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A RECESSION NOTION FOR A CLASS OF MONOTONE BIVARIATE FUNCTIONS

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Abstract. Using monotone bifunctions, we introduce a recession concept for general equilibrium problems relying on a variational convergence notion. The interesting purpose is to extend some results of P. L. Lions on variational problems. In the process we generalize some results by H. Brézis and H. Attouch relative to the convergence of the resolvents associated with maximal monotone operators.

1. Introduction and Preliminaries. Equilibrium problems theory has emerged as a branch of applicable mathematics permitting to have a general and unified view on a large number of problems arising in mathematical economics, optimization and operation research. Recently much attention has been given to develop different monotonicity notions and various compacity conditions to obtain existence results. Following a general approach initiated by

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R. T. Rockafellar [10], we propose a recession analysis for general equilibrium problems. This approach relies on the concept of recession function. Recall that given $f : X \to \mathbb{R} \cup \{+\infty\}$ convex, lower semicontinuous and proper, its recession function f_∞ is defined by

$$f_\infty(x) = \lim_{t \to +\infty} \frac{f(x_0 + tx) - f(x_0)}{t}$$

when x_0 is taken arbitrarily in Dom f. This concept was extended to general maximal monotone operators by P. L. Lions [8] and Attouch, Chbani & Moudafi [2]. More precisely given an operator A, they showed the existence of a recession operator A_∞. The surprise comes from the fact that A_∞ is a subdifferential operator. Indeed $A_\infty = \partial f^A_\infty$ where $f^A_\infty(x) = \sup_{y \in \overline{R(A)}} \langle y, x \rangle$, i.e., the support function of the range of A. Our main purpose is first to show the existence of a recession bifunction F_∞ which captures the behavior of F at infinity, then to construct a recession function which, in the convex optimization and monotone inclusion cases reduces to the above classical recession concepts.

In section 1 we will recall an existence result for equilibrium problems and introduce some new definitions which will be some of the keys for proving Theorem 3.1. Section 2 is devoted to some primilary results on the variational convergence of bifunctions and the pointwise convergence of their resolvent. These results will be used in force in section 3. We will also show how the already known recession formulas for convex functions and maximal monotone operators can be derived from Theorem 3.1. We end this section with a characterization of the solvability of problem (1.1) in terms of the boundedness of the sequence of solutions to the associated Tikhonov regularization problems.

Let X be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and which is identified with its dual. The associated norm will be denoted by $|\cdot|$. Throughout, we use the following concepts, which are of common use in the context of convex function and optimization. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called convex (resp. lower semicontinuous) provided its epigraph

$$\operatorname{epi} f = \{(x, \lambda) \in X \times \mathbb{R}; f(x) \leq \lambda\}$$

is a convex (resp. closed) subset of $X \times \mathbb{R}$. Furthermore, f is called proper if its epigraph is nonempty. Again, the domain of f, Dom f, is the set of points in X for which $f(x)$ is finite.

It is worthwhile to introduce the following definitions.

Definition 1.1. *Let K be a subset of X and $F : K \times K \to \mathbb{R}$ be a given function.*
(i) We define its domain and graph as follows

$$\operatorname{Dom} F = \{x \in K;\ \textit{exists } z \in X \mid F(x,y) + \langle z, x-y\rangle \geq 0 \quad \forall y \in K\}$$

and

$$\operatorname{gph} F = \{(x,z) \in K \times X \mid F(x,y) + \langle z, x-y\rangle \geq 0 \quad \forall y \in K\}\,.$$

(ii) The inverse F^{-1} of F is defined by

$$F^{-1}(z,y) + \langle x, z-y\rangle \geq 0 \quad \textit{for each } y \in K$$

if and only if

$$F(x,y) + \langle z, x-y\rangle \geq 0 \quad \textit{for each } y \in K.$$

(iii) Let $\{F, F_n : K \times K \to \mathbb{R}; n \in \mathbb{N}\}$ be a sequence of bivariate functions. The sequence $\{F_n\}$ is said to be variational convergent to F, if

$$\operatorname{gph}(F) \subset \liminf_{n\to+\infty} \operatorname{gph}(F_n),$$

in other words, for all $(x,z) \in K \times X$ such that

$$F(x,y) + \langle z, x-y\rangle \geq 0 \quad \textit{for each } y \in K,$$

there exists $(x_n, z_n) \in K \times X$ such that

$$F_n(x_n,y) + \langle z_n, x_n - y\rangle \geq 0 \quad \textit{for each } y \in K \quad \textit{with} \quad x_n \to x \textit{ and } z_n \to z.$$

We write $F = V - \lim_{n\to+\infty} F_n$.

Let us now recall some classical definitions.

Definition 1.2.
(i) A function F is said to be monotone, if

$$F(x,y) + F(y,x) \leq 0, \quad \textit{for each } x, y \in K.$$

(ii) It is said to be strictly monotone if

$$F(x,y)+F(y,x)<0, \quad \textit{for each } x,y\in K, \ \textit{with } x\neq y,$$

(iii) F is upper-hemicontinuous, if for each $x,y,z\in K$

$$\limsup_{t\to 0^+} F(tz+(1-t)x,y)\leq F(x,y).$$

The following result due to Blum-Otteli [3] will be used in Remark 1.2.

Theorem 1.1. *If the following conditions hold true:*
(i) F is monotone and upper hemicontinuous,
(ii) $F(x,.)$ *is convex and lower semicontinuous for each* $x\in K$*,*
(iii) there exists a compact subset B of X and there exists $y_0\in B\cap K$ *such that* $F(x,y_0)<0$*, for each* $x\in K\backslash B$*.*
Then, the set of solutions to the following problem

$$\textit{find} \quad \overline{x}\in K \ \textit{such that} \ F(\overline{x},y)\geq 0 \ \ \forall y\in K \tag{1.1}$$

is nonempty convex and compact.

Remark 1.1. If F is strictly monotone, then the solution of (1.1) is unique.

Let us now recall the extension of the resolvent and the Yosida approximate notions introduced in [9]. Let $\mu>0$ be a positive number. For a given bivariate function F the associated Yosida approximate, A^F_μ, over K and the corresponding resolvent operator, J^F_μ, are defined as follows

$$A^F_\mu(x):=\frac{1}{\mu}\left(x-J^F_\mu(x)\right), \tag{1.2}$$

in which $J^F_\mu(x)\in K$ is the unique solution of

$$F(J^F_\mu(x),y)+\mu^{-1}\left\langle J^F_\mu(x)-x,y-J^F_\mu(x)\right\rangle\geq 0 \qquad \forall y\in K. \tag{1.3}$$

The Yosida approximate of parameter $\mu>0$ is $\frac{1}{\mu}$-Lipschitz continuous, that is

$$\left|A^F_\mu(x)-A^F_\mu(y)\right|\leq\frac{1}{\mu}\left|x-y\right| \qquad \forall x,y\in X$$

and the resolvent is nonexpansive, namely

$$\left|J_\mu^F(x) - J_\mu^F(y)\right| \leq |x-y| \qquad \forall x, y \in X.$$

Remark 1.2. The existence and uniqueness of the solution of problem (1.3) above follow by invoking Theorem 1.1 and Remak 1.1 Observe that in the case where $F(x,y) = \sup_{\zeta \in Ax} \langle \zeta, y-x \rangle$, A being a maximal monotone operator, it directly yields: $\operatorname{Dom} F = \operatorname{Dom} A$, $J_\mu^F(x) = (I+\mu A)^{-1}x$ and $A_\mu^F(x) := A_\mu(x) = \frac{1}{\mu}\left(I - (I+\mu A)^{-1}\right)$, and we recover the classical concepts.

2. Convergence results.

In the sequel, we will consider a class of bivariate functions F satisfying the following conditions: F is upper hemicontinuous monotone over a closed convex K. $F(x, .)$ is convex and lower semicontinuous for all $x \in K$. We summarize them as assumption $(\mathcal{H})$.

To begin with, let us state the following result which will be needed in the proof of Theorem 3.1.

Proposition 2.1. *For all $x \in X$, one has*

$$\lim_{\mu \to 0} J_\mu^F(x) = \operatorname{proj}_{\overline{\operatorname{Dom} F}}(x).$$

Proof. Let $u \in \operatorname{Dom} F$. By Definition 1.1, there exists $v \in X$, such that

$$F(u,y) + \langle v, u-y \rangle \geq 0 \quad \forall y \in K. \tag{2.4}$$

On the other hand, for any $x \in X$, we have

$$F(J_\mu^F(x), y) + \mu^{-1}\left\langle J_\mu^F(x) - x, y - J_\mu^F(x)\right\rangle \geq 0 \qquad \forall y \in K. \tag{2.5}$$

Setting $y = J_\mu^F(x)$ in (2.4) and $y = u$ in (2.5) and adding the resulting inequalities, we obtain, thanks to the monotonicity of F,

$$\left\langle J_\mu^F(x) - x, u - J_\mu^F(x)\right\rangle + \mu\left\langle v, u - J_\mu^F(x)\right\rangle \geq 0. \tag{2.6}$$

Equivalently we have

$$\left\langle J_\mu^F(x) - x, u - x - (J_\mu^F(x) - x)\right\rangle + \mu\left\langle v, u - x + (J_\mu^F(x) - x)\right\rangle \geq 0,$$

that is

$$\left|J_\mu^F(x) - x\right|^2 \le \left|J_\mu^F(x) - x\right| (|u-x| + \mu\,|v|) + \mu\,|v|\,|u-x|\,.$$

Let us write $Q := \left|J_\mu^F(x) - x\right|$. Then

$$Q^2 \le Q\,(|u-x| + \mu\,|v|) + \mu\,|v|\,|u-x|$$

which implies

$$Q \le \frac{1}{2}\left(|u-x| + \mu\,|v| + \sqrt{(|u-x| + \mu\,|v|)^2 + 4\mu\,|u-x|\,|v|}\right),$$

from which follows that

$$Q \le |u-x| + \mu\,|v| + \sqrt{\mu\,|u-x|\,|v|}.$$

Finally

$$\left|J_\mu^F(x)\right| \le |x| + |u-x| + \mu\,|v| + \sqrt{\mu\,|u-x|\,|v|},$$

this clearly implies that $\left\{J_\mu^F(x)\right\}$ is bounded. Moreover, (2.6) yields

$$\left|J_\mu^F(x)\right|^2 \le \left\langle x - \mu v, J_\mu^F(x) - u\right\rangle + \left\langle J_\mu^F(x), u\right\rangle. \tag{2.7}$$

Now, let us choose a sequence $\mu_\nu \to 0$ such that $J_{\mu_\nu}^F(x)$ converges weakly to some p. Then passing to the limit in (2.7) with $\mu = \mu_\nu$, we obtain

$$|p|^2 \le \liminf_{\nu\to\infty} \left|J_{\mu_\nu}^F(x)\right|^2 \le \langle x, p-u\rangle + \langle p, u\rangle\,, \quad \text{for all } u \in \operatorname{Dom} F$$

that is

$$\langle x-p, u-p\rangle \le 0 \;\; \text{for all } u \in \operatorname{Dom} F. \tag{2.8}$$

The latter inequality still holds true for all $u \in \overline{\operatorname{Dom} F}$.

Using the fact that $\operatorname{Dom} F$ is convex, $\overline{\operatorname{Dom} F}$ is weakly closed and that $J_\mu^F(x) \in \overline{\operatorname{Dom} F}$, for all $\mu > 0$, we infer that $p \in \overline{\operatorname{Dom} F}$. In view of (2.8) we easily deduce that

$$p = \operatorname{proj}_{\overline{\operatorname{Dom} F}}(x).$$

The weak cluster point being unique, we obtain that the whole sequence $\left\{J_\mu^F(x)\right\}$ weakly converges to $\operatorname{proj}_{\overline{\operatorname{Dom} F}}(x)$ as $\mu \to 0$. In order to prove the strong convergence of the sequence $\left\{J_\mu^F(x)\right\}$, we need only to show its convergence in norm.

Passing to the limit superior in (2.7), we have

$$\limsup_{\mu\to 0} \left|J_\mu^F(x)\right|^2 \le \langle x, p-u\rangle + \langle p, u\rangle \quad \forall u \in \overline{\operatorname{Dom} F}.$$

It then follows, by taking $u = p$, that

$$\limsup_{\mu\to 0} \left|J_\mu^F(x)\right| \le |p|\,,$$

thus

$$\lim_{\mu\to 0} \left|J_\mu^F(x)\right| = |p|\,, \tag{2.9}$$

which completes the proof. □

It is worth mentioning that in the case where $F(x,y) = \sup\limits_{\zeta\in Ax} \langle \zeta, y-x\rangle$, A being a maximal monotone operator, we recover a result by Brézis [5], namely

$$\lim_{\mu\to 0} J_\mu^A(x) = \operatorname{proj}_{\overline{\operatorname{Dom} A}}(x).$$

Before stating the next Proposition, let us define the approximate Yosida bifunction, F_λ, of a given bifunction F, as: $F_\lambda(x,y) = \langle A_\lambda^F(x), y-x\rangle$.

Proposition 2.2. *Let F be a given bifunction, then the following variational convergence holds true*

$$F = V - \lim_{\lambda\to 0} F_\lambda.$$

P r o o f. Indeed, let $(x,y) \in K \times X$ such that

$$F(x,y) + \langle z, x-y\rangle \ge 0 \quad \forall y \in K.$$

This can be rewritten as

$$F(x,y) + \lambda^{-1}\langle x - (x+\lambda z), y-x\rangle \ge 0 \quad \forall y \in K,$$

which in the light of (1.3) gives $x = J_\lambda^F(x+\lambda z)$. By setting $x_\lambda = x + \lambda z$ and

$z_\lambda = z$, the following inequality is always satisfied

$$F_\lambda(x_\lambda, y) + \langle z_\lambda, x_\lambda - y\rangle \geq 0 \quad \forall y \in K.$$

Furthermore, we have $x_\lambda \to x$ and $z_\lambda \to z$, that is $F = V - \lim_{\lambda\to 0} F_\lambda$. □

It is worth mentioning that some similar results have been proved by Brézis [5] in the context of convex functions and maximal monotone operators.

Let us now describe the variational convergence of sequences of bivariate functions with the help of their resolvents.

Proposition 2.3. *For any sequence $\{F, F_n;\ n \in \mathbb{N}\}$ the following equivalences hold true:*

(i) $F = V - \lim_{n\to\infty} F_n$;

(ii) $\forall \mu > 0$, $\forall x \in X$, $J_\mu^F(x) = \lim_{n\to\infty} J_\mu^{F_n}(x)$;

(iii) $\exists \mu_0 > 0$, $\forall x \in X$, $J_{\mu_0}^F(x) = \lim_{n\to\infty} J_{\mu_0}^{F_n}(x)$.

Proof. (iii) ⇒(i), let $x \in K$, $z \in X$ such that

$$F(x, y) + \langle z, x - y\rangle \geq 0 \quad \forall y \in K,$$

this can be rewritten as

$$F(x, y) + \mu_0^{-1} \langle x - (\mu_0 z + x), y - x\rangle \geq 0 \quad \forall y \in K, \tag{2.10}$$

from which we infer that

$$x = J_{\mu_0}^F(x + \mu_0 z).$$

By invoking (ii), we have

$$x = \lim_{n\to\infty} x_n \quad \text{with} \quad x_n := J_{\mu_0}^{F_n}(x + \mu_0 z),$$

it then follows from the definition of the resolvent that

$$F_n(x_n, y) + \langle z_n, x_n - y\rangle \geq 0 \quad \forall y \in K, \tag{2.11}$$

with

$$z_n = \frac{x - x_n}{\mu_0} + z \text{ and } x_n \to x,\ z_n \to z,$$

which immediately yields that $F = V - \lim_{n\to\infty} F_n$.

(i) ⇒(ii) By setting $z := J^F_\mu x$, we can write

$$F(z, y) + \left\langle \frac{x - z}{\mu}, x - y \right\rangle \geq 0 \quad \forall y \in K. \tag{2.12}$$

Hypothesis (i) ensures the existence of (z_n, v_n) satisfying

$$F_n(z_n, y) + \langle v_n, z_n - y\rangle \geq 0 \quad \forall y \in K \tag{2.13}$$

with

$$z_n \to z \text{ and } v_n \to \frac{x - z}{\mu}.$$

Equation (2.13) is equivalent to

$$z_n = J^{F_n}_\mu (z_n + \mu v_n).$$

Finally, since

$$\left|z_n - J^{F_n}_\mu (x)\right| = \left|J^{F_n}_\mu (z_n + \mu v_n) - J^{F_n}_\mu (x)\right| \leq |z_n - x + \mu v_n|$$

and

$$\lim_{n\to\infty} (z_n - x + \mu v_n) = z - x + \mu \frac{x - z}{\mu} = 0, \tag{2.14}$$

we obtain the desired result.

(ii) ⇒(iii) obvious. □

This extends an earlier result by Attouch (see for example [2]).

3. A recession concept. To begin with, let us highlight the important relationship between the Yosida approximate of F and the resolvent of F^{-1}.

Lemma 3.1. *Let F be a given bifunction, then*

$$A^F_\mu (x) = J^{F^{-1}}_{\mu^{-1}} \left(\frac{x}{\mu}\right) \quad \forall \mu > 0 \;\; \textit{and } x \in X. \tag{3.15}$$

P r o o f. From the definition of the resolvent we obtain

$$F(J_\mu^F(x),y)+\langle A_\mu^F(x),J_\mu^F(x)-y\rangle\geq 0,$$

which is equivalent to

$$F(A_\mu^F(x),y)+\langle J_\mu^F(x),A_\mu^F(x)-y\rangle\geq 0, \tag{3.16}$$

Taking into account the fact that $x=J_\mu^F(x)+\mu A_\mu^F(x)$, it follows from the definition of F^{-1} that (3.16) can be rewritten as

$$F^{-1}(A_\mu^F(x),y)+\langle x-\mu A_\mu^F(x),A_\mu^F(x)-y\rangle\geq 0.$$

Thus

$$F^{-1}(A_\mu^F(x),y)+\mu\left\langle\frac{x}{\mu}-A_\mu^F(x),A_\mu^F(x)-y\right\rangle\geq 0,$$

from which we deduce the announced result. □

Now we address the following question: does the filtered sequence $\{F_t:=F(t.,.)\}$ variational converges as $t\to+\infty$?

Theorem 3.1. *Let F be a bivariate function. Then the variational limit of the sequence $\{F(t.,.);\ t\to+\infty\}$ exists, we write $F_\infty=V-\lim_{t\to+\infty}F_t$. F_∞ is still a bivariate function satisfying $(\mathcal{H})$. More precisely, it is given by*

$$F_\infty(x,y)=i^*_{\mathrm{Dom}\,F^{-1}}(y)-i^*_{\mathrm{Dom}\,F^{-1}}(x), \tag{3.17}$$

where $i^*_{\mathrm{Dom}\,F^{-1}}$ stands for the conjugate of the indicatrice function of $\mathrm{Dom}\,F^{-1}$*. The associated recession function of F is $f_\infty^F:=i^*_{\mathrm{Dom}\,F^{-1}}$ and we have:*

$$F_\infty(x,y)=f_\infty^F(y)-f_\infty^F(x).$$

P r o o f. A simple calculation involving the definition of the resolvent shows that

$$J_1^{F_t}(x)=\frac{1}{t}J_t^F(tx). \tag{3.18}$$

According to Proposition 2.3, it suffices to show the existence of the $\lim_{t\to+\infty}J_1^{F_t}(x)$, for every $x \in X$. Thanks to (3.18), we can write

$$x - J_1^{F_t}(x) = \frac{1}{t}(tx - J_t^F(tx)).$$

Using Lemma 3.1, we obtain

$$x - J_1^{F_t}(x) = J_{t^{-1}}^{F^{-1}}(x). \tag{3.19}$$

Letting $t \to +\infty$ in (3.19) and using Proposition 2.1, we infer

$$\lim_{t\to+\infty} J_1^{F_t}(x) = x - \mathrm{proj}_{\overline{\mathrm{Dom}\, F^{-1}}}(x). \tag{3.20}$$

Then it is easy to check that

$$p := \mathrm{proj}_{\overline{\mathrm{Dom}\, F^{-1}}}(x) = J_1^{\Psi}(x) \text{ with } \Psi(x,y) = i_{\overline{\mathrm{Dom}\, F^{-1}}}(y) - i_{\overline{\mathrm{Dom}\, F^{-1}}}(x).$$

Indeed, the characterization of the projection yields

$$i_{\overline{\mathrm{Dom}\, F^{-1}}}(y) - i_{\overline{\mathrm{Dom}\, F^{-1}}}(p) + \langle p - x, y - p\rangle \geq 0 \quad \forall y \in \overline{\mathrm{Dom}\, F^{-1}},$$

that is $p = J_1^{\Psi}(x)$.

This with (3.20) and Lemma 3.1 gives the desired result. □

Now we are going to show how the famous classical recession formulas can be deduced from Theorem 3.1.

Proposition 3.1. *(i) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $F(x,y) = f(y) - f(x)$. Then $f_\infty^F = f_\infty$.*

(ii) Let A be a maximal monotone operator and $F(x,y) = sup_{z\in Ax}\langle z, y - x\rangle$, then it holds that $f_\infty^F = f_\infty^A$.

P r o o f. (i) It is easy to check that $x \in \mathrm{Dom}\, F^{-1}$ if and only if $x \in \mathrm{Dom}\, \partial f^*$ where f^* stands for the conjugate function of f and ∂f^* its subdifferential operator. On the other hand it is well-known that $\overline{\mathrm{Dom}\, \partial f^*} = \mathrm{Dom}\, f^*$. Thanks to [7, Propositon 6.8.5] we immediately obtain that $f_\infty^F(x) = \sup_{y \in dom f^*} \langle y, x\rangle$.

(ii) From the definition of Dom F^{-1}, we have that $\mathrm{Dom}\, F^{-1} = \mathrm{Dom}\, A^{-1}$. Since $\mathrm{Dom}\, A^{-1} = R(A)$, this implies that $f_\infty^F = f_\infty^A$. □

Note that the results above justify the "recession function" appellation for f_∞^F.

We end this section with a characterization of the solvability of the problem (1.1).

Proposition 3.2. *The solvability of* (1.1) *is equivalent to the property for the sequence* $\{x_\varepsilon\}$*, defined as*

$$F(x_\varepsilon, y) + \varepsilon \langle x_\varepsilon, x - x_\varepsilon \rangle \geq 0 \quad \forall x \in K, \tag{3.21}$$

to be bounded.

Proof. Indeed, assume that the sequence $\{x_\varepsilon\}$ remains bounded and let $\widetilde{x}$ be any weak cluster point of $\{x_\varepsilon\}$. Using (3.21) and to the fact that F is monotone, we can write

$$-F(x, x_\varepsilon) + \langle \varepsilon x_\varepsilon, x - x_\varepsilon \rangle \geq 0 \quad \forall x \in K.$$

Passing to the limit, on a subsequence, in the last inequality and according to the weak lower semicontinuity of F, we obtain

$$F(x, \widetilde{x}) \leq 0 \; \forall x \in K. \tag{3.22}$$

Now, let $x_t = tx + (1-t)\widetilde{x}, 0 < t \leq 1$. From the properties of F, it follows that for all t

$$0 = F(x_t, x_t) \leq tF(x_t, x) + (1-t)F(x_t, \widetilde{x}) \leq tF(x_t, x).$$

Dividing by t and letting $t \downarrow 0$, we obtain $x_t \to \widetilde{x}$ which together with the upper hemicontinuity of F yields

$$F(\widetilde{x}, x) \geq 0 \; \forall x \in K.$$

That is $\widetilde{x}$ is a solution to the problem (1.1).

The converse is true, that is, if (1.1) has a solution, then the sequence $\{x_\varepsilon\}$ remains bounded: Assume $\overline{x}$ is a solution of (1.1). By setting $x = \overline{x}$ in (3.21), $x = x_\varepsilon$ in (1.1) and adding the resulting inequalities, we obtain

$$F(\overline{x}, x_\varepsilon) + F(x_\varepsilon, \overline{x}) + \langle \varepsilon x_\varepsilon, \overline{x} - x_\varepsilon \rangle \geq 0.$$

Thanks to the monotonicity of F, we obtain

$$\|x_\varepsilon\| \leq \|\overline{x}\| \,.$$

This implies the boundedness of the sequence $\{x_\varepsilon\}$. In fact we can show that $\{x_\varepsilon\}$ strongly converges to the element of minimal norm of the solution set of (1.1). □

Remark 3.1. It is an interesting question to give an existence result for problem (1.1) under compatibility conditions involving f^F_∞ and its kernel. This is naturally suggested by previous results obtained by Baiocchi, Buttazzo, Gastaldi and Tomarelli [3] in the convex case, by Attouch, Chbani & Moudafi [2] for general variational problems and by Adly, Goeleven and Théra [1] in the context of noncoercive variational inequalities.

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