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ON THE VARIOUS BISECTION METHODS DERIVED FROM VINCENT'S THEOREM

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Dedicated to Professors Alberto Alesina and Massimo Galuzzi.[1]

Abstract. In 2000 A. Alesina and M. Galuzzi presented Vincent's theorem "from a modern point of view" along with two new bisection methods derived from it, `B` and `C`. Their profound understanding of Vincent's theorem is responsible for simplicity — the characteristic property of these two methods. In this paper we compare the performance of these two new bisection methods — i.e. the time they take, as well as the number of intervals they examine in order to isolate the real roots of polynomials — against that of the well-known Vincent-Collins-Akritas method, which is the first bisection method derived from Vincent's theorem back in 1976. Experimental results indicate that `REL`, the fastest implementation of the Vincent-Collins-Akritas method, is still the fastest of the three bisection methods, but the number of intervals it examines is almost the same as that of `B`. Therefore, further research on speeding up `B` while preserving its simplicity looks promising.

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Key words: Vincent's theorem, real root isolation method, bisection method, continued fraction method, Descartes' method, modified Uspensky's method.

[1]For presenting Vincent's theorem in their own, *unique* way!

1. Introduction. We begin with a review of Descartes' rule of signs [15].

Consider the polynomial $p(x) \in \mathbb{R}[x]$, $p(x) = a_n x^n + \cdots + a_1 x + a_0$ and let $var(p)$ represent the number of sign *variations* or *changes* (positive to negative and vice-versa) in the sequence of coefficients $a_n, a_{n-1}, \ldots, a_0$.

Descartes' rule of signs: The number $\varrho_+(p)$ of real roots — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ in the open interval $]0, +\infty[$ is bounded above by $var(p)$; that is, we have $var(p) \geq \varrho_+(p)$.

According to Descartes' rule of signs if $var(p) = 0$ it follows that $\varrho_+(p) = 0$. Additionally, according to Descartes' rule of signs, the Mean Value Theorem and the fact that the polynomial functions are continuous, if $var(p) = 1$ it follows that $\varrho_+(p) = 1$.

Therefore, Descartes' rule of signs yields the *exact* number of positive roots *only* in the two special cases mentioned above.[2]

These two special cases of Descartes' rule are used in Vincent's theorem of 1836, [28], which we present in a way, due to Alessina and Galuzzi [13].

Theorem 1 (Vincent's theorem — "bisection" version)**.** *Let $f(z)$, be a real polynomial of degree n, which has only simple roots. It is possible to determine a positive quantity δ so that for every pair of positive real numbers a, b with $|b - a| < \delta$, every transformed polynomial of the form*

$$\phi(z) = (1+z)^n f\left(\frac{a+bz}{1+z}\right)$$

has exactly 0 *or* 1 *variations. The second case is possible if and only if $f(z)$ has a simple root within $]a, b[$.*

We call this the *bisection* version of Vincent's theorem, since all three bisection methods studied below are derived from it; see also [11], [12] and [21]. Also note that throughout this paper we are interested in polynomials with rational coefficients; see also [20], [23].

Moreover, the Vincent-Akritas-Strzeboński (`VAS`) continued fractions method is also derived from Theorem 1, since the Möbius transformation $M(z) = \dfrac{a+bz}{1+z}$, where $a, b \in \mathbb{Q}$, is equivalent to a continued fraction of the form

[2] These two special cases were known to Cardano; in other words, what Descartes did was to generalize "Cardano's *special* rule of signs". This detail is mentioned in [5].

$$\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \cfrac{1}{\ddots + \cfrac{1}{\alpha_\ell + \frac{1}{z}}}}},$$

where $\alpha_1 \geq 0$ is an arbitrary non-negative integer and $\alpha_2, \alpha_3, \ldots, \alpha_\ell$ are arbitrary positive integers, $\alpha_i > 0$, $i > 1$, [1], [2], [3], [4], [14].

It is well known, [7], [8], [9], [10], [24], and [26], that the `VAS` continued fractions method is, so far, the fastest method for the isolation of the real roots of polynomials; however, no data exists on the number of intervals it examines during the isolation process. This is achieved in this paper, where the `VAS` continued fractions method is used as a benchmark in the empirical results presented in the sequel.

The rest of the paper is structured as follows:

In Section 2 we present the three bisection methods derived from Vincent's theorem; to wit,

– the first bisection method: Vincent-Collins-Akritas (`VCA`), [15], developed in 1976 by Collins and Akritas [16],

– the second bisection method: `B`, suggested by Alesina and Galuzzi [13], and

– the third bisection method: `C`, suggested by Saeli [13].

In Section 3 we present experimental data comparing the performance of the three bisection methods mentioned above juxtaposing it with that of the (`VAS`) continued fractions method.

2. The Three Bisection Methods Derived from Vincent's Theorem.

The fact that bisection methods can be derived from Vincent's theorem is missing from major works such as ([29], pp. 470–478) and almost every paper on the subject; to our knowledge, it appears *only* in [6] and the papers by Alesina and Galuzzi [11], [13].

2.1. The First Bisection Method: Vincent-Collins-Akritas (`VCA`).

This method seems to be the most "complicated" of the three. It was originally presented in a way that obscured its relation to Vincent's theorem. Due to that obfuscation it was originally called "modified Uspensky's method" and recently "Descartes' method" ([19], [18], [17], [25]), which is totally misleading. Akritas set the record straight, [6], by presenting the algorithm in a way that reveals its relation to Vincent's theorem, justifying thus the name "Vincent-Collins-Akritas" given to it in France, [15].

A brief description of the `VCA` bisection method is presented below according to [6]; see also [27]. However, we first need the following definition:

Uspensky's test: The number $\varrho_{01}(p)$ of real roots in the open interval $]0,1[$ — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by $var_{01}(p)$, where

$$var_{01}(p) = var\left((x+1)^{deg(p)} p\left(\frac{1}{x+1}\right)\right), \tag{1}$$

and we have $var_{01}(p) \geq \varrho_{01}(p)$.

As in the case of Descartes' rule of signs if $var_{01}(p) = 0$ it follows that $\varrho_{01}(p) = 0$ and if $var_{01}(p) = 1$ it follows that $\varrho_{01}(p) = 1$.

Therefore, Uspensky's test yields the *exact* number of positive roots *only* in the two special cases mentioned above; to wit, whenever $var_{01}(p) = 0$ or $var_{01}(p) = 1$.

Please note in equation (1) that, *after* the substitution $x \longleftarrow \dfrac{1}{x+1}$, the positive roots of $p(x)$ that were in the interval $]0,1[$ are now in $]0,+\infty[$, in which case Descartes' rule of signs *can* be applied.

Finally, we note that Uspensky's test is a special instance of the powerful "Vincent's test", which is based on Theorem 1, applies to any interval $]a,b[$ and is described in Section 2.2.

We are now ready for the `VCA` bisection method.

Let $p(x)$ be the polynomial whose roots we want to isolate and let ub be an upper bound on the values of its positive roots. Then all the positive roots of $p(ub \cdot x)$ lie in the interval $]0,1[$ and the `VCA` method isolates them by repeatedly bisecting the interval $]0,1[$, while using in the process Uspensky's test to make inferences about the number of positive roots certain transformed polynomials have in the interval $]0,1[$. Finally, the isolating intervals of the roots of $p(x)$ are easily computed from the bijection:

$$\alpha_{]0,ub[} = a + \alpha_{]0,1[}(b-a), \tag{2}$$

that exists between the roots $\alpha_{]0,1[} \in]0,1[$ of the *transformed* polynomial $p(ub \cdot x)$ and the roots $\alpha_{]0,ub[} \in]a,b[=]0,ub[$ of the *original* polynomial $p(x)$.

Below is a recursive description of the `VCA` bisection method as was originally presented by Collins and Akritas:

The first bisection method: `VCA`

Input: A univariate, square-free polynomial $p(ub \cdot x) \in \mathbb{Z}[x], p(0) \neq 0$, and the open interval $]a,b[=]0,ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$. (The positive roots of $p(ub \cdot x)$ are all in the open interval $]0,1[$.)
Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $(x+1)^{deg(p)} p\left(\dfrac{1}{x+1}\right)$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a,b[\}$;
4 $p_{0\frac{1}{2}} \longleftarrow 2^{deg(p)} p\left(\dfrac{x}{2}\right)$ `// Look for real roots in` $\left]0, \dfrac{1}{2}\right[$;
5 $m \longleftarrow \dfrac{a+b}{2}$ `// Is` $\dfrac{1}{2}$ `a root?` ;
6 $p_{\frac{1}{2}1} \longleftarrow 2^{deg(p)} p\left(\dfrac{x+1}{2}\right)$ `// Look for real roots in` $\left]\dfrac{1}{2}, 1\right[$;
7 **if** $p\left(\dfrac{1}{2}\right) \neq 0$ **then**
8 | **RETURN** `VCA` $\left(p_{0\frac{1}{2}},]a,m[\right) \bigcup$ `VCA` $\left(p_{\frac{1}{2}1},]m,b[\right)$
9 **else**
10 | **RETURN** `VCA` $\left(p_{0\frac{1}{2}},]a,m[\right) \bigcup \{[m,m]\} \bigcup$ `VCA` $\left(p_{\frac{1}{2}1},]m,b[\right)$
11 **end**

Algorithm 1. The *original* version of the `VCA`$(p,]a,b[)$ "*bisection*" algorithm, where the second argument is the open interval $]a,b[$ associated with $p(x)$. The isolating intervals of the roots of $p(x)$ are computed directly, without using bijection (2).

To obtain the isolating intervals of the positive roots of $p(x)$ we could have also used the interval $]a,b[=]0,1[$ along with bijection (2). An excellent discussion of this algorithm can be found in [15]. Please note the following:

- Uspensky's test is a *crucial component* of the `VCA`$(p,]a,b[)$ bisection algorithm — lines 1–3.
- The substitutions in lines 4 and 6 are performed only on the polynomial

$p(x)$, whereas at the same time — in line 5 — the interval $]a, b[$ is divided into two equal parts $]a, m[$ and $]m, b[$, to be used in line 8 (or 10).

– To isolate the real roots of $p(x)$ in the open interval $]0, 1[$ we proceed as follows:

- we first isolate the real roots in the interval $\left]0, \frac{1}{2}\right[$ — lines 4 and 8 (or 10),
- we then deal with the case where $\frac{1}{2}$ is a root of $p(x)$ — lines 5, 7 and 10,
- and, finally, we isolate the real roots in the interval $\left]\frac{1}{2}, 1\right[$ — lines 6 and 8 (or 10).

– The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 10.

Algorithm 1 does not make clear the relation of the `VCA` bisection method with Vincent's theorem. That relation is revealed if instead of the interval $]a, b[$ we associate with the polynomial $p(x)$ the Möbius transformation $M(x) = ub \cdot x$ — or $M(x) = x$ if we use bijection (2). For details see [6].

Please note that the fastest implementation of the `VCA` bisection method, `REL`, is due to Rouillier and Zimmermann, [22].

2.2.The Second Bisection Method: `B`. This method is a direct implementation of Theorem 1. However, we first need the following definition, which is also used in Section 2.3:

Vincent's test: If $a \geq 0$ and $b > a$ then the number $\varrho_{ab}(p)$ of real roots in the open interval $]a, b[$, — multiplicities counted — of the polynomial $p(x) \in \mathbb{R}[x]$ is bounded above by $var_{ab}(p)$, where

$$var_{ab}(p) = var\left((1+x)^{deg(p)} p\left(\frac{a+bx}{1+x}\right)\right), \tag{3}$$

and we have $var_{ab}(p) = var_{ba}(p) \geq \varrho_{ab}(p)$.

Vincent's test is a direct consequence of Theorem 1, presented by Alesina and Galuzzi [13]. It is a *very* powerful test and can be applied in all cases — including the case $]a,b[=]1,0[$, from which we obtain Uspensky's test.[3]

As in the case of Descartes' rule of signs if $var_{ab}(p) = 0$ it follows that $\varrho_{ab}(p) = 0$ and if $var_{ab}(p) = 1$ it follows that $\varrho_{ab}(p) = 1$.

Therefore, Vincent's test yields the *exact* number of positive roots *only* in the two special cases mentioned above; to wit, whenever $var_{ab}(p) = 0$ or $var_{ab}(p) = 1$.

Below is a recursive description of the second bisection method derived from Vincent's theorem; its simplicity is unsurpassed, but we pay for it by using a much more complicated test. Obviously, there is a trade off between simplicity of the method and complexity of the termination test.

The second bisection method: `B`

Input: A univariate, square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the open interval $]a,b[=]0,ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$.

Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $(1+x)^{deg(p)} p\left(\frac{a+bx}{1+x}\right)$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a,b[\}$;
4 $m \longleftarrow \frac{a+b}{2}$ `// Subdivide the interval` $]a,b[$ `in two equal parts` ;
5 **if** $p(m) \neq 0$ **then**
6 | **RETURN** $\mathtt{B}(p,]a,m[) \bigcup \mathtt{B}(p,]m,b[)$
7 **else**
8 | **RETURN** $\mathtt{B}(p,]a,m[) \bigcup \{[m,m]\} \bigcup \mathtt{B}(p,]m,b[)$
9 **end**

Algorithm 2. The $\mathtt{B}(p,]a,b[)$ "*bisection*" algorithm, proposed by Alesina and Galuzzi [13]; the second argument is the open interval $]a,b[$, whose endpoints a,b are used in Vincent's test in line 1. The isolating intervals of the roots of $p(x)$ are computed directly, without using bijection (2).

Please note the following:

[3] By comparison, Uspensky's test in Section 2.1, is rather weak as it applies *only* in the case $]a,b[=]0,1[$.

– Vincent's test is a *crucial component* of the $\texttt{B}(p,]a, b[)$ bisection algorithm — lines 1–3.

– In line 4 the interval $]a, b[$ is divided into two equal parts $]a, m[$ and $]m, b[$, to be used in lines 6 (or 8). Note that there are no polynomial transformations at all; only polynomial evaluations in line 1.

– To isolate the real roots of $p(x)$ in the open interval $]a, b[$ we proceed as follows:

  - we first isolate the real roots in the interval $\left]a, \frac{a+b}{2}\right[$ — lines 6 (or 8),
  - we then deal with the case where $\frac{a+b}{2}$ is a root of $p(x)$ — lines 5 and 8,
  - and, finally, we isolate the real roots in the interval $\left]\frac{a+b}{2}, b\right[$ — lines 6 (or 8).

– The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 8.

2.3. The Third Bisection Method: C. As Alesina and Galuzzi mention in their paper ([13], p. 189), it was Donato Saeli who suggested that they look at the connection between Farey sequences and Vincent's theorem. We quote from their paper: "It is possible that Vincent's theorem might be connected with Farey series in the same fruitful way it was connected with continued fractions." Saeli's proposal is evaluated in Section 3.

Recall that if $a \geq 0$ and $b, c, d > 0$ a *mediant* of the two fractions $\frac{a}{b} < \frac{c}{d}$ is given by $\frac{a+c}{b+d}$ and we have $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

A Farey series of order N, denoted by $\mathfrak{F}_N$, is the set of all reduced fractions between 0 and 1 whose denominators are N or less, arranged in increasing order, [13]. We can obtain $\mathfrak{F}_N$ from $\mathfrak{F}_{N-1}$ by inserting mediants whenever it is possible to do so without getting a denominator greater than N. We begin with $\mathfrak{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$.

Instead of using Farey sequences in $]0, 1[$, along with bijection (2), what we have done with the third bisection method derived from Vincent's theorem is to start with $]a, b[= \left] \frac{0}{1}, \frac{ub}{1} \right[$ and keep inserting mediants until we isolate the positive roots of a polynomial.

Below is a recursive description of the third bisection method derived from Vincent's theorem:

The third bisection method: `C`

Input: A univariate, square-free polynomial $p(x) \in \mathbb{Z}[x], p(0) \neq 0$, and the open interval $]a, b[=]0, ub[$, where ub is an upper bound on the values of the positive roots of $p(x)$.
Output: A list of isolating intervals of the *positive* roots of $p(x)$

1 $var \longleftarrow$ the number of sign changes of $(1+x)^{deg(p)} p\left(\frac{a+bx}{1+x}\right)$;
2 **if** $var = 0$ **then RETURN** $\emptyset$;
3 **if** $var = 1$ **then RETURN** $\{]a, b[\}$;
4 $m_f \longleftarrow \frac{num}{den}$ `// The mediant of the interval` $]a, b[$`, as in Farey sequences` ;
5 **if** $p(m_f) \neq 0$ **then**
6 | **RETURN** $\mathtt{C}(p,]a, m_f[) \bigcup \mathtt{C}(p,]m_f, b[)$
7 **else**
8 | **RETURN** $\mathtt{C}(p,]a, m_f[) \bigcup \{[m_f, m_f]\} \bigcup \mathtt{C}(p,]m_f, b[)$
9 **end**

Algorithm 3. The $\mathtt{C}(p,]a, b[)$ "*bisection*" algorithm, proposed by Donato Saeli, [13]; the second argument is the open interval $]a, b[$, whose endpoints a, b are used in Vincent's test in line 1. The isolating intervals of the roots of $p(x)$ are computed directly, without using bijection (2).

To obtain the isolating intervals of the positive roots of $p(x)$ we could have also used the polynomial $p(ub \cdot x)$ and the interval $]a, b[=]0, 1[$ along with bijection (2). Please note the following:

– Vincent's test is a *crucial component* of the $\mathtt{C}(p,]a, b[)$ bisection algorithm — lines 1–3.

– In line 4 the interval $]a, b[$ is divided — according to Farey's sequences — into two parts $]a, m_f[$ and $]m_f, b[$, to be used in lines 6 (or 8). Note that, again, there are no polynomial transformations at all; only polynomial evaluations in line 1.

– To isolate the real roots of $p(x)$ in the open interval $]a, b[$ we proceed as follows:

  - we first isolate the real roots in the interval $]a, m_f[$ — lines 6 (or 8),
  - we then deal with the case where m_f is a root of $p(x)$ — lines 5 and 8,
  - and, finally, we isolate the real roots in the interval $]m_f, b[$ — lines 6 (or 8).

– The isolating intervals are directly obtained from line 3 — except for those roots that happen to coincide with the midpoint of an interval that gets bisected, in which case they are computed in lines 5 and 8.

3. Empirical Results. In this section, we present some statistics using various classes of polynomials, as in [7]. We compare the performance of `REL`, the fastest implementation of Vincent-Collins-Akritas (`VCA`) bisection method, [22], with the `B` and `C` bisection methods described in Sections 2.2 and 2.3, respectively; as a benchmark we use the Vincent-Akritas-Strzeboński (`VAS`) continued fractions method.

In Table 1, we present statistics on some special classes of polynomials regarding: (a) the number of intervals ($\#I$) the methods use in order to isolate the real roots of the given polynomial, (b) the time in seconds ($T(s)$) they need for doing so, and (c) the number of the roots ($\#R$).

In Table 2, we present statistics regarding some types of random polynomials; "rp" indicates a random polynomial with randomly generated coefficients, "rmp" indicates a monic polynomial with randomly generated coefficients; whereas "rr" denotes products of factors (x-randomly generated integer root).

From Table 1 we see that bisection method `C` examines the greatest number of intervals — save for the case of Mignotte polynomials, whose roots are extremely close. The number of intervals examined by `VCA` and `B` is of the same order. By contrast, of all methods `VAS` examines the smallest number of intervals.

With respect to time, `VCA` is the fastest of the three bisection methods as it examines the smallest number of intervals.

Table 2 confirms the observations made in Table 1.

4. Conclusion. In this paper we have presented three bisection methods derived from Vincent's theorem for the isolation of the real roots of polynomials

Table 1. Comparison of various root isolation statistics on special polynomials

Polynomial	Method	Degrees 20 $\#I$	$T(s)$	$\#R$	50 $\#I$	$T(s)$	$\#R$	100 $\#I$	$T(s)$	$\#R$
ChebyshevI	VCA	48	0.000	20	106	0.024	50	208	0.236	100
	B	50	0.320	20	106	3.440	50	222	51.88	100
	C	98	0.336	20	422	16.49	50	1230	419.4	100
	VAS	38	0.004	20	90	0.024	50	176	0.204	100
Laguerre	VCA	44	0.004	20	108	0.032	50			
	B	48	0.388	20	112	5.548	50			
	C	2150	9.305	20	33140	2116	50			
	VAS	35	0.000	20	92	0.028	50			
Wilkinson	VCA	29	0.000	20	62	0.012	50	114	0.132	100
	B	46	0.320	20	110	4.080	50	218	54.57	100
	C	212	0.748	20	1284	50.60	50	5066	1894	100
	VAS	21	0.000	20	51	0.000	50	101	0.028	100
Mignotte	VCA	40	0.004	4	94	0.040	4	181	0.544	4
	B	52	0.240	4	124	6.872	4	240	162.5	4
	C	20	0.068	4	20	0.672	4	20	3.776	4
	VAS	5	0.000	4	5	0.000	4	5	0.008	4

with rational coefficients and we have compared their performance against the Vincent-Akritas-Strzeboński (`VAS`) continued fractions method — the benchmark.

The three bisection methods — `VCA`, `B` and `C` — use either Uspensky's or Vincent's test[4]:

- the more complex `VCA` method uses the simpler of these two tests, *viz.*, Uspensky's,

whereas

- the very simple methods `B` and `C` use the more complex — and very powerful — Vincent's test.

The simplest of all termination tests is Descartes' rule of signs, which is employed by the `VAS` method, the most "complex" method derived from Vincent's theorem.

[4]Remember that the former is a special instance of the latter.

Table 2. Comparison of various root isolation statistics on random polynomials

Polynomial	Method	50 #I	50 T(s)	50 #R	100 #I	100 T(s)	100 #R	200 #I	200 T(s)	200 #R
rp—(10)	VCA	13.2	0.000	2.4	19.2	0.008	4.4	26	0.100	6
	B	8.4	0.166	2.4	16	2.767	4.4	19.2	32.51	6
	C	8	0.218	2.4	31.2	6.855	4.4	20.4	46.44	6
	VAS	4.8	0.002	2.4	7.2	0.010	4.4	12.2	0.082	6
rp—(1000)	VCA	12	0.003	3.2	16.8	0.011	4	24.4	0.113	5.6
	B	10.4	1.740	3.2	12	7.901	4	17.2	36.64	5.6
	C	10.8	2.112	3.2	12.8	9.041	4	22	62.46	5.6
	VAS	4.8	0.001	3.2	5.8	0.010	4	11.2	0.085	5.6
rmp	VCA	31.2	0.006	4.8	28.6	0.017	4.8			
	B	30.4	1.286	4.8	27.6	6.103	4.8			
	C	2556.8	149.6	4.8	1244.8	457.6	4.8			
	VAS	8.8	0.007	4.8	7	0.014	4.8			
rr	VCA	58.4	0.002	20	138.2	0.046	50			
	B	63.6	0.346	20	149.6	5.964	50			
	C	101.2	0.422	20	428	19.11	50			
	VAS	35.6	0.003	20	82.4	0.033	50			

From the empirical results presented we have the following rule of thumb: the speed and "structural" complexity of the real root isolation algorithms derived from Vincent's theorem are *inversely* proportional to the "structural" complexity of the "termination test" they employ.

Therefore, it is no longer a surprise that `VAS` is the fastest real root isolation method, whereas `VCA` comes second and is followed by `B` and `C`.

The fact that methods `VCA` and `B` examine approximately the same number of intervals indicates that further research on speeding up `B` is warranted.

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