)) \sim v_t m_B(t) x, \quad t \to \infty. \tag{4.18}$$

Similarly, we estimate the second sum

$$\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j + q(t,y))) \sim v_t \Big(m_B(q(t,x) + q(t,y)) - m_B(q(t,y)) \Big), \quad t \to \infty.$$

Now using (4.15), one can see that

$$m_B(q(t,x) + q(t,y)) \sim \max(x,y) m_B(t), \quad t \to \infty.$$

Thus,

$$\sum_{j=0}^{[q(t,x)]} v_{t-j}(1 - B(j + q(t,y))) \sim v_t m_B(t)(\max(x,y) - y), \quad t \to \infty. \tag{4.19}$$

Finally, (4.17), (4.18), (4.19) and (3.3) prove (4.2). To prove (4.3) and (4.4), we let $x \to 1$ or $y \to \infty$, respectively.

iii) From (2.12), we have

$$\begin{aligned} &\mathcal{P}(\sigma(t) \le x, \tau(t) \le y | \sigma(t) \ge 0) \\ &= \left(\sum_{0 \le j \le x} v_{t-j}(1 - B(j)) - \sum_{0 \le j \le x} v_{t-j}(1 - B(j + y)) \right) / \mathcal{P}(\sigma(t) \ge 0). \end{aligned} \tag{4.20}$$

Since x is fixed and v_t is regularly varying at infinity, then $v_{t-j} \sim v_t$ when $t \to \infty$, uniformly on $0 \le j \le x$. Hence,

$$\sum_{0 \le j \le x} v_{t-j}(1 - B(j + y))) \sim v_t \sum_{0 \le j \le x} (1 - B(j + y))), \quad t \to \infty,$$

and

$$\sum_{0 \le j \le x} v_{t-j}(1 - B(j))) \sim v_t \sum_{0 \le j \le x} (1 - B(j))), \quad t \to \infty.$$

Now (3.4), (4.20) and the last two relations yield

$$\lim_{t \to \infty} \mathcal{P}(\sigma(t) \le x, \tau(t) \le y | \sigma(t) \ge 0) = \frac{1}{m_B(\infty)} (\sum_{0 \le j \le x} (1 - B(j)) - \sum_{0 \le j \le x} (1 - B(j + y))),$$

which is equivalent to (4.5).

Letting $y \to \infty$ in (4.5), we obtain (4.6).

To prove (4.7) one can write the right-hand side of (4.5), as follows

$$\frac{1}{m_B(\infty)} \sum_{0 \le j \le x} (B(j+y) - B(j))$$
$$= \frac{1}{m_B(\infty)} \{ \sum_{0 \le j \le x} (1 - B(j)) + \sum_{0 \le j \le y} (1 - B(j)) - \sum_{0 \le j \le x+y} (1 - B(j)) \}.$$

Letting $x \to \infty$ in the last relation and remembering that $\sum_{j=0}^{\infty}(1 - B(j)) = m_B(\infty) < \infty$, we prove (4.7).

The proof of (4.8) is evident by (2.6) and (3.4).

The theorem is proved. □

5. Limit distributions for the process $\boldsymbol{Z(t)}$. Let us remember that $Z_i(t)$ and $Z(t)$ are m-dimensional vectors with non-negative components. We denote the components by an upper index. So,

$$Z(t) = (Z^1(t), \dots, Z^m(t)), \quad Z_i(t) = (Z_i^1(t), \dots . Z_i^m(t)), \quad i = 1, 2, \dots .$$

The relations $Z(t) \le z, \quad Z(t) < z$ will be understood as

$$(Z^1(t) \le z^1, \dots, Z^m(t) \le z^m), \quad (Z^1(t) < z^1, \dots, Z^m(t) < z^m).$$

For some function $G : \mathbf{R} \to \mathbf{R}$ we denote

$$\frac{Z(t)}{G(t)} = \left(\frac{Z^1(t)}{G(t)}, \dots, \frac{Z^m(t)}{G(t)} \right), \quad G(Z(t)) = (G(Z^1(t)), \dots, G(Z^m(t))).$$

For a real number c and a vector $z = (z^1, \dots, z^m)$, we denote $z.c = (z^1 c, \dots, z^m c)$.

In this section, additionally to (1.7), (C.1) and (C.2), we suppose the following condition

(C.3) *There exists a proper m-dimensional distribution function $P(z)$, $z \in \mathbf{R}_+^m$, such that*

$$\lim_{t \to \infty} \mathcal{P}(Z_i(t)/M(t) \le z | T_i > t) = P(z),$$

where $M(.)$ is a monotone increasing, regularly varying function with exponent $\gamma \ge 0$.

After this introduction, we can formulate the main results of the section.

Theorem 5.1. *Let* (1.7), (C.1) *and* (C.3) *hold.*

i) If (1.11) *with* $1/2 < \beta < 1$ *holds, then for* $z = (z^1, \dots, z^m) \geq 0$

$$(5.1)\ \lim_{t\to\infty} \mathcal{P}(\frac{Z(t)}{M(t)} \leq z | Z(t) \neq 0) = \frac{1}{\mathrm{B}(\alpha, 1-\beta)} \int_0^1 P(z.y^{-\gamma})(1-y)^{\alpha-1} y^{-\beta} dy.$$

ii) If (1.11) *with* $\beta = 1$ *holds and the exponent of* $M(.)$, $\gamma > 0$, *then for* $0 < z = (z^1, \dots z^m) \leq 1$

$$(5.2)\qquad \lim_{t\to\infty} (\frac{m_B(M^{-1}(Z(t)))}{m_B(t)} \leq z | Z(t) \neq 0) = \min_{1\leq i\leq m} z^i,$$

where $M^{-1}(.)$ *denote the inverse function of* $M(.)$.

iii) If (1.10) *holds, then for* $z = (z^1, \dots z^m) \geq 0$

$$(5.3)\ \lim_{t\to\infty} \mathcal{P}(Z(t) \leq z | Z(t) \neq 0) = \frac{1}{m_B(\infty)} \sum_{k=0}^{\infty} \mathcal{P}(Z_1(k) \leq z | T_1 > k)(1 - B(k)).$$

Theorem 5.2. *Assume* (1.7), (C.2) *and* (C.3).

i) If $1/2 < \beta < 1$, *then for* $z = (z^1, \dots, z^m) \geq 0$

$$(5.4)\quad \lim_{t\to\infty} \mathcal{P}(\frac{Z(t)}{M(t)} \leq z) = \frac{c}{1+c} + \frac{\sin \pi\beta}{\pi(1+c)} \int_0^1 P(z.y^{-\gamma})(1-y)^{\beta-1} y^{-\beta} dy.$$

ii) If $\beta = 1$, *then*

$$(5.5)\qquad \lim_{t\to\infty} \mathcal{P}(\frac{m_B(M^{-1}(Z(t)))}{m_B(t)} \leq z) = \frac{c}{1+c} + \frac{1}{1+c}(\min_{1\leq i\leq m} z^i).$$

P r o o f o f T h e o r e m 5.1. Construct independent sequences Ξ and H (cf. Lemma 2.7), such that

$$\mathcal{P}(\xi_k \leq z) = \mathcal{P}(Z_1(k) \leq z | T_1 > k),$$

and

$$\mathcal{P}(\eta_t = k) = \begin{cases} \mathcal{P}(\sigma(t) = k | \sigma(t) \geq 0) & 0 \leq k \leq t, \\ 0 & k > t. \end{cases}$$

Now, for the sum in the right-hand sides of (2.14) and (2.15), we get (cf. (2.28))

$$(5.6)\qquad \begin{aligned} &\sum_{k=0}^{t} \mathcal{P}(Z_1(k) \leq z | T_1 > k) \mathcal{P}(\sigma(t) = k | \sigma(t) \geq 0) \\ &= \sum_{k=0}^{\infty} \mathcal{P}(\xi_k \leq z) \mathcal{P}(\eta_t = k) = \mathcal{P}(\xi_{\eta_t} \leq z). \end{aligned}$$

The rest of the proof is a simple application of the lemmas 2.7, 2.8 and 2.9, taking into account (C.3) and Theorem 4.1 *i*), *ii*) and *iii*), for the cases *i*), *ii*) and *iii*), respectively. □

The proof of Theorem 5.2 is similar.

6. Branching processes with state-dependent immigration. In this section we consider some applications of the results obtained above to the multitype Galton-Watson (MGW) branching processes with state-dependent immigration.

Let $Z_i(t) = (Z_i^1(t), \ldots, Z_i^m(t))$, $i = 1, 2, \ldots$, be independent, identically distributed MGW branching processes with m type of particles, starting with random number of ancestors at time $t = 0$, $Z_i(0) \neq 0$, a.s.

Define the life-period T_i by the event $\{Z_i(t) \neq 0,\ t = 0, 1, 2, \ldots T_i - 1,\ Z_i(T_i) = 0\}$ and let X_i be as in the previous sections.

Now, the process $Z(t)$ defined by (1.4) is a MGW branching process with m type of particles and immigration only in the state zero.

Denote,

$$s = (s^1, s^2, \ldots, s^m) \in [0,1]^{\times m}, \quad e_l = (\delta_l^1, \ldots, \delta_l^m),$$

$$g(s) = \mathcal{E}(s^{Z_i(0)}) = \mathcal{E}((s^1)^{Z_i^1(0)} \ldots (s^m)^{Z_i^m(0)}),$$

$$f^l(s) = \mathcal{E}(s^{Z_i(t)} | Z_i(t-1) = e_l), \quad f(s) = (f^1(s), \ldots, f^m(s)).$$

Assume,

$$g^p = \frac{\partial g(s)}{\partial s^p}|_{s=1} < \infty, \quad a_p^l = \frac{\partial f^l(s)}{\partial s^p}|_{s=1} < \infty, \quad l, p = 1, 2, \ldots, m,$$

and denote $g = (g^1, \ldots, g^m)$.

Let the matrix $A = \|a_p^l\|$ be irreducible and has the Perron root $\rho = 1$ with corresponding right and left eigenvectors $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m)$, such that $u > 0$, $v > 0$, $(u, 1) = 1$, $(u, v) = 1$, where $(.,.)$ is a scalar product, (i.e. MGW process is critical.)

For the second moments, we consider separately the cases when they are finite or infinite.

6.1. Finite second moments. Assume,

$$b_{pq}^l = \frac{\partial f^l(s)}{\partial s^p \partial s^q}|_{s=1} < \infty, \quad l, p, q = 1, 2, \ldots m,$$

and

$$b = \frac{1}{2} \sum_{l,p,q} v^l b^l_{pq} u^p u^q \in (0, \infty).$$

It is well-known, that under these conditions, for critical MGW processes the following results are true (see e.g. [6]):

$$1 - B(t) = \mathcal{P}(T_i > t) \sim \frac{(g, u)}{bt}, \quad t \to \infty, \tag{6.1}$$

and for $z = (z^1, \dots, z^m) \geq 0$

$$\lim_{t\to\infty} \mathcal{P}(\frac{Z_i(t)}{t} \leq z | T_i > t) = P(z), \tag{6.2}$$

where

$$P(z) = 1 - exp(-\min(\frac{bz^j}{v^j})).$$

So, $M(t) = t$ and from (6.1), it follows that

$$m_B(t) = \int_0^t \mathcal{P}(T_i > y)dy \sim \frac{(c, u)}{b} \log t, \quad t \to \infty.$$

If we suppose that

$$(1 - A(t))t \to \infty, \quad t \to \infty,$$

then the conditions of Theorem 5.1 *ii*) hold, and, we obtain for $0 < z \leq 1$

$$\lim_{t\to\infty} (\frac{\log Z(t)}{\log t} \leq z | Z(t) \neq 0) = \min(z^1, \dots, z^m).$$

On the other hand, if we assume $\mathcal{E}X_i < \infty$, then the conditions of Theorem 5.2 *ii*) are fulfilled, and, the limit distribution is an unconditional one , i.e.

$$\lim_{t\to\infty} (\frac{\log Z(t)}{\log t} \leq z) = \min(z^1, \dots, z^m).$$

6.2. Infinite second moments. Assume for $x > 0$

$$(v, 1 - f(1 - x.u)) = x - x^{1+\xi} L_1(x),$$

where $\xi \in (0, 1]$ and $L_1(.)$ is a function slowly varying at zero.

In this case, (see [14])

$$1 - B(t) = \mathcal{P}(T_i > t) \sim (g, u) t^{-\frac{1}{\xi}} L_B(t), \quad t \to \infty, \tag{6.3}$$

where $L_B(.)$ is a function slowly varying at infinity.

Further on, if we denote $M(t) = t^{\frac{1}{\xi}}/L_B(t)$, then for $z = (z^1, \ldots, z^m) \geq 0$

$$\lim_{t\to\infty} \mathcal{P}(\frac{Z_i(t)}{M(t)} \leq z | T_i > t) = P(z), \tag{6.4}$$

where $P(z)$ is a proper distribution on $\mathbf{R}_+^m$ with L.-S. transform

$$\int_{\mathbf{R}_+^m} e^{-(\lambda,z)} dP(z) = 1 - (v,\lambda)(1 + (v,\lambda)^\xi)^{-\frac{1}{\xi}}, \quad \lambda = (\lambda^1, \ldots, \lambda^m) \geq 0.$$

If $\xi < 1$ or $\xi = 1$ but $\mathcal{E}T_i = m_B(\infty) < \infty$, then from Theorem 5.1 *iii*), we obtain

$$\lim_{t\to\infty} \mathcal{P}(Z(t) \leq z | Z(t) \neq 0) = \frac{1}{m_B(\infty)} \sum_{k=0}^{\infty} \mathcal{P}(Z_1(k) \leq z | T_1 > k) \mathcal{P}(T_1 > k).$$

If $\xi = 1$ and $\mathcal{E}T_i = \infty$ then

$$m_B(t) = \int_0^t \mathcal{P}(T_i > y) dy \uparrow \infty, \quad t \to \infty,$$

and $m_B(t)$ is slowly varying at infinity.

If we denote by $M^{-1}(.)$ the inverse function of $M(.)$, from Theorem 5.1 *ii*) or from Theorem 5.2 *ii)* it follows that for $0 < z \leq 1$

$$\lim_{t\to\infty} \mathcal{P}(\frac{m_B(M^{-1}(Z(t)))}{m_B(t)} \leq z | Z(t) \neq 0) = \min(z^1, \ldots, z^m),$$

independently of α.

If, additionally, we assume $\mathcal{E}X_i < \infty$, then Theorem 5.2 *ii)* is valid with $c = 0$. Hence, the limit distribution is unconditional.

These results are new for the MGW branching processes with state-dependent immigration.

The analogical results in the single-type case are obtained by Kurbanov [7] using the methods of probability generating functions.

It is clear, that other limit theorems known for the critical MGW branching processes without immigration, can be transferred to the processes with state-dependent immigration in similar way.

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