VECTOR COMBINATORIAL PROBLEMS IN A SPACE OF COMBINATIONS WITH LINEAR FRACTIONAL FUNCTIONS OF CRITERIA

Natalia Semenova, Lyudmyla Kolechkina, Alla Nagirna

Abstract: The paper considers vector discrete optimization problem with linear fractional functions of criteria on a feasible set that has combinatorial properties of combinations. Structural properties of a feasible solution domain and of Pareto–optimal (efficient), weakly efficient, strictly efficient solution sets are examined. A relation between vector optimization problems on a combinatorial set of combinations and on a continuous feasible set is determined. One possible approach is proposed in order to solve a multicriteria combinatorial problem with linear-fractional functions of criteria on a set of combinations.

Keywords: vector optimization, discrete optimization, linear fractional functions, set of combinations.

ACM Classification Keywords: G 2.1 Combinatorics (F2.2), G 1.6 Optimization

Introduction

A large number of management, planning, designing and other problems are solved with the use of combinatorial optimization models. Such a situation resulted in appearance of variety works devoted to research of combinatorial optimization problems [1,2,4]. Most of the practical decision-making problems are multicriteria. Therefore, it is very much interesting to examine different aspects of solution to multiobjective optimization problems, in particular, to discrete problems [1-8]. The interest in researching a solution to multicriteria discrete optimization models is due to their wide application when important economy, management, ecology, planning problems are dealt with, when different complicated systems are designed, decisions are made under uncertainty and so on. Vector optimization problems emerge if one needs to optimize more than single objective function. Fractional linear objective functions, being the result of division of two linear forms, are widely applied within the problems when it is necessary to optimize some relative indices of quality like a cost price, profitability, productivity and laboriousness. The models using such criteria show the tendencies in continuous lowering of a cost price level as for a unit of products and in increasing of high-quality production indices when production scale is broader. The union of the mentioned problems leads to investigation of multicriteria problems on combinatorial sets. The same problems are complicated each one by itself and they were not investigated earlier when taken totally, and, therefore, to it is quite important and urgent to solve them. It is interesting to consider the latter problems as the ones of combinatorial optimization on a feasible set of combinations.

Most of combinatorial optimization problems are known to be reduced to integer programming problems, but such a case is not always justified since the possibility to take combinatorial properties of a problem into account is lost [1]. The monographs [2,4] show that the convex hull of a combination set is the combination polyhedron, in which the set of vertices $vert\Pi$ is equal to the combination set $C(A)$ of, i.e. $vert\Pi = vert(convC_A)$. Such a combination polyhedron property allows to reduce a solution to an initial problem to a problem solution on a continuous feasible set. Thus, it is quite important and urgent to investigate and construct the methods that solve multicriteria problems and that are based on examining convex hulls of such sets when combinatorial properties of feasible solution domain are taken into consideration, in particular, the same concerns combinations.

At present, there exist many multicriteria problem solution methods, but none of them in the existing form are applicable for combinatorial problems on combinations. That is why it is quite important for the authors to consider the possible approaches used for multicriteria problems based on the combinatorial set of combinations.

The present paper investigates the structural properties of a feasible solution domain and the ones of the sets of Pareto-optimal (efficient), strictly efficient and weakly efficient solutions.

The relation between vector optimization problems on a combinatorial set of combinations and on a continuous feasible set is established. To solve the combinatorial problems on combinations, one possible approach is proposed.

Problem statement. Basic definitions

To state the problem, we use the concept of the multiset A determined by the foundation $S(A)$ and the element multipleness $k(a)$ (the number of the same elements *a*).

Let the multiset $A=\{a_1,a_2,...,a_g\}$ be specified, assume $S(A)=\{e_1,e_2,...,e_n\}$ as its foundation, where $e_j \in R_1$ $\forall j \in N_n=\{1,2,...,n\}$, and the multipleness $k(a_j)=r_j$, $j \in J_n$, $r_1+r_2+...+r_n=g$, is given.

Take an arbitrary $k \in N_g$. Call the ordered k - *selection* from the multiset A as the set

$$a=\left(a_{i_1},a_{i_2},...,a_{i_k}\right), \tag{1}$$

where $a_{i_j} \in A$ $\forall i_j \in N_k$, $\forall j \in N_k$, $i_s \neq i_t$, if $s \neq t$ $\forall s \in N_k$, $\forall t \in N_k$.

Definition. [2] A set *E(*A*)*, the elements of which are the k – *selections* of type (1) from the multiset *A*, is called the Euclidean combinatorial set, if the conditions $(a' \neq a'') \Leftrightarrow (\exists\, j \in J_k : a'_j \neq a''_j)$ are satisfied for its arbitrary elements $a'=(a'_1,a'_2,...,a'_k)$, $a''=(a''_1,a''_2,...,a''_k)$, that is, the set $E(A)$ the following property: two elements of the set $E(A)$ differ from each other, if, regardless of other differences, they differ in the order of location of the characters that form them.

Let the multiset A be the set. The set of all the k -*selections* from the set $S(A)$ of the type $\left(a_{i_1},a_{i_2},...,a_{i_k}\right)$ is called the Euclidean set of the k -combinations without reiteration of n different real numbers, if the condition $e_{i_1}<e_{i_2}<...<e_{i_k}$ is satisfied. Denote such a combination set by $C_n^k(A)$.

The set of all the k -selections of type (1) from the multiset A is called the general combination set, if the condition $a_{i1} \leq a_{i2} \leq ... \leq a_{ik}$ is satisfied.

The interest in Euclidean combinatorial sets is related to the possibility to consider them as the points of the space R^k, that is, with the possibility to immerse them into the arithmetic Euclidean space R^k. Let $E(A)$ be the combinatorial set and let its element be described by formula (1). The mapping $f: E(A) \rightarrow E_f(A) \subset R^k$ is called immersion of the set $E(A)$ into the arithmetic Euclidean space, if f brings the set $E(A)$ into one-to-one correspondence to the set $E_f(A) \subset R^k$ according to the rule: for $a=\left(a_{i_1}, \ \ldots, \ a_{i_k}\right) \in E(A)$, $x=(x_1...,x_k) \in E_f(A)$, have $x_j=a_{i_j}$. $\forall j \in N_k$

The convex hull for the points of the Euclidean combinatorial set of combinations with reiterations is the polyhedron Π of combinations with reiterations, and such a polyhedron is described the set of the inequalities

$$\begin{cases} e_1 \leq x^1, \\ x^i \leq x^{i+1}, \ \forall i \in N_{k-1}, \\ x^k \leq e^k, \end{cases} \tag{2}$$

where e_i are the elements of the basis $S(A)$ for the multiset A.

Consider the multicriteria optimization problem defined on the combinatorial set of combinations that is immersed into the arithmetic Euclidean space, and the problem has the following form:

$$Z_l(F,X): \ \max\{F(x) | \, x \in X\},$$

where *F(x)* is the vector criterion, $F(x)=(f_1(x),f_2(x),...,f_l(x)),$

$$f_i(x) = \frac{\left\langle c^i, x \right\rangle + c_0^i}{\left\langle d^i, x \right\rangle + d_0^i}, \ i \in N_l, \tag{3}$$

$X = \Pi \cap D$, where $\Pi = conv\ S_{gn}^k(A)$, D = $\{x \in R^k \mid Ax \le b\}$, D *is* the convex polyhedron in R^k , $A \in R^{m \times k}$, $b \in R^m$.

The solution to the problem $Z_l(F, X)$ is understood as searching for the elements from one of the following sets: $P(F, X)$, i.e. the Pareto-optimal (efficient) solution set, *Sl(F,X)*, i.e. the Slater-optimal (weakly efficient) solution set, $Sm(F, X)$, i.e. the Smale-optimal (strictly efficient) solution set. According to [3, 6], for any $x \in X$, the following statements are true:

$$x \in Sl(F, X) \Leftrightarrow \{ y \in X \mid F(y) > F(x) \} = \varnothing, \tag{4}$$

$$x \in P(F, X) \Leftrightarrow \{ y \in X \mid F(y) \ge F(x), F(y) \ne F(x) \} = \varnothing, \tag{5}$$

$$x \in Sm(F, X) \Leftrightarrow \{ y \in X \mid y \ne x, F(y) \ge F(x) \} = \varnothing. \tag{6}$$

Evidently,

$$Sm(F, X) \subset P(F, X) \subset Sl(F, X) . \tag{7}$$

The finiteness of the feasible domain X yields the non-emptiness of the set $P(F, X)$ and its external stability that means that $\forall y \in X\ \exists x \in P(F, X) : F(x) \ge F(y)$.

Take the combinatorial properties of the combinations polyhedron into account, and the criteria for its vertex, face and adjacency of vertex are formulated in [2], that are expedient to be used for the problem solution method construction.

Vertex criterion. The points $x^i = \left(x_1^i, x_2^i, ..., x_k^i\right) \in R^k$, where $x_j^i = e_j\ \forall j \in N_{k-i+1}$, and $x_{j+k-i+1}^i = e_n$ $\forall j \in N_{i-1}$, $\forall i \in N_{k+1}$, and only they can be the vertices of a polyhedron of combinations without reiterations.

Criterion face. The set of points $x^i = \left(x_1^i, x_2^i, ..., x_k^i\right) \in R^k$ is i-face of polyhedron Π then only then, when it is the decision of the system of arbitrary $k - i$ equalizations, $i \in N_{k-1}^0$ from a set:

$$x_1^i = e_1, x_j^i = x_{j+1}^i, \forall\ j \in N_{k-1}, x_k^i = e_n.$$

Adjacency of vertex. Arbitrary two vertices $x^i, x^j \in \Pi, i, j \in N_{k+1}$ are contiguous.

Properties of sets of Pareto of optimal (efficient), strictly efficient, weakly efficient solutions

Theorem 1. The elements of $Sm(F, X)$, i.e. the sets of strictly efficient solutions, $P(F, X)$, i.e. the sets of Pareto-optimal solutions, $Sl(F, X)$, i.e. the sets of weakly efficient solutions to multicriteria problems on the combinatorial sets of combinations are at the vertices of the combination polyhedron Π .

Proof. Consider relation (6) between the defined efficient solution set, take into account the fact that the feasible solution set X is the subset of the combination set $C_{gn}^k(A)$, and the following inclusions are true: $Sm(F, X) \subset P(F, X) \subset Sl(F, X) \subset C_{gn}^k(A)$. In accordance with Theorem 2.4 in [2], the combination set $C_{gn}^k(A)$ coincides with the set of vertices of general combination polyhedron $C_{gn}^k(A)$ = $vert\,\Pi_{gn}^k(A)$. Therefore, the inclusions $Sm(F, X) \subset P(F, X) \subset Sl(F, X) \subset vert\,\Pi_{gn}^k(A)$ are true. Theorem is proved.

The present property the combination polyhedron allows reducing the solution to the problem $Z_l(F,X)$ to the solution to the problem $Z_l(F,G)$ on the continuous feasible set $G = \Pi \cap D$. For any point $y \in vert\Pi$ in such a problem, the sufficient optimality conditions [3] for all the mentioned types of the efficient solutions are true.
The following theorems are valid.

Theorem 2. $Sl(F,G) \cap vert\Pi \subset Sl(F,X),\ P(F,G) \cap vert\Pi \subset P(F,X),\ Sm(F,G) \cap vert\Pi \subset Sm(F,X).$

Proof. Since $\mathrm{vert}\,\Pi \cap D \subset G$, the inclusion $Sl(F,G) \cap \mathrm{vert}\,\Pi \cap D \subset Sl(F, G \cap \mathrm{vert}\,\Pi \cap D) = Sl(F,X)$ is true.

The relations $P(F,X) = P(F, \mathrm{vert}\,\Pi \cap D) \supset P(F,G) \cap \mathrm{vert}\,\Pi \cap D,$
$Sm(F,X) = Sm(F, D \cap \mathrm{vert}\,\Pi) \supset Sm(F,G) \cap \mathrm{vert}\,\Pi$ are roved by analogy.

Theorem 3. $\forall x \in \mathrm{vert}\,\Pi : x \in P(F,\Pi) \cap D \Rightarrow x \in P(F,X),\quad x \in Sl(F,\Pi) \cap D \Rightarrow x \in Sl(F,X)$, $x \in Sm(F,\Pi) \cap D \Rightarrow x \in Sm(F,X)$.

Proof. Since $G = \Pi \cap D$, then the following implications are true:
$\forall x \in \mathrm{vert}\,\Pi : x \in P(F,\Pi) \cap D \Rightarrow x \in P(F, \Pi \cap D) = P(F,G) \Rightarrow x \in P(F,X),$
$x \in Sl(F,\Pi) \cap D \Rightarrow x \in Sl(F,X)$, $x \in Sm(F,\Pi) \cap D \Rightarrow x \in Sm(F,X)$.

Therefore, Theorems 1-3 establish the relation between the problems $Z_l(F,X)$ and $Z_l(F,G)$, and the latter one is defined on the continuous feasible set. It becomes possible to apply the classical continuous optimization methods for solutions to vector combinatorial problems on combinations. One can develop the new original solution methods using the properties of combinatorial sets and their convex hulls.
As for the strictly quasi-convex objective criterion function, the following theorem is proved in [5].

Theorem 4. If the vector criterion functions $f_i(x),\ i \in N_l$, are strictly quasi-convex and semicontinuous from below on linear segments *X*, then the set *Sl(F,X)* of the weakly efficient solutions to the problem is the union of the efficient sets $P(F,X)$ of the solutions to the subproblems $Z_I(F,X), I \subset N_l, I \neq \varnothing$, i.e. $Sl(F,X) = \bigcup \{ P_I(F,X) : I \subset N_l, |I| \leq k+1 \}$.

The linear fractional functions are known to be strictly quasi-convex on a convex set, and, therefore, Theorem 4 is valid for the set $Sl(F,X)$ of weakly efficient solutions to the problem *Zl(F,X)*.

Let x^0 be some efficient (Pareto-optimal) solution. Introduce the following denotations:
$P = (p_1, p_2, ..., p_l) \in R^{k \times l}$, $Q = (q_1, q_2, ..., q_l) \in R^{k \times l}$, $p_i = \left(\left\langle d_i, x^0 \right\rangle c_i - \left\langle c_i, x^0 \right\rangle d_i \right) \in R^k, i \in L,$
$q_i = \left(d_i^0 c_i - c_i^0 d_i \right) \in R^k, i \in L, y \in R^l, \alpha > 0,$ and $\alpha \in R^l$ is some continuous vector, consider the linear programming (LP) of such a form:

(LP): $\max \left\{ \langle \alpha, y \rangle \mid Q^T x^0 - (P+Q)^T x + y = 0, x \in \Pi \cap D \right\}.$

For the problem $Z_l(F,X)$, the following theorem is valid that establishes the interrelations between the solutions to it and the ones to the LP problem.

Theorem 5. The point $x^0 \in X$ is the efficient solution to the problem $Z_l(F,X)$ if and only if the LP problem has the optimal solution (x^*, y^*) with $y^* = 0$.

There exist a number of the methods used to solve multicriteria problems that allow to find the elements of the indicated of the indicated solution sets. But the problem $Z_l(F,X)$ is becomes more complicated in the presence of the criteria that are the linear fractional functions of type (2). This is the first specific feature of the aforesaid.

The second feature of the problem $Z_l\left(F,X\right)$ is the presence of the combinatorial properties that the feasible solution domain has. Therefore, to solve the problem, it is necessary to take all its features into account.

As it is generally known, to solve problems with a linear fractional goal function, there exist many methods conventionally falling into linearization methods, parametric methods, simplex-method modifications, among which one knows Charnse and Kuper method, Gilmori and Gomori algorithm and others. But none of them takes combinatorial terms of feasible solution domain into account.

On the basis of the proved theorems, continuing and developing works [2-5] one possible approach to the considered problem solution is proposed that means to implement the followings stages:

1) at the initial stage, consider some problem with a linear fractional objective function on a combination set;
2) for a separately taken problem, go by turns from a linear fractional objective function to a linear function;
3) a multicriteria problem with linear objective functions is examined, but on a changed feasible solution set;
4) a transition is carried out a multicriteria problem with linear objective functions to the the same thingcriterion with a linear objective function.

Consider the second stage of the proposed approach in more detail. Tho lolve the problem *Z(F,X)*, pass to the problems with linear objective functions. Use the denotations:

$$y_0=\frac{1}{\left\langle d^i,x\right\rangle+d_0}\ ,\ z_i=x_iy_0\ ,\qquad \forall i\in N_k\ . \tag{8}$$

Relation (8) specifies the mapping $\psi(X)=X^t\subset R^{k+1}$ for the feasile solution set. If $\left\langle d^i,x\right\rangle+d_0>0$, then $y_0>0$, $x_i\ge 0,\forall i\in N_k$, and $\psi(x)=z=(y_0,z_1,...,z_k)$.

In this case, the problem with a linear fractional objective function is reduced to the linear objective function

$$f'(\mathrm{x}^*)=f(\mathrm{z}^*)=\left\langle c^i,z\right\rangle, \tag{9}$$

under the condition

$$X'=\Pi'\cap D'\ . \tag{10}$$

It should be noted that, if is $z=(y_0,z_1,...,z_k)$ is the solution to problem (9), (10) with a linear objective function, the latter meets condition (8), and $x=(x_1^*,...,x_k^*)$ satisfies the constraints for the problem $Z_l\left(F,X\right)$.

Thus, on the general combination set, a combinatorial problem with a linear-fractional function can be reduced to a problem with a linear objective function. The number of variables is increased here by one, and to the system of limitations another equality is added: $\left\langle d^i,z\right\rangle+d_0y_0^*=1$.

Conclusions

The paper considers the statement of the multicriteria problem when the combinatorial properties of the feasible solution domain and linear-fractional functions of criteria are taken into account. The paper also establishes the interrelation between the problem with a linear objective function and the vector problem with linear-fractional functions of criteria defined on a feasible combinatorial set of combinations.

It becomes possible to apply the classical optimization methods for solutions to vector combinatorial problems on a combination set and, hence, to develop the new original solution methods when the properties of combinatorial sets and their convex hulls are used.

Proceeding from the proved theorems, the ongoing research and the results developed in [2-5], the approach is proposed that is used to solve the problem $Z_l\left(F,X\right)$. The basis for such an approach means to reduce searching for a solution to an initial problem to solving a number of scalar (single-criterion) problems and to verifying optimality of an obtained solution. The single-criterion problem solution methods rely on the ideas of decomposition, Kelly’s cutting plane method, relaxation.

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ONE CLASS OF STOCHASTIC LOCAL SEARCH ALGORITHMS

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***Abstract**: Accelerated probabilistic modeling algorithms, presenting stochastic local search (SLS) technique, are considered. General algorithm scheme and specific combinatorial optimization method, using "golden section" rule (GS-method), are given. Convergence rates using Markov chains are received. An overview of current combinatorial optimization techniques is presented.*

***Keywords**: combinatorial optimization, stochastic local search, simulated annealing, Markov chains*

Introduction

Approximate algorithms are well-known to solve different combinatorial problems. This based on some facts: firstly, mostly each problem is NP-hard; secondly, result functions have lots of local extremes; and finally, real data often are given with some inaccuracy and this makes serious calculations unnecessary. Also, it must be noticed that key ideas of these algorithms (metaheuristics) allow us to design algorithms, which can solve not one but some classes of optimization problems.

The most known definition of combinatorial optimization problem (according to Papadimitriou and Steiglitz [1,2]) is following: find $x_* \in X$