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DIFFERENTIAL EQUATIONS IN ABSTRACT CONES

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Abstract. We extend the method of quasilinearization to differential equations in abstract normal cones. Under some assumptions, corresponding monotone iterations converge to the unique solution of our problem and this convergence is superlinear or semi–superlinear.

1. Introduction. Denote by B a real Banach space with a norm $\|\cdot\|$ and let B^* denote the dual of B. Let K be a cone in B. We assume that K is closed convex subset of B such that $\lambda K \subset K$ for every $\lambda \geq 0$ and $K \cap \{-K\} = \{0\}$, where 0 denotes the null element of B. The cone K induces the order relation in B defined by $x \leq y,\ x, y \in B$ if and only if $y - x \in K$. We let $K^* = \{\phi \in B^* : \phi(u) \geq 0$ for all $u \in K\}$. We assume in this paper that K is a normal cone i.e. there exists a real number $c > 0$ such that $0 \leq u \leq v$ implies $\|u\| \leq c\|v\|$, where c is independent of u and v. A subset B_0 of B is said to be a distance set if for each $u \in B$ there corresponds a point $v \in B_0$ such that $d(u, B_0) = \|u - v\|$.

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Put $C_* = C(J \times B, B)$, $C_1 = C^1(J, B)$ with $J = [0, T]$. For $N \in C_*$, let us consider the problem

$$\begin{cases} x'(t) = N(t, x(t)), & t \in J, \\ x(0) = x_0. \end{cases} \tag{1}$$

Recently, the method of quasilinearization has been used so as to be applicable to a much larger class of nonlinear problems, (see, for example [9]). In this paper, we apply this method to differential problems of type (1) in a normal cone of the Banach space B (see, for example [3, 4, 6, 7, 9]). In [4, 7], some properties of measure of noncompactness are used to show that corresponding monotone sequences are convergent to the unique solution of (1). Quadratic and superlinear convergence of monotone iterations for problem (1) are obtained in [3, 9]. The purpose of this paper is to generalize that results when $N = f + g + h$. We assume that $f_x + \Phi_x$ is nondecreasing and $g_x + \Psi_x$ is nonincreasing for some nondecreasing function Φ_x and for some nonincreasing function Ψ_x. If h satisfies the Lipschitz condition, then corresponding monotone sequences converge to the unique solution of (1) and this convergence is superlinear or semi–superlinear. Note that, problem (1) is considered in [3] when $h = g = \Psi = 0$, and in [9] if $h = \Phi = g = \Psi = 0$.

2. Assumptions.

2. Assumptions. A function $v \in C_1$ is said to be a lower solution of problem (1) if

$$\begin{cases} v'(t) \le N(t, v(t)), & t \in J, \\ v(0) \le x_0, \end{cases}$$

and an upper solution of (1) if the inequalities are reversed.

Let us introduce some assumptions for later use.

(A_1) $f, g, h, \Phi, \Psi \in C_*$,

(A_2) N is quasimonotone nondecreasing in the second variable relative to K for each $t \in J$ i.e. if $u_1 \le u_2$ and $\phi(u_2 - u_1) = 0$ for some $\phi \in K^*$, then $\phi(N(t, u_1)) \le \phi(N(t, u_2))$,

(A_3) $y_0, z_0 \in C_1$ are lower and upper solutions of (1) such that $y_0(t) \le z_0(t)$, $t \in J$,

(A_4) $\|f(t,x) - f(t,y)\| \le L_1\|x-y\|,\ L > 0,\ y \in \delta K$, where δK denotes the boundary of K,

(A_5) K is a distance set,

(A_6) the Frechet derivative h_x exists, is continuous and $\|h_x(t,x)\| \le \overline{M}$ for $(t,x) \in J \times \Omega$ with $\overline{M} > 0$, where $\Omega = \{u \in B : y_0(t) \le u \le z_0(t),\ t \in J\}$,

(A_7) the Frechet derivatives f_x, g_x, Φ_x, Ψ_x exist, are continuous, and

(a) F_x, Φ_x are nondecreasing in the second variable, [i.e. $F_x(t,u)v \le F_x(t,\overline{u})v$ for $u, \overline{u} \in \Omega,\ v \in K$ and $u \le \overline{u}$], G_x, Ψ_x are nonincreasing in the second variable with $F = f + \Phi,\ G = g + \Psi$,

(b) $\|f_x(t,x)\| \le B_1,\quad \|g_x(t,x)\| \le B_2,\quad \|\Phi_x(t,x)\| \le B_3,\quad \|\Psi_x(t,x)\| \le B_4, x \in \Omega$,

(c) $\|f_x(t,x) - f_x(t,y)\| \le A_1\|x-y\|^\alpha,\ \|g_x(t,x) - g_x(t,y)\| \le A_2\|x-y\|^\beta,$
$\|\Phi_x(t,x) - \Phi_x(t,y)\| \le A_3\|x-y\|^\gamma,\ \|\Psi_x(t,x) - \Psi_x(t,y)\| \le A_4\|x-y\|^\delta$ for $(t,x),(t,y) \in J \times \Omega$ with $A_i, B_i > 0,\ i = 1,2,3,4$ and $\alpha, \beta, \gamma, \delta \in [0,1]$,

(A_8) there exists a constant $M \ge 0$ such that for $x, y \in \Omega$

$$h(t,x) - h(t,y) \le M[y-x] \text{ if } x \le y,$$

(A_9) $[h_x(t,\alpha_1) + F_x(t,\alpha_2) + G_x(t,\alpha_3) - \Phi_x(t,\alpha_3) - \Psi_x(t,\alpha_2)]v$ is quasimonotone nondecreasing in v relative to K for each $t \in J$, where $\alpha_1, \alpha_2, \alpha_3 \in C(J,B)$.

Remark 1. Clearly when assumption A_8 holds, then h is quasimonotone nondecreasing in the second variable relative to K.

Remark 2. In assumtion A_9 it is assumed, for example, that $F_x(t,\alpha_2)v$ is quasimonotone nondecreasing in v. Instead of it, by Lemma 4.5.2 [6], if we assume that F_x exists, F is convex and F is quasimonotone nondecreasing in the second variable, then $F_x(t,\alpha_2)v$ is quasimonotone nondecreasing in v for $(t,\alpha_2) \in J \times C(J,B)$.

3. Superlinear convergence.

Theorem 1 [6]. *Let K be a cone in B. Assume that $f \in C_*$, and*

1° $u, v \in C_1,\ u, v \in \Omega$ *satisfy* $u'(t) \le f(t,u),\ \ v'(t) \ge f(t,v),\ \ t \in J,$

2° *f is a quasimonotone nondecreasing in the second variable relative to K for each $t \in J$, and f satisfies assumption A_4,*

3° *K is a distance set.*

Then $u(0) \le v(0)$ implies $u(t) \le v(t)$ on J.

Now, we can formulate main results. The first theorem gives supelinear convergence while the second theorem semi–superlinear one.

Theorem 2. *Let K be a normal cone. Let assumptions A_1, A_2, A_3, A_5, A_6, A_7, A_9 hold for $N = f + g + h$. Then there exist monotone sequences which converge uniformly and monotonically to the unique solution x of problem (1) and the convergence is superlinear.*

Proof. First observe that, for $u, v \in \Omega$, $u \le v$, in view of $A_7(a)$,

$$\begin{cases} f(t,u) \le f(t,v) + [F_x(t,u) - \Phi_x(t,v)][u - v], \\ g(t,u) \le g(t,v) + [G_x(t,v) - \Psi_x(t,u)][u - v], \end{cases}$$

and

$$\mathcal{F}(t,u) - \mathcal{F}(t,v) \le V(t,u,v)(u - v) \tag{2}$$

with $\mathcal{F} = f + g$ and $V(t,u,v) = F_x(t,u) + G_x(t,v) - \Phi_x(t,v) - \Psi_x(t,u)$.

Using (2) and a mean value theorem we see that

$$\begin{aligned} &\mathcal{F}(t,u) + h(t,w) - \mathcal{F}(t,v) - h(t,v) + V(t,u,v)(w - u) \\ &\quad \le \textstyle\int_0^1 \left[h_x(t, sw + (1-s)v)ds + V(t,u,v)\right](w - v), \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\mathcal{F}(t,u) + h(t,u) - \mathcal{F}(t,v) - h(t,w) - V(t,u,v)(w - v) \\ &\quad \le \textstyle\int_0^1 \left[h_x(t, su + (1-s)w)ds + V(t,u,v)\right](u - w) \end{aligned} \tag{4}$$

for $u, v, w \in C_1$, $u, v \in \Omega$ and $u \le v$.

Let y_{n+1}, z_{n+1} be the solutions of IVPs

$$\begin{cases} y'_{n+1}(t) = \mathcal{F}(t, y_n) + h(t, y_{n+1}) + V_n(t)[y_{n+1}(t) - y_n(t)], \quad y_{n+1}(0) = x_0, \\ z'_{n+1}(t) = \mathcal{F}(t, z_n) + h(t, z_{n+1}) + V_n(t)[z_{n+1}(t) - z_n(t)], \quad z_{n+1}(0) = x_0 \end{cases}$$

for $n = 0, 1, \cdots$, where $V_n(t) = V(t, y_n, z_n)$. Note that y_{n+1} is a solution of the following nonlinear problem

$$y'(t) = \mathcal{F}(t, y_n) + h(t, y) + V_n(t)[y(t) - y_n(t)] \equiv Ay(t), \;\; y(0) = x_0. \tag{5}$$

By A_6 and $A_7(b)$, it is easy to conclude that the operator A satisfies a Lipschitz condition in y, and consequently there exists a unique solution y_{n+1} of (5). It means that the members y_{n+1} and z_{n+1} are well–defined.

In the first step, we need to show that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t) \;\text{ on }\; J. \tag{6}$$

To show (6) it is convenient to introduce $p = y_0 - y_1$ on J, so $p(0) \le 0$. Using the mean value theorem for h we obtain

$$\begin{aligned} p'(t) &\le \mathcal{F}(t, y_0) + h(t, y_0) - \mathcal{F}(t, y_0) - h(t, y_1) - V_0(t)[y_1(t) - y_0(t)] \\ &= \left[\int_0^1 h_x(t, sy_0 + (1-s)y_1)ds + V_0(t) \right] p(t), \;\; t \in J. \end{aligned}$$

Assumptions A_9, $A_7(b)$ and Theorem 1 yield $p(t) \le 0$ on J proving that $y_0(t) \le y_1(t)$ on J. Now, let $p = y_1 - z_0$ on J. Then, by (3), we get

$$\begin{aligned} p'(t) &\le \mathcal{F}(t, y_0) + h(t, y_1) + V_0(t)[y_1(t) - y_0(t)] - \mathcal{F}(t, z_0) - h(t, z_0) \\ &\le \left[\int_0^1 h_x(t, sy_1 + (1-s)z_0)ds + V_0(t) \right] p(t), \; t \in J, \;\; p(0) \le 0. \end{aligned}$$

Hence, by Theorem 1, $y_1(t) \le z_0(t), \; t \in J$ showing that $y_0(t) \le y_1(t) \le z_0(t)$ on J.

Let $p = y_0 - z_1$. Then $p(0) \le 0$, and, by (4),

$$\begin{aligned} p'(t) &\le \mathcal{F}(t, y_0) + h(t, y_0) - \mathcal{F}(t, z_0) - h(t, z_1) - V_0(t)[z_1(t) - z_0(t)] \\ &\le \left[V_0(t) + \int_0^1 h_x(t, sy_0 + (1-s)z_1)ds \right] p(t), \;\; t \in J. \end{aligned}$$

Hence, $y_0(t) \le z_1(t), \; t \in J$. Now, we put $p = z_1 - z_0$. Then

$$\begin{aligned} p'(t) &\le \mathcal{F}(t, z_0) + h(t, z_1) + V_0(t)[z_1(t) - z_0(t)] - \mathcal{F}(t, z_0) - h(t, z_0) \\ &= \left[\int_0^1 h_x(t, sz_1 + (1-s)z_0)ds + V_0(t) \right] p(t), \;\; t \in J, \;\; p(0) \le 0, \end{aligned}$$

so $z_1(t) \le z_0(t),\ t \in J$ showing that $y_0(t) \le z_1(t) \le z_0(t),\ t \in J$.

Obviously, basing on (2) and assumption $A_7(a)$, we have

$$V(t,u,\overline{v})w \le V(t,\overline{u},v)w \quad \text{if} \ \ u \le \overline{u},\ v \le \overline{v},\ u,\overline{u},v,\overline{v} \in \Omega,\ w \in K. \tag{7}$$

Next, we have to show that $y_1(t) \le z_1(t),\ t \in J$. We do this by showing that y_1 and z_1 are lower and upper solutions of (1), respectively. Basing on (2) and (7), we have

$$\begin{aligned} y_1'(t) &= \mathcal{F}(t,y_0) + h(t,y_1) + V_0(t)[y_1(t) - y_0(t)] - \mathcal{F}(t,y_1) + \mathcal{F}(t,y_1) \\ &\le N(t,y_1) + [V_0(t) - V(t,y_0,y_1)][y_1(t) - y_0(t)] \le N(t,y_1),\ \ t \in J, \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &= \mathcal{F}(t,z_0) + h(t,z_1) + V_0(t)[z_1(t) - z_0(t)] - \mathcal{F}(t,z_1) + \mathcal{F}(t,z_1) \\ &\ge N(t,y_1) + [V(t,z_1,z_0) - V_0(t)][z_0(t) - z_1(t)] \ge N(t,y_1),\ \ t \in J. \end{aligned}$$

Hence, by Theorem 1, $y_1(t) \le z_1(t),\ t \in J$. It proves that (6) holds.

Now, we assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_k(t) \le z_k(t) \le \cdots \le z_1(t) \le z_0(t),\ \ t \in J,$$

and let y_k, z_k be lower and upper solutions of (1) for some $k > 1$. We shall prove that

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t),\ \ t \in J. \tag{8}$$

Hence setting $p = y_k - y_{k+1}$ on J it follows as before

$$\begin{aligned} p'(t) &\le \mathcal{F}(t,y_k) + h(t,y_k) - \mathcal{F}(t,y_k) - h(t,y_{k+1}) - V_k(t)[y_{k+1}(t) - y_k(t)] \\ &= \left[\int_0^1 h_x(t, sy_k + (1-s)y_{k+1})ds + V_k(t)\right] p(t),\ \ t \in J,\ \ p(0) = 0 \end{aligned}$$

which again implies that $p(t) \le 0$ on J proving that $y_k(t) \le y_{k+1}(t)$ on J. On the other hand, letting $p = y_{k+1} - z_k$ on J, yields

$$\begin{aligned} p'(t) &\le \mathcal{F}(t,y_k) + h(t,y_{k+1}) + V_k(t)[y_{k+1}(t) - y_k(t)] - \mathcal{F}(t,z_k) - h(t,z_k) \\ &\le \left[\int_0^1 h_x(t, sy_{k+1} + (1-s)z_k)ds + V_k(t)\right] p(t),\ t \in J,\ \ p(0) = 0. \end{aligned}$$

This proves that $y_{k+1}(t) \le z_k(t),\ t \in J$ and hence $y_k(t) \le y_{k+1}(t) \le z_k(t)$ on J. Similarly as before we can show that $y_k(t) \le z_{k+1}(t) \le z_k(t),\ t \in J$.

Moreover, by (2) and (7), we have

$$\begin{aligned} y'_{k+1}(t) &= \mathcal{F}(t, y_k) + h(t, y_{k+1}) + V_k(t)[y_{k+1} - y_k(t)] - \mathcal{F}(t, y_{k+1}) + \mathcal{F}(t, y_{k+1}) \\ &\le N(t, y_{k+1}) + V(t, y_k, y_{k+1})[y_k(t) - y_{k+1}(t)] + V_k(t)[y_{k+1}(t) - y_k(t)] \\ &\le N(t, y_{k+1}),\ t \in J, \end{aligned}$$

and

$$\begin{aligned} z'_{k+1}(t) &= \mathcal{F}(t, z_k) + h(t, z_{k+1}) + V_k(t)[z_{k+1}(t) - z_k(t)] - \mathcal{F}(t, z_{k+1}) + \mathcal{F}(t, z_{k+1}) \\ &\ge N(t, z_{k+1}) - V(t, z_{k+1}, z_k)[z_{k+1}(t) - z_k(t)] + V_k(t)[z_{k+1}(t) - z_k(t)] \\ &\qquad \ge N(t, z_{k+1}),\quad t \in J \end{aligned}$$

showing that y_{k+1}, z_{k+1} are lower and upper solutions of (1), respectively. Hence, by Theorem 1, $y_{k+1}(t) \le z_{k+1}(t),\ t \in J$. It proves that (8) holds which means that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t),\quad t \in J$$

for all n, by mathematical induction.

In the next step we need to show that the sequences $\{y_n, z_n\}$ converge uniformly and monotonically on J. Note that the sequences are uniformly bounded on J since K is a normal cone. It remains to show that these sequences are Cauchy. For $M_0 = 2(B_1+B_2+B_3+B_4),\ \ L = \overline{M}+B_1+B_2+2B_3+2B_4,\ \ m_n(t) = \|y_{n+1}(t) - y_n(t)\|$, we put

$$u_n = \max_{t \in J} \left[e^{-Pt} m_n(t)\right] \qquad \text{with } \ P > L \ \text{ and } \ \frac{M_0}{P - L} \le q < 1.$$

Note that $m_n(0) = 0$. By assumptions A_6 and $A_7(b)$, we have

$$\begin{aligned} D^+ m_n(t) &\le \|y'_{n+1}(t) - y'_n(t)\| \\ &= \|\mathcal{F}(t, y_n) + h(t, y_{n+1}) + V_n(t)[y_{n+1}(t) - y_n(t)] - \mathcal{F}(t, y_{n-1}) - h(t, y_n) \\ &\quad - V_{n-1}(t)[y_n(t) - y_{n-1}(t)]\| \\ &\le L m_n(t) + M_0 m_{n-1}(t),\quad t \in J. \end{aligned}$$

Here D^+m denotes the right–hand upper Dini's derivative of m. Hence

$$m_n(t) \le M_0 \int_0^t e^{L(t-s)} m_{n-1}(s)ds, \quad t \in J,$$

and finally

$$u_n \le M_0 \max_{t\in J} \left[e^{-Pt} \int_0^t e^{L(t-s)} m_{n-1}(s)ds \right] < q\, u_{n-1}, \quad n = 1, 2, \cdots .$$

Basing on the above, we obtain

$$\begin{aligned} \max_{t\in J} \left[e^{-Pt} \|y_{n+k+1}(t) - y_{n+1}(t)\| \right] &\le \sum_{i=1}^{k} \max_{t\in J} \left[e^{-Pt} m_{n+i}(t) \right] = \sum_{i=1}^{k} u_{n+i} \\ &< \sum_{i=1}^{k} q^{n+i} u_0 \le \frac{q}{q-1} q^n u_0 \end{aligned}$$

which proves that $\{y_n\}$ is a Cauchy sequence on J. Hence $\{y_n\}$ converges monotonically and uniformly on J to $y \in \Omega$, where y is a solution of problem (1). Similarly, we can prove that $z_n \to z \in \Omega$, where z is a solution of (1). Note that problem (1) has a unique solution x since N satisfies a Lipschitz condition and therefore $y = z = x$.

It remains to show that convergence is superlinear. Put $p_{n+1} = x - y_{n+1} \ge 0$, $q_{n+1} = z_{n+1} - x \ge 0$, so $p_{n+1}(0) = q_{n+1}(0) = 0$. Note that

$$\begin{aligned} \int_0^1 [\mathcal{F}_x(t, sx + (1-s)y_n) - V_n(t)]\, ds = \int_0^1 [f_x(t, sx + (1-s)y_n) - f_x(t, y_n) \\ +g_x(t, sx + (1-s)y_n) - g_x(t, x) + g_x(t, x) - g_x(t, z_n) \\ +\Phi_x(t, z_n) - \Phi_x(t, x) + \Phi_x(t, x) - \Phi_x(t, y_n) \\ +\Psi_x(t, y_n) - \Psi_x(t, x) + \Psi_x(t, x) - \Psi_x(t, z_n)]ds. \end{aligned}$$

Hence, by assumption $A_7(c)$, we have

$$\left\| \int_0^1 [\mathcal{F}_x(t, sx + (1-s)y_n) - V_n(t)]\, ds\; p_n(t) \right\| \le A, \tag{9}$$

where

$$\begin{aligned} A = \max_{t\in J} [A_1 \|p_n(t)\|^{\alpha+1} + A_2 \|p_n(t)\|^{\beta+1} + A_2 \|q_n(t)\|^{\beta} \|p_n(t)\| + A_3 \|p_n(t)\|^{\gamma+1} \\ +A_3 \|q_n(t)\|^{\gamma} \|p_n(t)\| + A_4 \|p_n(t)\|^{\delta+1} + A_4 \|q_n(t)\|^{\delta} \|p_n(t)\|]. \end{aligned}$$

Using this and assumptions $A_6, A_7(b)$, we see that

$$\begin{aligned}
D^+\|p_{n+1}(t)\| &\le \|p'_{n+1}(t)\| = \|\mathcal{F}(t,x) + h(t,x) - \mathcal{F}(t,y_n) - h(t,y_{n+1}) \\
&\quad - V_n(t)[y_{n+1}(t) - y_n(t)]\| \\
&= \left\| \int_0^1 [\mathcal{F}_x(t, sx + (1-s)y_n) - V_n(t)]ds\; p_n(t) \right. \\
&\quad \left. + \left[\int_0^1 h_x(t, sx + (1-s)y_{n+1}ds + V_n(t) \right] p_{n+1}(t) \right\| \\
&\le A + L\|p_{n+1}(t)\|, \quad t \in J.
\end{aligned}$$

Hence

$$\|p_{n+1}(t)\| \le A \int_0^t e^{L(t-s)} ds \le A\; S \quad \text{with } S = \frac{1}{L} e^{LT},$$

and finally

$$\begin{aligned}
\max_{t\in J} \|p_{n+1}(t)\| \le S \max_{t\in J} \Big[& A_1\|p_n(t)\|^{\alpha+1} + A_2\|p_n(t)\|^{\beta+1} + A_2\|q_n(t)\|^{\beta}\|p_n(t)\| \\
& + A_3\|p_n(t)\|^{\gamma+1} + A_3\|q_n(t)\|^{\gamma}\|p_n(t)\| + A_4\|p_n(t)\|^{\delta+1} \\
& + A_4\|q_n(t)\|^{\delta}\|p_n(t)\| \Big].
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\max_{t\in J} \|q_{n+1}(t)\| \le S \max_{t\in J} \Big[& A_1\|q_n(t)\|^{\alpha+1} + A_2\|q_n(t)\|^{\beta+1} + A_1\|p_n(t)\|^{\alpha}\|q_n(t)\| \\
& + A_3\|q_n(t)\|^{\gamma+1} + A_3\|p_n(t)\|^{\gamma}\|q_n(t)\| + A_4\|q_n(t)\|^{\delta+1} \\
& + A_4\|p_n(t)\|^{\delta}\|q_n(t)\| \Big].
\end{aligned}$$

The proof is complete. □

Remark 3. If $\alpha = \beta = \gamma = \delta = 1$, then the convergence is quadratic.

4. Semi–superlinear convergence. Note that Theorem 2 gives superlinear convergence if the members of sequences $\{y_n\}$, $\{z_n\}$ are unique solutions of corresponding nonlinear problems of type (5). It is disadvantage in practice to construct them. If we do the linearization of those previous iterates, then we lost the superlinear convergence obtaing only semi–superlinear convergence. The next theorem deals with this case.

Theorem 3. *Let all assumptions of Theorem 2 with $h_x = 0$ in as-*

sumption A_9 hold. Moreover, we assume that assumption A_8 is satisfied. Then there exist monotone sequences which converge uniformly and monotonically to the unique solution x of problem (1) and the convergence is semi–superlinear.

Proof. Let I denote the unit element in B such that $Iu = u$ for any $u \in B$. Note that

$$N(t,u) - N(t,v) \le [V(t,u,v) - MI]\,[u-v], \text{ if } u \le v,\ u,v \in \Omega, \tag{10}$$

where V is defined as in Theorem 2. To prove (10) use (2) and Assumption A_8 to the following relation

$$N(t,u) - N(t,v) = \mathcal{F}(t,u) - \mathcal{F}(t,v) + h(t,u) - h(t,v). \tag{11}$$

Let

$$\begin{cases} y'_{n+1}(t) = N(t,y_n) + [V_n(t) - MI][y_{n+1}(t) - y_n(t)], & y_{n+1}(0) = x_0, \\ z'_{n+1}(t) = N(t,z_n) + [V_n(t) - MI][z_{n+1}(t) - z_n(t)], & y_{n+1}(0) = x_0, \end{cases}$$

where $V_n(t)$ is defined as in Theorem 2. Note that the elements y_{n+1} and z_{n+1} are well defined.

We shall show that

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t) \text{ on } J. \tag{12}$$

Let $p = y_0 - y_1$, so $p(0) \le 0$. Then

$$p'(t) \le N(t,y_0) - N(t,y_0) - [V_0(t) - MI][y_1(t) - y_0(t)] = [V_0(t) - MI]p(t),\ t \in J.$$

Theorem 1 gives $p(t) \le 0$ on J proving that $y_0 \le y_1$ on J. Let $p = y_1 - z_0$ on J, so $p(0) \le 0$. Then, by (10),

$$\begin{aligned} p'(t) &\le N(t,y_0) + [V_0(t) - MI][y_1(t) - y_0(t)] - N(t,z_0) \\ &\le [V_0(t) - MI][y_0(t) - z_0(t)] + [V_0(t) - MI][y_1(t) - y_0(t)] = [V_0(t) - MI]p(t). \end{aligned}$$

Hence $y_1(t) \le z_0(t),\ t \in J$ proving that $y_0(t) \le y_1(t) \le z_0(t),\ t \in J$.

Put $p = y_0 - z_1$, hence $p(0) \le 0$. Then

$$p'(t) \le N(t,y_0) - N(t,z_0) - [V_0(t) - MI][z_1(t) - z_0(t)] \le [V_0(t) - MI]p(t),\ t \in J$$

showing that $y_0(t) \le z_1(t),\ t \in J$. Put $p = z_1 - z_0$, so $p(0) \le 0$.

$$p'(t) \le N(t, z_0) + [V_0(t) - MI][z_1(t) - z_0(t)] - N(t, z_0) = [V_0(t) - MI]p(t),$$

hence $z_1(t) \le z_0(t)$ on J showing that $y_0(t) \le z_1(t) \le z_0(t),\ t \in J$.

In the next step we will show that y_1, z_1 are lower and upper solutions of (1), respectively. Indeed, we have

$$\begin{aligned} y_1'(t) &= N(t, y_0) + [V_0(t) - MI][y_1(t) - y_0(t)] - N(t, y_1) + N(t, y_1) \\ &\le N(t, y_1) + [V_0(t) - V(t, y_0, y_1)][y_1(t) - y_0(t)] \le N(t, y_1),\ t \in J, \end{aligned}$$

and

$$\begin{aligned} z_1'(t) &= N(t, z_0) + [V_0(t) - MI][z_1(t) - z_0(t)] - N(t, z_1) + N(t, z_1) \\ &\ge N(t, z_1) + [V(t, z_1, z_0) - V_0(t)][z_0(t) - z_1(t)] \ge N(t, z_1),\ t \in J. \end{aligned}$$

Again, by Theorem 1, $y_1(t) \le z_1(t),\ t \in J$. It means that (12) holds.

Let us assume that

$$y_0(t) \le y_1(t) \le \cdots \le y_k(t) \le z_k(t) \le \cdots \le z_1(t) \le z_0(t), \quad t \in J,$$

and let y_k, z_k be lower and upper solutions of (1) for some $k > 1$. We shall prove that

$$y_k(t) \le y_{k+1}(t) \le z_{k+1}(t) \le z_k(t), \quad t \in J. \tag{13}$$

Let $p = y_k - y_{k+1}$ on J, so $p(0) = 0$. Then

$$\begin{aligned} p'(t) &\le N(t, y_k) - N(t, y_k) - [V_k(t) - MI][y_{k+1}(t) - y_k(t)] \\ &= [V_k(t) - MI]p(t), \quad t \in J. \end{aligned}$$

Theorem 1 gives $p(t) \le 0$ on J proving that $y_k(t) \le y_{k+1}(t)$ on J.

Now, let $p = y_{k+1} - z_k$ on J. Then, by (10),

$$\begin{aligned} p'(t) &\le N(t, y_k) + [V_k(t) - MI][y_{k+1}(t) - y_k(t)] - N(t, z_k) \\ &\le [V_k(t) - MI][y_k(t) - z_k(t)] + [V_k(t) - MI][y_{k+1}(t) - y_k(t)] \\ &= [V_k(t) - MI]p(t),\ t \in J. \end{aligned}$$

Hence, $y_{k+1}(t) \le z_k(t),\ t \in J$ showing that $y_k(t) \le y_{k+1}(t) \le z_k(t)$ on J. By the similar argument, we can obtain $y_k(t) \le z_{k+1}(t) \le z_k(t),\ t \in J$.

Obviously,

$$\begin{aligned}y'_{k+1}(t) &= N(t,y_k) + [V_k(t) - MI][y_{k+1} - y_k(t)] - N(t,y_{k+1}) + N(t,y_{k+1})\\ &\le N(t,y_{k+1}) + [V(t,y_k,y_{k+1}) - MI][y_k(t) - y_{k+1}(t)]\\ &+ [V_k(t) - MI][y_{k+1}(t) - y_k(t)] \le N(t,y_{k+1}),\ t\in J,\end{aligned}$$

and

$$\begin{aligned}z'_{k+1}(t) &= N(t,z_k) + [V_k(t) - MI][z_{k+1}(t) - z_k(t)] - N(t,z_{k+1}) + N(t,z_{k+1})\\ &\ge N(t,z_{k+1}) - [V(t,z_{k+1},z_k) - MI][z_{k+1}(t) - z_k(t)]\\ &+ [V_k(t) - MI][z_{k+1}(t) - z_k(t)] \ge N(t,z_{k+1}),\ \ t\in J\end{aligned}$$

showing that y_{k+1}, z_{k+1} are lower and upper solutions of (1), respectively. Hence, by Theorem 1, $y_{k+1}(t) \le z_{k+1}(t),\ t\in J$. It proves that (13) holds. It means that

$$y_0(t) \le y_1(t) \le \cdots \le y_n(t) \le z_n(t) \le \cdots \le z_1(t) \le z_0(t),\ \ t\in J$$

for all n, by mathematical induction.

Using the method from Theorem 2, we see that the sequences $\{y_n\},\ \{z_n\}$ converge uniformly and monotonically to the unique solution x of (1). It remains to show that this convergence is semi-superlinear. Put $p_{n+1} = x - y_{n+1} \ge 0,\ q_{n+1} = z_{n+1} - x \ge 0$, so $p_{n+1}(0) = q_{n+1}(0) = 0$. Then, by (9) and assumptions $A_6, A_7(b)$,

$$\begin{aligned}D^+\|p_{n+1}(t)\| \le \|p'_{n+1}(t)\| &= \|\mathcal{F}(t,x) + h(t,x) - \mathcal{F}(t,y_n) - h(t,y_n)\\ &+ [V_n(t) - MI][p_{n+1}(t) - p_n(t)]\|\\ &\le \left\|\int_0^1 [\mathcal{F}_x(t,sx+(1-s)y_n) - V_n(t) + MI]ds\; p_n(t)\right\|\\ &+ \overline{M}\|p_n(t)\| + \|V_n(t) - MI\|\|p_{n+1}(t)\|\\ &\le A_0 + L\|p_{n+1}(t)\|,\ \ t\in J,\end{aligned}$$

where $A_0 = A + \left(M + \overline{M}\right)\max_{t\in J}\|p_n(t)\|,\ \ L = M + B_1 + B_2 + 2B_3 + 2B_4$ with A defined as in the proof of Theorem 2. Hence

$$\|p_{n+1}(t)\| \le A_0\int_0^t e^{L(t-s)}ds \le A_0\ S \quad \text{with}\ \ S = \frac{1}{L}e^{LT},$$

and finally

$$\max_{t\in J}\|p_{n+1}(t)\| \le S\max_{t\in J}\Big[A_1\|p_n(t)\|^{\alpha+1} + A_2\|p_n(t)\|^{\beta+1} + A_2\|q_n(t)\|^{\beta}\|p_n(t)\| \\ +A_3\|p_n(t)\|^{\gamma+1} + A_3\|q_n(t)\|^{\gamma}\|p_n(t)\| + A_4\|p_n(t)\|^{\delta+1} \\ +A_4\|q_n(t)\|^{\delta}\|p_n(t)\| + (M+\overline{M})\max\nolimits_{t\in J}\|p_n(t)\|\Big].$$

Similarly, we can show that

$$\max\nolimits_{t\in J}\|q_{n+1}(t)\| \le S\max\nolimits_{t\in J}\big[A_1\|q_n(t)\|^{\alpha+1} + A_2\|q_n(t)\|^{\beta+1} + A_1\|p_n(t)\|^{\beta}\|q_n(t)\| \\ +A_3\|q_n(t)\|^{\gamma+1} + A_3\|p_n(t)\|^{\gamma}\|q_n(t)\| + A_4\|q_n(t)\|^{\delta+1} \\ +A_4\|p_n(t)\|^{\delta}\|q_n(t)\| + (M+\overline{M})\max\nolimits_{t\in J}\|q_n(t)\|\big].$$

It ends the proof. □

Example. Consider the initial value problem of an infinite system for scalar differential equations of type

$$(14)\qquad \begin{cases} u_n'(t) = \dfrac{1}{4n}[t-u_n(t)]^3 + \dfrac{t}{4}[u_n^3(t) + u_{n+1}^3(t)], & t\in J=[0,1],\\ u_n(0)=0 \end{cases}$$

for $n=1,2,\cdots$. Here $B=\{u=(u_1,\cdots,u_n,\cdots): u_n\in R\}$ with the norm $\|u\| = \sup\limits_n\{|u_n(t)|: t\in J\}$ and $K=\{u\in B: u_n\ge 0, n=1,2,\cdots\}$. Indeed, K is a normal cone in B. In this case $N=(N_1,\cdots,N_n,\cdots)$, $f=(f_1,\cdots,f_n,\cdots)$, $g=(g_1,\cdots,g_n,\cdots)$, $h=(h_1,\cdots,h_n,\cdots)$ and $N_n(t,u)=g_n(t,u)+h_n(t,u)$ with

$$f_n(t,u)=0,\;\; g_n(t,u)=\frac{1}{4n}(t-u_n)^3,\;\; h_n(t,u)=\frac{t}{4}(u_n^3+u_{n+1}^3)\;\; t\in J,\;\; n=0,1,\cdots.$$

Indeed, $N\in C(J\times B,B)$. Let $y_0(t)=(0,\cdots,0,\cdots)$, $z_0(t)=\left(t,\frac{t}{2},\cdots,\frac{t}{n},\cdots\right)$. Then $y_0(t)\le z_0(t)$, $t\in J$. Moreover $y_0(0)=(0,\cdots,0,\cdots)=z_0(0)$, $y_0'(t)=(0,\cdots,0,\cdots)$, $z_0'(t)=\left(1,\frac{1}{2},\cdots,\frac{1}{n},\cdots\right)$, and

$$N_n(t,y_0(t)) = \frac{1}{4n} > 0 = y_{0n}'(t),\;\; t\in J,$$

$$N_n(t,z_0(t)) = \frac{t^3}{4n}\left(1-\frac{1}{n}\right)^3 + \frac{t^4}{4}\left(\frac{1}{n^3}+\frac{1}{(n+1)^3}\right) < \frac{1}{n} = z_{0n}'(t),\; t\in J.$$

It proves that y_0, z_0 are lower and upper solutions of problem (14), respectively.

Let $y_0(t) \le u(t) \le v(t) \le z_0(t),\ t \in J$. Then

$$N_n(t,u) - N_n(t,v) = \frac{1}{4n}[(t-u_n)^3 - (t-v_n)^3] + \frac{t}{4}[u_n^3 + u_{n+1}^3 - v_n^3 - v_{n+1}^3]$$

$$\le \frac{1}{4n}(v_n - u_n)[(t-u_n)^2 + (t-u_n)(t-v_n) + (t-v_n)^2] \le \frac{3}{4}(v_n - u_n),$$

so assumption A_2 holds. Moreover, $\overline{M} = \frac{3}{2}$ and $\Phi_n(t,u) = 0,\ t \in J,\ n = 1, 2, \cdots$. Put $\Psi_n(t,u) = -\frac{3t}{4n}u_n^2,\ t \in J,\ n = 1, 2, \cdots$. Let $y_0(t) \le u \le \overline{u} \le z_0(t),\ t \in J$ and $v \in K$. Then

$$\Psi_{nx}(t,u)v_n - \Psi_{nx}(t,\overline{u})v_n = \frac{6t}{4n}(\overline{u}_n - u_n)v_n \ge 0,$$

$$G_{nx}(t,u)v_n - G_{nx}(t,\overline{u})v_n = \frac{3}{4n}v_n[(\overline{u}_n - t)^2 - (u_n - t)^2 + 2t(\overline{u}_n - u_n)]$$

$$= \frac{3}{4n}(\overline{u}_n - u_n)(\overline{u}_n + u_n) \ge 0$$

for $t \in J,\ n = 1, 2, \cdots$. It proves that assumption $A_7(a)$ holds. Moreover, it is simple to see that $B_1 = 0,\ B_2 = \frac{3}{4},\ B_3 = 0,\ B_4 = \frac{3}{2},\ A_1 = \alpha = A_3 = \gamma = 0,\ A_2 = A_4 = \frac{3}{2},\ \beta = \delta = 1$.

Put

$$(15)\ \begin{cases} V_k(t) = g_x(t,z_k) + \Psi_x(t,z_k) - \Psi_x(t,y_k), \\ y'_{k+1}(t) = g(t,y_k) + h(t,y_{k+1}) + V_k(t)[y_{k+1}(t) - y_k(t)], \quad y_{k+1}(0) = y_0(0), \\ z'_{k+1}(t) = g(t,z_k) + h(t,z_{k+1}) + V_k(t)[z_{k+1}(t) - z_k(t)], \quad z_{k+1}(0) = z_0(0) \end{cases}$$

for $t \in J,\ \ k = 0, 1, \cdots$. Then, by Theorem 2, the monotone sequences $\{y_k, z_k\}$, $y_k = (y_{1k}, \cdots, y_{nk}, \cdots) \in B,\ z_k = (z_{1k}, \cdots, z_{nk}, \cdots) \in B$ converge (if $k \to \infty$) to the unique solution x of problem (14) and this convergence is quadratic i.e.

$$\|p_{k+1}\| \le a_1\|p_k\|^2 + a_2\|q_k\|^2, \quad \|q_{k+1}\| \le a_3\|p_k\|^2 + a_4\|q_k\|^2, \quad k = 0, 1, \cdots$$

for some nonnegative constants a_1, a_2, a_3, a_4.

Note that, by Theorem 3, the convergence of sequences $\{y_k, z_k\}$ to x is semi-superlinear, i.e.

$$\|p_{k+1}\| \le b_1\|p_k\|^2 + b_2\|q_k\|^2 + b_0\|p_k\|,$$
$$\|q_{k+1}\| \le b_3\|p_k\|^2 + b_4\|q_k\|^2 + b_0\|q_k\|, \ \ b_s \ge 0, \ s = 0,1,2,3,4, \ k = 0,1,\cdots$$

where

$$(16) \quad \begin{cases} y'_{k+1}(t) = g(t,y_k) + h(t,y_k) + V_k(t)[y_{k+1}(t) - y_k(t)], \ \ y_{k+1}(0) = y_0(0), \\ z'_{k+1}(t) = g(t,z_k) + h(t,z_k) + V_k(t)[z_{k+1}(t) - z_k(t)], \ \ z_{k+1}(0) = z_0(0) \end{cases}$$

for $t \in J, \ \ k = 0,1,\cdots$. Note that the rate of convergence for sequences (15) is higher than the corresponding one for (16) but to apply (15) we need to find the members of y_{k+1}, z_{k+1} solving corresponding nonlinear equations.

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