Serdica J. Computing **2** (2008), 331–348

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXTENDABILITY OF TERNARY LINEAR CODES

Kei Okamoto

Abstract. We give the necessary and sufficient conditions for the extendability of ternary linear codes of dimension $k \geq 5$ with minimum distance $d \equiv 1$ or $2 \pmod 3$ from a geometrical point of view.

1. Introduction. Let $V(n,q)$ denote the vector space of n-tuples over GF(q), the finite field of order q. A linear code C is an $[n,k,d]_q$ code over GF(q) of length n with dimension k whose minimum Hamming distance is d. The *weight* of a vector $\boldsymbol{x} \in V(n,q)$, denoted by $wt(\boldsymbol{x})$, is the number of nonzero coordinate positions in $\boldsymbol{x}$. Let A_i be the number of codewords of C with weight i. We only consider *non-degenerate* codes having no coordinate which is identically zero.

The code obtained by deleting the same coordinate from each codeword of C is called a *punctured code* of C. If there exists an $[n+1,k,d+1]_q$ code C' which gives C as a punctured code, C is called *extendable* (to C') and C' is an *extension* of C. It is well known that a binary linear code with odd d is extendable by adding an overall parity check. The extendability of linear codes has been studied by Hill

ACM Computing Classification System (1998): E.4.
Key words: Ternary linear codes, extensions, diversity, projective spaces.

[1, 2], van Eupen and Lisonek [13], Simonis [12] and Maruta [5, 6, 7, 8]. Recently, Kohnert [3] investigates how to get an $[n+l, k, d+s]_q$ code from a non-extendable $[n, k, d]_q$ code ((l, s)-extension).

Let C be an $[n, k, d]_3$ code with $k \geq 3$, $\gcd(3, d) = 1$. We define three non-negative integers $\Phi_0, \Phi_1, \Phi_{\mathrm{e}}$ as follows:

$$\Phi_0 = \frac{1}{2} \sum_{3|i, i \neq 0} A_i, \quad \Phi_1 = \frac{1}{2} \sum_{i \not\equiv 0, d \pmod 3} A_i, \quad \Phi_{\mathrm{e}} = \frac{1}{2} \sum_{d < i \equiv d (\mathrm{mod}\ 3)} A_i,$$

where the notation $x|y$ means that x is a divisor of y. The pair of integers (Φ_0, Φ_1) is called the *diversity* of C. Let $\mathcal{D}_k$ be the set of all possible diversities of such codes. $\mathcal{D}_k$ has been determined in [8] for $k \leq 6$ and in [10] for $k \geq 7$. For $k \geq 3$, let $\mathcal{D}_k^*$ and $\mathcal{D}_k^+$ be as follows:

$$\begin{aligned} \mathcal{D}_k^* &= \{(\theta_{k-2}, 0), (\theta_{k-3}, 2 \cdot 3^{k-2}), (\theta_{k-2}, 2 \cdot 3^{k-2}), (\theta_{k-2} + 3^{k-2}, 3^{k-2})\}, \\ \mathcal{D}_k^+ &= \mathcal{D}_k \setminus \mathcal{D}_k^*, \end{aligned}$$

where $\theta_j = (3^{j+1} - 1)/2$. It is known that $\mathcal{D}_k^*$ is included in $\mathcal{D}_k$ and that C is extendable if $(\Phi_0, \Phi_1) \in \mathcal{D}_k^*$ ([8]). Hence it suffices to investigate the extendability of C for $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$. It is also known that $\mathcal{D}_3^+ = \{(4, 3)\}$ and that an $[n, 3, d]_3$ code with diversity (4,3) is extendable if and only if $\Phi_{\mathrm{e}} > 0$ ([8]). The necessary and sufficient conditions for the extendability of C with $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$ are given in [10] for $k = 4$ and in [11] for $k = 5$. In this paper, we give the necessary and sufficient conditions for the extendability of an $[n, k, d]_3$ code with $\gcd(3, d) = 1$, general $k \geq 5$, whose diversity is in $\mathcal{D}_k^+$. It is expected that our results would be applicable to (l, s)-extension of ternary linear codes (e.g. see [14]). We also survey the known results about the extendability of ternary linear codes before giving our main theorem (Theorem 4.5).

2. Geometric preliminaries. We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathrm{GF}(q)$. A *j-flat* is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. 0-flats, 1-flats, 2-flats, 3-flats and $(r-1)$-flats are called *points, lines, planes, solids* and *hyperplanes* respectively as usual. We denote by $\mathcal{F}_j$ the set of j-flats of $\mathrm{PG}(r, q)$ and denote by θ_j the number of points in a j-flat, i.e. $\theta_j = |\mathrm{PG}(j, q)| = (q^{j+1} - 1)/(q - 1)$, where $|T|$ denotes the number of elements in T for a given set T. We set $\theta_j = 0$ when $j < 0$ for convenience.

For an $[n, k, d]_q$ code C with a generator matrix G, the columns of G can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted by $\bar{G}$. An *i-point* is a point of Σ which has multiplicity i in $\bar{G}$. Let Σ_i be the set of i-points in Σ. For any subset S of Σ we define *the multiplicity of S with respect to C* as

$$m_C(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \Sigma_i|,$$

where $\gamma_0 = \max\{i \mid \text{an } i\text{-point exists}\}$.

Then we obtain the partition $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_{\gamma_0}$ such that

$$\begin{aligned} n &= m_C(\Sigma), \\ n-d &= \max\{m_C(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition of Σ as above gives an $[n,k,d]_q$ code in the natural manner. Since $(n+1)-(d+1) = n-d$, we get the following.

Lemma 2.1. *C is extendable if and only if there exists a point $P \in \Sigma$ such that $m_C(\pi) < n-d$ for all hyperplanes π through P.*

Let Σ^* be the dual space of Σ (considering $\mathcal{F}_{k-2}$ as the set of points of Σ^*). Then Lemma 2.1 is equivalent to the following:

Lemma 2.2. *C is extendable if and only if there exists a hyperplane Π of Σ^* such that*

$$\Pi \subset \{\pi \in \mathcal{F}_{k-2} \mid m_C(\pi) < n-d\}.$$

Now, let C be an $[n,k,d]_3$ code with diversity (Φ_0, Φ_1), $\gcd(3,d) = 1$, $k \geq 3$, and let $\mathcal{F}_j^*$ be the set of j-flats of Σ^*, i.e., $\mathcal{F}_j^* = \mathcal{F}_{k-2-j}, 0 \leq j \leq k-2$. We define F_0, F_1, F_{e}, F and $\bar{F}$ as follows:

$$\begin{aligned} F_0 &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) \equiv n \pmod 3\}, \\ F_1 &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) \not\equiv n,\ n-d \pmod 3\}, \\ F_{\mathrm{e}} &= \{\pi \in \mathcal{F}_0^* \mid m_C(\pi) < n-d,\ m_C(\pi) \equiv n-d \pmod 3\}, \\ F &= F_0 \cup F_1, \quad \bar{F} = F \cup F_{\mathrm{e}}. \end{aligned}$$

Then we have $\Phi_0 = |F_0|, \Phi_1 = |F_1|, \Phi_{\mathrm{e}} = |F_{\mathrm{e}}|$ since $|\{\pi \in \mathcal{F}_{k-2} \mid m_C(\pi) = i\}| = A_{n-i}/(q-1)$. Lemma 2.2 implies the following:

Lemma 2.3. *C is extendable if and only if $\bar{F}$ contains a hyperplane of Σ^*.*

We consider the extendability of C from this geometrical point of view. A t-flat Π of Σ^* with $|\Pi \cap F_0| = i$, $|\Pi \cap F_1| = j$ is called an $(i,j)_t$ *flat.* A $(1,0)_0$ flat is just a point of F_0. An $(i,j)_1$ flat, an $(i,j)_2$ flat and an $(i,j)_3$ flat are called an (i,j)-*line*, an (i,j)-*plane* and an (i,j)-*solid* respectively.

Let Λ_1 be the set of all possible (i,j) for which an (i,j)-line exists in $\mathcal{F}_1^*$. Then we have

$$\Lambda_1 = \{(1,0),(0,2),(2,1),(1,3),(4,0)\},$$

see [8]. Assume $2 \le t \le k-1$ and let $\Pi \in \mathcal{F}_t^*$. Denote by $c_{i,j}^{(t)}$ the number of $(i,j)_{t-1}$ flats in Π and let $\varphi_s^{(t)} = |\Pi \cap F_s|$, $s = 0,1$. The pair $(\varphi_0^{(t)}, \varphi_1^{(t)})$ is called the *diversity of* Π and the list of $c_{i,j}^{(t)}$'s is called its *spectrum.* Let Λ_t be the set of all possible $(\varphi_0^{(t)}, \varphi_1^{(t)})$. Λ_t and the corresponding spectra are determined as in Table 1 for $t=2$ and as in Table 2 for $t=3$.

Table 1

$\varphi_0^{(2)}$	$\varphi_1^{(2)}$	$c_{1,0}^{(2)}$	$c_{0,2}^{(2)}$	$c_{2,1}^{(2)}$	$c_{1,3}^{(2)}$	$c_{4,0}^{(2)}$
4	0	12	0	0	0	1
1	6	2	9	0	2	0
4	3	4	3	6	0	0
4	6	0	3	6	4	0
7	3	1	0	9	1	2
4	9	0	0	0	12	1
13	0	0	0	0	0	13

Table 2

$\varphi_0^{(3)}$	$\varphi_1^{(3)}$	$c_{4,0}^{(3)}$	$c_{1,6}^{(3)}$	$c_{4,3}^{(3)}$	$c_{4,6}^{(3)}$	$c_{7,3}^{(3)}$	$c_{4,9}^{(3)}$	$c_{13,0}^{(3)}$
13	0	39	0	0	0	0	0	1
4	18	2	36	0	0	0	2	0
13	9	4	3	27	0	6	0	0
10	15	0	10	15	15	0	0	0
16	12	0	0	12	12	16	0	0
13	18	0	3	0	27	6	4	0
22	9	1	0	0	0	36	1	2
13	27	0	0	0	0	0	39	1
40	0	0	0	0	0	0	0	40

For $t \geq 2$ we set

$$\Lambda_t^- = \{(\theta_{t-1}, 0), (\theta_{t-2}, 2\cdot 3^{t-1}), (\theta_{t-1}, 2\cdot 3^{t-1}), (\theta_{t-1}+3^{t-1}, 3^{t-1}), (\theta_{t-1}, 3^t), (\theta_t, 0)\}.$$

It is known that Λ_t^- is included in Λ_t for all $t \geq 2$ ([8]).

Lemma 2.4. ([8]). *For $t \geq 2$, the spectrum corresponding to each diversity in Λ_t^- is uniquely determined as follows:*
(1) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-1},0}) = (\theta_t - 1, 1)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 0)$*;*
(2) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}) = (2, \theta_t - \theta_1, 2)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-2}, 2\cdot 3^{t-1})$*;*
(3) $(c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}) = (3, \theta_t - \theta_2, 6, 4)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 2\cdot 3^{t-1})$*;*
(4) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-1}}, c^{(t)}_{\theta_{t-1},0}) = (1, \theta_t - \theta_1, 1, 2)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{t-1}, 3^{t-1})$*;*
(5) $(c^{(t)}_{\theta_{t-2},3^{t-1}}, c^{(t)}_{\theta_{t-1},0}) = (\theta_t - 1, 1)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^t)$*;*
(6) $c^{(t)}_{\theta_{t-1},0} = \theta_t$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_t, 0)$.

Set $\Lambda_t^+ = \Lambda_t \setminus \Lambda_t^-$. The diversities in Λ_t^+ and the corresponding spectra for $t \geq 4$ are determined as follows.

Lemma 2.5 ([10])**.** (1) *When t is odd (≥ 5):*
$\Lambda_t^+ = \{(\theta_{t-1}, 3^{t-1})\} \cup \{(\theta_{t-1} - 3^{T+1+s}, \theta_{t-1} + \theta_{T+s} + 1), (\theta_{t-1} + 3^{T+1+s}, \theta_{t-1} - \theta_{T+s}) \mid 0 \leq s \leq T\} \cup \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1) \mid 1 \leq i \leq T\}$,
where $T = (t-3)/2$. The spectrum corresponding to each diversity is uniquely determined as follows:
(A-1) $c^{(t)}_{\theta_{t-2}-3^{T+1},\theta_{t-2}+\theta_T+1} = \theta_{t-1} - 3^{T+1}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = \theta_{t-1} + \theta_T + 1$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1}, \theta_{t-1} + \theta_T + 1)$*;*
(A-2) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_T} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_T+1} = \theta_{t-1} - \theta_T$, $c^{(t)}_{\theta_{t-2}+3^{T+1},\theta_{t-2}-\theta_T} = \theta_{t-1} + 3^{T+1}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)$*;*
(A-3) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}) = (4, 3, \theta_t - \theta_2, 6)$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, 3^{t-1})$*;*
(A-4) $c^{(t)}_{\theta_{t-2}-3^{T+1+s},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} - 3^{T+1-s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} + \theta_{T-s} + 1$, $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_t - \theta_{t-2s}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1} - 3^{T+1+s}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \leq s \leq T$*;*

(A-5) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-1-2s} - \theta_{T-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+1+s},\theta_{t-2}-\theta_{T+s}} = \theta_{t-1-2s} + 3^{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_t - \theta_{t-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} + 3^{T+1+s}, \theta_{t-1} - \theta_{T+s})$, $1 \leq s \leq T$*;*

(A-6) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T+s}} = \theta_{t-2s}$, $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_{t-2s} - \theta_{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_{t-2s} + \theta_{T+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{T-1+s}} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} - \theta_{T+s})$, $1 \leq s \leq T$*;*

(A-7) $c^{(t)}_{\theta_{t-2}-3^{T+s},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_{t-2s} - \theta_{T+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{T+s},\theta_{t-2}-\theta_{T-1+s}} = \theta_{t-2s} + \theta_{T+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T+s}+1} = \theta_{t-2s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{T-1+s}+1} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1)$, $1 \leq s \leq T$.

(2) *When* t *is even* (≥ 4)*:*

$\Lambda^+_t = \{(\theta_{t-1}, 3^{t-1})\} \cup \{(\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1) \mid 0 \leq s \leq U\} \cup \{(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1), (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s}) \mid 1 \leq s \leq U + 1\}$,

where $U = (t-4)/2$. *The spectrum corresponding to each diversity is uniquely determined as follows:*

(B-1) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1}} = \theta_{t-1}$, $c^{(t)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = \theta_{t-1} - \theta_{U+1}$, $c^{(t)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = \theta_{t-1} + \theta_{U+1} + 1$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1})$*;*

(B-2) $c^{(t)}_{\theta_{t-2}-3^{U+1},\theta_{t-2}+\theta_U+1} = \theta_{t-1} - \theta_{U+1}$, $c^{(t)}_{\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U} = \theta_{t-1} + \theta_{U+1} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+1}+1} = \theta_{t-1}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1} + 1)$*;*

(B-3) $(c^{(t)}_{\theta_{t-2},0}, c^{(t)}_{\theta_{t-3},2\cdot 3^{t-2}}, c^{(t)}_{\theta_{t-2},3^{t-2}}, c^{(t)}_{\theta_{t-2}+3^{t-2},3^{t-2}}) = (4, 3, \theta_t - \theta_2, 6)$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1}, 3^{t-1})$*;*

(B-4) $c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} - 3^{U+2-s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} + \theta_{U+1-s} + 1$, $c^{(t)}_{\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U-1+s}+1} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1)$, $1 \leq s \leq U + 1$*;*

(B-5) $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-2s} - \theta_{U+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-2s} + 3^{U+2-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+s},\theta_{t-2}-\theta_{U-1+s}} = \theta_t - \theta_{t+1-2s}$ *for* $(\varphi^{(t)}_0, \varphi^{(t)}_1) = (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s})$, $1 \leq s \leq U + 1$*;*

$(B\text{-}6)$ $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+1+s}} = \theta_{t-1-2s}$, $c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-1-2s} - \theta_{U+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-1-2s} + \theta_{U+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}-\theta_{U+s}} = \theta_t - \theta_{t-2s}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} - \theta_{U+1+s})$, $1 \le s \le U$;

$(B\text{-}7)$ $c^{(t)}_{\theta_{t-2}-3^{U+1+s},\theta_{t-2}+\theta_{U+s}+1} = \theta_{t-1-2s} - \theta_{U+1-s}$, $c^{(t)}_{\theta_{t-2}+3^{U+1+s},\theta_{t-2}-\theta_{U+s}} = \theta_{t-1-2s} + \theta_{U+1-s} + 1$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+1+s}+1} = \theta_{t-1-2s}$, $c^{(t)}_{\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1} = \theta_t - \theta_{t-2s}$ *for* $(\varphi_0^{(t)}, \varphi_1^{(t)}) = (\theta_{t-1}, \theta_{t-1} + \theta_{U+1+s} + 1)$, $1 \le s \le U$.

3. Characterizations of $(i,j)_t$ flats in Σ^*. Let Π be a t-flat in Σ^*. An s-flat S in Π is called the *axis of* Π *of type* (a,b) if every hyperplane of Π not containing S has the same diversity (a,b) and if there is no hyperplane of Π through S whose diversity is (a,b). Then the spectrum of Π satisfies $c^{(t)}_{a,b} = \theta_t - \theta_{t-1-s}$ and the axis is unique if it exists. The axis is helpful to characterize the geometrical structure of Π.

The geometrical structure of Π whose diversity is in Λ_t^- can be seen as the following lemma by means of the axis of Π. As for the type (3) of Lemma 2.4 for $t = 2$, see [8].

Lemma 3.1 ([9])**.** *Let* Π *be a* t*-flat in* Σ^*.
(1) *For* $(i,j) = (\theta_{t-1}, 0)$ *or* $(\theta_{t-1}, 3^t)$, $t \ge 2$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-1}, 0)_{t-1}$ *flat which is the axis of type* $((i-1)/3, j/3)$.
(2) *For* $(i,j) = (\theta_{t-2}, 2 \cdot 3^{t-1})$ *or* $(\theta_{t-1} + 3^{t-1}, 3^{t-1})$, $t \ge 2$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-2}, 0)_{t-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$.
(3) *For* $(i,j) = (\theta_{t-1}, 3^{t-1})$ *or* $(\theta_{t-1}, 2 \cdot 3^{t-1})$, $t \ge 3$, Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{t-3}, 0)_{t-3}$ *flat which is the axis of type* $((i-1)/3, j/3)$.

Lemma 3.2. *Let* Π *be a* 4*-flat in* Σ^* *and let* $(i,j) \in \{(31,45), (49,36)\}$. *Then* Π *is an* $(i,j)_4$ *flat if and only if* Π *contains a point of* F_0 *which is the axis of type* $((i-1)/3, j/3)$.

P r o o f. We prove only for $(i,j) = (31,45)$. One can prove it for $(i,j) = (49,36)$ in a similar way.

"only if" part: Assume that Π is a $(31,45)_4$ flat. By Lemma 2.5 (B-4), the spectrum of Π is $(c^{(4)}_{4,18}, c^{(4)}_{13,9}, c^{(4)}_{10,15}, c^{(4)}_{13,18})$ =(10,15,81,15). There are exactly two $(13,9)$-solids π_1, π_2 and two $(13,18)$-solids π_3, π_4 through a fixed $(7,3)$-plane δ. From (3) of Lemma 3.1, a $(13,9)$-solid contains a point of F_0 which is the axis of type $(4,3)$, and a $(13,18)$-solid contains a point of F_0 which is the axis of type $(4,6)$. Since δ contains a point P of F_0 which is the axis of type (2,1) by (2) of Lemma 3.1, the axis of π_i $(1 \leq i \leq 4)$ coincides with P. Let Δ be a solid not containing P. Δ meets π_1, π_2 in a $(4,3)$-plane and π_3, π_4 in a $(4,6)$-plane. Therefore Δ is just a $(10,15)$-solid. There is no $(10,15)$-solid through P since $c^{(4)}_{10,15} = 81$. Hence P is the axis of Π of type $(10,15)$.

"if" part: Assume that a 4-flat Π contains a point P of F_0 which is the axis of type (10,15). From the definition of the axis, all the 4-flats not containing P are (10,15)-solids. Hence the number of (10,15)-solids is at least 81. Thus Π is just a $(31,45)_4$ flat. $\square$

Lemma 3.3. *Let* Π *be a* 5*-flat in* Σ^**, and let* $(i,j) \in \{(121,108),(121,135)\}$*. Then* Π *is an* $(i,j)_5$ *flat if and only if* Π *contains a point of* F_0 *which is the axis of type* $((i-1)/3, j/3)$.

Proof. We prove this only for $(i,j) = (121,108)$. The other case is proved similarly.

"only if" part: Assume that Π is a $(121,108)_5$ flat. By Lemma 2.5 (A-6), the spectrum of Π is $(c^{(5)}_{40,27}, c^{(5)}_{31,45}, c^{(5)}_{40,36}, c^{(5)}_{49,36}) = (40,36,243,45)$. There are exactly two $(31,45)_4$ flats π_1, π_2 and two $(49,36)_4$ flats π_3, π_4 through a fixed $(13,18)$-solid δ. From Lemma 3.2, a $(31,45)_4$ flat contains a point of F_0 which is the axis of type (10,15), and a $(49,36)_4$ flat contains a point of F_0 which is the axis of type $(16,12)$. Since δ contains a point P of F_0 which is the axis of type $(4,6)$ by (3) of Lemma 3.1, the axis of π_i $(1 \leq i \leq 4)$ coincides with P. Indeed, any solid Δ in π_1 not containing P is a $(10,15)$-solid, for $\Delta \cap \delta$ is a $(4,6)$-plane and there is exactly one $(13,18)$-solid through a fixed $(4,6)$-plane in π_1. Let π be a 4-flat not containing P. π meets π_1, π_2 in a $(10,15)$-solid and π_3, π_4 in a $(16,12)$-solid. Therefore π is just a $(40,36)_4$ flat. There is no $(40,36)_4$ flat through P since $c^{(5)}_{40,36} = 324$. Hence P is the axis of Π of type $(40,36)$. The "if" part is similar to the one in Lemma 3.2. $\square$

Lemma 3.4. (1) *Let* Π *be a* t*-flat in* Σ^* *with even* $t \geq 4$*, and let* $(i,j) \in \{(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1), (\theta_{t-1} + 3^{U+1+s}, \theta_{t-1} - \theta_{U+s}) \mid 1 \leq s \leq U+1\}$, $U = (t-4)/2$*. Then* Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{2s-2}, 0)_{2s-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$.
(2) *Let* Π *be a* t*-flat in* Σ^* *with odd* $t \geq 5$*, and let* $(i,j) \in \{(\theta_{t-1}, \theta_{t-1} - \theta_{T+s}), (\theta_{t-1}, \theta_{t-1} + \theta_{T+s} + 1) \mid 1 \leq s \leq T\}$, $T = (t-3)/2$*. Then* Π *is an* $(i,j)_t$ *flat if and only if* Π *contains a* $(\theta_{2s-2}, 0)_{2s-2}$ *flat which is the axis of type* $((i-1)/3, j/3)$.

P r o o f. We prove this only for $(\theta_{t-1} - 3^{U+1+s}, \theta_{t-1} + \theta_{U+s} + 1)_t$ flat of (1) and for $(\theta_{t-1}, \theta_{t-1} - \theta_{T+s})_t$ flat of (2). The other cases are proved similarly.

"only if" part: We prove this part by induction on t (≥ 4). First, (1) holds for $t = 4$ and (2) holds for $t = 5$. Next, we assume (1) for $t - 1$ and (2) for $t - 2$ to prove (2) for t.

<u>Claim 1.</u> Let π be a $(i, j)_{t-1}$ flat in Σ^* with $t \geq 5$, and let $(i, j) \in \{(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1), (\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})\}$. Then the axis of π coincides with the axis of a $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat in π.

To see Claim 1, let π be a $(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1)_{t-1}$ flat. The spectrum of π is $(c^{(t-1)}_{\theta_{t-3}-3^{T+s},\theta_{t-3}+\theta_{T-1+s}+1}, c^{(t-1)}_{\theta_{t-3},\theta_{t-3}-\theta_{T-1+s}}, c^{(t-1)}_{\theta_{t-3},\theta_{t-3}+\theta_{T-1+s}+1}, c^{(t-1)}_{\theta_{t-3}-3^{T-1+s},\theta_{t-3}+\theta_{T-2+s}+1}) = (\theta_{t-1-2s} - 3^{T+1-s},\ \theta_{t-1-2s} + \theta_{T-s} + 1,\ \theta_{t-1-2s} + \theta_{T-s}+1,\ \theta_{t-1}-\theta_{t-2s})$ by Lemma 2.5 (B-4) for $t-1$. From the induction hypothesis for $t-2$, a $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat Δ contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat ϖ which is the axis of type $(\theta_{t-4}, \theta_{t-4} + \theta_{T-2+s} + 1)$. A $(t-3)$-flat in Δ not containing ϖ is a $(\theta_{t-4}, \theta_{t-4}+\theta_{T-2+s}+1)_{t-3}$ flat, say δ. A $(t-2)$-flat Δ' in π containing δ is a $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat or a $(\theta_{t-3}-3^{T-1+s}, \theta_{t-3}+\theta_{T-2+s}+1)_{t-2}$ flat by the spectrum of π. However, there is only one $(\theta_{t-3}, \theta_{t-3}+\theta_{T-1+s}+1)_{t-2}$ flat through a fixed $(\theta_{t-4}, \theta_{t-4} + \theta_{T-2+s} + 1)_{t-3}$ flat. So Δ' is just a $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flat. Since the number of $(t-3)$-flats in Δ not containing ϖ is $\theta_{t-2} - \theta_{t-2s-1}$, the number of $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flats in π not containing ϖ is $\theta_{t-1} - \theta_{t-2s}$. From the spectrum of π and the definition of the axis, the axis ϖ of Δ coincides with the axis of π. Similarly, the axis of a $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat coincides with the axis of a $(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flat.

For $t \geq 7$, assume that Π is a $(\theta_{t-1}, \theta_{t-1} - \theta_{T+s})_t$ flat. From the spectrum of Π (see Lemma 2.5 (A-6)), there are exactly two $(\theta_{t-2} - 3^{T+s}, \theta_{t-2} + \theta_{T-1+s} + 1)_{t-1}$ flats π'_1, π'_2 and two $(\theta_{t-2} + 3^{T+s}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flats π'_3, π'_4 through a fixed $(\theta_{t-3}, \theta_{t-3} + \theta_{T-1+s} + 1)_{t-2}$ flat Δ'. By Claim 1, the axis of Δ', say δ', coincides with the axes of π'_1–π'_4. Let π' be a $(t-1)$-flat not containing δ'. Since π' meets π_1, π_2 in a $(\theta_{t-3} - 3^{T-1+s}, \theta_{t-3} + \theta_{T-2+s} + 1)_{t-2}$ flat and π_3, π_4 in a $(\theta_{t-3} + 3^{T-1+s}, \theta_{t-3} - \theta_{T-2+s})_{t-2}$ flat, π' is just a $(\theta_{t-2}, \theta_{t-2} - \theta_{T-1+s})_{t-1}$ flat. Hence Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of type $((i-1)/3, j/3)$.

Finally, we assume (2) for $t-1$ and (1) for $t-2$ to prove (1) for t.

<u>Claim 2.</u> Let π be a $(i, j)_{t-1}$ flat in Σ^* with $t \geq 5$, and let $(i, j) \in \{(\theta_{t-2}, \theta_{t-2} - \theta_{U+s}), (\theta_{t-2}, \theta_{t-2} + \theta_{U+s} + 1)\}$. Then the axis of π coincides with

the axis of a $(\theta_{t-3}+3^{U+s},\theta_{t-3}-\theta_{T-1+s})_{t-2}$ flat in π.
This claim is proved similarly to Claim 1.

For $t\geq 6$, assume that Π is a $(\theta_{t-1}-3^{U+1+s},\theta_{t-1}+\theta_{U+s}+1)_t$ flat. From the spectrum of Π (see Lemma 2.5 (B-4)), there are exactly two $(\theta_{t-2},\theta_{t-2}-\theta_{U+s})_{t-1}$ flats π'_1,π'_2 and two $(\theta_{t-2},\theta_{t-2}+\theta_{U+s}+1)_{t-1}$ flats π'_3,π'_4 through a fixed $(\theta_{t-3}+3^{U+s},\theta_{t-3}-\theta_{T-1+s})_{t-2}$ flat Δ'. By Claim 2, the axis of Δ', say δ', coincides with the axes of π'_1–π'_4. Let π' be a $(t-1)$-flat not containing δ'. Since π' meets π'_1,π'_2 in a $(\theta_{t-3},\theta_{t-3}-\theta_{U+s-1})_{t-2}$ flat and π'_3,π'_4 in a $(\theta_{t-3},\theta_{t-3}+\theta_{U+s-1}+1)_{t-2}$ flat, π' is just a $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flat. Hence Π contains a $(\theta_{2s-2},0)_{2s-2}$ flat which is the axis of type $((i-1)/3,j/3)$.

"if" part: We prove only for the axis of type $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)$ of (1). The others are proved similarly. Assume that a t-flat Π contains a $(\theta_{2s-2},0)_{2s-2}$ flat δ which is the axis of type $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)$. From the definition of the axis, all the $(t-1)$-flats not containing δ are $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flats. Hence the number of $(\theta_{t-2}-3^{U+s},\theta_{t-2}+\theta_{U+s-1}+1)_{t-1}$ flats are at least $\theta_t-\theta_{t-2s}$. Thus Π is just a $(\theta_{t-1}-3^{U+1+s},\theta_{t-1}+\theta_{U+s}+1)_t$ flat by Lemma 2.5. □

Lemma 3.5. (1) *Let Π be a t-flat in Σ^* with even $t\geq 4$, and let $(i,j)\in\{(\theta_{t-1},\theta_{t-1}-\theta_{U+s}),(\theta_{t-1},\theta_{t-1}+\theta_{U+s}+1)\mid 1\leq s\leq U\}$ $U=(t-4)/2$. Then Π is an $(i,j)_t$ flat if and only if Π contains a $(\theta_{2s-1},0)_{2s-1}$ flat which is the axis of type $((i-1)/3,j/3)$.*
(2) *Let Π be a t-flat in Σ^* with odd $t\geq 5$, and let $(i,j)\in\{(\theta_{t-1}-3^{T+1+s},\theta_{t-1}+\theta_{T+s}+1),(\theta_{t-1}+3^{T+1+s},\theta_{t-1}-\theta_{T+s})\mid 1\leq s\leq T\}$, $T=(t-3)/2$. Then Π is an $(i,j)_t$ flat if and only if Π contains a $(\theta_{2s-1},0)_{2s-1}$ flat which is the axis of type $((i-1)/3,j/3)$.*

The above lemma can be proved similarly to Lemma 3.4.

Lemma 3.6. (1) *Let Π be a t-flat in Σ^* with even $t\geq 4$ and $U=(t-4)/2$. Then Π is a $(\theta_{t-1},\theta_{t-1}-\theta_{U+1})_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4},\theta_{t-4}-\theta_U)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$ for $1\leq i\leq 4$.*
(2) *For a t-flat Π in Σ^* with odd $t\geq 5$ and $T=(t-3)/2$, Π is a $(\theta_{t-1}+3^{T+1},\theta_{t-1}-\theta_T)_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1\cdots\pi_4$ through a fixed $(\theta_{t-3}+3^{T+1},\theta_{t-3}-\theta_T)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4}+3^T,\theta_{t-4}-\theta_{T-1})$ and that P_i is the axis of π_i of type $(\theta_{t-3}+3^T,\theta_{t-3}-\theta_{T-1})$ for $1\leq i\leq 4$.*

P r o o f. "only if" part: We prove this part by induction on $t \geq 4$. Assume that Π is a $(49,36)_4$ flat. The spectrum of Π is $(c^{(4)}_{13,9}, c^{(4)}_{10,15}, c^{(4)}_{16,12}) = (40,36,45)$ from Lemma 2.5 (A-1). There are exactly four (13,9)-solids $\pi_1, \cdots, \pi_4$ through a fixed $(4,0)$-plane Δ in Π. A $(13,9)$-solid contains a point $P_i \in F_0$ which is the axis of type $(4,3)$ by (3) of Lemma 3.1. And a $(4,0)$-plane contains a $(4,0)$-line l which is the axis of type $(1,0)$ by (1) of Lemma 3.1. Therefore the axis $P_i \in F_0$ of π_i is on l. Suppose $P_1 = P_2$. Then there are two $(1,6)$-planes $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(1,0)$-line $l' \subset \Delta$ containing $P_1 (= P_2)$. $\pi = \langle \Delta_1, \Delta_2 \rangle$ is a $(13,9)$-solid or a $(10,15)$-solid since a $(16,12)$-solid contains no $(1,6)$-plane from Table 2. In a $(10,15)$-solid, there is only one $(1,6)$-plane through a fixed $(1,0)$-line, so π is not a $(10,15)$-solid. If π is a $(13,9)$-solid, then two $(13,9)$-solids meet in a $(1,6)$-plane in Π. However there is only one $(13,9)$-solid through a fixed $(1,6)$-plane in Π, a contradiction. So $P_1 \neq P_2$, and (1) holds for $t = 4$.

Next, we assume (1) for $t-1$ to prove (2). For $t \geq 5$, assume that Π is a $(\theta_{t-1} + 3^{T+1}, \theta_{t-1} - \theta_T)_t$ flat. By the spectrum of Π (see Lemma 2.5 (A-1)) there are exactly four $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{t-3} + 3^{T+1}, \theta_{t-3} - \theta_T)_{t-2}$ flat Δ. π_i contains a point $P_i \in F_0$ which is the axis of type $(\theta_{t-3} + 3^T, \theta_{t-3} - \theta_{T-1})$ by (1) of Lemma 3.4. And Δ contains a (4,0)-line l which is the axis of type $(\theta_{t-4} + 3^T, \theta_{t-4} - \theta_{T-1})$ by (2) of Lemma 3.5. Therefore the axis $P_i \in F_0$ of π_i is on l. Suppose $P_1 = P_2$. Then there are two $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flats $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(\theta_{t-4} + 3^{T-1+s}, \theta_{t-4} - \theta_{T-2+s})_{t-3}$ flat containing P_1 in Δ. $\langle \Delta_1, \Delta_2 \rangle = \pi$ is a $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat or a $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flat since a $(\theta_{t-2}, \theta_{t-2} + \theta_T + 1)_{t-1}$ flat has no $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flat. If π is a $(\theta_{t-2}, \theta_{t-2} - \theta_T)_{t-1}$ flat, then the axes of two such $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flats do not coincide from the induction hypothesis for $t-1$, a contradiction. If π is a $(\theta_{t-2} + 3^{T+1}, \theta_{t-2} - \theta_T)_{t-1}$ flat, then there is only one $(\theta_{t-3}, \theta_{t-3} - \theta_T)_{t-2}$ flat through a fixed $(\theta_{t-4} + 3^{T-1+s}, \theta_{t-4} - \theta_{T-2+s})_{t-3}$ flat in Π, a contradiction. So $P_1 \neq P_2$, and our assertion follows.

For $t-1$, we assume (2) to prove (1). For $t \geq 6$, assume that Π is a $(\theta_{t-1}, \theta_{t-1} - \theta_{U+1})_t$ flat. From Lemma 2.5 (B-1), there are exactly four $(\theta_{t-2}, \theta_{t-2} - \theta_{U+1})_{t-1}$ flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{t-3}, \theta_{t-3} - \theta_{U+1})_{t-2}$ flat Δ. π_i contains a point $P_i \in F_0$ which is the axis of type $(\theta_{t-3}, \theta_{t-3} - \theta_U)$ by (2) of Lemma 3.4. And Δ contains a (4,0)-line l which is the axis of type $(\theta_{t-4}, \theta_{t-4} - \theta_U)$ by (1) of Lemma 3.5. Therefore the axis $P_i \in F_0$ of π_i is on l for $1 \leq i \leq 4$. Suppose $P_1 = P_2$. Then there are two $(\theta_{t-3} + 3^{U+1}, \theta_{t-3} - \theta_U)_{t-2}$ flats $\Delta_1 \subset \pi_1, \Delta_2 \subset \pi_2$ through a fixed $(\theta_{t-4}, \theta_{t-4} - \theta_U)_{t-3}$ flat containing P_1 in Δ. $\langle \Delta_1, \Delta_2 \rangle = \pi$ is a $(\theta_{t-2}, \theta_{t-2} - \theta_{U+1})_{t-1}$ flat or a $(\theta_{t-2} + 3^{U+1}, \theta_{t-2} - \theta_U)_{t-1}$ flat since $(\theta_{t-2} - 3^{U+1}, \theta_{t-2} + \theta_{U+1})_{t-1}$ flat has no $(\theta_{t-3} + 3^{U+1}, \theta_{t-3} - \theta_U)_{t-2}$

flat. If π is a $(\theta_{t-2}+3^{U+1},\theta_{t-2}-\theta_U)_{t-1}$ flat, then the axes of such two $(\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U)_{t-2}$ flats do not coincide from the induction hypothesis for $t-1$, a contradiction. If π is a $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flat, then there is only one $(\theta_{t-3}+3^{U+1},\theta_{t-3}-\theta_U)_{t-2}$ flat through a fixed $(\theta_{t-4},\theta_{t-4}-\theta_U)_{t-3}$ flat, a contradiction. So $P_1\neq P_2$, and our assertion follows.

"if" part: Let Π be a t-flat in Σ^* with even $t\geq 4$ and $U=(t-4)/2$. Assume that Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4},\theta_{t-4}-\theta_U)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$. From (1) of Lemma 3.5, a $(t-1)$-flat containing a point $P_i\in F_0$ which the axis of type $(\theta_{t-3},\theta_{t-3}-\theta_U)$ is a $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flat, say π_i. Since there are four $(\theta_{t-2},\theta_{t-2}-\theta_{U+1})_{t-1}$ flats through a fixed $(\theta_{t-3},\theta_{t-3}-\theta_{U+1})_{t-2}$ flat, Π is just a $(\theta_{t-1},\theta_{t-1}-\theta_{U+1})_t$ flat. The other cases are proved similarly. $\square$

The following lemma can be proved similarly to Lemma 3.6.

Lemma 3.7. (1) *Let Π be a t-flat in Σ^* with even $t\geq 4$, $U=(t-4)/2$. Then Π is a $(\theta_{t-1},\theta_{t-1}+\theta_{U+1}+1)_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3},\theta_{t-3}+\theta_{U+1}+1)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4},\theta_{t-4}+\theta_U+1)$ and that P_i is the axis of π_i of type $(\theta_{t-3},\theta_{t-3}+\theta_U+1)$ for $1\leq i\leq 4$.*
(2) *Let Π be a t-flat in Σ^* with odd $t\geq 5$, $T=(t-3)/2$. Then Π is a $(\theta_{t-1}-3^{T+1},\theta_{t-1}+\theta_T+1)_t$ flat if and only if Π contains four $(t-1)$-flats $\pi_1,\cdots,\pi_4$ through a fixed $(\theta_{t-3}-3^{T+1},\theta_{t-3}+\theta_T+1)_{t-2}$ flat Δ such that Δ contains a $(4,0)$-line $l=\{P_1,P_2,P_3,P_4\}$ which is the axis of type $(\theta_{t-4}-3^T,\theta_{t-4}+\theta_{T-1}+1)$ and that P_i is the axis of π_i of type $(\theta_{t-3}-3^T,\theta_{t-3}+\theta_{T-1}+1)$ for $1\leq i\leq 4$.*

4. Main Results.

In this section, we give the geometric conditions and the main theorems on the extendability of ternary linear codes. For $k\geq 4$, let (C_k-0), (C_k-1) and (C_k-2) be the following conditions:

(C_k-0) there exists a $(\theta_{k-4},0)_{k-3}$ flat δ_1 in Σ^* satisfying $\delta_1\setminus F_0\subset F_\mathrm{e}$,

(C_k-1) there is a $(k-2)$-flat Π with $\Pi\setminus F\subset F_\mathrm{e}$ containing a $(\theta_{k-4},0)_{k-4}$ flat L such that L is the axis of Π of type $(\theta_{k-4}+3^{k-4},3^{k-4})$.

(C_k-2) there is a $(k-2)$-flat Π with $\Pi\setminus F\subset F_\mathrm{e}$ containing a $(\theta_{k-4},0)_{k-4}$ flat L such that L is the axis of Π of type $(\theta_{k-5},2\cdot 3^{k-4})$.

We denote by $\langle \chi_1, \chi_2, \cdots \rangle$ the smallest flat containing subsets $\chi_1, \chi_2, \cdots$ of Σ^*. For $k = 4$ we consider two more conditions:

(C_4-3) there are three non-collinear points $R_1, R_2, R_3 \in F_e$ such that the three lines $\langle R_1, R_2 \rangle$, $\langle R_2, R_3 \rangle$, $\langle R_3, R_1 \rangle$ are $(0, 2)$-lines,

(C_4-4) there are three non-collinear points $Q_1, Q_2, Q_3 \in F_1$ such that the three lines $\langle Q_1, Q_2 \rangle$, $\langle Q_2, Q_3 \rangle$, $\langle Q_3, Q_1 \rangle$ are $(0, 2)$-lines each of which contains two points of F_e.

For $k \geq 5$, let (C_k-3) and (C_k-4) be the following conditions:

(C_k-3) there is a $(k-2)$-flat Π with $\Pi \setminus F \subset F_e$ containing a $(\theta_{k-5}, 0)_{k-5}$ flat L such that L is the axis of Π of type $(\theta_{k-4}, 2 \cdot 3^{k-4})$.

(C_k-4) there is a $(k-2)$-flat Π with $\Pi \setminus F \subset F_e$ containing a $(\theta_{k-5}, 0)_{k-5}$ flat L such that L is the axis of Π of type $(\theta_{k-4}, 3^{k-4})$.

For $k = 5$ we consider two more conditions:

(C_5-5) there exist a $(4, 0)$-line l and four skew $(1, 0)$-lines l_1, l_2, l_3, l_4 such that each of $l_1, \ldots, l_4$ meets l and that $\langle l_1, l_2, l_3, l_4 \rangle \in \mathcal{F}_3^*$ and $(\cup_{i=1}^4 l_i) \setminus l \subset F_e$ hold,

(C_5-6) there exist a $(2, 1)$-line l_0 containing two points $P_1, P_2 \in F_0$ and two $(1, 0)$-lines l_1, l_2 (resp. l_1', l_2') through P_1 (resp. P_2) such that $l = \langle l_1, l_2 \rangle \cap \langle l_1', l_2' \rangle$ and $m_i = \langle Q_0, Q_i \rangle$ are $(0, 2)$-lines for $i = 1, 2$, where $l_0 \cap F_1 = \{Q_0\}$, $l \cap F_1 = \{Q_1, Q_2\}$ and that $(\cup_{i=1}^2 (l_i \cup l_i' \cup m_i)) \setminus F \subset F_e$ holds.

Lemma 4.1 ([11])**.** *Let Δ be a solid in Σ^*.*
(1) *Δ is a $(10, 15)$-solid with $\Delta \setminus F \subset F_e$ if and only if Δ satisfies* (C_5-6).
(2) *Δ is a $(16, 12)$-solid with $\Delta \setminus F \subset F_e$ if and only if Δ satisfies* (C_5-5).

We define the conditions (C_k-5–C_k-10) for even $k \geq 6$ and $1 \leq s \leq T = (k-4)/2$ as the existence of a $(k-2)$-flat Π with $\Pi \setminus F \in F_e$ satisfying the following conditions, respectively.

(C_k-5) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4} - 3^{T+s-1}, \theta_{k-4} + \theta_{T+s-2} + 1)$.

(C_k-6) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4} + 3^{T+s-1}, \theta_{k-4} - \theta_{T+s-2})$.

(C_k-7) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} - \theta_{T+s-1})$.

(C_k-8) Π contains $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} + \theta_{T+s-1} + 1)$.

(C_k-9) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5}, \theta_{k-5} - \theta_T)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6}, \theta_{k-6} - \theta_{T-1})$ and that P_i is the axis of π_i of type $(\theta_{k-5}, \theta_{k-5} - \theta_{T-1})$.

(C_k-10) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5}, \theta_{k-5} + \theta_T + 1)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6}, \theta_{k-6} + \theta_{T-1} + 1)$ and that P_i is the axis of π_i of type $(\theta_{k-5}, \theta_{k-5} + \theta_{T-1} + 1)$.

We define the conditions (C_k-5) – (C_k-10) for odd $k \geq 7$ and $1 \leq s \leq U+1$ where $U = (k-5)/2$ as the existence of a $(k-2)$-flat Π with $\Pi \setminus F \in F_{\mathrm{e}}$ satisfying the following conditions, respectively.

(C_k-5) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4} - 3^{U+s}, \theta_{k-4} + \theta_{U+s-1} + 1)$.

(C_k-6) Π contains a $(\theta_{2s-1}, 0)_{2s-1}$ flat which is the axis of Π of type $(\theta_{k-4} + 3^{U+s}, \theta_{k-4} - \theta_{U+s-1})$.

(C_k-7) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} - \theta_{U+s-1})$.

(C_k-8) Π contains a $(\theta_{2s-2}, 0)_{2s-2}$ flat which is the axis of Π of type $(\theta_{k-4}, \theta_{k-4} + \theta_{U+s-1} + 1)$.

(C_k-9) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5} - 3^{U+1}, \theta_{k-5} + \theta_U + 1)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6} - 3^U, \theta_{k-6} + \theta_{U-1} + 1)$ and that P_i is the axis of π_i of type $(\theta_{k-5} - 3^U, \theta_{k-5} + \theta_{U-1} + 1)$.

(C_k-10) Π contains four $(k-3)$-flats $\pi_1, \cdots, \pi_4$ through a fixed $(\theta_{k-5} + 3^{U+1}, \theta_{k-5} - \theta_U)_{k-4}$ flat Δ such that Δ contains a $(4,0)$-line $l = \{P_1, P_2, P_3, P_4\}$ which is the axis of Δ of type $(\theta_{k-6} + 3^U, \theta_{k-6} - \theta_{U-1})$ and that P_i is the axis of π_i of type $(\theta_{k-5} + 3^U, \theta_{k-5} - \theta_{U-1})$.

Let C be an $[n, k, d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_k^+$, $d \equiv 1$ or 2 (mod 3), $k \geq 3$. Since $\mathcal{D}_3^+ = \{(4, 3)\}$, $\mathcal{D}_4^+ = \{(13, 9), (10, 15), (16, 12)\}$ and $\mathcal{D}_k^+ = \Lambda_{k-1}^+$ for $k \geq 5$ ([8], [10]), we have $|\mathcal{D}_k| = 2k - 1$ for all $k \geq 3$. It is known that an $[n, 4, d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_4^+$ is not extendable if $\Phi_e < 3$ for $k = 4$ ([8]). The conditions (C_4-0-C_4-4) are used to check the extendability of $[n, 4, d]_3$ codes.

Theorem 4.2 ([10]). *Let C be an $[n, 4, d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_4^+$, gcd$(3, d) = 1$. Then C is extendable if and only if one of the conditions indicated in Table 3 holds.*

Table 3

(Φ_0, Φ_1)	conditions
(13,9)	(C_4-1), (C_4-4)
(10,15)	(C_4-2), (C_4-3), (C_4-4)
(16,12)	(C_4-0), (C_4-3)

For the case when $k = 5$, C is not extendable if $\Phi_e < 9$ when $(\Phi_0, \Phi_1) \neq (40, 36)$ or if $\Phi_e < 12$ when $(\Phi_0, \Phi_1) = (40, 36)$ ([8]). Otherwise, we need to check whether one of the conditions (C_5-0-C_5-6) holds or not according to the diversity of C.

Theorem 4.3 ([11]). *Let C be an $[n, 5, d]_3$ code with diversity $(\Phi_0, \Phi_1) \in \mathcal{D}_5^+$, gcd$(3, d) = 1$. Then C is extendable if and only if one of the conditions indicated in Table 4 holds.*

Table 4

(Φ_0, Φ_1)	conditions
(40,27)	(C_5-1), (C_5-4)
(31,45)	(C_5-2), (C_5-3), (C_5-4), (C_5-6)
(40,36)	(C_5-4), (C_5-5), (C_5-6)
(40,45)	(C_5-3), (C_5-5), (C_5-6)
(49,36)	(C_5-0), (C_5-3), (C_5-5)

Theorem 4.4 ([9]). *Let C be an $[n, k, d]_3$ code with diversity $(\theta_{k-2}, 3^{k-2})$, gcd$(3, d) = 1$, $k \geq 6$. Then C is extendable if and only if either the conditions (C_k-1) or (C_k-4) holds.*

Theorem 4.5. *Let C be an $[n,k,d]_3$ code with diversity $(\Phi_0,\Phi_1)\in\mathcal{D}_k^+$, gcd$(3,d)=1$ for $k\ge 6$. Then C is extendable if and only if one of the conditions indicated in Table 5 holds.*

Table 5

(Φ_0,Φ_1) k : even	(Φ_0,Φ_1) k:odd	conditions
$(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$	$(\theta_{k-2},\theta_{k-2}-\theta_{U+1})$	(C$_k$-5), (C$_k$-9), (C$_k$-10)
$(\theta_{k-2}+3^{T+1},\theta_{k-2}-\theta_T)$	$(\theta_{k-2},\theta_{k-2}+\theta_{U+1}+1)$	(C$_k$-6), (C$_k$-9), (C$_k$-10)
$(\theta_{k-2}-3^{T+s+1},\theta_{k-2}+\theta_{T+s}+1)$	$(\theta_{k-2}-3^{U+s+1},\theta_{k-2}+\theta_{U+s}+1)$	(C$_k$-5), (C$_k$-7), (C$_k$-8)
$(\theta_{k-2}+3^{T+s+1},\theta_{k-2}-\theta_{T+s})$	$(\theta_{k-2}+3^{U+s+1},\theta_{k-2}-\theta_{U+s})$	(C$_k$-6), (C$_k$-7), (C$_k$-8)
$(\theta_{k-2},\theta_{k-2}-\theta_{T+s})$	$(\theta_{k-2},\theta_{k-2}-\theta_{U+s+1})$	(C$_k$-5), (C$_k$-6), (C$_k$-7)
$(\theta_{k-2},\theta_{k-2}+\theta_{T+s}+1)$	$(\theta_{k-2},\theta_{k-2}+\theta_{U+s+1}+1)$	(C$_k$-5), (C$_k$-6), (C$_k$-8)

(T,U and s are defined as in Lemma 2.5)

Proof. We prove this only for $(\Phi_0,\Phi_1)=(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$ of Theorem 4.5. The others are proved similarly. Let C be an $[n,k,d]_3$ code with diversity $(\theta_{k-2}-3^{T+1},\theta_{k-2}+\theta_T+1)$, gcd$(3,d)=1$, even $k\ge 6$ where $T=(k-4)/2$.

"only if" part: Assume that C is extendable. Then there is an (i,j)-hyperplane Π satisfying $\Pi\setminus F\subset F_{\mathrm{e}}$, where $(i,j)\in\{(\theta_{k-3}-3^{T+1},\theta_{k-3}+\theta_T+1),(\theta_{k-3},\theta_{k-3}-\theta_T),(\theta_{k-3},\theta_{k-3}+\theta_T+1)\}$. If Π is a $(\theta_{k-3}-3^{T+1},\theta_{k-3}+\theta_T+1)_{k-2}$ flat, then (C$_k$-5) holds with $s=1$ by (2) of Lemma 3.5. If Π is a $(\theta_{k-3},\theta_{k-3}-\theta_T)_{k-2}$ flat, then (C$_k$-9) holds by (1) of Lemma 3.6. If Π is a $(\theta_{k-3},\theta_{k-3}+\theta_T+1)_{k-2}$ flat, then (C$_k$-10) holds by (1) of Lemma 3.7.

"if" part: Assume that one of the conditions (C$_k$-5), (C$_k$-9), (C$_k$-10) holds. From the definition of conditions (C$_k$-5-C$_k$-10), there exists a $(k-2)$-flat Π with $\Pi\setminus F\in F_{\mathrm{e}}$. Hence C is extendable by Lemma 2.3. □

Example. Let C be a $[14,5,7]_3$ code with a generator matrix

$$G=\begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&1&1&1&1\\ 0&1&0&0&0&0&1&1&0&0&1&1&2&2\\ 0&0&1&0&0&1&0&1&0&1&0&2&1&2\\ 0&0&0&1&0&1&1&0&1&0&2&2&0&1\\ 0&0&0&0&1&1&1&2&1&2&0&2&1&0 \end{bmatrix},$$

whose weight distribution is

$$0^1 7^{32} 8^{50} 9^{64} 10^{28} 11^{40} 12^{16} 13^{12} \quad (\text{diversity}(40,45),\ \Phi_{\mathrm{e}} = 9).$$

Take four points P, Q_1, Q_2, Q_3 in $\Sigma = \mathrm{PG}(4,3)$ as

$$P = (1,0,0,0,2), Q_1 = (0,1,1,1,0), Q_2 = (0,1,2,2,2), Q_3 = (0,1,2,1,1).$$

Since $wt(P \cdot G) \equiv 0 \pmod 3$, $wt(Q_i \cdot G) \equiv 1 \pmod 3$ and $wt(Q_i \cdot G) \neq 7$, we have $P \in F_0$ and $Q_1, Q_2, Q_3 \in F_{\mathrm{e}}$. One can easily see that $\delta_1 = \langle P, Q_1 \rangle$, $\delta_2 = \langle P, Q_2 \rangle$, $\delta_3 = \langle P, Q_3 \rangle$ are $(1,0)$-lines. It also turns out that the three lines $\langle Q_1, Q_2 \rangle$, $\langle Q_1, Q_3 \rangle$, $\langle Q_2, Q_3 \rangle$ are (0,2)-lines, and that all of the planes not containing P are (4,6)-planes. Thus, P is the axis of type (4,6), that is, (C_5-3) of Theorem 4.3 holds. Hence C is extendable. The solid $\langle P, Q_1, Q_2, Q_3 \rangle$ is represented as the variety $V(f)$ with $f = x_0 + 2x_1 + 2x_2 + 2x_3 + x_4$. Hence we can take $h = (1,2,2,2,1)^{\mathrm{T}}$ so that $[G,h]$ generates a $[15,5,8]_3$ code, whose weight distribution is $0^1 8^{60} 9^{40} 10^{62} 11^{20} 12^{40} 13^{10} 14^{10}$.

REFERENCES

[1] Hill R. An extension theorem for linear codes. *Des. Codes Cryptogr.* **17** (1999), 151–157.

[2] Hill R., P. Lizak. Extensions of linear codes. In: Proceedings of IEEE Int. Symposium on Inform. Theory, Whistler, Canada, 1995, 345.

[3] Kohnert A. (l,s)-extension of linear codes. *Discrete Math.* **309** (2009), 412–417.

[4] Landjev I. The geometric approach to linear codes. In: Proceedings of the Fourth Isle of Thorns Conference (Eds A. Blokhuis, J. W. P. Hirschfeld, D. Jungnickel, J. A. Thas), Finite Geometries, Developments in Mathematics, Vol. **3**, Kluwer Academic Publishers, Dordrecht, 2001, 247–256.

[5] Maruta T. On the extendability of linear codes. *Finite Fields Appl.* **7** (2001), 350–354.

[6] Maruta T. Extendability of linear codes over $\mathrm{GF}(q)$ with minimum distance d, $\gcd(d,q) = 1$. *Discrete Math.* **266** (2003), 377–385.

[7] Maruta T. A new extension theorem for linear codes. *Finite Fields Appl.* **10** (2004), 674–685.

[8] Maruta T. Extendability of ternary linear codes. *Des. Codes Cryptogr.* **35** (2005), 175–190.

[9] Maruta T., K. Okamoto. Geometric conditions for the extendability of ternary linear codes. In: Coding and Cryptography-WCC 2005O (Ed. Ytrehus), Lecture Notes in Computer Science, Vol. **3969**, Springer (2006), 85–99.

[10] Maruta T., K. Okamoto. Some improvements to the extendability of ternary linear codes. *Finite Fields Appl.* **13** (2007), 259–280.

[11] Okamoto K., T. Maruta. Extendability of ternary linear codes of dimension five. In:Proceedings of 9th International Workshop in Algebraic and Combinatorial Coding Theory (ACCT), Kranevo, Bulgaria, 2004, 312–318.

[12] Simonis J. Adding a parity check bit. *IEEE Trans. Inform. Theory* **46** (2000), 1544–1545.

[13] Eupen M., P. Lisonek. Classification of some optimal ternary linear codes of small length. *Des. Codes Cryptogr.* **10** (1997), 63–84.

[14] Yoshida Y., T. Maruta. On the $(2, 1)$-extendability of ternary linear codes. In: Proceedings of 11th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT), Pamporovo, Bulgaria, 2008, 305–311.

Kei Okamoto
Department of Mathematics
and Information Sciences
Osaka Prefecture University
Sakai, Osaka 599-8531, Japan
e-mail: `kk20@hotmail.co.jp`

Received November 26, 2008
Final Accepted January 29, 2009