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NEW BOUNDS FOR THE MAXIMUM SIZE OF TERNARY CONSTANT WEIGHT CODES

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Abstract. Optimal ternary constant-weight lexicogarphic codes have been constructed. New bounds for the maximum size of ternary constant-weight codes are obtained. Tables of bounds on $A_3(n, d, w)$ are given for $d = 3, 4, 6$.

1. Introduction. Let $q, n \in N, \quad q \geq 2$. Let Z_q denote the set $\{0, 1, \ldots, q-1\}$, and Z_q^n the set of all n-tuples over Z_q. The Hamming weight of a vector is the number of its nonzero positions. The Hamming distance between two vectors is defined as the number of coordinates in which they differ.

We call any subset C of Z_q^n a q-ary code of length n. The vectors of C are called codewords. An (n, M, d)-code is a q-ary code of length n, containing M codewords and having minimum Hamming distance d. We call a constant-weight code the (n, M, d)-code in which every codeword has Hamming weight w.

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The space of q-ary constant-weight vectors is called the q-ary Johnson space [6]. We denote by $A_q(n,d,w)$ the largest value of M for which there exists a q-ary constant-weight code of length n, minumum distance d and weight w . We call an $(n, A_q(n,d,w), d)$ constant-weight code optimal.

The binary constant-weight codes have been studied in detail [2, 6]. Tables of lower bounds on $A_2(n,d,w)$ can be found in [2]. However, the problem of finding $A_q(n,d,w)$ for alphabet sizes greater than two has not received the same amount of attention. The ternary codes of constant weight have been investigated by Tarnanen [11], Svanström [9, 10], Bogdanova [1].

Lexicographic codes were introduced by Levenshtein [8]. Later they are studied in [5, 3, 12, 7].

For finding new lower bounds we have used greedy algorithms (lexicographic codes) which are described in Section 2. In Section 3 we obtain ternary constant-weight lexicographic codes of length n ($5 \leq n \leq 10$) and a minimum distance $d = 3$, 4 and 6. All the bounds on $A_3(n,3,w)$ given in [9] are improved and new bounds on $A_3(n,4,w)$ are obtained and tables of bounds on $A_3(n,d,w)$ for $d = 3,4,6$ are presented.

2. Methods for construction of lexicographic codes. The new ternary constant-weight codes found in this paper are constructed as lexicographic codes by means of greedy algorithms.

The standard lexicographic codes of length n and Hamming distance d are obtained by starting with a zeroword, considering all q-ary vectors of the given length in lexicographic order, and adding them to the code if they have the desired Hamming distance from it [8, 5].

Several variations are possible.

General lexicographic codes [3, 12] are obtained by considering a list of all q-ary vectors of the given length, but ordered lexicographically with respect to an arbitrary ordered basis instead of the standard basis. For instance the vectors may be considered in Gray code order.

Lexicographic codes with a seed [2, 1] are obtained in a similar way as the lexicographic codes. The difference is that we use an initial set of vectors (called a seed) instead of the empty set. In the most cases lexicographic codes with a seed give good lower bounds which are equal or better to the best known bounds. Obviously they give better bounds than those which we obtain from standard lexicographic codes.

If we consider the constant-weight space, a code constructed in this way is called **a constant-weight lexicographic code**. We construct such codes using the following algorithm.

Let L be a list of all ternary vectors of length n and weight w lexicographically ordered(or another order) and let $\{v\}$ be the seed, where v is a word chosen from L. If we search the list, selecting the next vector of the list if and only if its Hamming distance to each previously chosen word is at least d, then we obtain a code C with minimum distance d.

3. New results. We investigate codes of length n ($5 \le n \le 10$) and a minimum distance d=3, 4 and 6. The upper and lower bounds are given in Table 3.1 ($d = 3$), Table 3.2 ($d = 4$) and Table 3.3 ($d = 6$).

• **Lower bounds**

In this paper we construct ternary constant-weight lexicographic codes and obtain new lower bounds for $A_3(n, d, w)$.

Optimal ternary constant-weight codes in the tables are denoted by a point. Most of the codes were constructed by lexicographic order. We denote by "g" the codes obtained by Gray code order.

All the codes are found using a seed consisting of only one vector. The seeds of the codes are given in the tables. The proper choice of a seed is of great importance for the construction of good codes. To find a seed three methods are used: exhaustive search if possible, random choice, and nonexhaustive search.

By the methods given in section 2 we obtain some lower bounds better than the bounds described in [9, 10]. In Table 3.1 all lexicographic codes with a seed give lower bounds better or equal (only one) to the known bounds (all the lower bounds on $A_3(n, 3, w)$ in [9] are improved). There are seven codes among them that are optimal. New lower bounds on $A_3(n, 4, w)$ are obtained (Table 3.2). Eleven bounds are new and five are equal to the known bounds [10] and there are ten lexicographic codes among them which are optimal. Most of the lower bounds in Table 3.3 are obtained in [10]. We construct these optimal codes using ternary lexicographic codes.

Table 3.1. Lower and upper bounds on $A_3(n,3,w)$

n	w	Upper Bounds	Lower Bounds in [9]	Size of lexicodes	The seeds of lexicodes
4	3	8	—	8.	0111
5	3	12	10	12.	00221
6	3	20	16	18	001101
7	3	28	24	26	0210010g
8	3	37	32	34	20002001
9	3	48	42	44	000012001
10	3	60	54	56	0001000011
5	4	10*	10	10.	01111
6	4	30*	24	30	020212g
7	4	70*	46	54	0200212g
8	4	112	80	89	00011202
9	4	166*	126	134	000011202
10	4	240	186	193	0000010121
5	5	4	3	4.	11111
6	5	24	15	24.	101222
7	5	84	47	70	0011121
8	5	224	106	148	00011222
9	5	403	213	284	000101221
10	5	664*	387	481	0000121011
6	6	8	—	8.	111111
7	6	56	—	56.	1111102g

• **Upper bounds**

For the calculation of the upper bounds which are marked by an $*$ we have used and the following theorems:

Theorem 3.1 (the Johnson bounds for the constant-weight codes) [10]**.** *The maximum number of codewords in a ternary constant-weight code satisfy the inequalities:*

$$A_3(n,d,w) \leq \left\lfloor \frac{n}{n-w} A_3(n-1,d,w) \right\rfloor ; \tag{1}$$

(2) $$A_3(n,d,w) \le \left\lfloor \frac{2n}{w} A_3(n-1,d,w-1) \right\rfloor ;$$

(3) $$A_3(n,d,w) \le \left\lfloor \frac{2nd}{3w^2 - 4nw + 2nd} \right\rfloor \qquad \text{if} \quad 3w^2 - 4nw + 2nd > 0.$$

Table 3.2. Lower and upper bounds on $A_3(n, 4, w)$

n	w	Upper Bounds	Lower Bounds in [10]	Size of lexicodes	The seeds of lexicodes
5	3	5	5	5.	02111
6	3	8*	—	8.	002011
7	3	14	14	14.	0002011
8	3	16*	16	16.	02100010g
9	3	24*	—	24.	020001002g
10	3	26*	—	26.	1000001200
11	3	35*	—	35.	00210002000g
12	3	40*	—	39	000100001100
5	4	5	5	5.	02111
6	4	15	—	15.	002121
7	4	28*	—	23	0002121
8	4	56	—	37	00010221
9	4	72*	—	56	000101021
10	4	120*	—	84	1000021002
6	5	12	12	12.	102111
7	5	42*	—	32	1002121

Theorem 3.2. $A_3(5, 3, 4) = 10$.

Proof. Assume there is a ternary $(5, 11, 3)$ constant-weight code with $w = 4$. Therefore there exist codewords c_1, c_2, c_3 having 0 in the same position, first for instance. Now we have $\sum_{i \neq j} d(c_i, c_j) \le 8$ since the contribution of each one of the last four coordinates is at most 2. On the other hand, $\sum_{i \neq j} d(c_i, c_j) \ge 9$, a contradiction. Hence a ternary $(5, 11, 3)$ constant-weight code with $w = 4$ does not exist and the upper bound is $A_3(5, 3, 4) \le 10$.

The lower bound is $A_3(5,3,4) \geq 10$ (Table 3.1). So $A_3(5,3,4) = 10$. $\square$

The remaining upper bounds in tables 3.1, 3.2, 3.3 are taken from [11, 9, 10].

Table 3.3. Lower and upper bounds on $A_3(n,6,w)$

n	w	Upper Bounds	Lower Bounds in [9]	Size of lexicodes	The seeds of lexicodes
8	4	5	5	5.	00001111
9	4	9	9	9.	000001111
10	4	15	15	15.	1002020002
11	4	16*	—	15	00010000111
12	4	18*	—	17	000200010120
8	5	8	8	8.	00011111
9	5	18	18	18.	102010202
10	5	36	36	36.	1002000122
11	5	66	66	66.	10001010022
12	5	76*	—	66	200002220001
8	6	8	8	8.	00111111
9	6	24	24	24.	210202110
10	6	60	60	60.	2021021100
8	7	4	4	4.	02211111
9	7	18	18	18.	112101210

• **Examples**

Example 1. In Table 3.1 there is a constant-weght $(5,12,3)$-code with $w = 3$. This code is obtained as a standard lexicographic one with a seed (00221). The codewords are

00221, 01011, 01102, 02022, 02110, 10012,
10101, 11020, 12200, 20120, 20202, 22001.

Hence $A_3(5,3,3) \geq 12$. The upper bound is $A_3(5,3,3) \leq 12$ after [10]. So $A_3(5,3,3) = 12$.

Example 2. In Table 3.2 there is a constant-weight $(9, 24, 4)$-code with $w = 3$. This code is obtained as a lexicographic code with a seed (020001002) and the Gray code order (denoted by g) is used. The codewords are

020001002, 020000110, 120010000, 102000001, 100200010, 100002100,
210000020, 201000200, 200100002, 200021000, 000000222, 000002011,
000010101, 000120020, 000101200, 000212000, 001020002, 001001020,
002200100, 002010010, 012002000, 010200001, 010020200, 021100000.

Hence $A_3(9, 4, 3) \geq 24$. The upper bound is $A_3(9, 4, 3) \leq 24$ by (1), because $A_3(8, 4, 3) = 16$. So $A_3(9, 4, 3) = 24$.

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