A Comparison of Arbitration Procedures for Risk Averse Disputants[1]

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ABSTRACT

We propose an arbitration model framework that generalizes many previous quantitative models of final offer arbitration, conventional arbitration, and some proposed alternatives to them. Our model allows the two disputants to be risk averse and assumes that the issue(s) in dispute can be summarized by a single quantifiable value. We compare the performance of the different arbitration procedures by analyzing the gap between the disputants' equilibrium offers and the width of the contract zone that these offers imply. Our results suggest that final offer arbitration should give results superior to those of conventional arbitration.

Subject Areas: Game theory, Negotiation, Decision analysis, Labor relations, Utility, Strategic decision-making.

INTRODUCTION

Consider a situation where two parties are attempting to reach agreement on some joint decision. If they are successful, they can engage in some mutually beneficial activity. If however their negotiations end in a standstill, then one or both parties may take retaliatory action that would be costly not only to the two sides involved in the dispute and but also to other stakeholders. One common example of this scenario is wage bargaining between management and labor, where a failure to reach agreement would lead to a strike or lockout. Other examples include disagreements between two countries over their respective fishing quotas in a bordering ocean, where a failure to agree could result in trade sanctions or military action; or contract disputes between two businesses, where a failure to agree could lead to lengthy civil court proceedings. In each example, it would be desirable to have some alternative way of obtaining a mutually acceptable decision so as to avoid the consequences of a disagreement; one such alternative is third party arbitration.

Conventional Arbitration

The original form of arbitration is what we refer to herein as Conventional Arbitration (ConA). In this procedure a third party arbitrator listens to the two sides present their cases and propose settlements, and then the arbitrator imposes what she sees as an appropriate settlement. This settlement may match one of the disputant's proposals, or it may be quite different; in principle almost any result is possible. Because the exact process for determining the settlement is unspecified, previous studies have proposed several possible models of arbitrator decision making for ConA.

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In the simplest model (used in e.g. Ashenfelter et al 1992, Bloom 1986), the arbitrator ignores disputants' offers and imposes a settlement s of her own devising. This could be formulated very simply as $s = c$, where c is the arbitrator's notion of a fair settlement based on the observable facts of the dispute (finances, precedents, etc). Since the decision is completely beyond the influence of the disputants, we refer to it here as the Autocratic Arbitration (AA) model of ConA.

A second model that has been used for ConA (Ashenfelter & Bloom 1984, Bloom 1986) assumes that the final settlement is based upon the proposals of the disputants as well as on the arbitrator's view of the facts. Thus the final settlement is modeled as an averaging process like $s = (1 - w)(a + b)/2 + wc$, where a and b are the settlements suggested by the disputants and w is a weighting factor, $0 \le w \le 1$. Since this model in a sense represents a compromise of the views of the three parties, we refer to it herein as the Compromise Arbitration (CA) model of ConA.

Another model for ConA that has appeared in the literature (Farber 1981, Farber & Bazerman 1986, Farber & Bazerman 1989) is known as Equitable Settlement (ES). Like CA, this model assumes a 3-way compromise amongst the parties but here the arbitrator's weighting factor varies according to the gap between the offers received; i.e. $w = J(b - a)$, where $0 \le J(\bullet) \le 1$ is an increasing function and $J(0) = 0$. Thus the arbitrator is assumed to place more emphasis on her own opinion if the disputants are far apart, but less if the two offers are close together.

Final Offer Arbitration

The second arbitration procedure commonly found in practice is Final Offer Arbitration (FOA), which was introduced in part to correct some of the perceived deficiencies of ConA. Stevens (1966) suggested FOA as a way to pressure the parties to make more "reasonable" offers. The resolution process in this type of arbitration is simple: each disputant submits a proposal (their "best and final" offer) and then the arbitrator chooses one of these as the settlement, presumably the one that seems to be the most "fair", i.e. is closer to the arbitrator's opinion of an appropriate settlement. Thus the settlement may be formulated as $s = closest\{a,b\}$.

The proponents of FOA had originally argued that the threat of having the other party's offer chosen would be sufficient to force the disputants' offers to converge to an agreement. However, work by Farber (1980), Crawford (1982) and Brams & Merrill (1983) showed that the final offers would typically be separated by a significant gap. Consequently FOA appeared to lose some of its original appeal, although it continues to be used in practice and to be studied both empirically (e.g. Pecorino & van Boening 2001) and analytically (e.g. Farmer & Pecorino 1998).

Other Arbitration Procedures

Several arbitration methods other than FOA and ConA have been proposed in the literature with the goal of encouraging the disputants to converge to an agreement. For example, Brams & Merrill (1986) proposed a combination of FOA with AA called Combined Arbitration (CBA). If the arbitrator's position lay between the final offers (e.g. $a \le c \le b$) or if the offers converged, then FOA would be used; but if both offers were either above or below the arbitrator's opinion, then AA would be used. Their analysis for risk neutral disputants showed that this process would lead to full convergence of the equilibrium offers.

Zeng (2003) recently took a different approach by suggesting the use of some ideas from second-price auction theory (Vickrey 1961). In its simplest form, the arbitration settlements could be formulated as $s = 2c - farthest\{a,b\}$; that is, under what we might call Second Price Arbitration (2PA) the arbitration award would be based primarily upon the arbitrator's own opinion, but *reduced* by whichever offer was *farthest* from that view.

Another idea for improving arbitration comes from Armstrong & Hurley (2001) in the form the Closest Offer Principle of Arbitration (COPA). They used this term to describe their finding that arbitration formulations provided better theoretical performance as the emphasis on the closer offer was increased.

While these and other alternative methods do have some future potential, they do not seem to have made their way into actual use yet. It is still the FOA and ConA procedures that seem to predominate in practice and so are worthy of continued research attention.

RESEARCH OBJECTIVES

Given a choice of procedures that could be used, potential disputants and other stakeholders will face several decisions with respect to an impending arbitration process. For example, which procedure should disputants choose to use, or should a governing body choose to impose? How should arbitrators in ConA make their settlement decisions in order to best serve the disputants? Given the use of a particular procedure, how much compromise is in a disputant's own best interests? In other dispute resolution methods such as court proceedings, what features might be worth adopting from arbitration? These questions are not only of interest to researchers, but also to governing bodies who must decide how arbitration will be used, to arbitrators who must make the actual arbitration decisions, and to disputants who must decide how best to act. Consequently, in this paper our primary objective is to compare the performance of the models for FOA, ConA, and alternative procedures in order to provide an assessment of their relative merits.

In performing this comparison we build upon a recent proposal by Armstrong & Hurley (2001) for a model of arbitrator decision-making that is more general than previous ones, such that it includes FOA, CA, and AA as special cases. Their brief analysis was restricted to risk neutral disputants; it showed that although FOA would not promote complete convergence of the disputants' equilibrium offers, those offers would always be less extreme than ones made under CA. In our work herein we provide a clearer framework for comparing models and the arbitration procedures they represent, extend the Armstrong & Hurley model to include a wider range of arbitration models, and perform our analysis so as to include risk averse disputants.

In the next section of the paper, we define our modeling framework and its relation to previous arbitration models. We then analyze the properties of this model, with particular emphasis on the cases where the disputant's utility functions take either exponential form or linear form. This is followed by a short numerical study for the exponential case. We conclude with a discussion of how our theoretical results compare with previous empirical studies, and the implications of these results for arbitration research and practice. An appendix contains proofs for our analytical results.

MODEL DEFINITION

Two disputants A and B are entering arbitration over some issue. As with most other treatments of this topic, we assume here that the issue(s) in dispute can reasonably be represented by a single quantifiable value. This could be e.g. the total financial value of a salary and benefits package, the size of an import quota for international trade, or an award for damages in a civil lawsuit. We do not consider arbitration over multiple distinct issues (Wittman 1986), nor to what degree a settlement might be economically efficient rather than a pragmatic compromise (Crawford 1979). In the arbitration hearing each disputant proposes a settlement for the dispute, presumably along with a variety of supporting information. We denote the settlement suggested by disputant A with *a*, and that of B with *b*.

We assume that the value of a settlement *s* to disputants A and B is determined by their respective utility functions $U(s)$ and $V(s)$ and that each disputant is seeking to maximize his respective expected utility from the settlement. Disputant A would like the settlement to be as small as possible, while B would like it to be as large as possible. We suppose that neither party is risk seeking, so that for disputant A we have $U'(s) \leq 0$ and $U''(s) \leq 0$ (decreasing and concave), and similarly for B we have $V'(s) \geq 0$ and $V''(s) \leq 0$ (increasing and concave). Note that these definitions include risk neutral disputants as a limiting case.

We assume that the arbitrator's initial opinion of a fair settlement can be modeled as a random variable C having a unimodal density function $f(c)$, cumulative distribution $F(c)$, median n, and mean $\mu = 0$. This distribution function is common knowledge to both disputants. It is also common knowledge that the arbitrator will choose a final settlement *s* according to

$$s(a,b) = \begin{cases} wc + (1-w)(x)a + (1-w)(1-x)b & \text{if} \quad |a-c| < |b-c| \\ wc + (1-w)(1-x)a + (1-w)(x)b & \text{if} \quad |a-c| > |b-c| \end{cases} . \qquad (1)$$

That is, the arbitrator is assumed to choose the final settlement *s* to be a weighted average of:

(a) The arbitrator's own assessment *c*, which is given weight *w*;
(b) The disputant offer that is closest to the arbitrator's position, i.e. seems most "reasonable", which is given weight $(1-w)(x)$; and,
(c) The disputant offer that is farthest from the arbitrator's position, i.e. seems least "reasonable", which is given weight $(1-w)(1-x)$.

In the event that the two offers are equidistant from the arbitrator's opinion (a tie), the arbitrator will randomly designate one (e.g. by flipping a fair coin) as being "closer" for evaluation purposes.

We can use parameters *w* and *x* to define a wide variety of possible arbitration formulations, as illustrated in the two-dimensional diagram of Figure 1. In particular, several previous arbitration models can be shown to be special cases of this framework:

(a) If $w = 0$ & $x = 1$ so that all of the weight is given to the closer disputant's offer, the model reduces to that of regular FOA (as in Farber 1980 and many subsequent works);

(b) If we fix $w = 1$ then the disputants' offers are ignored and the arbitrator imposes her own opinion as the settlement, as in the AA model of ConA (Ashenfelter et al 1992, Bloom 1986);
(c) If we choose $w \in [0,1)$ & $x = 1/2$ so that the two disputants' offers are given equal weight and then averaged with the arbitrator opinion, the model simplifies to the CA model of ConA used in Ashenfelter & Bloom (1984) and in Bloom (1986);
(d) By fixing $w = 0$ & $x = 2$ (i.e. twice the x value used in FOA) we get a procedure in the spirit of the COPA approach suggested by Armstrong & Hurley (2001);
(e) By fixing $w = 2$ & $x = 0$ we get the 2PA model as a simplified version of the approach used in Zeng (2003).

If we further permit the parameter weights w and x to vary according to the offers received, rather than being fixed in advance, then the general framework above could also be adapted to describe (though not technically include) some more complicated arbitration models:

(f) If we fix $x = 1/2$ but let w be a function of the two offers such that $w = J(b - a)$, $0 \leq J(\bullet) \leq 1$, $J(0) = 0$, and $J'(\bullet) \geq 0$, then we have the ES model of ConA (Farber 1981, Farber & Bazerman 1986);
(g) If we fix $x = 1$ and then choose $w = 0$ if $c \in [a, b]$ or if $a \geq b$, but $w = 1$ otherwise, then we have the CBA proposal of Brams & Merrill (1986).

This weighted average framework is similar to the one used in Armstrong & Hurley (2001), except that there the parameters were restricted to $w \in [0,1]$ and $x \in [1/2,1]$. Herein we instead allow these parameters to take any finite values, positive or negative; the three weights will sum to one for any given w and x, as $w + (1-w)(x) + (1-w)(1-x) = 1$.

This sort of weighted-average decision logic could conceivably be mandated as a formula in some governing statute or contract, but it might just as well arise out of the individual actions of a population of arbitrators. As professionals trying to earn their living, arbitrators clearly have significant motivation to render decisions that appear "reasonable" or "fair" in some sense so that they can remain acceptable for employment by future disputants (Farber & Bazerman 1986). One way to achieve this acceptability is for the arbitrators to render settlements similar to what they *believe* other arbitrators would choose under the same circumstances; the arbitrators may therefore tend to behave sufficiently alike that they become "statistically exchangeable" with each other (Ashenfelter 1987). In those situations where they have freedom to choose this can mean that the decision process (i.e. the formula parameter weights) actually used by the arbitrator might well differ from whatever had been officially proposed.

Likewise, rational disputants can be expected to act according to how they *believe* the arbitrator will make a decision in their dispute and how they *believe* their opponent will respond; in practice the beliefs of the two disputants may differ from each other (see e.g. McKenna & Sadanand 1995) and/or from the arbitrator's actual decision process. The remainder of the discussion herein is phrased as if all three of these views are identical, but technically the only requirement is that each disputant *believes* that they are identical and acts accordingly.

Given this assumed knowledge of w and x, the decision faced by each disputant is to determine the "best" offer to submit so as to maximize his expected utility, given that the other disputant will simultaneously be trying to do the same; i.e. the disputants are playing a competitive game. As in previous work (e.g. Brams & Merrill 1983) we want to find the equilibrium solution to this game; that is, a pair of offers $\{a,b\}$ from which neither side can depart without worsening its own situation on average. Defining $m \equiv (a+b)/2$ as the mean of the two offers, we can write the expected utility functions for disputants A and B respectively as

$$EU \equiv \int_{-\infty}^{m} U\big(wc+(1-w)[xa+(1-x)b]\big) f(c)dc + \int_{m}^{\infty} U\big(wc+(1-w)[(1-x)a+xb]\big) f(c)dc \qquad (2a)$$

$$EV \equiv \int_{-\infty}^{m} V\big(wc+(1-w)[xa+(1-x)b]\big) f(c)dc + \int_{m}^{\infty} V\big(wc+(1-w)[(1-x)a+xb]\big) f(c)dc \qquad (3a)$$

for the typical situation where $a \le b$. In the less common case where $a > b$ these become

$$EU \equiv \int_{-\infty}^{m} U\big(wc+(1-w)[xb+(1-x)a]\big) f(c)dc + \int_{m}^{\infty} U\big(wc+(1-w)[(1-x)b+xa]\big) f(c)dc \qquad (2b)$$

$$EV \equiv \int_{-\infty}^{m} V\big(wc+(1-w)[xb+(1-x)a]\big) f(c)dc + \int_{m}^{\infty} V\big(wc+(1-w)[(1-x)b+xa]\big) f(c)dc\,, \qquad (3b)$$

i.e., variables *a* and *b* trade places in each formula.

ANALYSIS

In this section we assess the performance of the different arbitration procedures via two measures that often appear in arbitration research: the gap *G* between equilibrium offers (e.g. Farber 1980, Zeng 2003); and the width *Z* of the contract zone implied by these offers (Farber & Bazerman 1989, Marburger & Scoggins 1996). We would prefer (all else being equal) to have an arbitration procedure where the gap is as narrow as possible and the contract zone is as wide as possible. A narrow gap $G \equiv b - a$ presumably makes it more likely that the arbitration award will be close to what each disputant considers reasonable, and therefore be more acceptable to both parties. It may also lead to the two parties finding themselves close enough together to conclude an agreement, either just before arbitration is begun or after it is underway but not yet concluded (the latter was specifically analyzed in Farmer & Pecorino 1998).

The contract zone is the range of settlements such that both parties would be better off accepting a known *certainty equivalent* settlement in advance during negotiations, rather than going to arbitration and obtaining an uncertain settlement; $Z \equiv s_{max} - s_{min}$, where $U(s_{max}) = EU(a,b)$ and $V(s_{min}) = EV(a,b)$ for the equilibrium offers *a* and *b*. A wide contract zone (all else being equal) should provide the two parties with more common ground in which to find a mutually acceptable agreement during their negotiations, knowing that there is a threat of subsequent arbitration. Of course a wide zone is not needed under the idealized assumptions of our *model*, where both sides are perfectly rational and have complete information. Rather, it would be of value in an *actual* arbitration process where information and decision making would be less than perfect.

We begin our analysis by noting that in any arbitration formulation with $w = 1$ (regardless of x) the disputants' offers are irrelevant to the settlement; so in this case there are no final offers nor gap as such to calculate. In particular this is true of the AA model of ConA and so this arbitration procedure could be said to neither encourage nor discourage compromise by the disputants. This $w = 1$ situation (shown in Figure 1 as a vertical band) is a natural dividing line for our analysis, which will consider $w < 1$ and $w > 1$ as separate cases. Another useful division is at $x = 1/2$ (a horizontal band in Figure 1); points above this line ($x > 1/2$) represent procedures where the arbitrator gives more consideration to the closer offer, whereas points below the line ($x < 1/2$) indicate that the farther offer receives more attention. These vertical and horizontal lines divide the area of Figure 1 into four quadrants that we discuss in the following sub-sections.

Cases I & II: Infinite Separation

In the lower left quadrant of Figure 1 we have $w < 1$ and $x \leq 1/2$; that is, the arbitrator gives positive weight to the submitted offers (as $1-w > 0$) and greater attention is paid to the offer that is farther from the arbitrator's position (as $x \leq 1/2$). In effect the arbitrator will reward whoever submits the more extreme offer. This situation is of interest in part because it includes the CA model of ConA, where $x = 1/2$.

Likewise, the upper right quadrant of Figure 1 has $w > 1$ and $x \geq 1/2$; here the arbitrator gives negative weight to the submitted offers (as $1-w < 0$) and greater attention is paid to the offer that is closer to the arbitrator's position (as $x \geq 1/2$). This implies that the arbitrator will punish whoever submits the more moderate offer.

These two cases are mirror images of each other, and so their analysis is similar.

Proposition 1. For general risk averse utility functions:

(a) When $w < 1$ and $x \leq 1/2$, the equilibrium offers will diverge completely, so that $a \to -\infty$, $b \to +\infty$, and $G \to +\infty$;

(b) When $w > 1$ and $x \geq 1/2$, the equilibrium offers crossover (overlap) completely, so that $a \to +\infty$, $b \to -\infty$, and $G \to -\infty$.

This proposition is mathematically quite simple, but is important in light of its generality because part (a) includes the CA model of ConA as a special case. Prior to this result one might have intuitively argued that at least highly risk-averse disputants would moderate their offers if the arbitrator were paying reasonable attention to them under CA (where $x = 1/2$); but here we see that complete divergence will result regardless of the degree of disputant risk aversion. Part (b) shows the reverse situation likewise provides incentive for extreme offers. Since the arbitrator will place negative weights on both offers, disputant B has an incentive to submit a low (conciliatory) offer; and since the stronger negative weight will be placed upon whichever offer is farthest from the arbitrator's opinion, disputant B has an incentive to submit an extremely negative offer. In both parts (a) and (b) each side is motivated to submit an extreme offer purely in the hope of skewing the arbitrator's ruling in their favor; consequently these offers are unlikely to contain any information of use in settling the dispute. We therefore suggest that arbitration procedures in these categories be avoided, as they may have perverse affects on both the arbitration process and the preceding negotiations.

Cases III & IV: Reduced Separation

In the upper left quadrant of Figure 1 we have $w < 1$ and $x > 1/2$; that is, the arbitrator gives positive weight to the submitted offers (as $1-w > 0$) and greater attention is paid to the offer that is closer to the arbitrator's position (as $x > 1/2$). In effect the arbitrator will reward whoever submits the more moderate offer. This situation includes FOA ($w = 0$ & $x = 1$) and COPA ($w = 0$ & $x = 2$) as special cases.

The lower right quadrant of Figure 1 has $w > 1$ and $x < 1/2$; here the arbitrator gives negative weight to the submitted offers (as $1-w < 0$) and greater attention is paid to the offer that is farther from the arbitrator's position (as $x < 1/2$). Thus the arbitrator will punish whoever submits the more extreme offer. This situation includes 2PA ($w = 2$ & $x = 0$) as a special case.

These two situations are more interesting than the previous ones, but their analysis is less straightforward. We begin with some special cases.

Proposition 2. For risk averse disputants:

(a) Suppose that both disputants know the exact value of the arbitrator's opinion c, i.e. the density function f(c) has all of its probability mass at a single point; then for any set of decision weights with either {w < 1, x > 1/2} or {w > 1, x < 1/2} the equilibrium offers converge exactly, so that a=b=c and G=0.

(b) Suppose that the disputants have equal but opposite utility functions, i.e. with U(-s) = V(s) ∀ s, and that the density function for the arbitrator's opinion is symmetric, i.e. with f(-c) = f(c). In this "symmetric" case it can be shown that for FOA (w = 0 & x = 1) the equilibrium offers partially converge with gap G>0, such that a = -b, b > 0 > a. These offers can be found by solving for b in

$$f(0)\left[V(b)-V(-b)\right]=V'\left(b\right); \tag{4}$$

Likewise it can be shown that for 2PA (where w=2 and x=0) the equilibrium offers cross over with overlap G<0, such that a = -b, b < 0 < a. These offers can be found by solving for b in

$$f(0)\left[V(-b)-V(b)\right]=2\int_{-\infty}^{0}V'\left(2c-b\right)f(c)dc\ . \tag{5}$$

Part (a) is a generalization of an effect that was previously recognized for FOA. Part (b) likewise includes as a special case the result previously obtained for risk neutral disputants in FOA by Brams & Merrill (1983).

To continue our analysis for this case, we next focus our attention on two particular forms of utility functions: linear and exponential. Linear utilities cover the boundary situation where disputants are risk neutral; the utility functions can be simplified to $U(s) = -s$ and $V(s) = s$, and the expected utility EV for disputant B becomes simply

$$EV = (1-w)\left[xa + (1-x)b\right]F(m) + (1-w)\left[(1-x)a + xb\right]\left(1 - F(m)\right) \quad \text{for } a \le b \qquad (6a)$$
$$EV = (1-w)\left[xb + (1-x)a\right]F(m) + (1-w)\left[(1-x)b + xa\right]\left(1 - F(m)\right) \quad \text{for } a > b \qquad (6b)$$

with EU being the negative of these. This risk neutral case is quite tractable for analysis.

Proposition 3. For risk neutral disputants:

(a) When $w < 1$ & $x > 1/2$ (including FOA & COPA), the equilibrium offers have $a < b$ and $G > 0$;

(b) When $w > 1$ & $x < 1/2$ (including 2PA), the equilibrium offers have $b < a$ and $G < 0$;

(c) In each of these cases the distance between the offers is greatest when weight x is close to 1/2, i.e., $G \to 0$ as $x \to \pm\infty$ and $G \to \pm\infty$ as $x \to 1/2$. The equilibrium offers are equidistant from the median n of the arbitrator's distribution (so $b-n = n-a$) and can be calculated using

$$b = n + \frac{1}{2(2x-1)f(n)}. \qquad (7)$$

These results show that the equilibrium offers tend to move closer together as the arbitrator places stronger emphasis on the closer one. For example, Equation 7 shows that moving from FOA (where $x = 1$) to COPA (where $x = 2$) shrinks the gap G between equilibrium offers to only 1/3 of its original size. This increased closeness would presumably induce more negotiated settlements before arbitration, as well as more conciliatory behavior during the process.

Next we consider exponential utility functions, which are often used to represent risk averse utilities (e.g. Farber & Katz 1979). For analytical convenience, we shall assume that the arbitrator's distribution function is symmetric about the mean $\mu = 0$ (i.e. is "unbiased" towards the disputants), and likewise we shall use the simple "mirror image" utility functions $U(s) = 1-e^{s/r}$ and $V(s) = 1-e^{-s/r}$. Here r is a measure of risk tolerance: highly risk averse disputants would have small values of r, whereas those more tolerant of risk would have larger r values. The expected utility for disputant B now becomes

$$EV = \int_{-\infty}^{m}\left(1 - e^{-\left[wc+(1-w)\left(xa+[1-x]b\right)\right]/r}\right)f(c)dc + \int_{m}^{\infty}\left(1 - e^{-\left[wc+(1-w)\left([1-x]a+xb\right)\right]/r}\right)f(c)dc \quad \text{for } a \le b \qquad (8a)$$
$$EV = \int_{-\infty}^{m}\left(1 - e^{-\left[wc+(1-w)\left(xb+[1-x]a\right)\right]/r}\right)f(c)dc + \int_{m}^{\infty}\left(1 - e^{-\left[wc+(1-w)\left([1-x]b+xa\right)\right]/r}\right)f(c)dc \quad \text{for } a > b \qquad (8b)$$

with a similar modification to EU for disputant A.

While our assumption here of identical risk tolerances r for both parties is unlikely to occur in practice, we do not believe it limits the usefulness of our analysis as our main interest is to make comparisons *across* many different types of arbitration procedures rather than focus on the details of any particular one. Later in this article we present a brief numerical study where the disputants are allowed to differ in risk aversion. We note that the effect of differing risk aversion for FOA in particular has already been explored (e.g. Farber 1980).

Proposition 4. For arbitration where the arbitrator's opinion follows a symmetric distribution and the disputants are risk averse with utility functions $U(s) = 1\text{-}e^{s/r}$ *and* $V(s) = 1\text{-}e^{-s/r}$:

(a) When $w < 1$ & $x > 1/2$ *(including FOA & COPA), the equilibrium offers have* $a < b$ *and* $G > 0$;

(b) When $w > 1$ & $x < 1/2$ *(including 2PA), the equilibrium offers have* $b < a$ *and* $G < 0$;

(c) In each of these cases the distance between the offers is greatest when weight x is close to 1/2, i.e., $G \to 0$ *as* $x \to \pm\infty$ *and* $G \to \pm\infty$ *as* $x \to 1/2$. *The equilibrium offers are equidistant from the mean* $\mu=0$ *of the arbitrator's distribution (so* $a = -b$*) and can be calculated using*

$$b = \frac{r}{2(1-w)(2x-1)} \ln\left[\frac{rf(0) + 2(1-w)x\int_0^{\infty} e^{-wc/r} f(c)dc}{rf(0) - 2(1-w)(1-x)\int_{-\infty}^{0} e^{-wc/r} f(c)dc}\right]; \tag{9}$$

(d) In the particular case of FOA (with w=0 & x=1) the gap G is increasing in risk tolerance r (i.e. is decreasing in risk averseness) but this is not true in general for other values of w and x.

This proposition suggests that most of the results previously obtained in the risk neutral case (Proposition 3) also apply to risk averse disputants, albeit in a less straightforward form. Part (d) presents an interesting addition. It begins by repeating previous work on FOA that showed equilibrium offers converge more as risk aversion increases, but then goes on to show that contrary to what one might expect this principle does not necessarily follow for other arbitration variations (the appendix contains a counter example). Under some circumstances the equilibrium offers can actually grow farther apart as risk aversion increases. Mathematically it appears that the gap *G* between offers is concave in risk tolerance *r*; consequently, an increase in risk aversion can lead either to a smaller or a larger gap, depending on the values of the decision weights and other parameters in the specific case. Putting this more intuitively, we could describe risk averse disputants as choosing their offers to mitigate against not just the uncertainty surrounding their opponent's offer but also against the uncertainty surrounding the arbitrator's opinion, resulting in a trade-off that could favor either direction.

Our last result in this sub-section concerns the contract zone *Z*.

Proposition 5. For arbitration with $w < 1$ *and* $x > 1/2$ *(including FOA and COPA) and further assuming that the arbitrator's opinion follows a symmetric distribution and the disputants are risk averse with utility functions* $U(s) = 1\text{-}e^{s/r}$ *and* $V(s) = 1\text{-}e^{-s/r}$:

(a) The expected utilities EU and EV are increasing in closest offer weight x, while the contract zone width Z is decreasing in x;

(b) In particular, when w=0 the width Z of the contract zone can be calculated as

$$Z = \ln[1 + 1/(\,4[rf(\mu) + x - 1][rf(\mu)+x]\,)] \qquad (10)$$

and this zone will always be narrower than the gap between offers, i.e. Z < G.

Part (a) shows that the more the arbitrator rewards the closer offer, the more attractive the arbitration procedure will be (due to a higher expected utility) but the more difficult it will be to find a mutually acceptable settlement prior to arbitration (due to a narrower contract zone). Part (b) shows that at least in the $w = 0$ case (which includes FOA) the gap between equilibrium offers will always be wider than the contract zone, which implies that negotiated settlements should be less extreme than arbitrated ones. We conjecture that this is true in general, but have only proven it for the case with $w = 0$ where it is possible to simplify the expressions for G and Z.

Combined Arbitration

In this sub-section we analyze the CBA procedure of Brams & Merrill (1986). Their suggested combination of FOA with AA leads to a different form for the expected utility functions. For side B these would be

$$EV_{CBA} = \int_{-\infty}^{a} V(c)f(c)dc + \int_{a}^{m} V(a)f(c)dc + \int_{m}^{b} V(b)f(c)dc + \int_{b}^{\infty} V(c)f(c)dc \quad \text{when } a \le b, \quad (11a)$$

$$EV_{CBA} = \int_{-\infty}^{b} V(c)f(c)dc + \int_{b}^{m} V(b)f(c)dc + \int_{m}^{a} V(a)f(c)dc + \int_{a}^{\infty} V(c)f(c)dc \quad \text{when } a > b. \quad (11b)$$

Brams & Merrill (1986) showed that in this procedure the offers of risk neutral disputants would converge to the arbitrator's mean. The following extends their finding to a risk averse context.

Proposition 6. For arbitration where the arbitrator's opinion follows a symmetric distribution and the disputants have equal but opposite utility functions, i.e. U(-s) = V(s) ∀s: suppose we fix x=1, and then use w=0 if the arbitrator opinion c falls in the interval between offers (a<c<b) or if the offers converge (a=b) or crossover (a>b), but switch to using w=1 otherwise. In this case the equilibrium offers converge to the arbitrator's mean, so a=b=0, and G=0.

Because this method provides full convergence to the arbitrator's mean and then uses only that value in calculating a settlement, the contract zone is just a single point at the mean.

Equitable Settlement

In the ES model of ConA (Farber 1981) the weight w on the arbitrator's opinion depends on how far apart the two offers are. Substitution of $w = J(b - a)$ and $x = 1/2$ into either Equation 3a or 3b leads to an expected utility function for disputant B of

$$EV_{ES} = \int_{-\infty}^{\infty} V\Big(c\big\langle J(b-a)\big\rangle + m\big\langle 1 - J(b-a)\big\rangle\Big) f(c)dc \tag{12}$$

in the general risk averse case and of

$$EV_{ES} = m[1 - J(b-a)] \tag{13}$$

in the simpler risk neutral case. Our analysis is limited to the latter.

Proposition 7. For risk neutral disputants the equilibrium offers under ES will diverge completely, so that $a \to -\infty$, $b \to +\infty$, and $G \to +\infty$.

We conjecture that a similar result holds for ES in the risk averse case, but have so far been unable to prove this.

NUMERICAL EXAMPLE

This section presents a short numerical illustration of the behavior of the different arbitration procedures; for the sake of the discussion, we shall think of this as a dispute between management (side A) and labor (side B) over a forthcoming salary adjustment. Each disputant was assumed to have an exponential utility function with one of two possible values for risk aversion r. We used a normal distribution with mean μ = \$0 and standard deviation σ = \$100 for the uncertain arbitrator's opinion. For the decision process itself we fixed arbitrator weight w=0 and then varied the closer offer weight x. In each case we calculated the equilibrium offers as well as the location of the contract zone using Excel and Maple software. Note that we are ignoring other possible costs involved in the arbitration procedures; including such costs would give wider contract zones.

Table 1 shows the arbitration results when the disputants have utility functions $U(s) = 1-e^{+s/100}$ and $V(s) = 1-e^{-s/100}$; that is, both sides have r=100 and so are equally risk averse. For example, the column where x=1.0 (i.e. FOA) shows management's proposed salary cut a=-\$63, labor's proposed raise b=\$63, and the a total gap between them of \$126. In terms of the arbitrator's distribution these offers are 1.26 standard deviations apart, so about 47% of the time the arbitrator's actual preference c could be expected to fall between them. The contract zone implied by these offers runs from -\$19 to \$19, a total width of just \$38. The gap between offers is 3.32 times wider than this contract zone, which implies that an arbitrated settlement is likely to be much more extreme (higher or lower) than a negotiated one, just as suggested by Proposition 5.

Looking across the columns we see that as the weight x on the closer offer decreases (i.e. as we move away from FOA and towards the CA model) the gap between optimal offers quickly grows wider, as indicated by Proposition 4. For example, with x = 0.7 we have G = \$602, a separation of 6 standard deviations; even though the disputants know that there is almost no possibility that the arbitrator would agree with offers that extreme, it is still in their best interests to propose them in order to sway the final award. The contract zone Z is also decreasing in x, but at a different rate; thus we see that the relative size of the gap (i.e. the ratio G/Z) is also decreasing in x.

Table 2 shows that the same general patterns hold when the risk tolerance of both disputants is increased to r=200.

Table 3 and Figure 2 illustrate arbitration behavior when the two disputants differ in their degree of risk aversion; here management is relatively more risk tolerant (r=200) while labor is more risk averse (r=100). The figures here show the same overall patterns found in Tables 1 and 2, except that they are no longer symmetric. A comparison across the tables reveals an interesting effect of this asymmetry. Labor (disputant B) is more risk averse in this case and so naturally can be expected to submit less extreme offers. Table 3 shows that labor's offers are not only more conciliatory than management's, but are also more conciliatory than the ones labor would have submitted in either of the symmetric cases, even where it would have had the same risk tolerance (i.e. in Table 1). Similarly, the offers made by management (disputant A) are all more extreme than in the equivalent symmetric cases. This is an example of one disputant rationally choosing to "take advantage" of the other when they both know that there is an imbalance between their tolerances for risk. In cases that place a heavy emphasis on the closer offer (so that x>1) the contract zone of mutually acceptable settlements lies entirely below zero; i.e., risk averse labor is willing to accept a small negotiated pay cut so as to avoid the risk of a bigger loss in arbitration.

DISCUSSION

The previous sections have described the performance of a wide range of possible arbitration formulations. We can summarize this for existing models as follows:

(a) The AA model of ConA has no equilibrium offers to speak of, since it ignores them;
(b) The CA and ES models of ConA have their equilibrium offers diverging completely, $G \rightarrow +\infty$;
(c) The FOA and COPA models provides partial convergence of the offers, $G > 0$;
(d) The CBA model provides full convergence, $G = 0$;
(e) The 2PA model encourages offer crossover and overlap, so $G < 0$.

These results indicate that most existing models for ConA not only fail to encourage compromise, but may instead encourage even risk averse disputants to submit extreme offers in order to try to sway the settlement in their favor. In contrast, arbitration following the FOA model encourages at least some compromise. If arbitrators working in ConA *actually* behave like existing ConA *models* suggest (or even if disputants believe that to be the case), then it would appear that they are doing the disputants a disservice and would be better to use FOA instead.

There is however another possibility, which is that previous models are incomplete representations of how ConA arbitrators make their decisions. Perhaps at least some real arbitrators instead take into account the reasonableness of the offers in rendering their judgments. In other words they may implicitly employ some variation of our more general model and so provide performance that is superior to that of CA.

Our analytical results herein can draw some support from previous empirical studies. For example, it has been noted (Farber & Bazerman 1989) that the proportion of disputes settled prior to arbitration is historically much higher with FOA than with ConA. This clearly fits our model's predictions: we found that the offer gaps with FOA tend to be narrower than in CA, which implies that the parties in FOA would be closer together at the time of arbitration. Another possible reason for the higher dispute rate with ConA may be that the two disputants have differing beliefs about how the arbitrator will make her decision (i.e. what weights are implicitly being used) and consequently have different expectations of how well they would do if their dispute went to arbitration under ConA (see McKenna & Sadanand 1995 for a study on this topic).

Another empirical observation (Burgess & Marburger 1993) is that arbitrated awards under FOA tend to be more extreme than negotiated settlements. This fits the prediction of our model that the gap between final offers will tend to be larger than the width of the inherent contract zone; thus arbitrated settlements would fall across a wider range than would negotiated settlements.

While our study suggests that FOA is superior to ConA, it also points out that in principle we could do even better with an alternative procedure. One could put extra weight on the closer offer (COPA, with $w = 0$ & $x = 2$), actively penalize the farther offer (2PA, with $w = 2$ & $x = 0$), or carefully combine FOA with AA (as in CBA). All of these methods offer better theoretical convergence than does FOA; the challenge is to successfully introduce them in practice. Imperfect though they may be, the existing ConA and FOA procedures have the advantage of already being familiar to arbitrators and disputants.

CONCLUSION

In this article we have described a framework for arbitration procedures that generalizes previous models of Final Offer Arbitration (FOA) and Conventional Arbitration (ConA), as well as several proposed alternatives to them. The framework divides arbitration formulations into four categories (the quadrants of Figure 1), each with its own distinctive characteristics. Our evaluation of arbitration procedure performance was based upon the gap between the disputants' equilibrium offers and upon the width of the contract zone that these offers implied.

We found that both the gap between offers and the width of the contract zone would tend to be narrower in situations where the arbitrator places heavier weight on the closer offer, and that the gap would be much wider than the contract zone. In particular our analysis showed that while FOA only encourages partial convergence of the disputants' offers, existing models for ConA lead to either non-convergence (at best) or extreme divergence (at worst). These results suggest that although FOA is imperfect it should be superior to ConA at encouraging compromise by disputants and is therefore recommended for application. To the extent that arbitrators in practice behave the way previous models suggest, ConA does a disservice to disputants by hindering compromise. The reverse may also be true however; to the extent that ConA is successful in practice, previous models of the procedure may be insufficient to capture its key characteristics.

There are some limitations to this study, most of which follow from the assumptions made in defining its basic structure. One of these is the assumption that the disputants have full and common knowledge regarding all the procedure's elements: the utility functions, the arbitrator's settlement formula, and the distribution of the arbitrator's preferred settlement. In reality much of this information may be unknown, approximate, or not equally shared (Samuelson 1991). We would argue that our results are interesting in spite of this limitation, as an understanding of the relative performance of different arbitration alternatives under idealistic conditions is a useful prerequisite to understanding them in the more complicated real world.

Another consideration that the work herein ignores is whether there might be some other reason (beyond the settlement itself) for preferring one form of arbitration to another. If for example there is some social or business advantage to ensuring that the two disputants never know exactly what settlement c the arbitrator ideally favored, then a procedure with w=0 such as FOA might be preferred regardless of its other properties. For example, arbitrators of some professional baseball player contracts follow FOA and are specifically forbidden from discussing their decision rationale (Burgess & Marburger 1993).

Finally, like most previous studies our model does not explicitly include the negotiation stage that would precede any arbitration procedure. This kind of multi-stage modeling has been the focus of some recent work (Manzini & Mariotti 2001, Farmer & Pecorino 1998), and we suggest that more of this research is needed to explore how the linkages between negotiation and arbitration actually function. We believe our "single-stage" work here nevertheless makes a useful contribution, as it provides a broad basis for understanding the arbitration component of what will eventually need to be a larger and more integrative process model.

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APPENDIX: PROOFS

Note: it is assumed here that the arbitrator's density function is unimodal with mean μ=0.

Proposition 1. For part (a) there is an incentive for B to submit a large $b>0>a$. When $b>a$ take the derivative of EV in Equation 3a with respect to b.

$$\frac{\partial}{\partial b}EV = \frac{1}{2}f(m)\{V(wm+(1-w)[xa+(1-x)b]) - V(wm+(1-w)[(1-x)a+xb])\} \quad \text{(A1)}$$

$$+(1-w)\left\{(1-x)\int_{-\infty}^{m} V'(wc+(1-w)[xa+(1-x)b])f(c)dc + x\int_{m}^{\infty} V'(wc+(1-w)[(1-x)a+xb])f(c)dc\right\}$$

Given $b>a$, the 1st braced term is positive due to $(1-x) \geq x$ and $V(s)$ being increasing, while the 2nd is positive due to $V(s)$ being increasing. Thus with $w<1$ & $x\leq 1/2$ we have $\partial/\partial b\ EV>0\ \forall\ b$, so it is optimal to choose $b\to+\infty$. The argument showing $\partial/\partial a\ EU < 0\ \forall\ a$ is similar.

For the alternative case where $a>b$, take the derivative with respect to b of Equation 3b.

$$\frac{\partial}{\partial b}EV = \frac{1}{2}f(m)\{V(wm+(1-w)[xb+(1-x)a]) - V(wm+(1-w)[(1-x)b+xa])\}$$

$$+(1-w)\left\{x\int_{-\infty}^{m} V'(wc+(1-w)[xb+(1-x)a])f(c)dc + (1-x)\int_{m}^{\infty} V'(wc+(1-w)[(1-x)b+xa])f(c)dc\right\}$$

Given $a>b$, the 1st braced term is positive due to $(1-x) \geq x$ and $V(s)$ being increasing, while the 2nd is positive due to $V(s)$ being increasing. Thus again we have optimal $b\to+\infty$. A similar parallel holds for most of our proofs, so we only show both parts again if the $a>b$ and $b>a$ cases differ.

For part (b) there is incentive for B to choose a small $b<0<a$. The proof proceeds as in part (a) above, but now with $w>1$ & $x\geq 1/2$ we have the signs of the braced terms reversed, so $\partial/\partial bEV < 0\ \forall\ b$ and it is optimal to choose $b\to-\infty$.

Proposition 2. In part (a) both disputants know c. With either $(1-w)(x)>1/2>(1-w)(1-x)$ or $(1-w)(x)>-1/2>(1-w)(1-x)$ the side submitting the closer offer will benefit more. Suppose A selects $a=c-\Delta$; if B were to submit an equivalent offer $b=c+\Delta$ then it would receive expected utility

$$EV = [V(wc-(1-w)(2x-1)\Delta) + V(wc+(1-w)(2x-1)\Delta)]/2$$

whereas B could submit a slightly closer offer $b=c+\Delta-\varepsilon$ and receive a greater expected utility

$$EV = V(wc+(1-w)(2x-1)(\Delta-\varepsilon)).$$

Thus the only stable equilibrium is for both sides to submit the same offer, $a=b=c$.

For part (b) use Equation A1 for $\partial/\partial bEV$. Fix w=0 & x=1 (for FOA) and $a=-b$ & m=0 (by symmetry); noting that $F(m)= F(0)=1/2$, simplify and set equal to zero to get

$$f(0)[V(-b)-V(b)]/2 + V'(b)/2 = 0.$$

The second part is likewise obtained from Equation A1 by fixing w=2, x=0, $a=-b$ and m=0 before setting the whole equal to zero.

$$-2\int_{-\infty}^{0} V'(2c-b)f(c)dc + V(-b)f(0) - V(b)f(0) = 0$$

Proposition 3. Take derivatives of the expected utilities, e.g. Equation 6A for side B.

$$\frac{\partial}{\partial b}EV = (1-w)\left\{-\frac{1}{2}(2x-1)(b-a)f(m) - (2x-1)F(m) + (x)\right\}$$

$$\frac{\partial}{\partial a}EU = (1-w)\left\{\frac{1}{2}(2x-1)(b-a)f(m) - (2x-1)F(m) + (1-x)\right\}$$

Note that w affects the signs of these expressions but not the points at which they equal zero. Set both equal to zero and solve simultaneously to get

$$F(m) = 1/2 \quad \text{and} \quad b-a = \frac{1}{(2x-1)f(m)}.$$

The 1st relation indicates that $m = (a+b)/2 = n$, i.e. the equilibrium offers are equidistant about median n. Substitute this into the 2nd relation to get the expression for equilibrium offers.

Proposition 4. In Equation A1 insert the exponential utility functions, then by symmetry fix a=-b and m=0 and set the equation equal to zero.

$$\frac{\partial}{\partial b}EV = 0 = \frac{(1-w)(1-x)}{r}\int_{-\infty}^{0} e^{-[wc-(1-w)[2x-1]b]/r} f(c)dc + \frac{(1-w)(x)}{r}\int_{0}^{\infty} e^{-[wc+(1-w)[2x-1]b]/r} f(c)dc$$
$$-\frac{f(0)}{2}\left[e^{[(1-w)(2x-1)b]/r} - e^{-[(1-w)(2x-1)b]/r}\right].$$

Multiply by $2re^{[(1-w)(2x-1)b]/r}$ and then re-arrange terms.

$$e^{2[(1-w)(2x-1)b]/r} = \left[rf(0) + 2(1-w)(x)\int_{0}^{\infty} e^{-wc/r} f(c)dc\right]\left[rf(0) - 2(1-w)(1-x)\int_{-\infty}^{0} e^{-wc/r} f(c)dc\right]^{-1}$$

Take the natural logarithm of each side and re-arrange to get the desired expression.

To show that b and thus G are decreasing in x, take $\partial/\partial x$ of Equation 9 for equilibrium b. Define N and D as the numerator and denominator respectively of the fraction inside the logarithm in the expression for b.

$$N \equiv rf(0) + 2(1-w)(x)\int_{0}^{\infty} e^{-wc/r} f(c)dc \qquad D \equiv rf(0) - 2(1-w)(1-x)\int_{-\infty}^{0} e^{-wc/r} f(c)dc$$

Take derivatives with respect to x to get N' and D'.

$$N' \equiv \frac{\partial}{\partial x}N = 2(1-w)\int_{0}^{\infty} e^{-wc/r} f(c)dc \qquad D' \equiv \frac{\partial}{\partial x}D = -2(1-w)\int_{-\infty}^{0} e^{-wc/r} f(c)dc$$

Using this notation write the derivative with respect to x of the general expression for b as

$$\frac{\partial}{\partial x}b = \frac{-2r}{(1-w)(2x-1)^2}\ln\left[\frac{N}{D}\right] \quad + \quad \frac{r}{(1-w)(2x-1)}\frac{\{N'D - ND'\}}{ND}.$$

The 1st half of this expression is clearly negative. In the 2nd half we need to determine the sign of the expression in braces; working this out and simplifying yields the following.

$$2(1-w)rf(0)\left[\int_0^{\infty} e^{-wc/r} f(c)dc - \int_{-\infty}^{0} e^{-wc/r} f(c)dc\right] - 4(1-w)^2\left(\int_0^{\infty} e^{-wc/r} f(c)dc\right)\left(\int_{-\infty}^{0} e^{-wc/r} f(c)dc\right)$$

This is negative, so the entire derivative must be negative; thus the optimal *b* is decreasing in *x*.

For part (d) start with Equation 9, fix *w*=0 & *x*=1 (FOA), then take the 1st & 2nd derivatives with respect to *r*.

$$\frac{\partial}{\partial r} b = \frac{1}{2}\ln\left(\frac{rf(0)+1}{rf(0)}\right) - \frac{1}{2rf(0)+1} \qquad \frac{\partial^2}{\partial r^2} b = \frac{-1}{2r[rf(0)+1]^2}$$

The 2nd derivative is negative, so *b* is concave in *r* and ∂/∂*r* *b* is monotonically decreasing in *r*. Since the 1st derivative goes to zero from above as *rf*(0) → ∞ and is therefore positive, we have that *b* (and so *G*) is concave increasing in *r*.

A counter-example shows that the above result for FOA does not hold in general for other {*w*, *x*} combinations. Choose *w* = 0, *x* = 0.6, *f*(0) = 1, and suppose *r* = 1.1; then calculation yields *G*=2(2.442)=4.884. Suppose next that *r* = 1.0, so that we calculate *G*=2(2.455)=4.910. Hence here a decrease in *r* leads to an increase in *G*.

Proposition 5. In Equation 8a for *EV*, by symmetry substitute *a*=-*b* and *m*=0.

$$EV = \int_{-\infty}^{0}\left(1-e^{-[wc-(1-w)[2x-1]b]/r}\right)f(c)dc + \int_0^{\infty}\left(1-e^{-[wc+(1-w)[2x-1]b]/r}\right)f(c)dc$$

Insert Equation 9 for the equilibrium *b* and then simplify the combinations of exponentials and logarithms (using notation *N* and *D* similarly to before).

$$EV^* = 1-\left(\frac{N}{D}\right)^{1/2}\int_{-\infty}^{0} e^{-wc/r} f(c)dc - \left(\frac{D}{N}\right)^{1/2}\int_0^{\infty} e^{-wc/r} f(c)dc$$

Take the derivative with respect to *x*.

$$\frac{\partial}{\partial x} EV^* = -\frac{N'D-ND'}{N^{1/2}D^{3/2}}\frac{1}{2}\int_{-\infty}^{0} e^{-wc/r} f(c)dc - \frac{D'N-DN'}{N^{3/2}D^{1/2}}\frac{1}{2}\int_0^{\infty} e^{-wc/r} f(c)dc$$

Cancel terms and create a common denominator to yield the following.

$$\frac{\partial}{\partial x} EV^* = -\frac{1}{2N^{3/2}D^{3/2}}\left\{(NN'D-NND')\int_{-\infty}^{0} e^{-wc/r} f(c)dc + (DD'N-DDN')\int_0^{\infty} e^{-wc/r} f(c)dc\right\}$$

We need to show that the expression in braces is negative. Multiply out the terms and simplify.

$$-2(1-w)\left(rf(0)\left[\int_0^{\infty} e^{-wc/r} f(c)dc - \int_{-\infty}^{0} e^{-wc/r} f(c)dc\right]\right)^2 - 8(1-w)^3\left(\int_{-\infty}^{0} e^{-wc/r} f(c)dc\right)^2\left(\int_0^{\infty} e^{-wc/r} f(c)dc\right)^2$$

$$-4(1-w)^2 rf(0)\left(\int_{-\infty}^{0} e^{-wc/r} f(c)dc\right)\left(\int_0^{\infty} e^{-wc/r} f(c)dc\right)\left[\int_{-\infty}^{0} e^{-wc/r} f(c)dc - \int_0^{\infty} e^{-wc/r} f(c)dc\right]$$

That this is negative shows that *EV* is increasing in *x*. It follows in turn that *Z* is decreasing in *x*, since $V(s_{min})$ = *EV* and *V*(*s*) is increasing and concave.

For part (b), we want the certainty equivalent settlement with the same utility as the expected utility in equilibrium. Fix $w = 0$ in the above expression for EV^* and then set this equal to the utility of the certainty equivalent $s_{min} = (- Z/2)$ we are seeking.

$$1 - e^{-Z/2r} = 1 - \frac{1}{2}\left(\frac{rf(0)+x}{rf(0)+x-1}\right)^{1/2} - \frac{1}{2}\left(\frac{rf(0)+x-1}{rf(0)+x}\right)^{1/2}$$

Cancel common terms and take logarithms of both sides.

$$\frac{Z}{2r} = \ln\left[\frac{1}{2}\left(\frac{rf(0)+x}{rf(0)+x-1}\right)^{1/2} + \frac{1}{2}\left(\frac{rf(0)+x-1}{rf(0)+x}\right)^{1/2}\right]$$

Multiply out and simplify the term inside the logarithm to solve for Z.

$$Z = r\ln\left[\frac{(rf(0)+x-1)(rf(0)+x)+1/4}{(rf(0)+x-1)(rf(0)+x)}\right]$$

To compare G with Z for $w = 0$, start with the ratio of the two expressions (where G=2b).

$$G/Z = \frac{r}{2x-1}\ln\left[\frac{(rf(0)+x)}{rf(0)+x-1}\right] \Big/ r\ln\left[\frac{(rf(0)+x-1)(rf(0)+x)+1/4}{(rf(0)+x-1)(rf(0)+x)}\right]$$

Cancel r, change the fractions to have a common denominator, and re-arrange terms.

$$G/Z = \ln\left[\frac{(rf(0))^2 + x^2 + 2rf(0)x}{(rf(0)+x-1)(rf(0)+x)}\right] \Big/ \ln\left[\left(\frac{(rf(0))^2 + x^2 + 2rf(0)x - rf(0) - x + 1/4}{(rf(0)+x-1)(rf(0)+x)}\right)^{(2x-1)}\right]$$

This is greater than 1 since $x > 1/2$.

Proposition 6. Take the derivative of Equation 11a for EV_{CBA} with respect to b and set equal to zero.

$$\frac{\partial}{\partial b}EV_{CBA} = 0 + \frac{1}{2}V(a)f(m) - \frac{1}{2}V(b)f(m) + V(b)f(b) + \int_m^b V'(b)f(c)dc - V(b)f(b)$$

$$= \frac{1}{2}f(m)[V(a) - V(b)] + V'(b)[F(b) - F(m)] = 0$$

By symmetry substitute a=-b and re-arrange.

$$\frac{F(b) - 1/2}{f(0)} = \frac{V(b) - V(-b)}{2V'(b)}$$

Since $F(c)$ is unimodal and symmetric, the left-hand side is less than 2b for b>0; conversely, since $V(s)$ is concave the right-hand side is greater than 2b for b>0; thus equality holds only if b=0.

Proposition 7. Take the derivative of Equation 13 for EV_{ES} with respect to b; likewise take the derivative of the equivalent expression for $EU_{ES} = -EV_{ES}$ with respect to a.

$\partial/\partial b\ EV_{ES} = ½[1 - J(b-a)] - m\ J'(b-a)$ $\qquad$ $\partial/\partial a\ EU_{ES} = -½[1 - J(b-a)] - m\ J'(b-a)$

Set these equal to zero, then add and subtract them to find $m = 0$ and $J(b-a) = 1$. The first result confirms that the equilibrium offers will be equidistant about the arbitrator's mean (a=-b), while the second implies that $G = (b-a) \rightarrow +\infty$ i.e. the offers diverge completely.

Table 1: **Results with risk tolerance *r*=100 for A and *r*=100 for B**.

Weight *x*	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
Offer *a*	-301	-150	-92	-63	-46	-35
Zone Low	-60	-36	-25	-19	-15	-12
Zone High	60	36	25	19	15	12
Offer *b*	301	150	92	63	46	35
Gap *G*	602	300	184	126	92	70
Zone *Z*	120	72	50	38	30	24
Ratio *G/Z*	5.02	4.17	3.68	3.32	3.07	2.92

Table 2: **Results with risk tolerance *r*=200 for A and *r*=200 for B**.

Weight *x*	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
Offer *a*	-275	-164	-111	-81	-62	-50
Zone Low	-29	-23	-19	-16	-14	-12
Zone High	29	23	19	16	14	12
Offer *b*	275	164	111	81	62	50
Gap *G*	550	328	222	162	124	100
Zone *Z*	58	46	38	32	28	24
Ratio *G/Z*	9.48	7.13	5.84	5.06	4.43	4.17

Table 3: **Results with risk tolerance *r*=200 for A and *r*=100 for B**.

Weight *x*	**0.7**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**
Offer *a*	-291	-189	-138	-97	-75	-60
Zone Low	-56	-50	-45	-36	-31	-27
Zone High	28	6	0	0	-2	-3
Offer *b*	269	113	65	45	31	22
Gap *G*	560	302	203	142	106	82
Zone *Z*	84	56	45	36	29	24
Ratio *G/Z*	6.67	5.39	4.51	3.94	3.66	3.42

Figure 1: Arbitration models by arbitrator weight *w* (horizontal) and closer offer weight *x* (vertical).

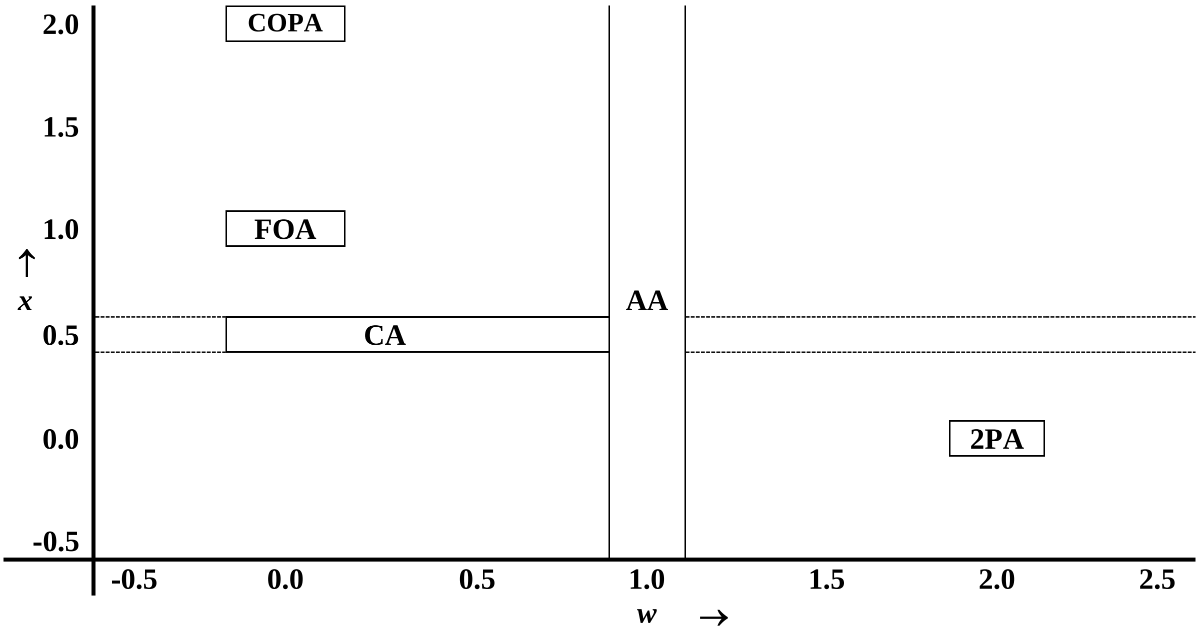

Figure 2: Plot showing gap between offers (top & bottom solid curves) and contract zone (middle two dashed curves) between disputants with unequal risk tolerances *r*=200 for A and *r*=100 for B.