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THE SPACE OF DIFFERENCES OF CONVEX FUNCTIONS ON $[0, 1]$

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Abstract. The space $K[0,1]$ of differences of convex functions on the closed interval [0, 1] is investigated as a dual Banach space. It is proved that a continuous function f on [0, 1] belongs to $K[0,1]$ if, and only if,

$$(*) \qquad \infty > \|f\| = |f(0)| + |f(1)| + 2\sup_n \Big\{ \sum_{i=1}^{2^{n-1}-1} i|f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n})| + \sum_{i=2^{n-1}}^{2^n-1} (2^n - i)|f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n})| \Big\}.$$

Under the norm $\| \ \|$, $K[0,1]$ has a predual isometric to $C(F)$, the space of continuous functions on $F = \{-1\} \cup [0,1] \cup \{2\}$. The isometry between the $L_1(\mu)$-space $C(F)^*$ and $K[0,1]$ maps the positive cone of $L_1(\mu)$ onto the set of all non positive convex functions on [0, 1].

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1. Introduction. The space $BVN[0,1]$ of differences $f = g - h$ of monotonely increasing functions g and h on [0, 1] has been thoroughly investigated as a Banach space under the norm of the total variation. It is well-known that $BVN[0,1]$ is isometric to the dual space of $C[0,1]$, the space of continuous functions; and, by Kakutani's axiomatic characterization of L-spaces [1], it is an $L_1(\mu)$ space. Less familiar is the space $K[0,1]$ of differences $f = g - h$ of convex functions on the closed interval [0, 1]. The purpose of this paper is to investigate $K[0,1]$ as a dual Banach space under certain natural norms. The space $BVN[0,1]$ draws attention mainly because of the relation $BVN[0,1] = C[0,1]^*$ and the fact that its natural norm is the precise dual norm. Analogously we wish to construct a space $\mathcal{A}(S)$ of affine continuous functions on a metrizable Choquet simplex S such that $K[0,1] = \mathcal{A}(S)^*$. This relation suggests a "natural" norm on $K[0,1]$, namely, the dual norm, which happens to be

$$(*)\qquad |f| = |f(0)| + |f(1)| + 2\sup_n \left\{ \sum_{i=1}^{2^{n-1}-1} i \left| f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n}) \right| + \sum_{i=2^{n-1}}^{2^n-1} (2^n - i) \left| f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n}) \right| \right\}.$$

As the dual of $\mathcal{A}(S)$, the space $K[0,1]$ is known to be an $L_1(\mu)$ space (see e.g. [2]). It turns out that the natural isometry of $L_1(\mu)$ onto $K[0,1]$ maps the positive cone onto the cone Π of non-positive convex functions on [0, 1]. The cone Π determines a natural lattice structure on $K[0,1]$. Since any convex function $\hat{g}$ on $[0,1]$ is continuous in the open interval, $\hat{g}$ can be represented as a linear combination $\hat{g} = \alpha_0 W_0 + \alpha_1 W_1 + g$, where g is a continuous convex function on $[0,1]$ while W_0 and W_1 are the (convex) indicator functions of the singleton subsets $\{0\}$ and $\{1\}$ of $[0,1]$, respectively. Denoting by $K^C[0,1]$ the space of differences of continuous convex functions on $[0,1]$ we have that $K[0,1] = K^C[0,1] + [W_0, W_1]$.

At this point we do not know what the "natural" norm on $K[0,1]$ should be. But we certainly wish the norm to express the nature of the space elements, namely, differences of convex functions. A reasonable choice of a temporary norm for $K[0,1]$ is

$$(1.1)\qquad |||f||| = \inf\{\|g\|_\infty + \|h\|_\infty : f = g - h, \quad \text{where } g \text{ and } h \text{ are convex on } [0,1]\}$$

with $\|\ \|_\infty$ denoting the sup norm. However, an equivalent but slightly different norm is easier to work with.

Let M denote the convex cone of non negative convex functions on [0, 1]. It will somewhat simplify our computations in the sequel if we norm $K[0,1]$ by

$$\|f\| = \inf\{\|g\|_\infty + \|h\|_\infty : f = g - h \quad \text{where } g, h \in M\} \tag{1.2}$$

instead of (1.1). Clearly, these two norms are equivalent with $|||f||| \leq \|f\| \leq 3|||f|||$ for all $f \in K[0,1]$. At this point we do not know that the normed space $K[0,1]$ is complete. Completeness will be established in Section 2. For each $0 \leq t \leq 1$ let $\delta(t)$ denote the point evaluation functional at t. Let Δ denote the norm closed linear span of $\{\delta(t) : 0 \leq t \leq 1\}$ in $K[0,1]^*$ and put $\Gamma = [\delta(t) : 0 < t < 1]$.

In Section 2 we prove that, under the norm $\| \ \|$, the spaces $K^C[0,1]$ and $K[0,1]$ are isometric to the spaces Γ^* and Δ^*, respectively. Sections 3 and 4 are devoted to the study of certain pavements of finite dimensional subspaces of $K[0,1]$ and Δ. In Section 5 we construct the "simplex space" $\mathcal{A}(S)$ which is isomorphic to Δ and induces the natural structure of $K[0,1]$. In Section 6 we investigate the simplex S itself and show that its extreme boundary ∂S (= the set of extreme points of S) is homeomorphic to the space $F = [0,1] \cup \{-1\} \cup \{2\}$. It is concluded that $K[0,1]$ is isometric to $C(F)^*$, the space of regular Borel measures on F. Let $G(0,0) = 0 = G(1,1), G(x,-1) = x - 1$ and $G(x,2) = -x$ for all $0 \leq x \leq 1$ and let

$$G(x,t) = \begin{cases} x - 1 & \text{if either } 0 \leq t < x \leq \frac{1}{2} \quad \text{or } 0 \leq t \leq \frac{1}{2} \leq x \leq 1 \\ -x & \text{if either } 0 \leq x \leq \frac{1}{2} \leq t \leq 1 \quad \text{or } \frac{1}{2} \leq x < t \leq 1 \\ (1 - t^{-1})x & \text{if } 0 \leq x \leq t \leq \frac{1}{2} \\ (1 - t)^{-1}t(x - 1) & \text{if } \frac{1}{2} \leq t \leq x \leq 1. \end{cases} \tag{1.3}$$

We show that the map $\tau : C(F)^* \to K[0,1]$, defined for every $0 \leq x \leq 1$ by

$$\tau(\mu)(x) = \int_F G(x,t) d\mu(t) \tag{1.4}$$

is a surjective isometry which maps the positive cone of $C(F)^*$ onto the cone of non positive convex functions on [0, 1]. In Section 7 we present an algorithm which constructs a unique pair of non positive convex functions g and h for a given $f \in K[0,1]$, satisfying $f = g - h$ and $\|f\| = \|g\| + \|h\|$.

The space of differences of convex functions with *bounded one side derivatives* at the boundary has been studied in the literature (see. e.g., [5, 4]). This space is denoted by $BC[0,1]$ and its norm, as suggested by F. Riesz's paper, is the following

$$(**)\qquad \|f\| = |f(0)| + |f'_r(0)| + \sup_P\left\{\sum_{i=1}^{n-1}\left|\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i} - \frac{f(x_i)-f(x_{i-1})}{x_i-x_{i-1}}\right| : P=\{x_i\}_{i=0}^n \quad \text{a partition of } [0,1]\right\}.$$

However, $BC[0,1]$ does not include important classes of convex functions. Our main tools are taken from the theory of $L_1(\mu)$ preduals. For information about these spaces the reader is referred to [3], [2] and [6].

Notation. We use standard Banach space notation as can be found in [3]. In particular, if $M = \{x_\alpha\}_{\alpha\in A}$ is a subset of a Banach space X then $[M]$ and $[x_\alpha]_{\alpha\in A}$ denote the norm closed linear span of $\{x_\alpha\}_{\alpha\in A}$. Ball (X) denotes the closed unit ball of X. The Banach-Mazur distance between the isomorphic spaces X and Y is denoted by $d(X,Y)$. We consider only real Banach spaces.

2. The dual space of Δ. The facts described in this section depend on the special properties of the classical Schauder basis of the space $C[0,1]$. Let $\{f_j\}_{j=0}^\infty$ denote the Schauder basis and let $\{\psi_j\}_{j=0}^\infty$ be the corresponding biorthogonal functionals.

Recall that $f_0 \equiv 1$, $f_1(t) = t$, $\psi_0 = \delta(0), \psi_1 = \delta(1) - \delta(0)$, for $n \geq 0$ and $1 \leq i \leq 2^n$, $f_{2^n+i}(t) = 0$ at the points 0, $(2i-2)2^{-n-1}, (2i)2^{-n-1}$ and 1; $f_{2^n+i}((2i-1)2^{-n-1}) = 1$ and f_{2^n+i} is linear in between. Also,

$$\psi_{2^n+i} = \delta((2i-1)2^{-n-1}) - \frac{1}{2}\delta((2i-2)2^{-n-1}) - \frac{1}{2}\delta((2i)2^{-n-1}).$$

It is known that $\{f_j\}_{j=0}^\infty$ is a monotone basis of $C[0,1]$, i.e., if Q_j denotes the natural basis projection of $C[0,1]$ onto $[f_i]_{i=0}^j$ (that is, $Q_j(\sum_{i=1}^\infty a_i f_i) = \sum_{i=1}^j a_i f_i$) then $\|Q_j\|_\infty = 1$ for all $0 \leq j < \infty$. It is also known that, for each $j = 2^m + i$ and $f \in C[0,1], Q_j(f)$ is the unique continuous, piecewise linear function on [0, 1] which agrees with f at each of the points $0, 1, \frac{1}{2}, \ldots, 2^{-m}, 3\cdot 2^{-m}, \ldots, (2^m-1)2^{-m}, 2^{-m-1}, \ldots, (2i-1)2^{-m-1}$ and is linear in between. It follows that

(2.1) For each $j \geq 1, Q_j(f)$ is nonnegative if f is and $Q_j(f)$ is convex if f is convex.

It is easy to check that each f_j is in $K[0,1]$, however, the sequence $\{f_j\}_{j=0}^{\infty}$ does not span the whole space $K[0,1]$ as this space is non separable. Let $K_0=[f_j]_{j=0}^{\infty}\cap K[0,1]$.

Lemma 2.1. *For each $j\geq 0$ Q_j is a projection with norm 1 on $K^C[0,1]$. The Schauder functions $\{f_j\}_{j=0}^{\infty}$ form a monotone basis of K_0.*

Proof. Let $f\in K^C[0,1]$ and assume that $\|f\|=1$. Hence, given $\varepsilon>0$, there exist non negative continuous convex functions g and h such that $\|g\|_\infty+\|h\|_\infty<1+\varepsilon$ and $f=g-h$. By (2.1), for each j, Q_jg and Q_jh are non negative continuous convex functions with $\|Q_jg\|_\infty+\|Q_jh\|_\infty\leq\|g\|_\infty+\|h\|_\infty<1+\varepsilon$. Since $Q_jf=Q_jg-Q_jh$ we get that $\|Q_jf\|\leq 1$. □

Let us now discuss the properties of the subspace $\Delta_0=[\psi_j]_{j=0}^{\infty}$ of $K[0,1]^*$. It is easy to see that, for each $k\geq 1$, the subspace $\Delta_k=[\psi_j]_{j=0}^{2^k}=[\delta(j2^{-k})]_{j=0}^{2^k}$ hence $\Delta_0\subset\Delta$. In fact $\Delta_0=\Delta$ because of the following

Example 2.2. *For every* $0<s<t<1$

$$\|\delta(s)-\delta(t)\|=|s-t|\max\{t^{-1},(1-s)^{-1}\}. \tag{2.2}$$

Proof. Suppose that $g\in M\cap\mathrm{Ball}(K[0,1])$ then

$$|(\delta(s)-\delta(t))(g)|=|g(s)-g(t)|=|a|\ |s-t|,$$

where a is the slope of the chord joining $(s,g(s))$ and $(t,g(t))$. Since g is convex, we have that

$$t^{-1}(g(t)-g(0))\leq a\leq(1-s)^{-1}(g(1)-g(s))$$

and therefore, because g is non negative,

$$\begin{aligned}|(\delta(s)-\delta(t))(g)| &\leq |s-t|\max\{t^{-1}|g(0)-g(t)|,(1-s)^{-1}|g(1)-g(s)|\}\\ &\leq |s-t|\max\{t^{-1},(1-s)^{-1}\}.\end{aligned}$$

On the other hand, if $0\leq s<t\leq 1$ and $g(u)=\max\{1-t^{-1}u,\ 0\}$ for $0\leq u\leq 1$ then

$$g\in M\cap\mathrm{Ball}\ K[0,1]\quad\text{and } |(\delta(s)-\delta(t))(g)|=|g(s)-g(t)|=1-t^{-1}s=t^{-1}|s-t|.$$

Let $h(u)=\max\{(1-s)^{-1}(u-1)+1,\ 0\}$ then $h\in M\cap\mathrm{Ball}(K[0,1])$ and $|(\delta(s)-\delta(t))(h)|=|h(s)-h(t)|=|s-t|(1-s)^{-1}$. This proves (2.2). □

It follows from Lemma 2.1 that, for every $f \in K_0$ and $\varphi \in \Delta$,

(2.3) $$\|f\| = \sup\{\langle\psi, f\rangle : \psi \in \mathrm{Ball}(\Delta)\} \text{ and } \|\varphi\| = \sup\{\langle\varphi, g\rangle : g \in \mathrm{Ball}(K_0)\}.$$

Let $J : K[0,1] \to \Delta^*$ denote the natural embedding defined by $(Jf)(\delta(t)) = f(t)$. Then, by (2.3), $J|_{K_0}$ is an isometric embedding. Put $\Gamma = [\delta(t)]_{0<t<1}$. We will now prove

Proposition 2.3. (a) *The map $J : K[0,1] \to \Delta^*$ is an isometric isomorphism onto Δ^* hence $K[0,1]$ is complete. Moreover, there is projection Q of Δ^* onto $J(K^C[0,1])$ with $\|Q\| = 1$ and Kernel $(Q) = [\delta(t)]^{\perp}_{0<t<1} = \Gamma^{\perp}$.*
(b) $K^C[0,1]$*is isometric to* Γ^*.

Proof. The standard separation theorem shows that, by (2.3), Ball$(J(K_0))$ is ω^* dense in Ball (Δ^*). Put $J(K_0) = \hat{K}_0$ and let us denote $J(f)$ by $\hat{f}$ for every $f \in K[0,1]$. Let $x^* \in \Delta^*$ and assume that $\|x^*\| = 1$. Then there exists a sequence $\{\hat{e}_n\} \subset \hat{K}_0$ with $\|e_n\| = \|\hat{e}_n\| < 1$ so that each e_n is a finite linear combination of $\{f_j\}$ and $x^* = \omega^* \lim \hat{e}_n$. By the definition of $\|\ \ \|$, for each $n \geq 1$ there exist g_n and $h_n \in M \cap \mathrm{Ball}(K_0)$ so that $e_n = g_n - h_n$ and $\|g_n\|_\infty + \|h_n\|_\infty \leq 1$. By passing to a subsequence, we may assume the existence of $\omega^* \lim \hat{g}_n = g^*$ and $\omega^* \lim \hat{h}_n = h^*$. Clearly, $g^* - h^* = x^*$ and $\|g^*\| + \|h^*\| \leq 1$. Let $g_0(t) = g^*(\delta(t))$ and $h_0(t) = h^*(\delta(t))$ for all $0 \leq t \leq 1$. As pointwise limits of convex functions, g_0 and h_0 are convex. Therefore $x^* = \hat{g}_0 - \hat{h}_0 \in J(K[0,1])$ and $\|g_0\|_\infty + \|h_0\|_\infty \leq 1$. Hence $\|g_0 - h_0\| \leq 1$ and $x^* = J(g_0 - h_0)$. Let $f \in K[0,1]$ and assume that $\|J(f)\| = 1$. Put $x^* = J(f)$ then the above argument shows that $\|J(f)\| = \|f\|$. We have thus proved (a) and the completeness of $K[0,1]$.

Let us now correct the discontinuities of g_0 and h_0 at the end of points by defining $g(t) = g_0(t)$ and $h(t) = h_0(t)$ for $0 < t < 1$, $g(0) = \lim_{t\to 0+} g(t)$, $g(1) = \lim_{t\to 1-} g(t)$, $h(0) = \lim_{t\to 0+} h(t)$ and $h(1) = \lim_{t\to 1-} h(t)$. Then g and h are non negative continuous convex functions on [0, 1] and $\|g\|_\infty + \|h\|_\infty \leq 1$. Put $f_0 = g - h$ then $f_0 \in K^C[0,1]$ and

(2.4) $$x^*(\delta(t)) = f_0(t) \quad \text{for all } 0 < t < 1.$$

It follows that the map $Q : \Delta^* \to J(K^C[0,1])$ defined by $Qx^* = f_0$ is a projection with $\|Q\| = 1$ and Kernel $(Q) = [\delta(t)]^{\perp}_{0<t<1} = \Gamma^{\perp}$. Indeed, (2.4) ensures that $Q(x^*)$ is well defined (independently of the sequences $\{e_n\}, \{g_n\}$ and $\{h_n\}$) and the inequality $\|g\|_\infty + \|h\|_\infty \leq 1$ implies that $\|Q\| = 1$. To determine Γ^*, note that $\Gamma \subset \Delta$, hence Γ^* is naturally isometric to $\Delta^*/\Gamma^{\perp} = \Delta^*/(\mathrm{kernel}Q) \cong Q(\Delta^*) = J(K^C[0,1])$. This middle isomorphism is, in fact, an isometry because, as is easily checked, for every $x^* \in \Delta^*$, $\inf\{\|x^* - w\| : w = \alpha_0 W_0 + \alpha_1 W_1\} = \|Qx^*\|$. This proves Proposition 2.3. □

Remark 2.4. It follows from (2.2) that $\Delta = \text{span}\{\delta(0),\ \delta(1), \Gamma\}$. Hence $\Delta^* = \text{span}\{W_0, W_1, K^C[0,1]\}$ where, as mentioned above, W_0 and W_1 can be identified with the convex (discontinuous) indicator functions $W_0 = 1_{\{0\}}$ and $W_1 = 1_{\{1\}}$ of the singleton subsets $\{0\}$ and $\{1\}$ of [0, 1], respectively.

3. The structure of $\text{Ball}([f_j]_{j=0}^{2^n})$. Consider the subspace $K_n \overset{\text{def}}{=} Q_{2^n}(K_0) = \text{span}\{f_j\}_{j=0}^{2^n}$ in $K[0,1]$. In this section we investigate the structure of $\text{Ball}(K_n)$ as a convex set. We start with the identification of the extreme points of the convex set

$$A_n = \{f \in M \cap \text{Ball}(K_n) : f(1) = 0\}.$$

Lemma 3.1. *The extreme points of A_n are the functions $\{g_i^n\}_{i=1}^{2^n}$ where $g_i^n(t) = 1 - 2^n i^{-1} t$ if $0 \le t \le i2^{-n}$ and $g_i^n(t) = 0$ otherwise.*

Proof. It is easy to see that each g_i^n is an extreme point of A_n. Let us show that $A_n = \text{conv}\{g_i^n\}_{i=1}^{2^n}$. Pick $g \in A_n$ and assume that $g(0) = 1$. We will show that there exist $\alpha_i \ge 0 (1 \le i \le 2^n)$ with $\sum_{i=1}^{2^n} \alpha_i = 1$ so that

$$g = \sum_{i=1}^{2^n} \alpha_i g_i^n. \tag{3.1}$$

Since g and g_i^n are linear in each interval $[(j-1)2^{-n}, j2^{-n}]$, the problem is that of solving the following system of 2^n linear equations in 2^n variables $\{\alpha_i\}_{i=1}^{2^n}$:

$$\begin{cases} \sum_{i=1}^{2^n} \alpha_i = 1 \\ \sum_{i=1}^{2^n} \alpha_i g_i^n(j2^{-n}) = g(j2^{-n}) \quad 1 \le j \le 2^n - 1. \end{cases} \tag{3.2}$$

But $g_i^n(\frac{j}{2^n}) = 0$ if $j \ge i$, therefore (3.2) reduces to the triangular system

$$\begin{cases} \sum_{i=1}^{2^n} \alpha_i = 1 \\ \sum_{i=j+1}^{2^n} \alpha_i (1 - ji^{-1}) = g(j2^{-n}) \quad 1 \le j \le 2^n - 1. \end{cases} \tag{3.3}$$

Let us first agree that $g(t)$ denotes 0 whenever $t \geq 1$. Starting with the last equation we get $\alpha_{2^n} = 2^n g((2^n-1)2^{-n})$ and going down one easily proves by induction that for every $0 \leq k < 2^n$

$$\alpha_{2^n-k} = (2^n-k)[g((2^n-(k+1))2^{-n}) - 2g((2^n-k)2^{-n}) + g((2^n-k+1)2^{-n})] \tag{3.4}$$

Note that $\alpha_i \geq 0$ because g is convex. With these values of α_i,

$$\begin{aligned}\sum_{k=0}^{2^n-1} \alpha_{2^n-k} &= 2^n g((2^n-1)2^{-n}) + (2^n-1)[g((2^n-2)2^{-n}) - 2g(2^n-1)2^{-n})]+\\ &+(2^n-2)[g(2^n-3)2^{-n}) - 2g((2^n-2)2^{-n}) + g((2^n-1)2^{-n})]+\\ &+(2^n-3)[g((2^n-4)2^{-n}) - 2g((2^n-3)2^{-n}) + g((2^n-2)2^{-n})]+\\ &+\ldots+\\ &+3[g(2\cdot 2^{-n}) - 2g(3\cdot 2^{-n}) + g(4\cdot 2^{-n})]+\\ &+2[g(2^{-n}) - 2g(2\cdot 2^{-n}) + g(3\cdot 2^{-n})+\\ &+[g(0) - 2g(2^{-n}) + g(2\cdot 2^{-n})] =\\ &= g((2^n-1)2^{-n})[2^n - 2(2^n-1) + (2^n-2)]+\\ &+g((2^n-2)2^{-n})[2^n-1-2(2^n-2) + (2^n-3)]+\\ &+g((2^n-3)2^{-n})[2^n-2-2(2^n-3) + (2^n-4)] + \cdots +\\ &+g(2\cdot 2^{-n})[3-4+1]+\\ &+g(2^{-n})[2-2]+\\ &+g(0) = g(0) = 1\end{aligned}$$

This proves Lemma 3.1. □

The functions $g_i^n(t) = \max\{1 - 2^n i^{-1} t, 0\} \quad (1 \leq i \leq 2^n)$ and their symmetric images with respect to $\frac{1}{2}$,

$$\tilde{g}_i^n(t) = g_{2^n-i}^n(1-t) = \max\{(2^n-i)^{-1} 2^n (t-1) + 1, 0\} \quad (0 \leq i \leq 2^n - 1)$$

play an important role in the following computations.

Corollary 3.2. *For every $n \geq 1$*

$$M \cap \mathrm{Ball}(K_n) \subset 2\mathrm{conv}\{\{g_i^n\}_{i=1}^{2^n},\ \{\tilde{g}_i^n\}_{i=0}^{2^n-1}\} \tag{3.5}$$

and

$$\mathrm{Ball}(K_n) \subset 2\mathrm{conv}\{\{\pm g_i^n\}_{i=1}^{2^n},\ \{\pm\tilde{g}_i^n\}_{i=0}^{2^n-1}\}. \tag{3.6}$$

P r o o f. First note that the function $1 = g_{2^n}^n + \tilde{g}_0^n$. Let $f \in M \cap \mathrm{Ball}(K_n)$ and assume that $\min\{f(t) : 0 \leq t \leq 1\} = \beta = f(t_0)$. Then $\|f - \beta 1\| = \|f\| - \beta$ and

$f = \beta 1 + g + h$ where $g(t) = f(t) - \beta$ if $0 \le t \le t_0$, $g(t) = 0$ if $t_0 \le t \le 1$, $h(t) = f(t) - \beta$ if $t_0 \le t \le 1$ and $h(t) = 0$ if $0 \le t \le t_0$. Clearly, $|\beta| + \|g + h\| \le 1$. if g is not identically 0 then $g(0) = \|g\| \ne 0$ and $g(0)^{-1}g$ is a member of the set A_n defined in Lemma 3.1. If $h \ne 0$ then $h(1) = \|h\| \ne 0$ and the function $h(1)^{-1}h$ is a symmetric image of a member of A_n with respect to the point $t = \frac{1}{2}$. By Lemma 3.1, $g(0)^{-1}g$ is a convex combination of $\{g_i^n\}_{i=1}^{2^n}$ and $h(1)^{-1}h$ is a convex combination of $\{\tilde{g}_i^n\}_{i=0}^{2^n-1}$ hence $f = \beta 1 + g(0)\sum_{i=1}^{2^n} \alpha_i g_i^n + h(1)\sum_{i=0}^{2^n-1} \beta_i \tilde{g}_i^n$ with $\beta \ge 0$, $\alpha_i \ge 0$, $\beta_i \ge 0$ and $\beta + \sum_{i=1}^{2^n} g(0)\alpha_i + \sum_{i=0}^{2^n-1} h(1)\beta_i = \beta + g(0) + h(1) \le 2$. This proves (3.5).

By the definition of $\| \;\; \|$, each $f \in \mathrm{Ball}(K_n)$ is a convex combination of some f_1 and $-f_2$ with $f_1, f_2 \in M \cap \mathrm{Ball}(K_n)$ therefore, (3.5) implies (3.6).

This proves Corollary 3.2. □

4. The structure of Δ_n. Recall that, by (2.3), the subspace $\Delta_n = [\psi_j]_{j=0}^{2^n}$ is isometric to the dual of K_n (under the norm $\| \;\; \|$). For each $n \ge 1$ and $1 \le i \le 2^n - 1$ let $\tilde{w}_i^n = \delta(i2^{-n}) - \frac{1}{2}\delta((i-1)2^{-n}) - \frac{1}{2}\delta((i+1)2^{-n})$. Put $\tilde{w}_0^n = \delta(0)$ and $\tilde{w}_{2^n}^n = \delta(1)$ then, as is easily checked $\Delta_n = [\tilde{w}_i^n]_{i=0}^{2^n}$.

Proposition 4.1. *For every* $n \ge 1$,

$$d(\Delta_n, \ell_\infty^{2^n+1}) \le 8.$$

Proof. We wish to estimate $\left\| \sum_{i=0}^{2^n} a_i \tilde{w}_i^n \right\|$ from above and from below. This is done in two steps. We start with the computation of $\langle \tilde{w}_i^n, g_j^n \rangle, \langle \tilde{w}_i^n, \tilde{g}_j^n \rangle$ and $\|\tilde{w}_i^n\|$.

Lemma 4.2. *For all* $1 \le i \le 2^n - 1$, $1 \le j \le 2^n$ *and* $0 \le h \le 2^n - 1$

$$\begin{aligned} &\langle \tilde{w}_i^n, g_j^n \rangle = -(2i)^{-1}\delta_{i,j}, \;\; \langle \tilde{w}_i^n, \tilde{g}_h^n \rangle = -[2(2^n - i)]^{-1}\delta_{i,h} \\ &\|\tilde{w}_0^n\| = \langle \tilde{w}_0^n, g_{2^n}^n \rangle = 1, \;\; \|\tilde{w}_{2^n}^n\| = \langle \tilde{w}_{2^n}^n, \tilde{g}_0^n \rangle = 1 \end{aligned} \tag{4.1}$$

and

$$\frac{1}{2}\max\{i^{-1}, (2^n - i)^{-1}\} \le \|\tilde{w}_i^n\| \le \max\{i^{-1}, (2^n - i)^{-1}\}.$$

P r o o f. Clearly, $\|\tilde{w}_0^n\| = \|\delta(0)\| = 1 = \|\delta(1)\| = \|w_{2^n}^n\|$ and $\langle \tilde{w}_0^n, g_{2^n}^n\rangle = g_{2^n}^n(0) = 1 = \tilde{g}_0^n(1) = \langle \tilde{w}_{2^n}^n, \tilde{g}_0^n\rangle$. By (2.2), for each $1 \le i \le 2^n - 1$

$$\|\tilde{w}_i^n\| \le \frac{1}{2}\|\delta(i2^{-n}) - \delta((i-1)2^{-n})\| + \frac{1}{2}\|\delta(i2^{-n}) - \delta((i+1)2^{-n})\|$$

$$= \frac{1}{2}\max\{i^{-1}, (2^n - i + 1)^{-1}\} + \frac{1}{2}\max\{(i+1)^{-1}, (2^n - i)^{-1}\}.$$

Therefore $\|\tilde{w}_i^n\| \le i^{-1}$ if $1 \le i \le 2^{n-1}$ while $\|\tilde{w}_i^n\| \le (2^n - i)^{-1}$ if $2^{n-1} \le i \le 2^n - 1$. Clearly, $\langle \tilde{w}_i^n, g_j^n\rangle = 0 = \langle w_i^n, \tilde{g}_j^n\rangle$ whenever $i \ne j$. If $1 \le i \le 2^n - 1$ then

$$\begin{aligned}\langle \tilde{w}_i^n, g_i^n\rangle &= \left\langle -\frac{1}{2}\delta((i-1)2^{-n}), g_i^n\right\rangle \\ &= -\frac{1}{2}g_i^n((i-1)2^{-n}) = -\frac{1}{2}(1 - i^{-1}2^n(i-1)2^{-n}) = -(2i)^{-1}.\end{aligned}$$

Since $\|g_i^n\| = 1$, we get, for all $1 \le i \le 2^{n-1}$, that $1 = \langle -(2i)\tilde{w}_i^n, g_i^n\rangle \le \| -(2i)\tilde{w}_i^n\| \le 2$. Similarly, if $1 \le i \le 2^n - 1$ then

$$\begin{aligned}\langle \tilde{w}_i^n, \tilde{g}_i^n\rangle &= \left\langle -\frac{1}{2}\delta((i+1))2^{-n}), \tilde{g}_i^n\right\rangle \\ &= -\frac{1}{2}\tilde{g}_i^n((i+1)2^{-n}) = -\frac{1}{2}[(2^n - i)^{-1}2^n((i+1)2^{-n} - 1) + 1] \\ &= -[2(2^n - i)]^{-1}.\end{aligned}$$

Again, since $\|\tilde{g}_i\| = 1$, we have that, for $2^{n-1} \le i \le 2^n - 1$,

$$1 \le \langle -2(2^n - i)\tilde{w}_i^n, \tilde{g}_i^n\rangle \le \| -2(2^n - i)\tilde{w}_i^n\| \le 2.$$

This proves (4.1). □

Let us define

$$\begin{aligned} w_0^n &= -\tilde{w}_0^n = -\delta(0),\ w_{2^n}^n = -\tilde{w}_{2^n}^n = -\delta(1) \\ w_i^n &= -(2i)\tilde{w}_i^n \quad \text{if } 1 \le i \le 2^{n-1} \\ &\quad \text{and } w_i^n = -2(2^n - i)\tilde{w}_i^n \quad \text{if } 2^{n-1} < i \le 2^n - 1.\end{aligned}$$

In order to complete the proof of Proposition 4.1 we need

Lemma 4.3. *For every $n \ge 1$ and any sequence $\{a_i\}_{i=0}^{2^n}$ of numbers,*

$$\frac{1}{2}\max|a_i| \le \|\sum_{i=0}^{2^n} a_i w_i^n\| \le 4\max|a_i|. \tag{4.2}$$

P r o o f. By Corollary 3.2, for each $w \in \Delta_n$

$$\begin{aligned} ||w|| &\leq 2 \quad \max\{\max\{|\langle w, g_i^n\rangle| : 1 \leq i \leq 2^n\}, \\ &\qquad \max\{| < w, \tilde{g}_i^n\rangle| : 0 \leq i \leq 2^n - 1\}\}. \end{aligned}$$

If $w = \sum_{i=0}^{2^n} a_i w_i^n$ then, by (4.1),

$$\begin{aligned} &\max\{\langle w, g_i^n\rangle| : 1 \leq i \leq 2^n\} = \\ &\max\{\max\{| - a_0 + a_i| : 1 \leq i < 2^{n-1}\}, \max\{| - a_0 + a_i(2^n - i)i^{-1}| : \\ &2^{n-1} \leq i \leq 2^n - 1\}\} \leq 2\max\{|a_i| : 0 \leq i \leq 2^n\} \quad \text{while} \\ &\max\{|\langle w, \tilde{g}_i^n\rangle| : 0 \leq i \leq 2^n - 1\} = \\ &\max\{\max\{| - a_{2^n} + a_i(2^n - i)^{-1}i| : 1 \leq i \leq 2^{n-1}\}, \\ &\max\{| - a_{2^n} + a_i| : 2^{n-1} < i \leq 2^n\}\} \leq 2\max\{|a_i| : 0 \leq i \leq 2^n\} \end{aligned}$$

Therefore $\|w\| \leq 4\max\{|a_i| : 0 \leq i \leq 2^n\}$. On the other hand, for all $1 \leq i \leq 2^n$ and $0 \leq j \leq 2^n - 1$, $\|g_i^n\| = \|\tilde{g}_j^n\| = 1$ hence

$$\begin{aligned} |a_0| &= |a_0 g_{2^n}^n(0) = |a_0\langle g_{2^n}^n, \delta(0)|\rangle = |a_0\langle -g_{2^n}^n, w_0^n\rangle| \\ &= |\langle -g_{2^n}^n, w\rangle| \leq \|w\| \quad \text{and} \\ |a_{2^n}| &= |a_{2^n}\tilde{g}_0^n(1)| = |a_{2^n}\langle \tilde{g}_0^n, \delta(1)\rangle| = \ a_{2^n}^n\langle -\tilde{g}_0^n, \tilde{w}_{2^n}^n\rangle| \\ &= |\langle -\tilde{g}_0^n, w\rangle| \leq \|w\|. \end{aligned}$$

For $1 \leq i < 2^{n-1}$ let $v_i^n = g_i^n - g_{2^n}^n$ and, if $2^{n-1} \leq i < 2^n$, put $v_i^n = \tilde{g}_i^n - \tilde{g}_0^n$. Then $\langle v_i^n, w_0^n\rangle = \langle v_i^n, -\delta(0)\rangle = -v_i^n(0) = 0 = -v_i^n(1) = \langle v_i^n, -\delta(1)\rangle = \langle v_i^n, w_{2^n}^n\rangle$. Because both $\langle g_{2^n}^n, w_i\rangle = 0 = \langle \tilde{g}_0^n, w_i\rangle$ and $\|v_i^n\| \leq 2$ for all $1 \leq i \leq 2^n - 1$ we get that $|a_i| = |\langle v_i^n, w\rangle| \leq 2\|w\|$. This proves (4.2) and completes the proof of Proposition 4.1 □

Remark 4.4. *The biorthogonal functionals of* $\{w_i^n\}_{i=0}^{2^n}$.

We have established the fact that, for each $n \geq 1$, $\{w_i^n\}_{i=0}^{2^n}$ is a basis of Δ_n which is 8-equivalent to the unit vector basis of $\ell_\infty^{2^n+1}$. While proving Lemma 4.3 we have, in fact, constructed "natural" biorthogonal functionals $\{v_i^n\}_{i=0}^{2^n}$ of $\{w_i^n\}_{i=0}^{2^n}$ in K_n as follows: $v_0^n(t) = t - 1, v_{2^n}^n(t) = -t$, for each $1 \leq i < 2^{n-1}, v_i^n(t) = g_i^n(t) - g_{2^n}^n(t) = \max\{-(2^n - i)i^{-1}t, t - 1\}$ and for $2^{n-1} \leq i < 2^n, v_i^n(t) = \max\{-t, i(2^n - i)^{-1}(t - 1)\}$. Note that each of the functions $v_i^n(t)$ is convex and continuous and those v_i^n which are not linear vanish at the end points. Clearly, $\langle v_i^n, w_j^n\rangle = \delta_{i,j}, \|v_i^n\| \le 2$ for all $0 \le i,j \le 2^n$ and for every sequence $\{b_i\}_{i=0}^{2^n}$ of numbers

$$(4.3)\qquad 2\sum_{i=0}^{2^n}|b_i| \ge \|\sum_{i=0}^{2^n} b_i v_i^n\| \ge \frac{1}{4}\sum_{i=0}^{2^n}|b_i|.$$

Note that $v_i^n = v_{2i}^{n+1}$ for all $n \ge 1$ and $0 \le i \le 2^n$.

5. The structure of the space Δ. Most of the properties of $K[0,1]$ stated in the Introduction are based on the following

Theorem 5.1. *There exists a metrizable Choquet simplex S such that Δ is 8-isomorphic to the space $\mathcal{A}(S)$ of all affine continuous functions on S.*

Proof. The theory of $L_1(\mu)$ preduals makes it possible to show that $\Delta \sim \mathcal{A}(S)$ by studying the structure of Δ without even knowing what the simplex S is. By Theorem 5.2 of [2] it suffices to show that there exists a sequence $\{\Delta_n\}_{n=1}^\infty$ of subspaces of Δ with $\overline{\bigcup_{n=1}^{\infty}\Delta_n} = \Delta$ such that $\Delta_n \subset \Delta_{n+1}, \dim \Delta_n = 2^n+1$ for each $n \ge 1$, and, each Δ_n admits a basis $\{w_i^n\}_{i=0}^{2^n}$ satisfying the following three conditions:

$$(5.1)\qquad \begin{aligned} w_0^n &= w_0^{n+1}, w_{2^n}^n = w_{2^{n+1}}^{n+1} \quad \text{and for each } 1 \le i \le 2^n-1,\\ w_i^n &= w_{2i}^{n+1} + a_i^n w_{2i-1}^{n+1} + b_i^n w_{2i+1}^{n+1}\end{aligned}$$

where $1 \ge a_i^n, b_i^n \ge 0$,

$$(5.2)\qquad a_1^n = b_{2^n-1}^n = 1 \quad \text{and } a_i^n + b_{i-1}^n = 1 \quad \text{for all } 1 \le i \le 2^n-1$$

hence, there is an $w_0^0 \in \Delta$ so that, for every $n \ge 1$, $w_0^0 = \sum_{i=0}^{2^n} w_i^n$, and

$$(5.3)\qquad \text{For every}\quad n \ge 1 \text{ and any numbers} \{a_i\}_{i=0}^{2^n},$$
$$\frac{1}{2}\max|a_i| \le \|\sum_{i=0}^{2^n} a_i w_i^n\| \le 4\max|a_i|.$$

We will show that the bases $\{w_i^n\}_{i=0}^{2^n}$ of Δ_n constructed in Section 4 satisfy the above three conditions. Indeed, for every $n \ge 1$ $w_0^n = -\delta(0) = w_0^{n+1}$ and $w_{2^n}^n = -\delta(1) = w_{2^{n+1}}^{n+1}$. If $1 \le i \le 2^n-1$ then we have

$$\tilde{w}_i^n = \delta(i2^{-n}) - \frac{1}{2}\delta((i-1)2^{-n}) - \frac{1}{2}\delta((i+1)2^{-n})$$

$$\begin{aligned}
&= 2[\delta(2i2^{-n-1}) - \frac{1}{2}\delta(2i-1)2^{-n-1} - \frac{1}{2}\delta((2i+1)2^{-n-1})] \\
&= [\delta(2i-1)2^{-n-1} - \frac{1}{2}\delta((2i-2)2^{-n-1}) - \frac{1}{2}\delta((2i)2^{-n-1})] \\
&\quad +[\delta((2i+1)2^{-n-1}) - \frac{1}{2}\delta((2i)2^{-n-1}) - \frac{1}{2}\delta((2i+2)w^{-n-1})] \\
&= 2\tilde{w}_{2i}^{n+1} + \tilde{w}_{2i-1}^{n+1} + \tilde{w}_{2i+1}^{n+1}.
\end{aligned}$$

Therefore, if $1 \le i \le 2^{n-1} - 1$ then

$$\begin{aligned}
w_i^n &= -2i\tilde{w}_i^n = -(4i\tilde{w}_{2i}^{n+1} + 2i\tilde{w}_{2i-1}^{n+1} + 2i\tilde{w}_{2i+1}^{n+1}) \\
&= w_{2i}^{n+1} + (4i-2)^{-1}2iw_{2i-1}^{n+1} + (4i+2)^{-1}2iw_{2i+1}^{n+1} \\
&= w_{2i}^{n+1} + (2i-1)^{-1}iw_{2i-1}^{n+1} + (2i+1)^{-1}iw_{2i+1}^{n+1}.
\end{aligned}$$

If $2^{n-1} < i \le 2^n - 1$ then

$$\begin{aligned}
w_i^n &= -2(2^n-i)\tilde{w}_i^n = -(4(2^n-i)\tilde{w}_{2i}^{n+1} + 2(2^n-i)\tilde{w}_{2i-1}^{n+1} \\
&\quad +2(2^n-i)\tilde{w}_{2i+1}^{n+1}) = w_{2i}^{n+1} + (2^{n+1}-2i)(2^{n+2}-4i+2)^{-1}w_{2i-1}^{n+1} \\
&\quad +(2^{n+1}-2i)(2^{n+2}-4i-2)^{-1}w_{2i+1}^{n+1} \\
&= w_{2i}^{n+1} + (2^n-i)(2^{n+1}-(2i-1))^{-1}w_{2i-1}^{n+1} \\
&\quad +(2^n-i)(2^{n+1}-(2i+1))^{-1}w_{2i+1}^{n+1}.
\end{aligned}$$

In the case $i = 2^{n-1}$ we get that

$$\begin{aligned}
w_{2^{n-1}}^n &= -2^n\tilde{w}_{2^{n-1}}^n = -(2^{n+1}\tilde{w}_{2^n}^{n+1} + 2^n\tilde{w}_{2^n-1}^{n+1} + 2^n\tilde{w}_{2^n+1}^{n+1}) \\
&= w_{2^n}^{n+1} + 2^n(2^{n+1}-2)^{-1}w_{2^n-1}^{n+1} + 2^n(2^{n+2}-2^{n+1}-2)^{-1}w_{2^n+1}^{n+1} \\
&= w_{2^n}^{n+1} + 2^{n-1}(2^n-1)^{-1}w_{2^n-1}^{n+1} + 2^{n-1}(2^{n+1}-(2^n+1))^{-1}w_{2^n+1}^{n+1} \\
&= w_{2^n}^{n+1} + 2^{n-1}(2^n-1)^{-1}w_{2^n-1}^{n+1} + 2^{n-1}(2^n-1)^{-1}w_{2^n+1}^{n+1}.
\end{aligned}$$

We have thus proved (5.1) with

$$\begin{aligned}
a_i^n &= (2i-1)^{-1}i \quad \text{if } 0 \le i \le 2^{n-1}, \\
a_i^n &= (2^n-i)(2^{n+1}-(2i-1))^{-1} \quad \text{if } 2^{n-1} < i \le 2^n \\
b_i^n &= (2i+1)^{-1}i \quad \text{if } 0 \le i < 2^{n-1}, \\
\text{and } b_i^n &= (2^n-i)(2^{n+1}-(2i+1))^{-1} \quad \text{if } 2^{n-1} \le i \le 2^n.
\end{aligned}$$

Let us prove (5.2). Clearly,

$$\begin{aligned}
&a_1^n = 1 = b_{2^n-1}^n \quad \text{and, for } 1 \le i < 2^{n-1}, \\
&a_i^n + b_{i-1}^n = i(2i-1)^{-1} + (i-1)(2i-1)^{-1} = 1.
\end{aligned}$$

If $i = 2^{n-1}$ then $a^n_{2^{n-1}} + b^n_{2^{n-1}-1} = 2^{n-1}(2^n - 1)^{-1} + (2^n - 1)^{-1}(2^{n-1} - 1) = 1$. For $2^{n-1} < i < 2^n - 1$ we get that

$$a^n_i + b^n_{i-1} = (2^n - i)(2^{n+1} - (2i - 1))^{-1} + (2^n - i + 1)(2^{n+1} - 2i + 1)^{-1} = 1.$$

Finally, if $i = 2^{n-1} + 1$ we have

$$a^n_{2^{n-1}+1} + b^n_{2^{n-1}} = (2^{n-1} - 1)(2^n - 1)^{-1} + 2^{n-1}(2^n - 1)^{-1} = 1.$$

These equalities imply, in view of (5.1), that the element $w^0_0 = -2\delta(\frac{1}{2}) = \sum_{i=0}^{2^n} w^n_i$ for all $n \geq 1$; this w^0_0 is the unit function of $\mathcal{A}(S)$. This proves (5.2). Since condition (5.3) is exactly (4.2), the proof of Theorem 5.1 is complete. □

We now know that Δ is 8-isomorphic to the space $\mathcal{A}(S)$ where the natural norm of each affine continuous function $w(s)$ is $\|w\| = \max\{|w(s)| : s \in S\}$.

Remark 5.2. It is known (see e.g., [2]) that, for each $w \in \mathcal{A}(S)$

$$(5.4)\quad \|w\| = \sup_n \max\{|\langle w, v^n_i\rangle| : 0 \leq i \leq 2^n\} = \lim_n \max\{|\langle w, v^n_i\rangle| : i \leq 2^n\}$$

Since $\frac{1}{2}\|w\| \leq |\!|\!|w|\!|\!| \leq 4\|w\|$, we get that, for every $f \in K[0,1]$,

$$(5.5)\qquad \frac{1}{4}\|f\| \leq |\!|\!|f|\!|\!| = \lim_n \sum_{i=0}^{2^n} |\langle f, w^n_i\rangle| \leq 2\|f\|.$$

Hence, the dual norm $|\!|\!|f|\!|\!|$ in $\mathcal{A}(S)^* = K[0,1]$, can be expressed in the form

$$\begin{aligned}(5.6)\ |\!|\!|f|\!|\!| \;=\; & \lim_n [|f(0)| + |f(1)| \\ & +2\sum_{i=0}^{2^{n-1}-1} i|f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n})| \\ & +2\sum_{i=2^{n-1}}^{2^n-1} (2^n - i)|f(i2^{-n}) - \frac{1}{2}f((i-1)2^{-n}) - \frac{1}{2}f((i+1)2^{-n})|\end{aligned}$$

which is (∗)

6. The simplex *S*. We have used structure theorems in order to show that Δ is a space $\mathcal{A}(S)$ of all affine continuous functions on some Choquet simplex S, under the norm $|\!|\!|w|\!|\!|$ determined by (5.4). In this space the unit function is

$w_0^0 = -2\delta(\frac{1}{2})$. The theory of $\mathcal{A}(S)$ spaces suggests a simple way to present the simplex S itself: S can be identified with the set $\{f \in \mathcal{A}(S^*) : \|f\| = \langle f, w_0^0\rangle = 1\}$ equipped with the relative w^* topology (see e.g., Section 2 of [6]). Before investigating S we study the properties of some special members of S. Define the following functions on [0, 1]: $v_{-1}(x) = x - 1, v_2(x) = -x, v_0(x) = x - 1$ if $0 < x \leq 1$, $v_0(0) = 0$, $v_1(x) = -x$ if $0 \leq x < 1$ and $v_1(1) = 0$. For every $0 < t \leq \frac{1}{2}$ put $v_t(x) = \max\{\{-t^{-1}(1-t)x, x-1\}$ and, if $\frac{1}{2} < t \leq 1$, let $v_t(x) = \max\{-x, t(1-t)^{-1}(x-1)\}$. Let $V = \{v_t : 0 \leq t \leq 1\} \cup \{v_{-1}\} \cup \{v_2\}$.

Lemma 6.1. (a) *For each* $v \in V$, $\|v\| = \langle v, w_0^0\rangle = 1$ *and* $\langle v, w_i^n\rangle \geq 0$ *for all* $n \geq 1$ *and* $0 \leq i \leq 2^n$.

(b) *Each* $v \in V$ *is an extreme point of* S.

Proof. (a) A straightforward computation gives $\langle v, w_0^0\rangle = \langle v, -2\delta(\frac{1}{2})\rangle = 1$ for every $v \in V$. Let us compute the norm $\|v\|$. Since v_{-1} and v_2 are linear, $\langle v_{-1}, w_i^n\rangle = 0 = \langle v_2, w_i^n\rangle$ for all $n \geq 1$ and $1 \leq i \leq 2^n - 1$, while $\langle v_{-1}, w_{2^n}^n\rangle = \langle v_{-1}, -\delta(1)\rangle = 0 = \langle v_2, -\delta(0)\rangle = \langle v_2, w_0^n\rangle$. Therefore, by (5.6), $\|v_{-1}\| = |\langle v_{-1}, w_0^n\rangle| = \langle v_{-1}, -\delta(0)\rangle = -v_{-1}(0) = 1$ and $\|v_2\| = |\langle v_2, w_{2^n}^n\rangle| = \langle v_2, -\delta(1)\rangle = -v_2(1) = 1$. Consider v_0 and v_1: because of the linearity in the half-closed interval, $\langle v_1, w_i^n\rangle = 0$ for $n \geq 1$ and all $2 \leq i \leq 2^n$ while $\langle v_2, w_i^n\rangle = 0$ if $n \geq 1$ and $0 \leq i \leq 2^n - 2$. Since $\langle v_0, w_0^n\rangle = -v_0(0) = 0 = -v_1(1) = \langle v_1, w_{2^n}^n\rangle$ we get that $\|v_0\| = |\langle v_0, w_1^n\rangle| = |2v_0(2^{-n}) - v_0(2^{-n+1})| = |2^{-n+1} - 2 - 2^{-n+1} + 1| = 1$. Similarly one proves that $\|v_1\| = |\langle v_1, w_{2^n-1}^n\rangle| = 1$. Consider v_t with $0 \leq t \leq 1$. First note that if $t = i2^{-n}$ then $v_t = v_i^n$ and by Remark 4.4, $\|v_t\| = \|v_i^n\| = |\langle v_i^n, w_i^n\rangle| = 1$. If $0 \leq t \leq 1$ and $t \neq i2^{-n}$ then pick a numerical sequence $t(k) = i(k)2^{-n(k)}$ which converges to t. The definition of v_t implies that the pointwise limit $\lim_n v_{t(k)} = v_t$. Therefore $\langle v_{t(k)}, \delta(s)\rangle \to \langle v_t, \delta(s)\rangle$, i.e., $v_{t(k)}$ converges to v_t in the w^* topology. Since $\|v_{t(k)}\| = 1$, $\|v_t\| \leq 1$ and because $\langle v_t, w_0^0\rangle = 1$ we get that $\|v_t\| = 1$ for all $0 \leq t \leq 1$. It follows from (5.1) and (5.2) that, for every $v \in V$ and $n \geq 1$,

$$\begin{aligned} 1 &\geq \sum_{i=0}^{2^n} |\langle v, w_i^n\rangle| \geq \sum_{i=0}^{2^n} \langle v, w_i^n\rangle \\ &= \langle v, \sum_{i=0}^{2^n} w_i^n\rangle = \langle v, w_0^0\rangle = 1 \quad \text{therefore} \\ &\quad \langle v, w_i^n\rangle \geq 0 \quad \text{for all } 0 \leq i \leq 2^n. \end{aligned}$$

This proves (a).

(b). Suppose that $0 < t < \frac{1}{2}, c > 0$ and let $g(x) = -ct^{-1}x$ if $0 \le x \le t$ and $g(x) = (1-t)^{-1}c(x-1)$ if $t \le x \le 1$. Let us compute $\|g\|$. Let n be so large that $t \le \frac{1}{2} - 2^{-n+1}$ and let j be the integer for which $(j-1)2^{-n} < t \le j2^{-n}$. Because g is linear in each of the intervals [0, t] and [t, 1], $\langle g, w_i^n \rangle = 0$ for all $0 \le i \le 2^n$ except possibly for $i = j$ and $i = j-1$. Hence

$$\begin{aligned}\sum_{i=0}^{2^n} |\langle g, w_i^n\rangle| &= |\langle g, w_{j-1}^n\rangle| + |\langle g, w_j^n\rangle| \\ &= 2(j-1)|g((j-1)2^{-n}) - \frac{1}{2}g(j-2)2^{-n}) - \frac{1}{2}g(j2^{-n})| \\ &\quad +2j|g(j2^{-n}) - \frac{1}{2}g((j-1)2^{-n}) - \frac{1}{2}g((j+1)2^{-n})| \\ &= (j-1)c(1-t)^{-1}(j2^{-n}t^{-1}-1) + jc(1-t)^{-1}(1-2^{-n}t^{-1}(j-1)) \\ &= (1-t)^{-1}c.\end{aligned}$$

It follows that

$$\|g\| = (1-t)^{-1}c. \tag{6.1}$$

We are now ready to prove that every $v \in V$ is an extreme point of S. First suppose that t is either $-1, 2$ or a dyadic fraction $t = i2^{-n}$ with $1 \le i < 2^n$. Then $v_t = v_i^n$ for some $0 \le i \le 2^n$, which is an extreme point of S by Proposition 3.1 of [6]. It remains to consider $t = 0, 1$, and $0 < t < 1$, $t \ne i2^{-n}$. Suppose that $0 < t < \frac{1}{2}$, $t \ne i2^{-n}$. Let $g, h \in S$ and assume that $v_t = \frac{1}{2}(g+h)$. Recall that, because $\|g\| = \langle g, w_0^0\rangle = 1 = \langle h, w_0^0\rangle = \|h\|$, we have that $\langle g, w_i^n\rangle$, $\langle h, w_i^n\rangle \ge 0$ for all $n \ge 1$ and $0 \le i \le 2^n$. Since v_t is linear in each of the intervals $[0, t]$ and $[t, 1]$ g and h must also be linear in these intervals. Similarly, because $-v_t(0) = \langle v_t, w_0^n\rangle = 0 = \langle v_t, w_{2^n}^n\rangle = -v_t(1)$ also $g(0) = h(0) = 0 = h(1) = g(1)$. Suppose that $g(t) = -c$ and $h(t) = -d$ for some $c, d \ge 0$. Then by (6.1),

$$1 = \|g\| = (1-t)^{-1}c \quad \text{and } 1 = \|h\| = (1-t)d$$

hence $c = d = (1-t)^{-1}$ and therefore $v_t = h = g$. This proves that v_t is an extreme point of S if $0 < t < \frac{1}{2}$. The case $\frac{1}{2} < t < 1$ is proved in a similar fashion. Let us consider v_0 and let $v_0 = \frac{1}{2}(g+h)$ where $g, h \in S$. Again, because v_0 is linear in $(0, 1]$ so are g and h. Since $v_0(1) = 0$ also $-g(1) = \langle g, w_{2^n}^n\rangle = 0 = \langle h, w_{2^n}^n\rangle = -h(1)$. It follows that $g(x) = a(x-1)$ and $h(x) = b(x-1)$ for $0 < x \le 1$. Let $g(0) = c$ and $h(0) = d$. Then $c = g(0) = \langle g, \delta(0)\rangle = -\langle g, w_0^n\rangle \le 0$

and $d = h(0) = \langle h, \delta(0)\rangle = -\langle h, w_0^n\rangle \le 0$. Since $0 = v_0(0) = \frac{1}{2}(g(0)+h(0))$ we get that $c = d = 0$. If $0 < x \le 1$ then $x-1 = v_0(x) = \frac{1}{2}(g(x)+h(x)) = \frac{1}{2}(a+b)(x-1)$. Hence $\frac{1}{2}(a+b) = 1$. But, for each $n \ge 1$, by (a),

$$\begin{aligned} 1 &\ge \langle g, w_1^n\rangle = -2\langle g, \delta(2^{-n}) - \frac{1}{2}\delta(0) - \frac{1}{2}\delta(2^{-n+1})\rangle \\ &= -2g(2^{-n}) + g(2^{-n+1}) = -2a(2^{-n}-1) + a(2^{-n+1}-1) = a. \end{aligned}$$

Similarly, $b \le 1$ and therefore $a = b = 1$. It follows that $v_0 = g = h$. A similar argument works for v_1. This proves (b) and completes the proof of Lemma 6.1. □

We are now prepared to characterize the extreme boundary ∂S of S which is the set of extreme points of S.

Lemma 6.2. (a) $V = \partial S$
(b) *∂S is a w^* compact set.*

P r o o f. (a). In Lemma 6.1 we proved that each $v \in V$ is an extreme point of S. By Proposition 3.4 of [6] each extreme point of S is a w^* limit of a sequence $v_{i(k)}^{n(k)}$ with $0 \le i(k) \le 2^{n(k)}$. We know that every $w^* \lim v_{t(k)}$ with $\{t(k)\} \subset [0,1]$ is in V. Hence by Lemma 6.1 (b) $V = \partial S$. To prove (b), given a numerical sequence $\{t(k)\} \subset [0,1]$, pick a converging subsequence $\{t'(k)\}$ with $\lim t'(k) = t \in [0,1]$. Then the pointwise limit $\lim_k v_{t'(k)}(x) = v_t(x)$ hence $v_t = w^* \lim v_{t'(k)}$. This proves (b). □

A metrizable Choquet simplex with a compact set ∂S of its extreme points is called a Bauer Simplex. For information about Bauer simplices we refer the reader to Section 4 of [6]. A direct consequence of Lemma 6.2, Theorem 6.1 and Remark 5.2 is the following

Theorem 6.3. *Let F denote the compact metric space $[0,1]\cup\{-1\}\cup\{2\}$. Then the operator $T : \mathcal{A}(S) \to C(F)$ defined by*

$$(Tw)(t) = \langle w, v_t\rangle \tag{6.2}$$

is an isometry of $\mathcal{A}(S)$ onto $C(F)$. The dual surjective isometry $T^* : C(F)^* \to K[0,1]$ maps the measure $\mu \in C(F)^*$ (represented on $[0,1]$ by the $BVN[0,1]$ function $\tilde{\mu}$ with $\tilde{\mu}(0) = 0$) onto the function $f \in K[0,1]$ defined (see (1.3)) by

$$f(x) = \int_0^1 G(x,t)d\mu(t) + x(\mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) \tag{6.3}$$

$$= \begin{cases} x(2\tilde{\mu}(\frac{1}{2}) - \tilde{\mu}(1) + \mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) - \tilde{\mu}(x) - x \int\limits_x^{\frac{1}{2}} t^{-1} d\mu(t) \\ \textit{if } 0 \le x \le \dfrac{1}{2} \\ x(2\tilde{\mu}(\dfrac{1}{2}) - \tilde{\mu}(1) + \mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) - 2\tilde{\mu}(\dfrac{1}{2}) + \\ \tilde{\mu}(x) + (x-1) \int\limits_{\frac{1}{2}}^{x} (1-t)^{-1} d\mu(t) \quad \textit{if } \frac{1}{2} \le x \le 1 \end{cases}$$

P r o o f. Lemma 6.2 states that $\partial S = \{v_t : 0 \le t \le 1\} \cup \{v_{-1}\} \cup \{v_2\}$ which, under the topology of S (i.e., the w^* topology of Ball $\mathcal{A}(S)^*$), is homeomorphic to F. Since ∂S is compact, S is a Bauer simplex, hence (see e.g., Proposition 2.1 of [6]) the restriction map $R : \mathcal{A}(S) \to C(\partial S)$ defined by $(Rw)(t) = \langle w, v_t \rangle$ for all $v_t \in \partial S$ $(t \in F)$, is an isometry of $\mathcal{A}(S)$ onto $C(\partial S) = C(F)$. But, in view of Lemma 6.2 and (6.2), $R = T$, which proves the first part of Theorem 6.3. Let us now consider the dual surjective isometry $T^* : C(F)^* \to K[0,1]$ and put $T^* = \tau$. For each $x \in [0,1]$ we have that

$$\begin{aligned} (6.4) \quad \tau(\mu)(x) &= f(x) = \langle f, \delta(x) \rangle = \langle T^*\mu, \delta(x) \rangle = \langle \mu, T\delta(x) \rangle \\ &= \int_F \langle \delta(x), v_t \rangle d\mu(t) = v_{-1}(x)\mu(\{-1\}) + v_2(x)\mu(\{2\}) + \int_0^1 \langle \delta(x), v_t \rangle d\mu(t) \\ &= x(\mu(\{-1\}) - \mu(\{2\}) - \mu(\{-1\}) + \int_0^1 \langle \delta(x), v_t \rangle d\mu(t). \end{aligned}$$

Let us compute $\langle \delta(x), v_t \rangle$. We get that, if $0 < x \le \dfrac{1}{2}$, then

$$(6.5) \qquad \langle \delta(x), v_t \rangle = \begin{cases} x - 1 & \text{if } 0 \le t < x \\ t^{-1}(t-1)x & \text{if } x \le t \le \dfrac{1}{2} \\ -x & \text{if } \dfrac{1}{2} \le t \le 1 \end{cases}$$

while, for $\frac{1}{2} \le x < 1$, we have

$$\langle \delta(x), v_t \rangle = \begin{cases} x-1 & \text{if } 0 \le t \le \frac{1}{2} \\ (1-t)^{-1}t(x-1) & \text{if } \frac{1}{2} \le t \le x \\ -x & \text{if } x < t \le 1. \end{cases} \tag{6.6}$$

Also, $\langle \delta(0, v_t \rangle = 0 = \langle \delta(1), v_t \rangle$ for all $0 \le t \le 1$. Using (6.4) and (6.5) we get for all $0 \le x \le \frac{1}{2}$ that

$$\begin{aligned} \tau(\mu)(x) &= f(x) = x(\mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) + \int_0^1 G(x,t)d\mu(t) \\ &= x(\mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) \\ &\quad + (x-1)\tilde{\mu}(x) - x\int_x^{\frac{1}{2}} t^{-1}(1-t)d\mu(t) - x(\tilde{\mu}(1) - \tilde{\mu}(\tfrac{1}{2})) \\ &= x(\mu(\{-1\}) - \mu(\{2\}) + 2\tilde{\mu}(\tfrac{1}{2}) - \tilde{\mu}(1)) - \mu(\{-1\}) \\ &\quad - \tilde{\mu}(x) - x\int_x^{\frac{1}{2}} t^{-1}d\mu(t). \end{aligned}$$

If $\frac{1}{2} \le x \le 1$ then, a similar computation yields, by (6.4) and (6.6), the equality

$$\begin{aligned} \tau(\mu)(x) &= f(x) = x(\mu(\{-1\}) - \mu(\{2\})) - \mu(\{-1\}) \\ &\quad + \int_0^1 G(x,t)d\mu(t) = x(\mu(\{-1\}) - \mu(\{2\}) + 2\tilde{\mu}(\tfrac{1}{2}) - \tilde{\mu}(1)) \\ &\quad - 2\tilde{\mu}(\tfrac{1}{2}) - \mu(\{-1\}) + \tilde{\mu}(x) + (x-1)\int_{\frac{1}{2}}^x (1-t)^{-1}d\mu(t). \end{aligned}$$

Also, $\tau(\mu)(0) = f(0) = \langle \delta(x), T^*\mu \rangle = \int_F \langle \delta(0), v_t \rangle d\mu(t) - = -\mu(\{-1\})$ while $f(1) = \int_F \langle \delta(1), v_t \rangle \delta\mu(t) = -\mu(\{2\})$. This proves Theorem 6.3. □

The properties of the map $\tau = T^*$.

(6.7) *If* μ *is a finite positive measure on* F *then* $f = \tau(\mu)$ *is a non positive convex function on* $[0,1]$.

Proof. By (6.3), $f(x)$ is the sum of a non positive linear function and $\int_0^1 G(x,t)d\mu(t)$. For each $0 \le t \le 1$, $G(x,t) = v_t(x)$, which is a convex function vanishing at the end points. When $\mu(t)$ is a finite positive measure, $\int_0^1 G(x,t)d\mu(t)$ is also a convex function vanishing at 0 and at 1. □

(6.8) *Conversely, let* $T^*\mu = f$ *and assume that* f *is a nonpositive convex function. Then* μ *is a positive measure on* F.

Proof. Let $n \ge 1$ and pick $0 \le i \le 2^n$. Then $\langle f, w_i^n\rangle = \langle T^*\mu, w_i^n\rangle = \int_F \langle w_i^n, v_t\rangle d\mu(t)$. If $i = 0$ then $w_0^n = -\delta(0), v_{-1}(t) = t-1$ and v_t vanishes at 0. Therefore $0 \le \langle f, -\delta(0)\rangle = \int_F \langle -\delta(0), v_t\rangle d\mu(t) = -v_{-1}(0)\mu(\{-1\}) = \mu(\{-1\})$. Similarly, choosing $i = 2^n$ we get that $0 \le \mu(\{2\})$.

For $1 \le i \le 2^n$, the definition of v_t yields the equality

$$\langle w_i^n, v_t\rangle = (\text{negative constant})[\langle\delta(i2^{-n}) - \frac{1}{2}\delta(i-1)2^{-n})$$
$$-\frac{1}{2}\langle\delta((i+1)2^{-n}), v_t\rangle] = 0 \quad \text{if either } (i+1)2^{-n} \le t \quad \text{or } t \le (i-1)2^{-n}.$$

Hence, because f is convex,

$$\begin{aligned} 0 \;\le\; \langle f, w_i^n\rangle &= \int_F \langle w_i^n, v_t\rangle d\mu(t) = \int_0^1 \langle w_i^n, v_t\rangle d\mu(t) \\ &= \int_{(i-1)2^{-n}}^{(i+1)2^{-n}} \langle w_i^n, v_t\rangle d\mu(t). \end{aligned}$$

Since the integrand is positive (v_t is a convex function) and because this inequality holds for every $n \ge 1$ and $1 \le i \le 2^n - 1$ we get that the regular measure μ is positive. □

7. Construction of the difference $\boldsymbol{g - h = f}$**.** The purpose of this section is to present an algorithm which will construct, for each $f \in K[0,1]$, a unique pair of non positive convex functions g and h satisfying $f = g - h$ and $\|g\| + \|h\| = \|f\|$. This is done in the following three steps.

Step 1. *A commuting sequence of projections on* Δ.

We use the bases $\{w_i^n\}_{i=0}^{2^n}$ and their biorthogonal functionals $\{v_i^n\}_{i=0}^{2^n}$ described in Remark 4.4. For each $n \geq 1$ define the projection P_n of Δ onto Δ_n by $P_n w = \sum_{i=0}^{2^n} \langle w, v_i^n > w_i^n$. Then (5.4) yields the equality $\|P_n\| = 1$ and, by (5.1) we get that $P_{n+1}P_n = P_n$. Since $P_n^*(\Delta) = [v_i^n]_{i=0}^{2^n}$ and because (see Remark 4.4) $v_0^n = v_0^{n+1}$, $v_{2^n}^n = v_{2^{n+1}}^{n+1}$ and, for $1 \leq i \leq 2^n$, $v_i^n = v_{2i}^{n+1}$ we have that $P_{n+1}^* P_n^* = P_n^*$. It follows that $\{P_n\}_{n=1}^{\infty}$ is a sequence of commuting projections on Δ. Moreover, for every $w \in \bigcup_{n=1}^{\infty} \Delta_n$, $P_m w = w$ eventually, hence

$$\lim_{n\to\infty} P_n w = w \quad \text{for all } w \in \Delta. \tag{7.1}$$

It follows from (7.1) that, for every $f \in \Delta^*$ and $w \in \Delta$

$$\lim_n \langle P_n^* f,\ w\rangle = \langle f, w\rangle. \tag{7.2}$$

Hence $\{P_n^*\}_{n=1}^{\infty}$ converges to the identity I of Δ^* in the w^*-strong topology. In fact, it is easy to see that, for each $n \geq 1$, $P_n^* = Q_{2^n}$, the $(2^n + 1)$th Schauder basis projection.

Step 2. *Obtaining approximating differences of continuous convex functions.*

Given $f \in K[0,1]$ with $\|f\| = 1$, let

$$B(n) = \{0 \leq i \leq 2^n : \langle f, w_i^n\rangle \geq 0\} \quad \text{and}$$
$$C(n) = \{0 \leq i \leq 2^n : \langle f, w_i^n\rangle < 0\} \quad \text{and consider}$$

the functions $u_n = P_n^* f = \sum_{i=0}^{2^n} \langle f, w_i^n\rangle v_i^n$, $g_n = \sum_{i\in B(n)} \langle f, w_i^n\rangle v_i^n$ and $h_n = -\sum_{i\in C(n)} \langle f, w_i^n\rangle v_i^n$. Since each v_i^n is a non positive continuous convex function, so are g_n and h_n; moreover, $u_n = g_n - h_n$.

By (5.5), the norm $\|g_n\| + \|h_n\| = \|u_n\| = \|P_n f\| \leq \|f\| = 1$.

Step 3. *The limit process.*

Using the w^* compactness of Ball(Δ^*), by passing to a subsequence $\{n(k)\}$ we may assume that $g = w^* \lim g_{n(k)}$ and $h = w^* \lim h_{n(k)}$ exist with

$\|g\| + \|h\| \le \|f\| = 1$ and, by (7.3), $g - h = w^* \lim(g_{n(k)} - h_{n(k)}) = w^* \lim P^*_{n(k)} f = f$. As pointwise limits of sequences of non positive convex functions, both g and h are non positive convex functions with $\|g\| + \|h\| = \|f\|$. We claim that g and h are independent of the subsequence $\{n(k)\}$. Indeed, if $\{m(k)\}$ is another subsequence with $\tilde{g} = w^* \lim g_{m(k)}$ and $\tilde{h} = w^* \lim h_{m(k)}$ then, again, both $\tilde{g}$ and $\tilde{h}$ are non positive convex functions, $f = \tilde{g} - \tilde{h}$ and $\|\tilde{g}\| + \|\tilde{h}\| = \|f\|$. Since $K[0,1]$ is order-isometric to $C(F)^*$, an $L_1(\mu)$ - space, and because, in $L_1(\mu)$, the decomposition of each elements into the sum of its positive and negative parts is uniquely determined, we get by (6.7) and (6.8) that $g = \tilde{g}$ and $h = \tilde{h}$. This concludes the construction of the difference of convex functions $g - h = f$.

REFERENCES

[1] S. Kakutani. Concrete representation of abstract L-spaces and the mean ergodic theorem. *Ann. of Math.* **42** (1941), 523–537.

[2] A. J. Lazar, J. Lindenstrauss. Banach spaces whose duals are L_1 spaces and their representing matrices. *Acta Math.* **126** (1971), 165–193.

[3] J. Lindenstrauss, L. Tzafriri. Classical Banach spaces. Lectures Notes in Math. Berlin-Heidelberg-New York, Springer, 1973.

[4] W. Roberts, D. E. Varberg. Convex Functions. Academic Press, New York and London, 1973.

[5] F. Riesz. Sur cetain systems singuliers d'equations integrals. *Ann. Sci École Norm. Sup., Paris* **28** (1911), 33–62.

[6] Y. Stenfeld. Characterization of Bauer simplieces and some other classes of Choquet simplices by their representing matrices. Notes in Banach spaces, University of Texas Press, Austin and London, 1980, 306–358.

[7] M. Zippin. On some subspaces of Banach spaces whose duals are L_1 spaces. *Proc. Amer. Math. Soc.* **23** (1969), 378–385.

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