Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **25** (1999), 5-10

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

ON A CONDITIONAL CAUCHY-TYPE FUNCTIONAL EQUATION INVOLVING POWERS

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Communicated by V. Kanev

Abstract. We solve the functional equation $f(x^m + y) = f(x)^m + f(y)$ in the realm of polynomials with integer coefficients.

Introduction. The aim of this paper is the study of the conditional functional equation

$$f(x^m + y) = f(x)^m + f(y)$$

for any fixed positive integer m, greater than 1, and where f is a map from the polynomial ring $Z[X]$ into itself. First we cover the case $m = 2$. This functional equation belongs to the general framework which appears in section 16.6 of [1].

1. The functional equation $f(x^2 + y) = f(x)^2 + f(y)$ with f defined over $Z[X]$. In this section we show that the functional equation

$$f(x^2 + y) = f(x)^2 + f(y) \tag{1}$$

for $f : Z[X] \to Z[X]$ entails the Cauchy functional equation, but is not equivalent. In fact it has less solutions.

1991 *Mathematics Subject Classification:* 39B52
Key words: functional equation

Lemma 1.1. *Any map $f : Z[X] \to Z[X]$ satisfying* (1) *sends zero into zero, and also satisfies*

$$f(x^2) = f(x)^2 \tag{2}$$

and

$$f\left(\sum_{i=1}^{r} \alpha_i x_i^2 + y\right) = \sum_{i=1}^{r} \alpha_i f(x_i)^2 + f(y) \tag{3}$$

where $\alpha_1, \ldots, \alpha_r$ are integers and r is arbitrary.

Proof. Setting $x = 0$ in (1), we get $f(0) = 0$, and setting then $y = 0$ in (1), we get (2).

We shall now show that

$$f(\alpha x^2 + y) = \alpha f(x)^2 + f(y) \tag{4}$$

holds for any nonzero α in Z and any x, y in $Z[X]$. From (4), by induction, (3) follows at once.

In order to establish (4), assume first α positive. Using (1) inductively, we may write

$$f(\alpha x^2 + y) = f(\underbrace{x^2 + \cdots + x^2}_{\alpha} + y) = \underbrace{f(x)^2 + \cdots + f(x)^2}_{\alpha} + f(y) = \alpha f(x)^2 + f(y)$$

which proves (4) for positive α.

Now, if $z = -x^2 + y$, i.e. $y = x^2 + z$, we see (by (1)) that $f(y) = f(x)^2 + f(z)$, i.e.

$$f(-x^2 + y) = f(z) = -f(x)^2 + f(y).$$

Proceeding inductively as before for positive α we now also obtain (4) for negative α and the lemma is proved. $\square$

Proposition 1.2. *If $f : Z[X] \to Z[X]$ satisfies functional equation* (1) *then f also satisfies the Cauchy functional equation $f(x+y) = f(x) + f(y)$. And in case that $f(1) = 0$, then f is necessarily the zero map.*

Proof. By (2), if f satisfies equation (1) then the image of 1 is necessarily 0 or 1. On the other hand, for any $x \in Z[X]$, $2x = (1+x)^2 - x^2 - 1$, and then, by (3),

$$f(2X) = f(1+x)^2 - f(x)^2 - f(1)^2 = \big(f(1)^2 + f(x)\big)^2 - f(x)^2 - f(1)^2. \tag{5}$$

When $f(1) = 0$, we obtain $f(2x) = 0$ for any $x \in Z[X]$, and so (by the preceding lemma) $0 = f(2x)^2 = f(2^2x^2) = 4f(x)^2$, therefore $f(x)^2 = 0$, for all x in $Z[X]$, so that f is the zero map.

When $f(1) = 1$, from (5), we obtain $f(2x) = 2f(x)$ for any $x \in Z[X]$ and now we may show that f satisfies the Cauchy functional equation:

$$\begin{aligned} 2f(x+y) &= f(2x+2y) = f\left((1+x)^2 - x^2 - 1 + 2y\right) \\ &= \left(f(1)^2 + f(x)\right)^2 - f(x)^2 - f(1)^2 + f(2y) \\ &= 1 + 2f(x) + f(x)^2 - f(x)^2 - 1 + f(2y) \\ &= 2f(x) + f(2y) = 2f(x) + 2f(y) = 2(f(x) + f(y)) \end{aligned}$$

Obviously, if $f : Z[X] \to Z[X]$ is not the zero map, then $f(1) = 1$ and f satisfies the Cauchy functional equation, so it is $Z-$linear and therefore it will be determined by the images of $X, X^2, X^3, \ldots$, so that

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_01 + a_1f(X) + a_2f(X^2) + \cdots + a_nf(X^n)$$

However, if f satisfies (1), the images of $X, X^2, X^3, \ldots$ are related as the next lemma shows.

Lemma 1.3. *If $f : Z[X] \to Z[X]$ is a solution of* (1) *with* $f(1) = 1$ *then for any n,*

$$f(X^n) = f(X)^n.$$

P r o o f. From (2), $f(x^2) = f(x)^2$ for all $x \in Z[X]$, in particular for $x = X$. We proceed by induction on n, but have to distinguish the case n even (trivial) from n odd where we write $n = 2k+1$ and use $Z-$linearity (proved in Proposition 1.2) together with the expansions of $f\left((X^{2k} + X)^2\right) = f\left(X^{2k} + X\right)^2$. □

Thus, if f is not the trivial solution, then

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_01 + a_1u + a_2u^2 + \cdots + a_nu^n \tag{6}$$

where $u \in Z[X]$ may be chosen arbitrarily and since all maps given by (6) satisfy equation (1), we can assert the following.

Theorem 1.4. *A map $f : Z[X] \to Z[X]$ is a solution of the functional equation* (1) *if only if f is either the zero map, or, for any polynomial $P(X)$ of $Z[X]$,*

$$f(P(X)) = P(u)$$

where $u \in Z[X]$ may be chosen arbitrarily.

So, functional equation (1) has less solutions than the Cauchy equation, because in the latter it is possible to select the images of $X, X^2, X^3, \ldots$, arbitrarily.

2. The functional equation $f(x^m + y) = f(x)^m + f(y)$ for f defined over $Z[X]$. Here we show that the functional equation

$$f(x^m + y) = f(x)^m + f(y) \tag{7}$$

for any fixed m ($m > 2$) is equivalent to functional equation (1) when m is even, but has more solutions when m is odd.

To begin with, remark that a similar argument to that of Lemma 1.1 would now prove the following:

Lemma 2.1. *If $f : Z[X] \to Z[X]$ satisfies functional equation (7), for any m fixed ($m \geq 2$), then f sends zero into zero, and moreover, f satisfies:*

$$f(x^m) = f(x)^m \tag{8}$$

for all x in $Z[X]$, and

$$f\left(\sum_{i=1}^{r} \alpha_i x_i^m + y\right) = \sum_{i=1}^{r} \alpha_i f(x_i)^m + f(y) \tag{9}$$

where $\alpha_1, \ldots, \alpha_r$ are integers and r is arbitrary.

Observe that by (8), $f(1) = f(1)^m$, so that the image of 1 may be 0, 1 or even -1 when m is odd.

Proposition 2.2. *With the preceding hypotheses,in accordance with $f(1)$ being $0, 1$ or -1, we have that $f(\alpha + y)$ is either $f(y)$, $\alpha + f(y)$ or $-\alpha + f(y)$, respectively.*

Proof. Immediate, setting $r = 1$ and $x_1 = 1$ in (9). □

Remark that by setting $y = 0$ in the preceding proposition, we get that the image of α (for any integer α) is either 0, α, or $-\alpha$, depending on $f(1)$ being $0, 1$, or -1, respectively.

Now we shall show that the solutions of (7) satisfy the Cauchy functional equation, but before we need the following lemma.

Lemma 2.3. *For any integer $m \geq 1$, there are integers $\alpha, \delta, \alpha_0, \ldots, \alpha_{m-1}$, with $\alpha \neq 0$ and $\sum_{r=0}^{m-1} \alpha_r = 0$ such that,*

$$\alpha T + \delta = \sum_{r=0}^{m-1} \alpha_r (T + r)^m,$$

where T is an indeterminate. As a consequence, for any x in $Z[X]$, the following relation holds:

$$\alpha x + \delta = \sum_{r=0}^{m-1} \alpha_r (x + r)^m.$$

Proof. If g is a polynomial in the indeterminate T and $\Delta g(T) :=$

$g(T+1)-g(T)$, then iterating the operator Δ, we get easily by induction

$$\Delta^s g(T) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} g(T+s-i)$$

Using this result and directly computing $\Delta^s g(T)$ with $s = m-1$ and $g(T) = T^m$, we get (cf [2, Theorem 402]):

$$m!T + \delta = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (T+m-1-i)^m$$

which proves the lemma, because $\sum_{r=0}^{m-1} \alpha_r = \sum_{r=0}^{m-1} (-1)^{m-1-r} \binom{m-1}{r} = 0.$ $\square$

Using the preceding propositions and proceeding as in Proposition 1.2, (but working with αx instead of $2x$), we get

Proposition 2.4. *If $f : Z[X] \to Z[X]$ satisfies* (7)*, with m fixed ($m \geq 2$), then f also satisfies the Cauchy functional equation. And in the particular case that $f(1) = 0$, then f is necessarily the zero map.*

Therefore, if $f : Z[X] \to Z[X]$ is a non-trivial solution of (7), then

$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_0f(1) + a_1f(X) + a_2f(X^2) + \cdots + a_nf(X^n).$$

We next prove that the image $f(X)$ of X already determines the images of $X^2, X^3, \ldots$, but first we need the following result.

Lemma 2.5. *For any integer $m \geq 2$, there are integers $\beta, \gamma, \beta_0, \ldots, \beta_{m-1}$, with $\beta \neq 0$ such that for any $x \in Z[X]$, we have*

$$\beta x^2 + \gamma = \sum_{r=0}^{m-1} \beta_r (x+r)^m.$$

P r o o f. Similar that of Lemma 2.3, by setting $s = m-2$ and $g(T) = T^m$. $\square$

Proposition 2.6. *Let $f : Z[X] \to Z[X]$ be a solution of* (7) *for a fixed m, ($m > 2$). If $f(1) = 1$ then f also satisfies*

$$f(x^2 + y) = f(x)^2 + f(y)$$

and if $f(1) = -1$ then f satisfies

$$f(x^2 + y) = -f(x)^2 + f(y) \tag{10}$$

P r o o f. Analogous to the proof of Proposition 1.2, but working now with the expression βx^2 instead of $2x$. $\square$

Therefore, when $f(1) = 1$ and $u := f(X)$, we get
$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = a_0 + a_1u + a_2u^2 + \cdots + a_nu^n$$
and when $f(1) = -1$, we obtain
$$f(a_0 + a_1X + a_2X^2 + \cdots + a_nX^n) = -a_0 + a_1u - a_2u^2 + \cdots + (-1)^{n+1}a_nu^n$$
since (proceeding as in Proposition 1.3) when f satisfies (10), we obtain
$$f(X^n) = (-1)^{n+1}f(X)^n.$$
And as all these maps obviously satisfy the functional equation (7), for any $u \in Z[X]$, we finally get

Theorem 2.7. *The only maps $f : Z[X] \to Z[X]$ satisfying the functional equation*
$$f(x^m + y) = f(x)^m + f(y)$$
for a fixed m ($m \geq 2$), are the zero map and the maps given by
$$f(P(X)) = P(u)$$
or
$$f(P(X)) = -P(v)$$
(the latter occurring only in case m is odd) where $u = f(X)$ and $v = -f(X)$ may be chosen arbitrarily in $Z[X]$.

Acknowledgment. The author is grateful to professor J-L. García-Roig (Barcelona) for his suggestions and encouragement during the preparation of this paper.

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Received November 13, 1996