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ON FINITE ELEMENT METHODS FOR 2ND ORDER (SEMI–)PERIODIC EIGENVALUE PROBLEMS

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Abstract. We deal with a class of elliptic eigenvalue problems (EVPs) on a rectangle $\Omega \subset \mathbb{R}^2$, with periodic or semi–periodic boundary conditions (BCs) on $\partial\Omega$. First, for both types of EVPs, we pass to a proper variational formulation which is shown to fit into the general framework of abstract EVPs for symmetric, bounded, strongly coercive bilinear forms in Hilbert spaces, see, e.g., [13, §6.2]. Next, we consider finite element methods (FEMs) without and with numerical quadrature. The aim of the paper is to show that well–known error estimates, established for the finite element approximation of elliptic EVPs with *classical* BCs, hold for the present types of EVPs too. Some attention is also paid to the computational aspects of the resulting algebraic EVP. Finally, the analysis is illustrated by two non-trivial numerical examples, the exact eigenpairs of which can be determined.

1. Introduction. In this paper we consider a class of elliptic EVPs on a rectangle $\Omega \subset \mathbb{R}^2$,with periodic or semi-periodic BCs on $\partial\Omega$. Such EVPs may

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arise in the mathematical model of various physical problems. In particular, note that (semi–)periodic EVPs on a rectangle are basic to the study of EVPs in $\mathbb{R}^2$ with periodic coefficient functions, see, e.g., [9, §2.2] and [14, §7.1]. An example of the latter is the 2D-Schrödinger equation for an electron in a crystalline solid, cf. [6, §III.2.1]. By Fourier's expansion method, they may also be related to linear transient boundary value problems (BVPs) with (semi-)periodic BCs. A recent example of such a transient BVP is considered in [10]. A classical reference where EVPs and BVPs with periodic BCs play an important role is [15].

Without loss of generality, the rectangle Ω may be taken to be the unit square in $\mathbb{R}^2$. Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be its sides, numbered as in Fig. 1.1.

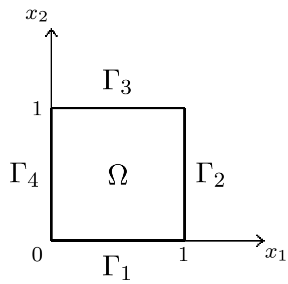

Fig. 1.1 The domain Ω

To fix the ideas, we consider the following model problem:

(P) : Find a pair $[\lambda, u] \in \mathbb{R} \times H^2(\Omega)$ which obeys, in a weak sense, the differential equation

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + a_0 u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

together with the BCs

$$u|_{\Gamma_1} = \alpha u|_{\Gamma_3}, \ \frac{\partial u}{\partial \nu_a}|_{\Gamma_1} = -\alpha \frac{\partial u}{\partial \nu_a}|_{\Gamma_3}, \tag{1.2}$$

$$u|_{\Gamma_2} = \alpha u|_{\Gamma_4}, \ \frac{\partial u}{\partial \nu_a}|_{\Gamma_2} = -\alpha \frac{\partial u}{\partial \nu_a}|_{\Gamma_4}, \tag{1.3}$$

where

$$\frac{\partial u}{\partial \nu_a} = \sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} n_i,$$

is the conormal derivative associated to the matrix (a_{ij}), n_i being the i-th component of the outward unit normal vector $\overline{n}$ to $\partial\Omega$. Here, $\alpha = 1$ or $\alpha = -1$, corresponding to periodic and semi–periodic BCs respectively. The data a_{ij}, a_0 obey the following conditions:

$$(1.4)\quad a_{ij}(x) \in L_\infty(\Omega); \qquad \exists\beta > 0, \forall\xi \in \mathbb{R}^2 : \sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j \geq \beta|\xi|^2 \text{ a.e. in } \Omega,$$

$$(1.5)\qquad a_{12}(x) = a_{21}(x) \text{ a.e. in } \Omega,$$

$$(1.6)\quad a_0(x) \in L_\infty(\Omega); \qquad \exists a > 0 : a_0(x) \geq a \text{ a.e. in } \Omega.$$

In the next section, we bring the EVP stated above into a suitable variational form, choosing a proper subspace V^α of the first order Sobolev space $H^1(\Omega)$ as the space of trial– and test functions. The variational EVP is shown to be formally equivalent to the underlying differential one and, moreover, to fit into the general framework of EVPs for coercive, symmetric and bounded bilinear forms in Hilbert spaces, see, e.g., [13, §6.2] and [3, §5.1].

In Section 3 we prove the density of $H^2(\Omega) \cap V^\alpha$ in V^α, a result which is crucial for the convergence of the FEMs considered.

In Section 4 we introduce suitable finite element approximation spaces V_h^α of the respective V^α. For the approximate eigenpairs, either obtained by the standard *consistent mass* FEM or by a *numerical quadrature* FEM, well–known error estimates obtained for EVPs with classical BCs (see, e.g., [13, §6.5], [3, §5.3-5.4], [2, Ch.II] and [16]) are shown to remain valid for the present type of EVP and of approximation space. Moreover, by a proper choice of the canonical basis of V_h^α, the stiffness and mass matrices, which appear in the resulting algebraic EVP, can be constructed in a transparent way starting from the respective matrices for an EVP with Neumann BCs.

Finally, in Section 5, we illustrate the analysis by two examples, the exact eigenvalues of which can be found. The theoretical order of convergence of the approximate eigenvalues is confirmed by the results. Moreover, in one of the examples, leaning upon the theory of block (skew) circulant matrices, see, e.g., [7, §5.8], we obtain the approximate eigenpairs from the consistent mass FEM in closed form. The resulting analytical formulae for the approximation error reveal the precise dependence of this error on the number of the exact eigenvalues (numbered in increasing order of magnitude). For numerical algorithms for algebraic eigenproblems associated with periodic BCs (in one space dimension), we refer to [4].

The natural norm and seminorm in the r-th order Sobolev space $H^r(\Omega) = W^{2,r}(\Omega)$, $r \in \mathbb{N}_0$, will be denoted as $\|\cdot\|_{r,\Omega}$ and $|.|_{r,\Omega}$ respectively; $|\cdot|_{0,\Omega}$ denotes the L_2-norm.

2. Variational EVPs.

2.1. Variational formulation. Let γ denote the usual trace–operator. Choosing

$$V^\alpha = \{v \in H^1(\Omega) \,|\, \gamma v|_{\Gamma_1} = \alpha\, \gamma v|_{\Gamma_3},\, \gamma v|_{\Gamma_2} = \alpha\, \gamma v|_{\Gamma_4}\}, \tag{2.1}$$

as the space of trial– and test functions, the variational formulation of the EVP (P) reads

$$(P_{var}): \text{ Find } [\lambda, u] \in \mathbb{R} \times V^\alpha : \ a(u,v) = \lambda(u,v)_{L_2(\Omega)}, \quad \forall v \in V^\alpha, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0 uv \right) dx, \quad \forall u, v \in V^\alpha. \tag{2.3}$$

Notice that V^1 is the space $H^1_{per}(\Omega)$, defined in [15, §1.3].

2.2. Formal equivalence. We aim at proving the *formal* equivalence of the variational and the classical EVP. To this end, we need two successive auxiliary lemmas. First, by a slight adaptation of a classical argument, see, e.g., [13, Théorème 1.4–3], we have

Lemma 2.1. *Let D be a bounded domain of $\mathbb{R}^n$ with Lipschitz–continuous boundary. Let Γ be a Lipschitz–continuous curve dividing D into the subdomains D_1 and D_2. Let $\Gamma = \overline{D}_1 \cap \overline{D}_2$ and $D = D_1 \cup D_2 \cup \Gamma$. Consider functions $w_1 \in H^1(D_1)$ and $w_2 \in H^1(D_2)$ for which $\gamma w_1|_\Gamma = \gamma w_2|_\Gamma$. Then the function w, defined a.e. on D by $w|_{D_1} = w_1$, $w|_{D_2} = w_2$, belongs to $H^1(D)$.*

Leaning upon the lemma above, we may show

Lemma 2.2. *The space $W = \{\gamma v|_{\Gamma_1} \,|\, v \in V^\alpha,\ \gamma v|_{\Gamma_2 \cup \Gamma_4} = 0\}$ is dense in $L_2(\Gamma_1)$.*

Proof. We start from the density of the space $W' = \{\gamma v|_{\Gamma_1} \,|\, v \in H^1(\Omega),$ $\gamma v|_{\Gamma_2 \cup \Gamma_4} = 0\}$ in $L_2(\Gamma_1)$, see, e.g., [18, $\mathcal{P}$94], and we show that $W' \subset W$. Let $w \in$

W', then, by definition, $w = \gamma v|_{\Gamma_1}$, for a function $v \in H^1(\Omega)$, with $\gamma v|_{\Gamma_2 \cup \Gamma_4} = 0$. Now, put

$$v'(x_1, x_2) = \begin{cases} v(x_1, 2x_2), & (x_1, x_2) \in]0,1[\times \left]0, \dfrac{1}{2}\right[, \\ \alpha v(x_1, 2(1 - x_2)), & (x_1, x_2) \in]0,1[\times \left]\dfrac{1}{2}, 1\right[. \end{cases}$$

Clearly, $v' \in H^1\left(]0,1[\times \left]0, \frac{1}{2}\right[\right)$ and $v' \in H^1\left(]0,1[\times \left]\frac{1}{2}, 1\right[\right)$. Moreover, in an obvious notation, $\gamma v'|_{\underset{\uparrow}{\Gamma}} = \gamma v'|_{\overset{\downarrow}{\Gamma}} = 0$, where Γ is the midline $x_2 = \dfrac{1}{2}$ of Ω. Thus, from Lemma 2.1, $v' \in H^1(\Omega)$. Moreover, *by construction*, $\gamma v'|_{\Gamma_2 \cup \Gamma_4} = 0$ and $\gamma v'|_{\Gamma_1} = \alpha \gamma v'|_{\Gamma_3}$, while $\gamma v'|_{\Gamma_1} = \gamma v|_{\Gamma_1} = w$. Hence, $w \in W$. □

Invoking Lemma 2.2 in order to recover the periodic BCs (1.3) from (2.2), we arrive at

Theorem 2.1. *The EVPs* (P), (1.1)–(1.3), *and* (P_{var}), (2.2), *are* formally *equivalent.*

2.3. Existence of exact eigenpairs. We first state some important, but direct properties of the bilinear form $a(\cdot,\cdot)$ and the space of trial and test functions V^α.

Proposition 2.1.

1. *Under the assumptions* (1.4)–(1.6), *the bilinear form* $a(\cdot,\cdot)$, (2.3), *is bounded, symmetric and strongly coercive on* $H^1(\Omega) \times H^1(\Omega)$.
2. V^α, (2.1), *is densily and compactly embedded in* $L_2(\Omega)$. *Moreover,* V^α *is a closed subspace of* $H^1(\Omega)$.

Proposition 2.1 allows us to recast the problems (P_{var}) into the framework of abstract elliptic EVPs in Hilbert spaces, considered, e.g., in [13, §6.2], from which it directly follows that

Theorem 2.2.

1. *The problem* (P_{var}), (2.2), *has an infinite number of eigenvalues, all being strictly positive, with finite multiplicity and without a finite accumulation point,*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty\,.$$

Here, each eigenvalue occurs as many times as given by its multiplicity.

2. The corresponding eigenfunctions u_1, u_2, ..., can be chosen to be orthonormal in $L_2(\Omega)$. They constitute a Hilbert basis for V^α as well as for $L_2(\Omega)$.

3. A density property in the space V^α. In this section we deal with a density property in V^α. In view of the convergence of the finite element methods for the EVP (2.2), such a property is crucial, cf. [13, §6.5].

Theorem 3.1. *$H^2(\Omega) \cap V^\alpha$ is dense in V^α ($\alpha = -1$ or $\alpha = 1$).*

Proof. Let $u \in V^\alpha$ be arbitrary. We will construct a sequence of functions $(\tilde{u}_\delta)_{\delta\to 0}$ in $C^\infty(\overline{\Omega}) \cap V^\alpha$ such that

$$\|u - \tilde{u}_\delta\|_{1,\Omega} \to 0 \quad \text{as } \delta \to 0\,. \tag{3.1}$$

Step 1. Consider the square $\tilde{\Omega} =]-a, 1+a[\times]-a, 1+a[\supset \Omega$. We want to construct a function $\tilde{u} \in H^1(\tilde{\Omega})$, with $\tilde{u}|_\Omega = u$. To this end, we first put

$$\tilde{u}(x_1,x_2) = \begin{cases} u(x_1,x_2), & (x_1,x_2) \in \Omega\,, \\ \alpha\, u(x_1-1,x_2), & (x_1,x_2) \in\,]1,1+a[\times]0,1[\equiv \tilde{S}\,. \end{cases}$$

As $u \in H^1(S)$, $S \equiv]0,a[\times]0,1[$, we have $\tilde{u} \in H^1(\tilde{S})$. From the periodicity of u, we get $\gamma\tilde{u}|_{\to\Gamma_2} = \gamma\tilde{u}|_{\Gamma_2\leftarrow}$. Hence, Lemma 2.1 assures that $\tilde{u} \in H^1(]0,1+a[\times]0,1[)$. Proceeding in an analogous way for the other 3 strips which may be 'glued' to Ω, we construct a function $\tilde{u} \in H^1(\tilde{\Omega}_1)$, with $\tilde{u}|_\Omega = u$, where $\tilde{\Omega}_1$ is indicated in Fig. 3.1(a). We still need to define $\tilde{u}$ in the four remaining squares at the corners of $\tilde{\Omega}$. First, take

$$\tilde{u}(x_1,x_2) = u(x_1-1,x_2-1), \quad (x_1,x_2) \in]1,1+a[\times]1,1+a[\equiv \tilde{Q}. \tag{3.2}$$

Then $\tilde{u} \in H^1(\tilde{Q})$. Moreover, by the construction of $\tilde{u}$ and by the periodicity of u, we get, with Γ_2' and Γ_3' being the open segments shown in Fig 3.1(b),

$$\gamma\tilde{u}|_{\to\Gamma_2'} = \gamma\tilde{u}|_{\Gamma_2'\leftarrow}, \quad \gamma\tilde{u}|_{\substack{\Gamma_3'\\ \uparrow}} = \gamma\tilde{u}|_{\substack{\downarrow\\ \Gamma_3'}}\,.$$

Hence, from Lemma 2.1, we get that $\tilde{u} \in H^1(\tilde{\Omega}_2)$, where $\tilde{\Omega}_2 \supset \tilde{\Omega}_1$ is indicated in Fig. 3.1(c). Repeating this procedure for the 3 remaining corner squares of $\tilde{\Omega}$, i.e. taking

$$\tilde{u}(x_1,x_2) = \begin{cases} u(x_1+1,x_2+1), & (x_1,x_2) \in]-a,0[\times]-a,0[\,, \\ u(x_1-1,x_2+1), & (x_1,x_2) \in]1,1+a[\times]-a,0[\,, \\ u(x_1+1,x_2-1), & (x_1,x_2) \in]-a,0[\times]1,1+a[\,, \end{cases}$$

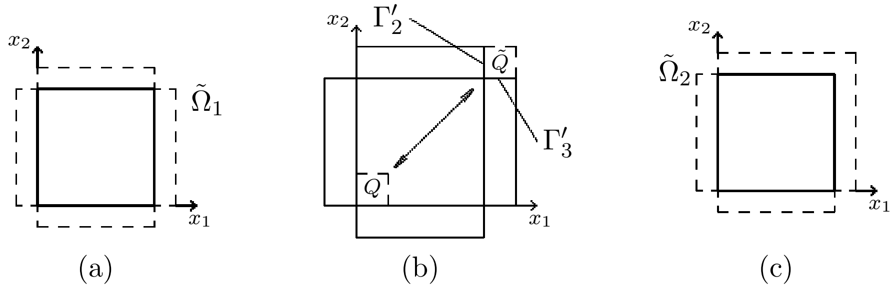

Fig. 3.1 (a) The domain $\tilde\Omega_1$ (enclosed by the dotted line);
(b) The 'diagonal translation' (3.2); (c) The domain $\tilde\Omega_2$ (enclosed by the dotted line)

we arrive at a function $\tilde u \in H^1(\tilde\Omega)$, with $\tilde u|_\Omega = u$, by Lemma 2.1.

STEP 2. Next, we use a well–known regularisation argument for L_2–functions (see, e.g., [12, p. 58–60]), in order to construct a sequence of functions $(\tilde u_\delta)_{\delta\to 0}$ in $C^\infty(\overline{\Omega})\cap V^{\alpha=1}$, whichs shows the desired property (3.1). Consider the function $\tilde u_\delta$, $\delta < a$, defined on $\overline{\Omega}$ by

$$(3.3)\quad \tilde u_\delta(x) = \int_{|x-\xi|\le\delta} \omega_\delta(|x-\xi|)\tilde u(\xi)\,d\xi, \qquad x=(x_1,x_2)\in\overline{\Omega},\ \xi=(\xi_1,\xi_2),$$

where ω_δ is a standard regularising kernel. As $\tilde u_\delta \in C^\infty(\overline{\Omega})$, $\forall\delta < a$, and $\tilde u_\delta \to u$ in $H^1(\Omega)$, for $\delta\to 0$, it only remains to argue that $\tilde u_\delta \in V^\alpha$. To fix the ideas, consider two corresponding points $x\in\Gamma_1$ and $x'\in\Gamma_3$. Then $\tilde u_\delta(x) = \alpha\tilde u_\delta(x')$, as *by construction*, the function $\tilde u$ takes the same (resp. opposite) values in each pair of points ξ and ξ' of $\tilde\Omega$, (including the corner regions), related to each other by a horizontal translation over the distance 1. $\square$

Remark 3.1. The density result, proved above, can also be read as

$$V^\alpha = \overline{W^\alpha}, \quad \text{where } W^\alpha = \{v\in C^\infty(\overline{\Omega}) \mid v|_{\Gamma_1} = \alpha v|_{\Gamma_3},\ v|_{\Gamma_2} = \alpha v|_{\Gamma_4}\},$$

the closure being taken in $H^1(\Omega)$. It is of a similar nature as the well-known –however non-evident– density of $W = \{v \in C^\infty(\overline{\Omega})\,|\, v = 0 \text{ on } \Gamma_1 \subset \partial\Omega\}$ in $V = \{v\in H^1(\Omega)\,|\,\gamma v = 0 \text{ on } \Gamma_1\}$, proved, e.g., in [18, pp.30–31].

4. Finite element approximations.

4.1. The approximation space V_h^α. To fix the ideas, let $(\tau_h)_h$ be a family of triangulations of Ω with identical triangular or rectangular elements K, see Fig. 4.1. Fix $k \in \mathbb{N}_0$. Consider the standard function spaces

$$(4.1) \qquad \begin{aligned} X_h &= \left\{ v \in C_0(\overline{\Omega}) \,|\, v_{|K} \in P(K),\, \forall K \in \tau_h \right\} \qquad \subset H^1(\Omega), \\ X_{0h} &= \left\{ v \in X_h \,|\, v = 0 \text{ on } \partial\Omega \right\}, \end{aligned}$$

where

$$P(K) = \begin{cases} P_k(K) & \text{for triangular elements,} \\ Q_k(K) & \text{for rectangular elements,} \end{cases}$$

$P_k(K)$ standing again for the set of polynomials on K of degree $\leq k$, and $Q_k(K)$ standing for the set of polynomials on K of degree $\leq k$ in each variable.

Let $(a_{i,h})$, $1 \leq i \leq N$, $(N \equiv N(h))$ denote the usual set of all finite element nodes associated with X_h. In each element they are chosen identically, according to the triangle of type k or the rectangle of type k, respectively, see [5, §2.2]. Hence, they will form a rectangular array, with r rows and c columns, say. The numbering of the nodes is done in the natural way, cf. Fig. 4.1 for the case $k = 2$.

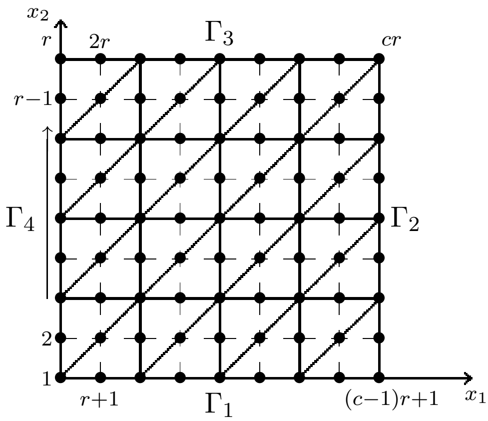

Fig. 4.1. Position and numbering of the finite element nodes in Ω (case $k = 2$)

Let $(\varphi_{i,h})$, $1 \leq i \leq N$, be the canonical basis of X_h, associated to the nodes $a_{i,h}$. To construct a proper subspace of V^α, we introduce some special linear combinations of the basis functions $\varphi_{i,h}$. First, put

$$(4.2) \qquad \psi_{l,h}^{hor,\alpha} = \varphi_{l,h} + \alpha \varphi_{l+(c-1)r,h}, \qquad l = 2, \ldots, r-1.$$

Likewise, introduce

$$(4.3) \qquad \psi_{kr+1,h}^{vert,\alpha} = \varphi_{kr+1,h} + \alpha \varphi_{(k+1)r,h}, \qquad k = 1, \ldots, c-2,$$

and

$$\psi_h^{corner,\alpha} = \varphi_{1,h} + \alpha\varphi_{r,h} + \alpha\varphi_{(c-1)r+1,h} + \varphi_{cr,h}\,. \tag{4.4}$$

One readily obtains

Proposition 4.1. *The space V_h^α, defined by*

$$V_h^\alpha = X_{0h} \oplus span\left(\psi_{l,h}^{hor,\alpha}\right)_{l=2}^{r-1} \oplus span\left(\psi_{kr+1,h}^{vert,\alpha}\right)_{k=1}^{c-2} \oplus span(\psi_h^{corner,\alpha}), \tag{4.5}$$

is a finite dimensional subspace of V^α (with dimension $I = (r-1)(c-1)$).

For simplicity in the notation, from now on, we omit the subindex h when there is no confusion possible.

Proposition 4.2. *The space V_h^α, (4.5), shows the standard approximation property*

$$\begin{aligned} \inf_{v_h \in V_h^\alpha} \{|v - v_h|_{0,\Omega} + h|v - v_h|_{1,\Omega}\} \ \leq\ & Ch^{r+1}||v||_{r+1,\Omega}, \\ & \forall v \in V^\alpha \cap H^{r+1}(\Omega),\ 1 \leq r \leq k. \end{aligned} \tag{4.6}$$

P r o o f. Take $v \in V^\alpha \cap H^{r+1}(\Omega)$ and recall that, by a Sobolev embedding theorem, $H^{r+1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. For the usual Lagrange interpolant $\Pi_h v$ of v in X_h, i.e. $\Pi_h v = \sum_{k=0}^{c-1}\sum_{l=1}^{r} v(a_{kr+l})\varphi_{kr+l}$, we have, invoking the periodic BCs incorporated in V^α, that $\Pi_h v \in V_h^\alpha$, on account of the direct sum (4.5). Then (4.6) follows by classical interpolation error estimates, established, e.g., in [5, §3.2]. □

Let $P : V^\alpha \to V_h^\alpha$ be the elliptic projection operator, associated with $a(\cdot,\cdot)$, i.e.

$$a(v - Pv, w) = 0 \quad \forall v \in V^\alpha, \forall w \in V_h^\alpha. \tag{4.7}$$

As a consequence of (4.6) this projection operator retains the properties of the classical projection operator, established, e.g., in [13, §6.5] for the case of a triangular FE–mesh and in [1], for the case of a rectangular FE–mesh.

4.2. The consistent mass EVP. The consistent mass approximation to the EVP (2.2) reads

$$\text{Find}\quad [\lambda_h, u_h] \in \mathbb{R} \times V_h^\alpha : a(u_h, v_h) = \lambda_h (u_h, v_h)_{L_2(\Omega)} \qquad \forall v_h \in V_h^\alpha, \tag{4.8}$$

where the bilinear form $a(\cdot,\cdot)$ is given by (2.3).

For this EVP, the discrete analogue of Theorem 2.2 holds, the number of eigenvalues (counted with their multiplicity) of course being $I = \dim V_h^\alpha$.

Theorem 3.1 and the properties of the elliptic projection (4.7), implied by Proposition 4.2, ensure that the error analysis, as outlined, e.g., in [13, §6.5] and [16] for EVPs with *classical* BCs, may be rephrased for the present type of EVP and of approximation space. Hence, well-known error estimates for the approximate eigenpairs $[\lambda_h, u_h]$, established there remain valid. Thus, optimally, $\lambda_{l,h} - \lambda_l = O(h^{2k})$. When λ_l is simple, for the corresponding eigenfunctions, (normalised in $L_2(\Omega)$), one optimally has $\|u_{l,h} - u_l\|_{1,\Omega} = O(h^k)$ and $|u_{l,h} - u_l|_{0,\Omega} = O(h^{k+1})$.

For the case of a multiple exact eigenvalue, similar estimates as, e.g., in [16] and [17] can be shown to remain valid, starting from Theorem 3.1 and Proposition 4.2.

4.3. Computational aspects. For a transparent construction of the algebraic version of (4.8), the basis functions of the approximation space V_h^α should be suitably renumbered. To this end, consider again the rectangular array of Fig. 4.1, renumber the nodes not belonging to Γ_2 and Γ_3 in the natural way and denote

$$\begin{aligned} \phi_1 &= \psi^{corner,\alpha}, \\ \phi_l &= \psi_l^{hor,\alpha}, \qquad l = 2, \ldots, r-1, \end{aligned}$$

and

$$\left.\begin{aligned} \phi_{k(r-1)+1} &= \psi_{kr+1}^{vert,\alpha}, \\ \phi_{k(r-1)+1+j} &= \varphi_{kr+1+j}, \quad j = 1, \ldots, r-2, \end{aligned}\right\} \quad k = 1, \ldots, c-2,$$

where the functions $\psi_l^{hor,\alpha}$, $\psi_{kr+1}^{vert,\alpha}$ and $\psi^{corner,\alpha}$ are defined by (4.2)–(4.4). Then, $(\phi_i)_{i=1}^I$, $I = (c-1)(r-1)$, constitutes a basis of V_h^α, (4.5). Putting $u_h = \sum_{i=1}^{I} c_{ih}\phi_i$, the EVP (4.8) may be rewritten in algebraic form as:

$$\text{Find } [\lambda_h, c_h] \in \mathbb{R} \times \mathbb{R}^I : \mathcal{K}c_h = \lambda_h \mathcal{M} c_h, \tag{4.9}$$

where $c_h = [c_{1,h}, \ldots, c_{I,h}]^T$ and where $\mathcal{K}$ and $\mathcal{M}$ are the symmetric stiffness and mass matrix respectively, defined in the usual way, i.e. $K_{ij} = a(\phi_i, \phi_j)$, $M_{ij} = (\phi_i, \phi_j)$, $i, j = 1, \ldots, I$.

Due to the foregoing numbering of the basis functions, and due to the particular form of the basis functions associated to the boundary nodes, the stiffness matrix $\mathcal{K}$, may *in practice* be constructed out of the classical stiffness matrix $\mathcal{K}^{cl}$, which corresponds to an EVP with a classical Neumann BC on $\partial\Omega$. Dividing the rows (resp. columns) of the matrix $\mathcal{K}^{cl}$ into c blocks of r rows (resp. columns), the l-th block is seen to correspond to the set of r nodes on the l-th column in the rectangular array of Fig. 4.1. Then the construction of $\mathcal{K}$ is effected as follows. First, add α times the last block of rows to the first one and then omit the last block. Next, within each remaining block, add α times the last row to the first one and omit the last row. Finally, perform the same two operations on the columns.

A similar remark applies to the mass matrix $\mathcal{M}$.

4.4. The effect of numerical quadrature. In practice, instead of (4.8), we consider the EVP

$$\text{(4.10)}\quad \text{Find } [\tilde{\lambda}_h, \tilde{u}_h] \in \mathbb{R} \times V_h^\alpha : a_h(\tilde{u}_h, v_h) = \tilde{\lambda}_h(\tilde{u}_h, v_h)_h \qquad \forall v_h \in V_h^\alpha,$$

where $a_h(\cdot,\cdot)$ and $(\cdot,\cdot)_h$ represent suitable approximations of $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ respectively, resulting from well chosen quadrature formulas, as described in some detail in [16] and the references therein.

On account of Theorem 3.1 and the properties of the elliptic projection (4.7), itselves implied by Proposition 4.2, the error analysis for numerical quadrature finite element methods for EVPs with classical BCs, outlined in [16] and [17], remains valid in the present case. Actually, the estimates for the eigenpairs of (4.10) will be formally the same as those for the eigenpairs of (4.8). However, to preserve the same rates of convergence, higher regularity of the data must be imposed.

5. Numerical examples. We aim at comparing the numerical eigenvalues with the analytical ones. This requires the example to be chosen in such a way that the analytical eigenvalues can indeed be computed. We begin with an EVP for the Laplacian where an explicit expression of the approximate eigenvalues, and hence of the error, can be found. To this end we apply the theory of (skew)circulant matrices. In the 2nd EVP, involving Bessel functions, we include numerical quadrature.

5.1. A problem for the Laplacian. Let Ω be the square of Fig. 1.1 and consider the following EVP with *semi–periodic BCs*:

$$
\text{(5.1)} \quad \text{Find } [\lambda, u] \in \mathbb{R} \times H^2(\Omega) \ : \quad \begin{cases} -\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2} = \lambda u \quad \text{in } \Omega \\ u|_{\Gamma_1} = -u|_{\Gamma_3} \, , \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_1} = \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_3} \\ u|_{\Gamma_2} = -u|_{\Gamma_4} \, , \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_2} = \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_4} \, . \end{cases}
$$

The exact eigenvalues of (5.1) are found to be

$$
\lambda_{m,l} = \big((2m+1)^2 + (2l+1)^2)\big)\pi^2 \qquad m, l \in \mathbb{N}. \tag{5.2}
$$

where $\lambda_{m,m}$ is 4–fold, and $\lambda_{m,l} = \lambda_{l,m}$, $m \neq l$, is 8–fold.

For the consistent mass FEM, we choose a *bilinear* mesh with N^2 square elements (i.e. $c = r = N + 1$ in the notations of Section 4.1). By the method described in §4.3, the stiffness– and mass matrix are found to be *skew block circulant matrices with skew circulant blocks*, cf. [7, p. 212–214]). Hence the eigenvalues of $\mathcal{M}^{-1}\mathcal{K}$ can be found in the following closed form:

$$
\lambda^h(m,l) \ = \ 3N^2 \frac{2 - \frac{1}{2}\left(\cos\frac{(2m+1)\pi}{N} + \cos\frac{(2l+1)\pi}{N}\right) - \cos\frac{(2m+1)\pi}{N}\cos\frac{(2l+1)\pi}{N}}{1 + \frac{1}{2}\left(\cos\frac{(2m+1)\pi}{N} + \cos\frac{(2l+1)\pi}{N}\right) + \frac{1}{4}\cos\frac{(2m+1)\pi}{N}\cos\frac{(2l+1)\pi}{N}},
$$

$$
m, l = 0, \ldots, N-1, \tag{5.3}
$$

where $\lambda^h(m,m)$ is 4–fold, and $\lambda^h(m,l) = \lambda^h(l,m)$ is 8–fold for $m \neq l$, except when $m = \dfrac{N-1}{2}$ or $l = \dfrac{N-1}{2}$ in the case that N is odd. Hence, for simplicity of notations, let N be even.

From (5.2)–(5.3) we easily find

$$
\lambda_{m,l} - \lambda^h_{m,l} \ = \ -\frac{1}{24}\left((2k+1)^2 + (2l+1)^2\right)\pi^4 h^2 + \mathcal{O}(h^4) \ \longrightarrow \ 0 \ \text{ for } \ h \to 0,
$$

$$
m, l = 0, \ldots, \frac{N}{2} - 1 \, . \tag{5.4}
$$

Thus, taking into account that $\lambda_{m,l}$ and $\lambda^h_{m,l}$ have the same multiplicity, by a classical argument, see, e.g., [11, p. 290], $\lambda^h_{m,l}$ turns out to be the consistent mass finite element approximation of the exact eigenvalue $\lambda_{m,l}$, $m, l = 0, \ldots, \dfrac{N}{2} - 1$.

Moreover, (5.4) confirms the theoretical order of convergence, viz. $\mathcal{O}(h^2)$, for the case of a bilinear mesh, as stated in Section 4.2. Again, $\lambda_{m,l}^h$ approximates $\lambda_{m,l}$ from above, as it should. Furthermore, the approximation error of the eigenvalues is seen to increase 'quadratically' with the number of the eigenvalues, numbered in increasing order of magnitude.

5.2. Consistent mass and numerical quadrature FEM for a problem with nonconstant coefficients. Let again Ω be the square of Fig. 1.1 and consider the following EVP with *periodic BCs*:

$$(5.5)\ \text{Find } [\lambda, u] \in \mathbb{R} \times H^2(\Omega) \ : \quad \begin{cases} -\dfrac{\partial}{\partial x_1}\left(4^{x_1}\dfrac{\partial u}{\partial x_1}\right) - \dfrac{\partial^2 u}{\partial x_2^2} = \lambda u \quad \text{in } \Omega \\ u|_{\Gamma_1} = u|_{\Gamma_3}\,, \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_1} = -\dfrac{\partial u}{\partial \nu_a}|_{\Gamma_3} \\ u|_{\Gamma_2} = u|_{\Gamma_4}\,, \dfrac{\partial u}{\partial \nu_a}|_{\Gamma_2} = -\dfrac{\partial u}{\partial \nu_a}|_{\Gamma_4} \end{cases}$$

Table 5.1. Numerical results for the EVP (5.5).
Case of a bilinear mesh.

	approximate eigenvalue		relative error in %	
N	CM	NQ	CM	NQ
2nd and 3rd exact eigenvalue: 39.478				
4	48.000	32.000	21.59	-18.94
8	41.457	37.490	5.01	-5.03
16	39.988	38.974	1.29	-1.28
32	39.605	39.352	0.32	-0.32
4th exact eigenvalue: 67.671				
4	79.663	51.500	17.72	-23.90
8	70.908	63.301	4.78	-6.46
16	68.490	66.568	1.21	-1.63
32	67.879	67.397	0.31	- 0.40
5th exact eigenvalue: 84.044				
4	112.671	54.617	34.06	-35.01
8	90.304	80.420	7.45	-4.31
16	85.561	83.105	1.81	-1.12
32	84.414	83.802	0.44	-0.29
6th exact eigenvalue: 107.149				
4	127.663	54.617	19.15	-49.03
8	112.454	90.525	4.95	-15.51
16	108.478	102.755	1.24	-4.10
32	107.484	106.038	0.31	-1.04

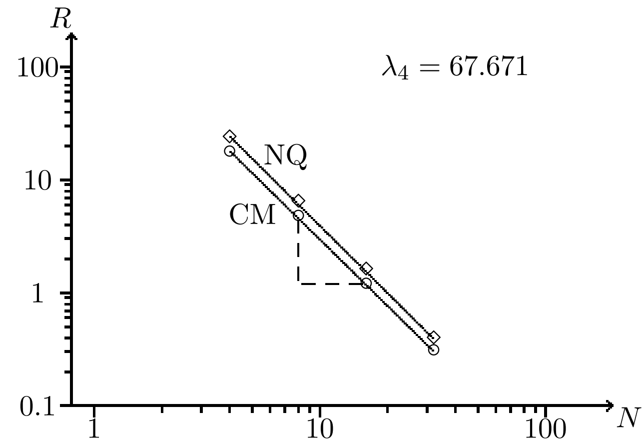

Fig. 5.1. R versus N (logarithmic scale) for λ_4. Case of a bilinear mesh.

By the method of separation of variables, the exact eigenvalues of (5.5) are found to be

$$\lambda_{m,l} = (\xi_m \ln 4)^2 + 4\pi^2 l^2, \quad m \in \mathbb{N}_0,\ l \in \mathbb{N},$$

where ξ_m is the m-th positive root of the equation

$$\begin{aligned}\xi \cdot [&2J_1(2\xi)Y_0(2\xi) - 2J_1(2\xi)Y_0(\xi) - J_1(\xi)Y_0(2\xi) + J_1(\xi)Y_0(\xi) \\ &-2Y_1(2\xi)J_0(2\xi) + 2Y_1(2\xi)J_0(\xi) + Y_1(\xi)J_0(2\xi) - Y_1(\xi)J_0(\xi)] = 0\,,\end{aligned}$$

J_n and Y_n standing for the n-th order Bessel functions of the first and the second kind, respectively, $n = 0, 1$. Here, $\lambda_{m,l}$ is seen to be simple when $l = 0$, and two–fold when $l \neq 0$.

For the finite element approximations, we again choose a *bilinear* mesh with N^2 square elements. For the numerical quadrature FEM, according to Theorem 4.2, we approximate $(\cdot,\cdot)$ by a 4–points formula which is exact for bilinear polynomials and $a(\cdot,\cdot)$ by a 9–points formula which is exact for bicubic polynomials, see, e.g., [8, §5.6]. The mass matrix of the corresponding algebraic EVP is diagonal.

According to the theoretical $\mathcal{O}(h^2)$–convergence, the relative error in the eigenvalues, both for the consistent mass and for the numerical quadrature FEM, is expected to decrease by a factor 4, when N is doubled. This is confirmed by the results shown in Table 5.1 below, as well as by Fig. 5.1, which reveals the relation $\log R \approx C - 2\log N$, R standing for the absolute value of the relative error and C denoting a constant, independent on h. In Table 5.1 we omit the first eigenvalue, viz. 0, as this eigenvalue is recovered exactly by both FEMs.

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