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A CLARKE–LEDYAEV TYPE INEQUALITY FOR CERTAIN NON–CONVEX SETS

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Abstract. We consider the question whether the assumption of convexity of the set involved in Clarke-Ledyaev inequality can be relaxed. In the case when the point is outside the convex hull of the set we show that Clarke-Ledyaev type inequality holds if and only if there is certain geometrical relation between the point and the set.

1. Introduction. F. Clarke and Yu. Ledyaev in their paper [2] discovered a mean value inequality of new feature. Let illustrate it with a simple partial case. Let $\varepsilon > 0$, $a \in \mathbb{R}^n$ and $C \subset \mathbb{R}^n$ be closed convex and bounded set. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable then there is $\xi \in \mathbb{R}^n$ that is almost between a and C and such that $\inf f(C) - f(a) < \inf \langle f'(\xi), C - a\rangle + \varepsilon$, where $\langle .,.\rangle$ denotes the inner product in $\mathbb{R}^n$.
In fact the theorem proved in [2] deals with arbitrary Hilbert space, function that is only lower semicontinuous and its smooth subdifferential. Later it has been extended to even more general spaces. Works in this direction are for example [1], [3], [4], [6], [5]. In the paper [5] we show that the requirement of boundedness

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below of the function is not really necessary. Further, in this paper we relax the assumption the set C to be convex.
As it is pointed out in the original paper [2] the assumption C convex can not easily be dropped. In the example given there $a \in \operatorname{co} C$. In this paper we consider the case $a \notin \overline{\operatorname{co}} C$ and work with a weaker geometrical assumption on C (see Definition 2.1).
It follows from Theorem 2.2 and Proposition 3.6 that if $a \notin \overline{\operatorname{co}} C$ then a Clarke-Ledyaev type inequality is valid for a and C if and only if C *seems convex* from a (Definition 2.1).
We work with smooth subdifferentials in the setting of smooth Banach spaces, but since everything that we use is Clarke-Ledyaev inequality for convex set the main result (Theorem 2.2) is true with appropriate changes for any subdifferential for which the Clarke-Ledyaev inequality is satisfied. For example, the result is true for the Clarke subdifferential of lower semicontinuous function on arbitrary Banach space.
We always work at some Banach space denoted by $(X, \|.\|)$. Let us fix some notation that will simplify the statements to follow. We denote by B_X the closed unit ball of the Banach space X, while B_X° is the open unit ball. For arbitrary set $A \subset X$ we put $\sup \|A\| = \sup\limits_{a \in A} \|a\|$ and $\inf \|A\| = \inf\limits_{a \in A} \|a\|$; $\overline{A}$ denotes the norm closure. For $\varepsilon > 0$ we set $A_\varepsilon = A + \varepsilon B_X^\circ$. We denote $[A, B] = \operatorname{co}\{A, B\}$ for $A, B \subset X$ (co stands for the convex hull). For simplicity in the previous formulae we write a instead of $\{a\}$ when the set A is a singleton, i.e. $[a, B] = \operatorname{co}(\{a\} \cup B)$, $[a, b] = \operatorname{co}\{a, b\}$, etc.
A bornology β on the Banach space X is a family of bounded subsets of X together with the properties: $\{x\} \in \beta$ for arbitrary $x \in X$; $A \in \beta, B \subset A \Rightarrow B \in \beta$. It is clear that the bornology G consisting of all singletons is contained in any bornology and the bornology F of all bounded sets contains any other bornology.
The Banach space X is said to be β-smooth with respect to certain bornology β if there exists a Lipschitz continuous bump (i.e. with non empty bounded support) function $b \in C^1_\beta(X) = \{f : X \to \mathbb{R}; f$ is Gâteaux differentiable and the derivative is a continuous mapping from X to the dual space X^*, equipped with the topology of uniform convergence on the members of the bornology $\beta\}$.
If β is some bornology on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous function then the β–smooth subdifferential of f at x is

$$D^-_\beta f(x) = \{u'(x) : \ u \in C^1_\beta(X) \text{ and } f - u \text{ has a local minimum at } x\}$$

if $x \in \operatorname{dom} f$ and $D^-_\beta f(x) = \emptyset$ if $f(x) = \infty$. As usual, $\operatorname{dom} f = \{x \in X; f(x) \in \mathbb{R}\}$.
We will use the following version of Clarke-Ledyaev inequality that is proved in [5].

Theorem 1.1 ([5]). *Let X be a β-smooth Banach space, $C \subset X$ be a closed, convex and bounded set. Given a lower semicontinuous function $f : X \to$*

$\mathbb{R} \cup \{+\infty\}$, $a \in \operatorname{dom} f$, real number r so that

$$r < \lim_{\varepsilon \downarrow 0} \inf f(C_\varepsilon) - f(a)$$

and arbitrary $\delta > 0$, one can find $x \in [a, C] + \delta B_X$ and $p \in D_\beta^- f(x)$ such that $r < \inf p(C - a)$ and $f(x) < f(a) + 2^{-1}(r + |r|) + \delta$.

2. Main result. In order to state our main theorem, we introduce the following relation.

Definition 2.1. *We say that the closed bounded set $C \subset X$ seems convex from the point $a \notin \operatorname{co} C$ if any sequence $\{x_n\}_{n=0}^\infty$ such that $x_0 = a$, $x_{n+1} - x_n \in \mathbb{R}^+ \operatorname{co}(C - a)$ and $[x_n, x_{n+1}] \cap C = \emptyset$ is bounded.*

It is easy to see that each closed convex and bounded set C seems convex from any point $a \notin C$.

We use few geometrical constants depending on the set and the point. Given an arbitrary bounded set C such that $0 \notin \overline{\operatorname{co}} C$ we set

$$J_0(C, 0) = \inf\{k \geq 1;\ k\operatorname{co} C \cap \operatorname{co} C \subset C\}.$$

For k large enough $k\operatorname{co} C \cap \operatorname{co} C = \emptyset$ hence, the above is well defined. For the bounded set C and $a \notin \overline{\operatorname{co}} C$ we put $J_0(C, a) = J_0(C - a, 0)$ and

$$J(C, a) = \liminf_{\delta \downarrow 0} J_0(C_\delta, a).$$

Let C and a satisfy same requirements as above. We set

$$j_0(C, 0) = \sup\{k \leq 1;\ k\operatorname{co} C \cap \operatorname{co} C \subset C\};$$

$j_0(C, a) = j_0(C - a, 0)$ and $j(C, a) = \limsup_{\delta \downarrow 0} j_0(C_\delta, a)$.

It is clear that if C is convex then for any $a \notin \overline{C}$ it is true that $j_0(C, a) = J_0(C, a) = 1$, i.e. the introduced constants measure in some sense the nonconvexity of C. It is also easy to see that $J(C, 0) \leq \dfrac{\sup \|\operatorname{co} C\|}{\inf \|\operatorname{co} C\|}$ and $j(C, 0) \geq \dfrac{\inf \|\operatorname{co} C\|}{\sup \|\operatorname{co} C\|}$.

Our main result is the following extension of Clarke-Ledyaev inequality.

Theorem 2.2. *Let X be a β-smooth Banach space, $C \subset X$ be a closed and bounded set, $D = \overline{\operatorname{co}} C$, the point $a \notin D$ and C seem convex from a. Given a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, $a \in \operatorname{dom} f$, real number r so that*

$$r < \lim_{\varepsilon \downarrow 0} \inf f(C_\varepsilon) - f(a)$$

and arbitrary $\delta > 0$*, one can find* $x \in [a, D] + \delta B_X$ *and* $p \in D^-_\beta f(x)$ *such that* $\varphi(r) < \inf p(C - a)$ *and* $f(x) < f(a) + 2^{-1}(r + |r|) + \delta$*, where*

$$\varphi(r) = \begin{cases} r/J(C,a) & , \quad r \geq 0 \\ r/j(C,a) & , \quad r < 0. \end{cases}$$

This result contains Theorem 1.1 as a partial case (since for C convex $J(C,a) = j(C,a) = 1$) and is really more general (since it is easy to construct nonconvex sets that seem convex from some points).
It is natural to ask what happens when $a \notin \overline{\mathrm{co}}\, C$ but the set C does not seem convex from a. In this case Proposition 3.6 shows that for all $n \in \mathbb{N}$ there is a lower semicontinuous function $f_n(x) = nf(x-a)$, where f is defined in Proposition 3.6, such that for some $\varepsilon > 0$, that does not depend on n, $\inf f_n(C_\varepsilon) - f_n(a) \geq n\varepsilon$ but $\forall p \in D^-_\beta f_n(X) \Rightarrow \inf p(C - a) \leq 0$. So, in general one can not estimate the infimum of the values of arbitrary function over C by the corresponding infimum of the elements of its subdifferential. In other words, no statement like Theorem 2.2 is true for sets that does not seem convex from the point considered.
Note also that the subdifferential estimate we obtain depends on C and a via the geometrical constants $J(C,a)$ or $j(C,a)$. Example 3.7 provides a case in which the constant $\varphi(r)$ is optimal.

3. Proof of the main result. We start with few preliminary steps. We show at the beginning the way that we are going to use the constants $J(C,a)$ and $j(C,a)$.

Lemma 3.1. *Let the set* $C \subset X$ *be bounded and such that* $0 \notin \overline{\mathrm{co}}\, C$*. If* $k > J_0(C,0)$ *then* $k\mathrm{co}\, C \cap [0, \mathrm{co}\, C] \subset C$*. If* $k \in (0, j_0(C,0))$ *then* $k\mathrm{co}\, C \cap \mathrm{co}\, C \subset C$*.*

Proof. By the definition of $J_0(C,0)$ there is $l \in [1,k)$ such that $l\mathrm{co}\, C \cap \mathrm{co}\, C \subset C$. Let $z \in k\mathrm{co}\, C \cap [0, \mathrm{co}\, C]$. That is $k^{-1}z \in \mathrm{co}\, C$ and there is $\alpha \in (0,1]$ such that $\alpha^{-1}z \in \mathrm{co}\, C$ ($\alpha \neq 0$ since $z \neq 0$). By the convexity of $\mathrm{co}\, C$ it follows that $[k^{-1}z, \alpha^{-1}z] \subset \mathrm{co}\, C$. But $[l^{-1}z, z] \subset [k^{-1}z, \alpha^{-1}z] \subset \mathrm{co}\, C$, i.e. $l^{-1}z \in \mathrm{co}\, C$ and $z \in \mathrm{co}\, C$, i.e. $z \in l\mathrm{co}\, C \cap \mathrm{co}\, C \subset C$.
Let now $k \in (0, j_0(C,0))$. By definition, there is $l \in (k, j_0(C,0))$ such that $l\mathrm{co}\, C \cap \mathrm{co}\, C \subset C$. Let $z \in k\mathrm{co}\, C \cap \mathrm{co}\, C$, i.e. $[z, k^{-1}z] \subset \mathrm{co}\, C$. But $[z, l^{-1}z] \subset [z, k^{-1}z]$ and consequently $z \in l\mathrm{co}\, C \cap \mathrm{co}\, C \subset C$. $\square$

The following is a key argument for our approach.

Lemma 3.2. *Let the set* $D \subset X$ *be closed, convex and bounded,* $0 \notin D$*;* $0 < \varepsilon < 2^{-1}\delta$*,* $\mu > 1$*. If* $y \in \mu D_\varepsilon$*,* $x \in [0, D_\delta]$ *and* $\|x - y\| < \varepsilon$ *then* $x \in D_\delta$*.*

Proof. Put $(\alpha, \beta) = \{t \in \mathbb{R}^+; ty \in D_\varepsilon\}$. Since $\mu^{-1}y \in D_\varepsilon$, we have that $\alpha < 1$.

Assume first that $\beta < 1$. Then $\|\beta x - \beta y\| = \beta\|x - y\| < \beta\varepsilon < \varepsilon$. Since $\beta y \in \overline{D_\varepsilon}$ and $2\varepsilon < \delta$, there is $y_1 \in D$ such that $\|\beta y - y_1\| < \delta - \varepsilon$. Hence $\|\beta x - y_1\| < \delta$, i.e. $\beta x \in D_\delta$. So, if we let $(\alpha_1, \beta_1) = \{t \in \mathbb{R}^+; tx \in D_\delta\}$, then $\beta \in (\alpha_1, \beta_1)$ i.e. $\alpha_1 < 1$. If $\beta_1 < 1$, then $x \notin [0, D_\delta]$ which is a contradiction. Thus $1 \in (\alpha_1, \beta_1)$ that is $x \in D_\delta$.
Assume now that $\beta \geq 1$. Then $y \in \overline{D_\varepsilon}$ and, since $2\varepsilon < \delta$, there is $y_1 \in D$ such that $\|y - y_1\| < \varepsilon + \delta - 2\varepsilon = \delta - \varepsilon$. Hence $\|x - y_1\| < \delta$, i.e. $x \in D_\delta$. □

We always work with some neighborhood of the set instead of the set itself. This is due to the lack of compactness in infinite dimensional case that generates some technical problems. The following Lemma contents some of the computations we need.

Lemma 3.3. *Let X be a Banach space, $C \subset X$ be a closed and bounded set, $D = \overline{\mathrm{co}}\, C$ and $0 \notin D_\delta$ for some $\delta > 0$. Assume that there exist $\varepsilon \in (0, \delta)$ such that $\varepsilon \dfrac{\sup\|D_\delta\|(1+\varepsilon)}{\inf\|D_\varepsilon\|} < \dfrac{\delta}{2}$ and $\varepsilon \sup\|D_\varepsilon\| < 4^{-1}\delta$ (recall that $\inf\|D_\varepsilon\| > 0$); and $y_0 = 0$, $y_1, \dots, y_N$ such that $y_n = \sum_{i=1}^{n} t_i x_i$, where $x_i \in D_\varepsilon$ and $t_i \in [0, \varepsilon]$; $y_N \notin [0, D_\delta]$, $y_n \in [0, D_\delta]$ for $n = 0, \dots, N-1$ and $y_n \notin C_\delta$ for $n = 0, \dots, N$. Then C does not seem convex from 0.*

P r o o f. Note that $\|y_n\| = \|\sum_{i=1}^{n} t_i x_i\| \leq \sup\|x_i\| \sum_{i=1}^{n} t_i \leq \sup\|D_\varepsilon\| \sum_{i=1}^{n} t_i$. Since $(\sum_{i=1}^{n} t_i)^{-1} y_n \in D_\varepsilon$ we have that $\|y_n\| \geq \inf\|D_\varepsilon\| \sum_{i=1}^{n} t_i$. So, we have the estimates

$$\inf\|D_\varepsilon\| \sum_{i=1}^{n} t_i \leq \|y_n\| \leq \sup\|D_\varepsilon\| \sum_{i=1}^{n} t_i. \tag{3.1}$$

For arbitrary $i = 1, \dots, N$ there is $z_i \in D$ such that $\|z_i - x_i\| < \varepsilon$ (recall that $x_i \in D_\varepsilon$). Put $\overline{y}_0 = 0$, $\overline{y}_j = \sum_{i=1}^{j} t_i z_i$, $j = 1, \dots, N$. Using (3.1) and the fact that $y_{N-1} \in [0, D_\delta]$, we compute $\inf\|D_\varepsilon\| \sum_{i=1}^{N} t_i \leq \|y_N\| \leq \|y_{N-1}\| + \|y_{N-1} - y_N\| = \|y_{N-1}\| + t_N\|x_N\| \leq \sup\|D_\delta\| + \varepsilon \sup\|D_\varepsilon\| \leq \sup\|D_\delta\|(1+\varepsilon)$. Moreover, for $j = 1, \dots, N$ it is fulfilled that $\|y_j - \overline{y}_j\| \leq \sum_{i=1}^{j} t_i\|z_i - x_i\| < \varepsilon \sum_{i=1}^{N} t_i \leq \varepsilon \dfrac{\sup\|D_\delta\|(1+\varepsilon)}{\inf\|D_\varepsilon\|} < \dfrac{\delta}{2}$ by the assumption. That is $\overline{y}_j \notin C_{2^{-1}\delta}$ for $j = 1, \dots, N$.

For $\alpha \in [0,1]$, $z \in C$ and $j = 1, \dots, N-1$ we have that $\|\alpha \overline{y}_j + (1-\alpha)\overline{y}_{j+1} - z\| \geq \|\overline{y}_{j+1} - z\| - \alpha t_{j+1}\|z_{j+1}\| \geq \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}$, since $t_{j+1}\|z_{j+1}\| \leq \varepsilon \sup \|D\| < 4^{-1}\delta$. We see now that $[\overline{y}_j, \overline{y}_{j+1}] \cap C = \emptyset$ for $j = 1, \dots, N-1$. Also, since, as we have already proved, $\|y_N - \overline{y}_N\| < 2^{-1}\delta$ and $y_N \notin [0, D_\delta]$, the point $\overline{y}_N$ is outside the closed and convex set $\overline{[0, D]}$. Let $q \in X^*$ be such that $q(\overline{y}_N) > \sup q([0, D]) \geq 0$. Put $d = (\sum_{i=1}^{N} t_i)^{-1}\overline{y}_N \in D$. Note that $q(d) > 0$ and hence $q(\overline{y}_N + td) > \sup q([0, D])$ for $t > 0$, i.e. $\overline{y}_N + td \notin D$. Now we can construct the sequence

$$y'_n = \begin{cases} \overline{y}_n & , \quad n = 0, \dots, N \\ \overline{y}_n + (n-N)d & , \quad n > N \end{cases}$$

which shows that C does not seem convex from 0. $\square$

As we can see from the statement of our main result the cases $r \geq 0$ and $r < 0$ are bit different. The next two Lemmas essentially solve these two cases.

Lemma 3.4. *Let X be a β-smooth Banach space, $C \subset X$ be a closed and bounded set that seems convex from 0, $D = \overline{\mathrm{co}}C$ and $0 \notin D_\delta$ for some $\delta > 0$. Given a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, $f(0) = 0$, real numbers $r \geq 0$ such that $r < \lim_{\varepsilon \downarrow 0} \inf f(C_\varepsilon)$ and $k > J_0(C_\delta, 0)$, one can find $x \in [0, D]_{2\delta}$ and $p \in D^-_\beta f(x)$ such that* $\inf p(C) > \frac{r}{k}$ *and $f(x) < r + 2\delta$.*

Proof. Choose $r_1 \in (r, r + 2^{-1}\delta)$ such that $\inf f(C_\delta) > r_1 > r$ and $k_1 > k$ such that $\frac{r_1}{k_1} > \frac{r}{k}$. Let $\varepsilon > 0$ be so small that

1) $\varepsilon < k_1 - k$, $\varepsilon < \delta$ and $r\varepsilon < 2^{-1}\delta$;

2) ε satisfies together with the given δ the conditions of Lemma 3.3.

For $y \in X$ define the set

$$A(y) = \{(t, x) \in [0, \varepsilon] \times D_\varepsilon;\ f(y + tx) \leq f(y) + \frac{r_1}{k_1}t\}.$$

Observe that for every $y \in X$ the set $A(y)$ is non-empty, since $(0, x) \in A(y)$ for any $x \in D_\varepsilon$. Define the function $s(y) = \sup\{\|tx\|; (t, x) \in A(y)\}$. From the definitions it is clear that for every $y \in X$ it is fulfilled that $0 \leq s(y) \leq \varepsilon\|D_\varepsilon\|$. Consider the sequences $\{y_n\}_{n=0}^\infty \subset X$, $\{x_n\}_{n=1}^\infty \subset D_\varepsilon$ and $\{t_n\}_{n=1}^\infty \subset [0, \varepsilon]$ such that $y_0 = 0$ and for $n \geq 1$ $(t_n, x_n) \in A(y_{n-1})$ and $t_n\|x_n\| \geq 2^{-1}s(y_{n-1})$; $y_n = \sum_{i=1}^{n} t_i x_i$. Such sequences exist by induction.

We claim that

(3.2) if $y_n \in C_\delta$ for some $n \in \mathbb{N}$ then either $y_{n-1} \in C_\delta$ or $y_{n-1} \notin [0, D_\delta]$.

To see this let $y_n \in C_\delta$. From

$$r_1 < \inf f(C_\delta) \le f(y_n) \le f(y_{n-1}) + t_n \frac{r_1}{k_1} \le \cdots \le f(0) + \frac{r_1}{k_1} \sum_{i=1}^{n} t_i$$

it follows that $\sum_{i=1}^{n} t_i > k_1$. So, $\sum_{i=1}^{n-1} t_i > k_1 - \varepsilon > k > J_0(C_\delta, 0)$ and, since $y_{n-1} \in \left(\sum_{i=1}^{n-1} t_i\right) D_\delta$, from Lemma 3.1 it follows that if $y_{n-1} \in [0, D_\delta]$ then $y_{n-1} \in C_\delta$. For the sequence $\{y_n\}_{n=0}^{\infty}$ there are exactly two possibilities.

Case 1: It is true that $y_n \in [0, D_\delta]$ for arbitrary $n \in \mathbb{N}$.

Then $\sup_{n\in\mathbb{N}} \|y_n\| < \infty$. From (3.1) it follows that $\sum_{n=1}^{\infty} t_i < \infty$ and consequently $y_n \underset{n\to\infty}{\longrightarrow} y = \sum_{i=1}^{\infty} t_i x_i$. Obviously $y \in \overline{[0, D_\delta]}$. If we assume that $\sum_{i=1}^{\infty} t_i > k$ then we can fix $n \in \mathbb{N}$ such that $\sum_{i=1}^{n} t_i > k$. Since by assumption $y_n \in [0, D_\delta]$ and $y_n \in (\sum_{i=1}^{n} t_i) D_\delta$ Lemma 3.1 shows that $y_n \in C_\delta$. From (3.2) it follows that $y_{n-1} \in C_\delta$. Similarly, $y_{n-2} \in C_\delta$, $\ldots$, $y_0 = 0 \in C_\delta$, which yields a contradiction. Therefore $\sum_{i=1}^{\infty} t_i \le k$.

Now, assume that we can find $x \in D_\varepsilon$ such that $f(y+\varepsilon x) < f(y) + \frac{r_1}{k_1}\varepsilon$. Put $\overline{x_n} = x + \varepsilon^{-1}(y - y_n)$ and note that $\overline{x_n} \underset{n\to\infty}{\longrightarrow} x \in D_\varepsilon$. Since D_ε is open set we have that $\overline{x_n} \in D_\varepsilon$ for sufficiently large n. Observe that $f(y_n + \varepsilon\overline{x_n}) = f(y+\varepsilon x) < f(y) + \frac{r_1}{k_1}\varepsilon$ and the lower semicontinuity of f at y gives $f(y_n + \varepsilon\overline{x_n}) < f(y_n) + \frac{r_1}{k_1}\varepsilon$ for sufficiently large n. Thus $(\varepsilon, \overline{x_n}) \in A(y_n)$ and $\overline{\lim_{n\to\infty}} s(y_n) \ge \overline{\lim_{n\to\infty}} \varepsilon\|\overline{x_n}\| = \varepsilon\|x\| > 0$, since $x \in D_\varepsilon \subset D_\delta \not\ni 0$ and $\varepsilon > 0$. But from the definition of (t_n, x_n) we have that $s(y_{n-1}) \le 2t_n\|x_n\| \underset{n\to\infty}{\longrightarrow} 0$, which yields a contradiction. Therefore, for arbitrary $x \in D_\varepsilon$ it follows that $f(y+\varepsilon x) \ge f(y) + \frac{r_1}{k_1}\varepsilon$. We will apply Theorem 1.1 to the point y and the set $y + \varepsilon D$ with δ. To this end we see that $f(y) \le \limsup_{n\to\infty} f(y_n) \le \frac{r_1}{k_1} \sum_{i=1}^{\infty} t_i \le \frac{r_1}{k_1} k < r_1$. Also, $\liminf_{\gamma\downarrow 0} f(y + \varepsilon D + \gamma B_X) - f(y) \ge$

$\inf f(y+\varepsilon D_\varepsilon) - f(y) \geq \dfrac{r_1}{k_1}\varepsilon > \dfrac{r}{k}\varepsilon$ by our previous consideration. According to Theorem 1.1 there are $x \in [y, y+\varepsilon D] + \delta B_X \subset [0, D]_{2\delta}$ and $p \in D^-_\beta f(x)$ such that $\inf p(\varepsilon C) > \dfrac{r}{k}\varepsilon$, i.e. $\inf p(C) > \dfrac{r}{k}$ and $f(x) < f(y) + \dfrac{r}{k}\varepsilon + \delta \leq r_1 + r\varepsilon + \delta < r + 2\delta$ because of the choice of r_1 and ε.

Case 2: We can find the smallest natural number N such that $y_N \notin [0, D_\delta]$. If there was $n_0 \in \{1, \dots, N-1\}$ such that $y_{n_0} \in C_\delta$ then (3.2) together with the fact that $y_n \in [0, D_\delta]$ for $n \in \{1, \dots, N-1\}$ would lead to a contradiction via the same argument as above. So, $y_n \notin C_\delta$ for all $n = 1, \dots, N$. Then by Lemma 3.3 it follows that C does not seem convex from 0 which is a contradiction. □

Lemma 3.5. *Let X be a β-smooth Banach space, $C \subset X$ be a closed and bounded set that seems convex from 0, $D = \overline{\mathrm{co}}\, C$ and $0 \notin D_\delta$ for some $\delta > 0$. Given a lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, $f(0) = 0$, real numbers $r < 0$ such that $r < \lim_{\varepsilon \downarrow 0} \inf f(C_\varepsilon)$ and $k \in (0, j_0(C_\delta, 0))$, one can find $x \in [0, D]_{2\delta}$ and $p \in D^-_\beta f(x)$ such that $\inf p(C) > \dfrac{r}{k}$ and $f(x) < 2\delta$.*

Proof. Most of the steps are similar to these in the proof of Lemma 3.4, so we just sketch them.

We choose in an appropriate way $r_1 < 0$ such that $\inf f(C_\delta) > r_1 > r$, $k_1 \in (0, k)$ such that $\dfrac{r_1}{k_1} > \dfrac{r}{k}$ and $\varepsilon > 0$ small enough.

We construct inductively in the same manner as in the proof of the previous Lemma the sequences $\{y_n\}_{n=0}^\infty \subset X$, $\{x_n\}_{n=1}^\infty \subset D_\varepsilon$ and $\{t_n\}_{n=1}^\infty \subset [0, \varepsilon]$.

Case 1: It is true that $y_n \in [0, D_\delta]$ for arbitrary $n \in \mathbb{N}$.

Then $y_n \underset{n\to\infty}{\longrightarrow} y = \sum_{i=1}^\infty t_i x_i \in \overline{[0, D_\delta]}$. By the lower semicontinuity of f it follows that $f(y) \leq 0$. The same arguments as that used in the proof of Lemma 3.4 and the Clarke-Ledyaev inequality (Theorem 1.1) complete the proof in this case.

Case 2: We can find the smallest natural number N such that $y_N \notin [0, D_\delta]$. Since C seems convex from 0 we can see from Lemma 3.3 that the set $\{n \in 1, \dots, N; y_n \in C_\delta\}$ is nonempty and we put m to be the maximum of this set. From

$$r_1 < \inf f(C_\delta) \leq f(y_m) \leq f(y_{m-1}) + t_m \frac{r_1}{k_1} \leq \ldots \leq f(0) + \frac{r_1}{k_1} \sum_{i=1}^m t_i$$

and $r_1 < 0$ it follows that $\sum_{i=1}^m t_i < k_1$. So, $\sum_{i=1}^{m+1} t_i < k < 1$ and in particular $y_{m+1} \in (\sum_{i=1}^{m+1} t_i) D_\varepsilon \subset [0, D_\varepsilon]$. Also, since $y_{m+1} \notin C_\delta$ Lemma 3.1 gives that

$y_{m+1} \notin D_\delta$. Let $(\alpha, \beta) = \{t \in \mathbb{R}^+; ty_{m+1} \in D_\delta\}$. Since $y_{m+1} \in [0, D_\varepsilon]$ the above set is nonempty and $\beta > 1$. Since $y_{m+1} \notin D_\delta$ we have that $\alpha > 1$. In other words $[0, y_{m+1}] \cap D_\delta = \emptyset$. Let $z_0 = 0, z_1, \ldots, z_l = y_{m+1}$ be partition of the interval $[0, y_{m+1}]$ such that $\|z_{i+1} - z_i\| < \varepsilon$. Then the sequence $z_0, \ldots, z_l, y_{m+1}, \ldots, y_N$ satisfies the conditions of Lemma 3.3 and consequently C does not seem convex from 0 which is a contradiction. $\square$

We are ready now to proceed with the proof of the main theorem.

Proof of Theorem 2.2. We may assume that $a = 0$ and $f(0) = 0$ by considering instead of $f(x)$ the function $f(x + a) - f(a)$.

Case 1: $r \geq 0$. Take $r_1 > r$ such that $\liminf_{\varepsilon \downarrow 0} f(C_\varepsilon) > r_1$ and $k > J(C, 0)$ such that $\dfrac{r_1}{k} > \dfrac{r}{J(C,0)} = \varphi(r)$ and $r_1 < r + 2^{-1}\delta$. We can find then $\delta_1 \in (0, 4^{-1}\delta)$ such that $\inf f(C_{\delta_1}) > r_1$ and $k > J_0(C_{\delta_1}, 0)$. Apply Lemma 3.4 with this k and $r = r_1$, $\delta = \delta_1$. We obtain $x \in [0, D]_{2\delta_1} \subset [0, D]_\delta$ and $p \in D^-_\beta f(x)$ such that $\inf p(C) > \dfrac{r_1}{k} > \varphi(r)$ and $f(x) < r_1 + 2\delta_1 < r + \delta$. The proof of this case is completed.

Case 2: $r < 0$. We apply Lemma 3.5 instead of Lemma 3.4. $\square$

We conclude by giving the two examples, the purpose of which has been explained after the statement of Theorem 2.2.

Proposition 3.6. *If the closed bounded set $C \subset X$, such that $0 \notin \overline{\text{co}}\, C$, does not seem convex from 0 then there is a continuous function $f : X \to \mathbb{R}$ such that $\inf f(C_\varepsilon) > 0$ for some $\varepsilon > 0$ but for arbitrary $x \in \operatorname{dom} f$ and $p \in D^-_\beta f(x)$ it follows that $\inf p(C) \leq 0$.*

Proof. From the definition we know that there exists an unbounded sequence $\{y_n\}_{n=0}^\infty$ such that $y_0 = 0$, $y_{n+1} - y_n \in \mathbb{R}^+ \text{co}\, C$ and $[y_n, y_{n+1}] \cap C = \emptyset$ for all $n \in \mathbb{N}$. Estimates similar to (3.1) show that in fact $\lim_{n\to\infty} \|y_n\| = \infty$. Define the function

$$f(x) = \operatorname{dist}\left(x, \bigcup_{i=0}^{\infty} [y_i, y_{i+1}]\right).$$

Since $\lim_{n\to\infty} \|y_n\| = \infty$, the set C is bounded and for arbitrary $n \in \mathbb{N}$ the set $\bigcup_{i=0}^{n} [y_i, y_{i+1}]$ is compact we have that $\operatorname{dist}\Big(C, \bigcup_{i=0}^{\infty} [y_i, y_{i+1}]\Big) > 2\varepsilon$ for some $\varepsilon > 0$. This means that $\inf f(C_\varepsilon) > 0$.

Fix arbitrary $x \in X$. There are at least one index $i(x) \in \mathbb{N} \cup \{0\}$ and point $P(x) \in [y_{i(x)}, y_{i(x)+1})$ such that $\|x - P(x)\| = f(x)$. Let $h = \|y_{i(x)+1} - y_{i(x)}\|^{-1}(y_{i(x)+1} - y_{i(x)})$. For $t > 0$ small enough $P(x) + th \in [y_{i(x)}, y_{i(x)+1})$, so $f(x+th) \leq \|(x+th) -$

$(P(x)+th)\| = \|x-P(x)\| = f(x)$. Therefore, $f^+(x,h) = \limsup_{t\downarrow 0} \frac{f(x+th)-f(x)}{t} \leq 0$. In particular, if $p \in D^-_\beta f(x)$ then $p(h) \leq 0$. Since $y_{i(x)+1} - y_{i(x)} \in \mathbb{R}^+ \mathrm{co}\, C$ there are $z \in \mathrm{co}\, C$ and $s > 0$ such that $h = sz$. Consequently $p(z) \leq 0$, i.e. $0 \geq \inf p(\mathrm{co}\, C) = \inf p(C)$. □

Example 3.7. Consider $X = \mathbb{R}^2 = \{(x,y); x, y \in \mathbb{R}\}$. Let $C = [(2,-1),(2,1)] \cup \{(1,4^{-1})\} \cup \{(1,-4^{-1})\}$ and $f(x,y) = 2^{-1}x + 2|y|$. Then C seems convex from 0, $J(C,0) = 2$ and $\min f(C) = 1$. If $p \in D^- f(x,y)$ then $\inf p(C) \leq p(1,0) \leq f'_x(x,y) = 2^{-1}$.

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