

# Optimal and Robust Designs of Step-stress Accelerated Life Testing Experiments for Proportional Hazards Models

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## **Abstract**

Accelerated life testing (ALT) is widely used to obtain reliability information about a product within a limited time frame. The Cox's proportional hazards (PH) model is often utilized for reliability prediction. My master thesis research focuses on designing accelerated life testing experiments for reliability estimation. We consider multiple step-stress ALT plans with censoring. The optimal stress levels and times of changing the stress levels are investigated. We discuss the optimal designs under three optimality criteria. They are D-, A- and Q-optimal designs. We note that the classical designs are optimal only if the model assumed is correct. Due to the nature of prediction made from ALT experimental data, attained under the stress levels higher than the normal condition, extrapolation is encountered. In such case, the assumed model cannot be tested. Therefore, for possible imprecision in the assumed PH model, the method of construction for robust designs is also explored.

**Key Words:** Step-stress accelerated life test, Cox's proportional hazards model, Maximum likelihood estimation, Robust design, Reliability estimation.

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# Chapter 1

## Introduction

### 1.1 Background

Reliability is a crucial factor which significantly influences the quality of product. The definition of the reliability is the probability that a unit will operate successfully at a particular point in time under normal conditions. Thus, many manufacturers make great effort to enhance the reliability of a product which also largely determines the competitiveness of their product. Such importance has brought many statisticians attention in evaluating reliability, assessing new designs and modifications, and identifying causes of failures. In the life data analysis, we have to collect the failure data of a product under normal design conditions to quantify the life characteristics of the product. However, such failure life data is very difficult to obtain in many situations, especially for the product with high reliability. Today lifetimes of many products are too long and the life testing time period between design and release is limited. Tests under normal design conditions are too lengthy to get any failures. To overcome this problem, accelerate life testing (ALT) has been developed.

ALT is widely used to quickly obtain the failure information about products since it can shorten the lifetime of a product in order to observe failures in a reasonable time period. In general, test units in ALT are subjected to higher than normal design level of stresses (e.g., temperature, vibration, voltage, pressure, etc.) since testing at high stress levels will result in early failures. The use of such accelerating stresses for a particular material is established by engineering practice. To avoid invalid reliability estimates, the range of stress levels should be restricted. The failure data obtained at accelerated conditions are extrapolated to estimate the characteristic of life distribution at normal design conditions. In practical situations, different types of stress loads can be considered when performing an ALT. The stress loading in an accelerated test can be applied in many ways, such as constant, cyclic, step, progressive, and random stress loading. All loading schemes can be classified into two types according to the dependency between the stress levels and testing time. They are time-independent and time-dependent stress loading. When the stress loading is time-independent such as constant stress loading, then the stress applied to a sample of units should be the same during testing period. But there is a drawback of using constant stress loading in ALT. When the inappropriate stress levels are used, the test experiment could still take too long to observe sufficient failures. To address this problem, a time-dependent stress loading scheme is often adopted to assure quick failures. When the stress loading is time-dependent, the test units are subjected to a stress level that changes with

respect to time. The most commonly used time-dependent stress loading scheme is the step-stress test. In step-stress ALT, all test units are subjected to stress levels increasing by steps. In general, all the test units start being subjected to a lower stress level for a specified length of time or a fixed number of failures, then the stress level increases and the test continues for all unfailed test units. The test is finished when all units have failed, when a certain number of failures are observed, or when a certain time has elapsed.

There are two types of accelerated life data: complete data and censored data. Complete data consist of the exact life failure time of each sample unit. Namely, the experiment producing complete data only finishes when all the test units have failed. On the other hand, there are two types of censored data: time censored and failure-censored data. Time-censoring is also called Type I censoring. Time-censoring occurs when the censoring time is predefined and the experiment stops at the censoring time. The lifetimes of all remaining test units which have not failed by the censoring time are said to be right-censored. Failure-censoring is also called Type II censoring. Data are singly failure censored if the experiment is stopped when a predetermined number of failures occurs. Time-censoring is more popular in practice while failure-censoring is more common in theoretical literature.

## 1.2 Our model, some definitions for step-stress ALT, and optimal design criteria

### 1.2.1 Cox's proportional hazards model

The Cox's proportional hazards (PH) model is one of the most important means for predicting the lifetime of a product. This model provides a flexible method for identifying the effects of covariates on failure rate. We consider a PH model that can be expressed as

$$\lambda(t; \mathbf{s}) = \lambda_0(t) \exp(\boldsymbol{\beta}^T \mathbf{s}), \quad (1.1)$$

where  $\lambda_0(t)$  is its baseline hazard function,  $\mathbf{s}$  is a column vector of the covariates (applied or transformed stresses) in the ALT test, and  $\boldsymbol{\beta}$  is a column vector of unknown parameters. In a PH model, the stresses are presumed having multiplicatively effects on the hazard rate. Furthermore, when testing two units at different stress levels, the ratio of the hazard rates for these two units will remain unchanged over time.

### 1.2.2 Step-stress ALT

A step-stress ALT allows changing stress levels during testing. In an ALT with step-stress loading, all the test units begin at a specified stress level. After a time fraction, the stress level is changed to a higher stress level. The stress level can be raised more than once before the end of the test. Simple step-stress ALT (Figure 1.1) only uses two stress levels during the testing. Simple step-stress ALT is also called two-step-stress ALT. However, in multiple step-stress ALT (see Figure 1.2 for a three-step-stress scheme), the stress level is raised twice or more before the end of the test. In this thesis, we consider multiple step-stress ALT plans.



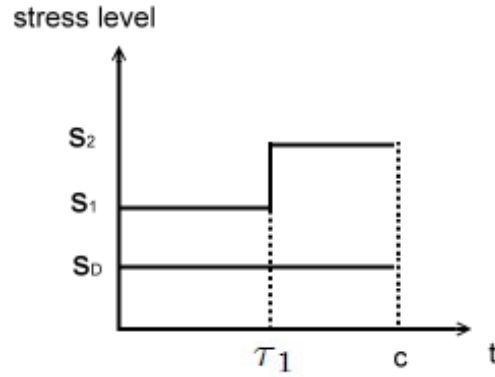


Figure 1.1 Simple step-stress test

( $s_D$  – the normal design stress level,  $s_1$  and  $s_2$  – the low and high stress levels,  $\tau_1$  – the stress changing time, and  $c$  – the censoring time)

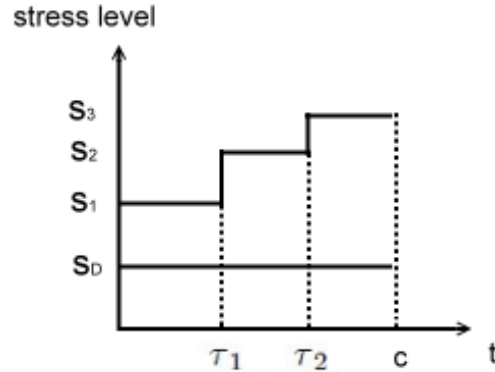


Figure 1.2 Three-step-stress test

( $s_D$  – the normal design stress level,  $s_1$ ,  $s_2$ ,  $s_3$  – the low, middle and high stress levels  $\tau_1$  and  $\tau_2$  – the stress changing times, and  $c$  – the censoring time)

### 1.2.3 Three design criteria

Optimal design criteria used in this thesis are D-optimality, A-optimality and Q-optimality. When the assumed model is correct, D-optimal design is obtained in order to minimize the determinant of covariance matrix of the estimators for model parameters, or equivalently maximize the determinant of Fisher's information matrix. An A-optimal design is obtained by minimizing the trace of covariance matrix of the estimators for the model parameters (or maximizing the trace of Fisher's information matrix). A Q-optimal design is attended by minimizing the asymptotic variance of the estimator for a specific quantity of interest. In this thesis, we are interested in (1) the hazard rate over a specific period of time at the normal design stress level and (2) the (transformed) reliability at a specified time and at the normal design stress level.

## 1.3 Literature review

In this section, we review the literature of optimal and robust designs for ALT models.

### 1.3.1 Optimal designs for simple step-stress ALT with a parametric model

Miller and Nelson (1983) first presented Q-optimal designs for simple step-stress tests with complete failure data from an exponential distribution. Their designs minimize the asymptotic variance (AVAR) of the maximum likelihood estimator (MLE) of the mean life at a normal design stress. Bai, Kim and Lee (1989) extended the theory of Miller and Nelson to censored data. They constructed the optimal simple step-stress ALT with time-censoring. However, in many cases the failure time distribution of the test units may not be exponential. Thus, the optimal simple step-stress ALT for Weibull distribution has also been developed. Bai and Kim (1993) presented an optimal simple step-stress ALT for a Weibull distribution under time-censoring. The optimal low stress level and stress change time are obtained by minimizing the AVAR of the MLE of a specific percentile at normal design stress. Fard and Li (2009) also presented a step-stress ALT for a Weibull distribution under time-censoring; however, their optimal stress-changing time is obtained by minimizing the AVAR of the MLE for reliability instead. Recent work on optimal designs for step-stress ALT has been reviewed by Hunt and Xu (2012). They adopted a generalized Khamis-Higgins model for the effect of changing stress levels and also assumed that the lifetime of a test unit follows a Weibull distribution; however, their designs are also optimal when both the shape and scale parameters are considered being functions of the stress levels. The resulting optimal design chooses the stress-changing time in order to minimize the asymptotic variance (AVAR) of the MLE of reliability at the normal design stress level and at a pre-specified time.

### 1.3.2 Optimal designs for multiple step-stress ALT with parametric model

Ma and Meeker (2008) extended the results of Bai and Kim (1993) to provide a general method for multiple step-stress ALTs for a log location-scale family of distributions. They presented an approach to calculate the large-sample approximate variance of the MLE for a quantile of the failure time distribution at normal design conditions from a step-stress ALT. The approach allows for multiple step-stress loading, censoring, and it is based on a cumulative exposure model. As an application of the approach from Ma and Meeker (2008), the minimum variance is expressed as a function of shape parameter for both Weibull and lognormal distributions. Graphical comparisons among different test plans including step-up, step-down, and constant-stress loading are also presented. Their results show that depending on the values of the model parameters and quantile of interest, one of the three test plans can be the most preferable in terms of optimum variance.

### 1.3.3 Optimal designs for simple step-stress ALT with Cox's proportional hazards model

Jiao (2001) developed the first simple step-stress ALT plan that utilized PH model for reliability prediction. Their optimal stress level is obtained by minimizing the variance of the MLE of hazard rate at the normal design stress level and over a pre-specified time. Elsayed and Zhang (2005) also presented an optimal simple step-stress ALT plan based on PH model to obtain the most accurate reliability function estimates at

normal design conditions. They also formulate a nonlinear programming problem to minimize the asymptotic variance of the hazard rate estimator over a prespecified the period at normal design stress.

### **1.3.4 Optimal designs for constant stress ALT with Cox's proportional hazards model**

Jiao (2001) developed the first constant stress ALT plan that utilized PH model for reliability prediction. They considered the selection of stress levels and proportion of units allocated to each stress level for a constant stress ALT such that the most accurate reliability estimate at normal design conditions could be obtained. An algorithm was also provided to solve the constrained nonlinear optimization problem for obtaining the optimal test designs.

### **1.3.5 Robust designs for ALT**

The general theory for the asymptotic distribution of MLEs under model misspecification was derived by White (1982). He examined the consequences and detection of model misspecification when the maximum likelihood technique has been used for estimation. The proposal quasi-MLE converges to a well-defined limit. Also, the properties of their quasi-MLE and the information matrix are exploited to yield several useful tests when model misspecification is suspected.

An ALT plan can be robust with respect to inaccuracy in assumed underlying stress-life relationship. Pascual (2006) presented the methodology for deriving the asymptotic distribution of MLEs of model parameters with constant stress ALT when the stress-life relationship is misspecified. If the possible departures from an assumed ALT model can be suspected, then his results can be used to investigate the ALT plans which provide protection against potential bias and against the possible loss in estimation efficiency. The methodology developed can be used to derive robust designs by minimizing asymptotic squared bias (ABIAS<sup>2</sup>), asymptotic variance (AVAR) or asymptotic mean squared error (AMSE) for constant stress ALTs.

### **1.3.6 Methods for model-robust test planning**

Pascual (2006) mentioned that research on robust designs is appreciated for the following two practical situations. First situation is that model to be fitted to the data is not known and is chosen only after data being collected and then a model selection process is applied. Thus, for this situation the implemented experimental design is expected to perform relatively well under different candidate models. Second situation is that assuming a particular model to be fit to data, but the possibility that the assumed model may not be the true one is recognized.

In a complete general setting, robust designs for one-point extrapolation which is the case for ALT have been discussed in Wiens and Xu (2008), for least squares estimation of a mean response. As Fang and Wiens (1999) pointed out: "Extrapolation to regions outside of that in which observations are taken is, of course, an inherently risky procedure and is made even more so by an over-reliance on stringent model assumptions." The classical optimal designs minimize the variance alone. However, when the fitted models are incorrect, the estimation is biased. A robust design should be obtained in an optimal way so that even when the fitted model was not exactly correct, the designs can still be relatively efficient with a small bias.

## 1.4 Motivation and problems of interest

In the literature, there hasn't been any work done on optimal designs for PH based ALT models with multiple step-stress plans. Since all of the previous works done are on simple step-stress for PH based ALT models, exploring optimal designs for PH based ALT models with multiple step-stress plans would be an area deserving of research. Due to the nature of the prediction made from ALT experimental data, attained under the stress levels higher than the normal design condition, extrapolation is encountered. In such case, the assumed model cannot be tested. For possible imprecision in an assumed PH model, the method of construction for robust designs is also needed to explore. Therefore, we are interested in constructing optimal and robust designs for multiple step-stress ALTs for PH models. For optimal designs, we consider the baseline hazard function to be either a simple linear or a quadratic form. For each case, we discuss the optimal multiple step-stress ALT designs under three different design criteria: D-optimality, A-optimality, and Q-optimality.

For robust designs, we consider three different optimal criteria: minimizing the asymptotic squared bias (ABIAS<sup>2</sup>), asymptotic variance (AVAR) and asymptotic mean squared error (AMSE) of the estimator for the quantity that experimenter is interested. We consider the situation where a PH model with a simple linear baseline hazard function is fitted, however the true baseline hazard function is a quadratic function. Robust designs will be obtained in order to protect against possible departure from the linear model. The changing times are derived by minimizing the ABIAS<sup>2</sup>, AVAR and AMSE of hazard rate estimation or transformed reliability estimation.

In order to demonstrate the performance of our resulting designs, we carry out a simulation study using given model parameters and the optimal design we constructed. The simulated data are used to compute the maximum likelihood estimates, simulation squared bias (SBIAS<sup>2</sup>), simulation variance (SVAR), and simulation mean squared error (SMSE) in order to evaluate the performance of our resulting designs.

## 1.5 Main results of this thesis

This thesis consists of five chapters. The main results of our research are presented in the following four chapters.

In Chapter 2, the optimal three-step-stress ALT designs have been constructed when the baseline hazard function in a PH model is assumed to be a simple linear function. We have solved the minimization problem for a nonlinear objective function with nonlinear constraints. Using two practical examples, we have demonstrated the construction of constrained optimal designs under each of the three criteria considered. The resulting optimal designs under three different criteria are quite similar in terms of optimal stress-changing times. In addition, we have found that the middle stress level should be kept as close to the lower bound of the middle stress level as possible as long as the constraint condition is satisfied. We conclude that optimal three-step-stress designs have gained efficiency at a minimum of 3%, maximum of 10% for hazard rate prediction respect to two-step-stress designs.

In Chapter 3, we have derived the optimal designs when the baseline hazard function in a PH model is assumed to be a quadratic function. The optimal stress-changing times and the optimal middle stress level are chosen in order to minimize the loss functions under some given nonlinear constraints. Under these constraints (minimum number of being fixed failures at each stress level), the optimal middle stress

level should be kept as close to the low stress level as possible as long as such constraints are satisfied. Six constrained optimal designs are constructed under each of the three criteria. We also point out that designing a three-step-stress ALT with a fixed middle stress seems only helping reduce the value of loss function under A-optimality. Thus, we against to use average middle stress level when the PH-based three-step-stress ALT with the quadratic baseline hazard.

In Chapter 4, robust designs have been derived against possible departure from the linear baseline hazard in a PH model. Optimal stress-changing times have been obtained in order to minimize the ABIAS<sup>2</sup>, AVAR and AMSE respectively of both hazard rate estimation and transformed reliability estimation. Under certain constraints, an iteration algorithm has been described to help determining the optimal stress-changing times and the optimal middle stress level simultaneously. We demonstrate this algorithm using a practical example, the optimal stress-changing times and the optimal middle stress level have been obtained for each of these two quantities that the experimenter may be interested.

In Chapter 5, we have evaluated the performance of our resulting designs obtained in Chapters 2-4 by simulations. The design for a three-step-stress ALT with an optimal middle stress level and two optimal stress-changing times has reduced all SBIAS<sup>2</sup>, SVAR, and SMSE of hazard rate estimator in comparison to three-step-stress ALT with fixed middle stress level. The optimal design with optimal middle stress outperforms that with middle stress is conveniently chosen as the average of other two stress levels. In addition, the middle stress level should also be chosen as close to the low stress level as possible provided the constraints are satisfied.

## Chapter 2

# Optimal designs when the baseline hazard function is a simple linear function

### 2.1 Introduction

The PH model plays a pivotal role in the reliability area. In this chapter, we present a method of finding optimal designs for PH-based three-step-stress ALT with the case where the baseline hazard function is a simple linear function. The information matrix will be derived and the optimal times of changing the stress levels will be investigated. We consider constructing optimal designs under three commonly used criteria. They are D-optimality, A-optimality and Q-optimality. This research extends the work of Elsayed and Zhang (2005), which has only found the optimal designs for PH-based simple step-stress ALT. Hunt and Xu (2012) has pointed out the optimal multiple step-stress ALT may further improve the accuracy of the reliability predicts. Therefore, we expand the work of Elsayed and Zhang (2005) and investigate the optimal designs of multiple step-stress ALT. In this chapter, we focus on constructing the procedures for determining the optimal PH-based three-step-stress ALT with time-censoring.

### 2.2 Model and test plan notation

There are  $n$  test units that first place at low stress level  $s_1$  and run until time  $\tau_1$ , and the surviving units are subjected to a higher stress level  $s_2$  and run until time  $\tau_2$ , and then the remaining units are subjected to the highest stress level  $s_3$  until a predetermined censoring time  $c$ . We have the following assumptions:

- (1) The PH model is satisfied under different stress level  $s$ :

$$\lambda(t; s) = \lambda_0(t) \exp(\boldsymbol{\beta}^T s), \quad (2.1)$$

where  $\lambda_0(t) = \gamma_0 + \gamma_1 t$  is the baseline hazard function.

(2) There are three test stress levels  $s_1, s_2$ , and  $s_3$  ( $s_1 < s_2 < s_3$ ) used in three-step-stress ALT test.

We use the following notations in order to denote the cumulative distribution function (cdf), the probability density function(pdf), the cumulative hazard function and the reliability function of failure time at a given stress level  $s$ :

$$\begin{aligned} F(t; s) &= 1 - \exp \left[ - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s) \right], \\ f(t; s) &= \lambda_0(t) \exp(\beta s) \exp \left[ - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s) \right], \\ \Lambda(t; s) &= \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s), \text{ and} \\ R(t; s) &= \exp \left[ - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s) \right]. \end{aligned}$$

Hazard rate is defined as the ratio of the density and reliability functions of time to failure, and it can be expressed as

$$\lambda(t; s) = \frac{f(t; s)}{R(t; s)}.$$

We adopt the most commonly used cumulative exposure model (CEM) to derive the cdf of the failure time. According to Nelson (1980), in the cumulative exposure models, the cdf of lifetime  $t$  under a step-stress

ALT is:

$$F_{CEM}(t) = \begin{cases} F_1(t) & \text{if } t \leq \tau_1, \\ F_2(a + t - \tau_1) & \text{if } \tau_1 \leq t \leq \tau_2, \\ F_3(b + t - \tau_2) & \text{if } \tau_2 \leq t \leq c, \\ P(t > c) & \text{if } t > c, \end{cases} \quad (2.2)$$

where  $a$  and  $b$  satisfy  $F_1(\tau_1) = F_2(a)$  and  $F_2(a + \tau_2 - \tau_1) = F_3(b)$ . Namely,  $a = F_2^{-1}[F_1(\tau_1)]$  and  $b = F_3^{-1}[F_2(a + \tau_2 - \tau_1)]$ .

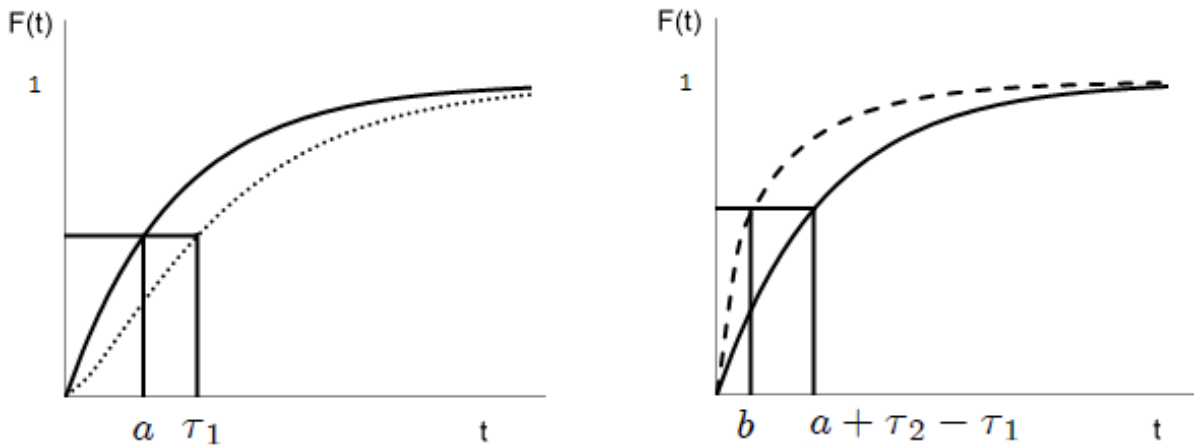


Figure 2.1 Graphical representation of cumulative exposure model  
(dotted line – cdf under  $s_1$ , solid line – cdf under  $s_2$ , broken line – cdf under  $s_3$ )

We also define the indicator functions  $I_1(t \leq \tau_1)$ ,  $I_2(t \leq \tau_2)$ , and  $I_3(t \leq c)$  in terms of the stress change point  $\tau_1$  and  $\tau_2$  :

$$I_1 = I_1(t \leq \tau_1) = \begin{cases} 1 & \text{if } t \leq \tau_1, \\ 0 & \text{if } t > \tau_1, \end{cases} \quad (2.3)$$

$$I_2 = I_2(t \leq \tau_2) = \begin{cases} 1 & \text{if } t \leq \tau_2, \\ 0 & \text{if } t > \tau_2, \end{cases} \quad \text{and} \quad (2.4)$$

$$I_3 = I_3(t \leq c) = \begin{cases} 1 & \text{if } t \leq c, \\ 0 & \text{if } t > c, \end{cases} \quad (2.5)$$

with  $\tau_1 < \tau_2 < c$ . The log likelihood function of an observed lifetime  $t$  can be expressed as:

$$\begin{aligned} \ln L(t; \xi) &= I_3 \left\{ \begin{array}{l} I_2 \{ I_1 [\ln f(t; s_1)] + (1 - I_1) [\ln f(x; s_2)] \} \\ + (1 - I_2) [\ln f(y; s_3)] \end{array} \right\} + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 \left\{ \begin{array}{l} I_2 I_1 [\ln f(t; s_1)] + I_2 (1 - I_1) [\ln f(x; s_2)] \\ + (1 - I_2) [\ln f(y; s_3)] \end{array} \right\} + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 I_2 I_1 [\ln f(t; s_1)] + I_3 I_2 (1 - I_1) [\ln f(x; s_2)] \\ &\quad + I_3 (1 - I_2) [\ln f(y; s_3)] + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 I_2 I_1 \left[ \ln(\gamma_0 + \gamma_1 t) + \beta s_1 - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right] \\ &\quad + I_3 I_2 (1 - I_1) \left[ \ln(\gamma_0 + \gamma_1 x) + \beta s_2 - \left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right] \\ &\quad + I_3 (1 - I_2) \left[ \ln(\gamma_0 + \gamma_1 y) + \beta s_3 - \left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right] \\ &\quad - (1 - I_3) \left[ \left( \gamma_0 z + \frac{\gamma_1}{2} z^2 \right) \exp(\beta s_3) \right], \end{aligned}$$

where  $x = a + t - \tau_1$ ,  $y = b + t - \tau_2$ ,  $z = b + c - \tau_2$ .

The first partial derivatives with respect to the model parameters are

$$\begin{aligned} \frac{\partial \ln L}{\partial \gamma_0} &= \frac{I_3 I_2 I_1}{\lambda_0(t)} - I_3 I_2 I_1 t \exp(\beta s_1) + \frac{I_3 I_2 (1 - I_1)}{\lambda_0(x)} - I_3 I_2 (1 - I_1) x \exp(\beta s_2) \\ &\quad + \frac{I_3 (1 - I_2)}{\lambda_0(y)} - I_3 (1 - I_2) y \exp(\beta s_3) - (1 - I_3) z \exp(\beta s_3), \\ \frac{\partial \ln L}{\partial \gamma_1} &= \frac{I_3 I_2 I_1 t}{\lambda_0(t)} - \frac{I_3 I_2 I_1 t^2}{2} \exp(\beta s_1) + \frac{I_3 I_2 (1 - I_1) x}{\lambda_0(x)} - \frac{I_3 I_2 (1 - I_1) x^2}{2} \exp(\beta s_2) \\ &\quad + \frac{I_3 (1 - I_2) y}{\lambda_0(y)} - \frac{I_3 (1 - I_2) y^2}{2} \exp(\beta s_3) - \frac{(1 - I_3) z^2}{2} \exp(\beta s_3), \\ \frac{\partial \ln L}{\partial \beta} &= I_3 I_2 I_1 s_1 - I_3 I_2 I_1 s_1 \Lambda(t; s_1) + I_3 I_2 (1 - I_1) s_2 - I_3 I_2 (1 - I_1) s_2 \Lambda(x; s_2) \\ &\quad + I_3 (1 - I_2) s_3 - I_3 (1 - I_2) s_3 \Lambda(y; s_3) - (1 - I_3) s_3 \Lambda(z; s_3). \end{aligned}$$



The second partial derivatives with respect to the model parameters are

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \gamma_0^2} &= -\frac{I_3 I_2 I_1}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1)}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2)}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_1^2} &= -\frac{I_3 I_2 I_1 t^2}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x^2}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y^2}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} &= -\frac{I_3 I_2 I_1 t}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \beta^2} &= -I_3 I_2 I_1 s_1^2 \Lambda(t; s_1) - I_3 I_2 (1 - I_1) s_2^2 \Lambda(x; s_2) \\
&\quad - I_3 (1 - I_2) s_3^2 \Lambda(y; s_3) - (1 - I_3) s_3^2 \Lambda(z; s_3).
\end{aligned}$$

The elements of the Fisher's information matrix for  $t$  are the negative expectations of second partial derivative equations. Each of them can be derived as below.

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} &= \int_0^{\tau_1} \frac{1}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{1}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{1}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{1}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt, \\
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} &= \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^2}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y^2}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt, \\
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} &= \int_0^{\tau_1} \frac{t}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt, \\
E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} &= \int_0^{\tau_1} s_1^2 \Lambda(t; s_1) f(t; s_1) dt + \int_{\tau_1}^{\tau_2} s_2^2 \Lambda(x; s_2) f(x; s_2) dt \\
&\quad + \int_{\tau_2}^c s_3^2 \Lambda(y; s_3) f(y; s_3) dt + \int_c^\infty s_3^2 \Lambda(z; s_3) f(z; s_3) dt.
\end{aligned}$$

The Fisher's information matrix,  $F$ , for the full sample with size  $n$  can be obtained as

$$F = n \begin{bmatrix} E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} & 0 \\ E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} & 0 \\ 0 & 0 & E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} \end{bmatrix}.$$

Since in the PH model the baseline hazard rate is assumed to be independent of the covariates. Thus, the correlations between the stress coefficient  $\beta$  and baseline parameters  $\gamma_0$  and  $\gamma_1$  are equal to zero.

The covariance matrix,  $\Sigma$ , of the maximum likelihood estimators (MLE) of  $\gamma_0, \gamma_1, \beta$  is the inverse matrix of  $F$  and it is

$$\Sigma = F^{-1} = \begin{bmatrix} \text{Var}(\widehat{\gamma}_0) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Var}(\widehat{\gamma}_1) & 0 \\ 0 & 0 & \text{Var}(\widehat{\beta}) \end{bmatrix}.$$

Denoting the non-zero elements of Fisher's information by  $F_0, F_1, F_{01}$  and  $F_\beta$ . They are

$$\begin{aligned} F_0 &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} = \int_0^{\tau_1} \frac{1}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{1}{\lambda_0^2(y)} f(y; s_3) dt \\ &= \int_0^{\tau_1} \frac{1}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right] dt \\ &\quad + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right] dt \\ &\quad + \int_{\tau_2}^c \frac{1}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right] dt \\ &= \exp(\beta s_1) \int_{\frac{\gamma_0}{\gamma_1}}^{\frac{\gamma_0}{\gamma_1} + \tau_1} \frac{1}{\gamma_1 z_1} \exp \left[ -\left( \frac{\gamma_1}{2} z_1^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_1) \right] dz_1 \\ &\quad + \exp(\beta s_2) \int_{\frac{\gamma_0}{\gamma_1} + a}^{\frac{\gamma_0}{\gamma_1} + \tau_2 - \tau_1 + a} \frac{1}{\gamma_1 z_2} \exp \left[ -\left( \frac{\gamma_1}{2} z_2^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_2) \right] dz_2 \\ &\quad + \exp(\beta s_3) \int_{\frac{\gamma_0}{\gamma_1} + b}^{\frac{\gamma_0}{\gamma_1} + c - \tau_2 + b} \frac{1}{\gamma_1 z_3} \exp \left[ -\left( \frac{\gamma_1}{2} z_3^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_3) \right] dz_3, \end{aligned}$$

$$\begin{aligned} F_1 &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} = \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\ &= \int_0^{\tau_1} \frac{t^2}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right] dt \\ &\quad + \int_{\tau_1}^{\tau_2} \frac{t^2}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right] dt \\ &\quad + \int_{\tau_2}^c \frac{t^2}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right] dt \\ &= \exp(\beta s_1) \int_{\frac{\gamma_0}{\gamma_1}}^{\frac{\gamma_0}{\gamma_1} + \tau_1} \frac{t^2}{\gamma_1 z_1} \exp \left[ -\left( \frac{\gamma_1}{2} z_1^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_1) \right] dz_1 \\ &\quad + \exp(\beta s_2) \int_{\frac{\gamma_0}{\gamma_1} + a}^{\frac{\gamma_0}{\gamma_1} + \tau_2 - \tau_1 + a} \frac{t^2}{\gamma_1 z_2} \exp \left[ -\left( \frac{\gamma_1}{2} z_2^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_2) \right] dz_2 \\ &\quad + \exp(\beta s_3) \int_{\frac{\gamma_0}{\gamma_1} + b}^{\frac{\gamma_0}{\gamma_1} + c - \tau_2 + b} \frac{t^2}{\gamma_1 z_3} \exp \left[ -\left( \frac{\gamma_1}{2} z_3^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_3) \right] dz_3, \end{aligned}$$

$$\begin{aligned}
F_{01} &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} = \int_0^{\tau_1} \frac{t}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{t}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{t}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right] dt \\
&= \exp(\beta s_1) \int_{\frac{\gamma_0}{\gamma_1}}^{\frac{\gamma_0}{\gamma_1} + \tau_1} \frac{t}{\gamma_1 z_1} \exp \left[ -\left( \frac{\gamma_1}{2} z_1^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_1) \right] dz_1 \\
&\quad + \exp(\beta s_2) \int_{\frac{\gamma_0}{\gamma_1} + a}^{\frac{\gamma_0}{\gamma_1} + \tau_2 - \tau_1 + a} \frac{t}{\gamma_1 z_2} \exp \left[ -\left( \frac{\gamma_1}{2} z_2^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_2) \right] dz_2 \\
&\quad + \exp(\beta s_3) \int_{\frac{\gamma_0}{\gamma_1} + b}^{\frac{\gamma_0}{\gamma_1} + c - \tau_2 + b} \frac{t}{\gamma_1 z_3} \exp \left[ -\left( \frac{\gamma_1}{2} z_3^2 - \frac{\gamma_0^2}{2\gamma_1} \right) \exp(\beta s_3) \right] dz_3, \text{ and}
\end{aligned}$$

$$\begin{aligned}
F_\beta &= E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} = \int_0^{\tau_1} s_1^2 \Lambda(t; s_1) f(t; s_1) dt + \int_{\tau_1}^{\tau_2} s_2^2 \Lambda(x; s_2) f(x; s_2) dt \\
&\quad + \int_{\tau_2}^c s_3^2 \Lambda(y; s_3) f(y; s_3) dt + \int_c^\infty s_3^2 \Lambda(z; s_3) f(z; s_3) dt.
\end{aligned}$$

## 2.3 Loss functions

### 2.3.1 The loss function under D-optimality

The first criterion we considered is D-optimality, which seeks to minimize the determinant of covariance matrix of the model parameter estimators. The loss function under D-optimality is

$$\begin{aligned}
L_D &= \det \begin{bmatrix} \text{Var}(\widehat{\gamma}_0) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Var}(\widehat{\gamma}_1) & 0 \\ 0 & 0 & \text{Var}(\widehat{\beta}) \end{bmatrix} \\
&= \det \left( \begin{bmatrix} nF_0 & nF_{01} & 0 \\ nF_{01} & nF_1 & 0 \\ 0 & 0 & nF_\beta \end{bmatrix}^{-1} \right) \\
&= \frac{1}{n^3(F_0 F_1 - F_{01}^2) F_\beta}.
\end{aligned} \tag{2.6}$$

### 2.3.2 The loss function under A-optimality

The second criterion we considered is A-optimality, which minimizes the trace of covariance matrix of the model parameter estimators. The loss function under A-optimality is used as

$$\begin{aligned}
L_A &= \text{tr} \begin{bmatrix} \text{Var}(\widehat{\gamma}_0) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Var}(\widehat{\gamma}_1) & 0 \\ 0 & 0 & \text{Var}(\widehat{\beta}) \end{bmatrix} \\
&= \text{tr} \left( \begin{bmatrix} nF_0 & nF_{01} & 0 \\ nF_{01} & nF_1 & 0 \\ 0 & 0 & nF_\beta \end{bmatrix}^{-1} \right) \\
&= \frac{F_0 + F_1}{n(F_0F_1 - F_{01}^2)} + \frac{1}{nF_\beta}.
\end{aligned} \tag{2.7}$$

### 2.3.3 The loss function under Q-optimality

The third criterion we considered is the Q-optimality. We search for an optimal design that minimizes the asymptotic variance (AVAR) of the hazard rate estimator over a specific period of time,  $T$ , at a normal design stress level,  $s_D$ . The loss function under Q-optimality can be derived as

$$\begin{aligned}
L_Q &= \int_0^T \text{Var} \left[ \widehat{\lambda}(t; s_D) \right] dt \\
&= \int_0^T \text{Var} \left[ (\widehat{\gamma}_0 + \widehat{\gamma}_1 t) \exp(\widehat{\beta} s_D) \right] dt \\
&= \int_0^T \left\{ \begin{bmatrix} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{bmatrix} F^{-1} \begin{bmatrix} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{bmatrix}^T \right\}_{\widehat{\gamma}_0=\gamma_0, \widehat{\gamma}_1=\gamma_1, \widehat{\beta}=\beta} dt \\
&= \int_0^T \left\{ \begin{bmatrix} \exp(\beta s_D) & t \exp(\beta s_D) & s_D (\gamma_0 + \gamma_1 t) \exp(\beta s_D) \\ nF_0 & nF_{01} & 0 \\ nF_{01} & nF_1 & 0 \\ 0 & 0 & nF_\beta \end{bmatrix}^{-1} \begin{bmatrix} \exp(\beta s_D) \\ t \exp(\beta s_D) \\ s_D (\gamma_0 + \gamma_1 t) \exp(\beta s_D) \end{bmatrix} \right\} dt \\
&= \int_0^T \frac{\exp(2\beta s_D) \left( F_1 + F_0 t^2 + \frac{F_0 F_1 - F_{01}^2}{F_\beta} s_D^2 (\gamma_0 + \gamma_1 t)^2 - 2F_{01} t \right)}{n(F_0 F_1 - F_{01}^2)} dt.
\end{aligned} \tag{2.8}$$

where  $s_D$  is the specified design stress and  $T$  is the specific period of time.

## 2.4 Optimal designs

Elsayed and Zhang (2005) has made a requirement on the minimum number of failures at  $s_1$  to avoid the resulting optimal stress-changing time too close to zero, so that the optimal stress-changing time would provide a reasonable simple step-stress ALT. Therefore, we make this practical consideration of having the requirements on the minimum number of failures at each stress level of our optimal design. Under the constraints of minimum number of failures at each stress level, we optimally allocate stress-changing times and the middle stress level so that each the loss functions can be minimized respectively. Namely, the

optimal decision variables  $(\tau_1, \tau_2, s_2)$  are chosen by minimizing the loss function (2.6), (2.7), or (2.8) with the constraints defined in (2.9), (2.10) and (2.11) as below:

The constraints of minimum number of failures at  $s_i$  stress level ( $MNF_i$ ) are

- (1) The expected number of failures at stress level  $s_1$  has a minimum value  $MNF_1$ :

$$n \Pr [t \leq \tau_1 | s_1] \geq MNF_1. \quad (2.9)$$

- (2) The expected number of failures at stress level  $s_2$  has a minimum, called  $MNF_2$ :

$$(n - n_1) \Pr [a + t - \tau_1 \leq \tau_2 | s_2] \geq MNF_2. \quad (2.10)$$

- (3) The expected number of failures at stress level  $s_3$  has a minimum, named  $MNF_3$ :

$$(n - n_1 - n_2) \Pr [b + t - \tau_2 \leq c | s_3] \geq MNF_3, \quad (2.11)$$

where  $n_i$  is the number of failures under  $s_i$ ,  $i = 1, 2, 3$ .

The optimal designs are obtained by minimizing the loss functions with respect to variables  $(\tau_1, \tau_2, s_2)$  in the presence of three constraints (2.9), (2.10) and (2.11). Namely, the corresponding designs  ${}^D\xi$ ,  ${}^A\xi$  and  ${}^Q\xi$  can be described as

$${}^D\xi = \arg \{ \min (L_D | \text{given } (2.9), (2.10) \text{ and } (2.11)) \}, \quad (2.12)$$

$${}^A\xi = \arg \{ \min (L_A | \text{given } (2.9), (2.10) \text{ and } (2.11)) \}, \quad (2.13)$$

and

$${}^Q\xi = \arg \{ \min (L_Q | \text{given } (2.9), (2.10) \text{ and } (2.11)) \}. \quad (2.14)$$

#### 2.4.1 Optimal designs when the middle stress level is fixed

We revisit an example discussed in Elsayed and Zhang (2005), two-step-stress ALT was conducted for Metal Oxide Semiconductor (MOS) capacitors in order to estimate its hazard rate over a 10 year period of time at design temperature of  $50^\circ C$ . The test needs to be completed in 300 hours and the total number of test items placed for the test is 200. To avoid any unexpected change in failure mechanisms within the design temperature range, it has been decided by engineer's judgment that the testing temperatures cannot exceed  $250^\circ C$ . The initial values for the model parameters has provided as:  $\gamma_0 = 0.0001$ ,  $\gamma_1 = 0.5$  and  $\beta = -3800$ . Their resulting Q-optimal low accelerated stress level is  $145^\circ C$ .

In our example, we compare the optimal designs obtained from two-step-stress ALT with the optimal design obtained three-step-stress. For two-step-stress ALT, we suppose that the high accelerated stress level is  $250^\circ C$ , low accelerated stress level is  $145^\circ C$ . For three-step-stress ALT, the high accelerated stress level and low accelerated stress level remain the same as two-step-stress ALT and the middle accelerated stress level is the average of high and low accelerated stress level. Thus, these three accelerated stress levels are specified as:  $s_1 = 145^\circ C = \frac{1}{418.16K}$ ,  $s_3 = 250^\circ C = \frac{1}{523.16K}$  and  $s_2 = 197.5^\circ C = \frac{1}{470.66K}$ . In addition, we let the normal design stress level  $s_D = 50^\circ C = \frac{1}{323.16K}$ , the censoring time  $c = 300$  hours and total number of test items  $n = 200$ . From the example in Elsayed and Zhang (2005), the initial values for the

model parameters are:  $\gamma_0 = 0.0001$ ,  $\gamma_1 = 0.5$  and  $\beta = -3800$ . The optimal designs are determined under above mentioned three criteria. In ALT practice, a certain number of failures under each test stress level is often required. Such requirement is made to avoid that the stress changing times occur too soon to provide a reasonable step-stress ALT having the same number of stress levels as planned. Please see Elsayed and Zhang (2005) for an example of simple step-stress ALT, they use  $MNF_1=40$ ,  $MNF_2=30$ . We also make several cases of different constraints with given  $MNF_1$ ,  $MNF_2$  and  $MNF_3$ .

Table 2.1 Constraint cases for two-step-stress ALT

Notation of the constraint plans	Constraint parameters	Minimum expected total failures
$C_{12}$	$MNF_1=40, MNF_2=30.$	70
$C_{34}$	$MNF_1=30, MNF_2=40.$	70
$C_5$	$MNF_1=40, MNF_2=20.$	60
$C_6$	$MNF_1=20, MNF_2=40.$	60

Table 2.2 Constraint cases for three-step-stress ALT

Notation of the constraint plans	Constraint parameters	Minimum expected total failures
$C_1$	$MNF_1=40, MNF_2=20, MNF_3=10.$	70
$C_2$	$MNF_1=40, MNF_2=15, MNF_3=15.$	70
$C_3$	$MNF_1=30, MNF_2=30, MNF_3=10.$	70
$C_4$	$MNF_1=30, MNF_2=20, MNF_3=20.$	70
$C_5$	$MNF_1=40, MNF_2=10, MNF_3=10.$	60
$C_6$	$MNF_1=20, MNF_2=20, MNF_3=20.$	60

We denote  ${}^D\xi_{C_k}^2, {}^A\xi_{C_k}^2, {}^Q\xi_{C_k}^2$  as the D-, A- and Q-optimal design obtained from two-step-stress ALT under the given constraint  $C_k$ , where  $k = 12, 34, 5, 6$ , and  ${}^D\xi_{C_k}^{3(i)}, {}^A\xi_{C_k}^{3(i)}, {}^Q\xi_{C_k}^{3(i)}$  as the D-, A- and Q-optimal design obtained for three-step-stress ALT under the given constraint  $C_k$ , where  $k = 1, \dots, 6$ ;  $i = 1, 2, 3$ ,  $i$  referring to the different situation of  $s_2$  ( $i = 1$  when  $s_2$  is fixed at  $197.5^\circ C$ ,  $i = 2$  when  $s_2$  is fixed at  $191.65^\circ C$  (the average of transformed  $s_1$  and  $s_3$ ), and  $i = 3$  when  $s_2$  being unfixed). Besides, we also define the efficiency of  $\xi_{C_k}^{3(i)}$  relative to  $\xi_{C_k}^2$  is

$$eff(3(i), 2) = \frac{\text{The value of loss function under } \xi_{C_k}^2}{\text{The value of loss function under } \xi_{C_k}^{3(i)}}$$

and the efficiency of  $\xi_{C_k}^{3(i)}$  relative to  $\xi_{C_k}^{3(j)}$

$$eff(3(i), 3(j)) = \frac{\text{The value of loss function under } \xi_{C_k}^{3(j)}}{\text{The value of loss function under } \xi_{C_k}^{3(i)}}$$

The optimal designs for two-step and three-step-stress ALT obtained by minimizing  $L_D$ ,  $L_A$  and  $L_Q$  are displayed respectively in Table 2.3, Table 2.4 and Table 2.5.

Table 2.3 D-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$D_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$D_{\xi_{C_{12}}}^2$	183	1.06
$D_{\xi_{C_2}}^{3(1)}$	177.96	209.16			1.04
$D_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$D_{\xi_{C_{34}}}^2$	156	1.10
$D_{\xi_{C_4}}^{3(1)}$	162.05	194.83			1.06
$D_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$D_{\xi_{C_5}}^2$	201	1.04
$D_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$D_{\xi_{C_6}}^2$	156	1.06

Table 2.4 A-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$A_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$A_{\xi_{C_{12}}}^2$	183	1.06
$A_{\xi_{C_2}}^{3(1)}$	177.96	209.16			1.03
$A_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$A_{\xi_{C_{34}}}^2$	156	1.09
$A_{\xi_{C_4}}^{3(1)}$	162.05	194.83			1.06
$A_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$A_{\xi_{C_5}}^2$	201	1.04
$A_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$A_{\xi_{C_6}}^2$	156	1.06

Table 2.5 Q-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$Q_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$Q_{\xi_{C_{12}}}^2$	183	1.04
$Q_{\xi_{C_2}}^{3(1)}$	177.96	209.16			1.03
$Q_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$Q_{\xi_{C_{34}}}^2$	156	1.08
$Q_{\xi_{C_4}}^{3(1)}$	162.05	194.83			1.05
$Q_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$Q_{\xi_{C_5}}^2$	201	1.03
$Q_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$Q_{\xi_{C_6}}^2$	156	1.05

From Tables 2.3, 2.4, and 2.5, we have gained efficiency by average of 5.44%. The resulting optimal stress-changing times are the same for these three different optimal criteria.

#### 2.4.2 Optimal designs when the middle stress level is not fixed

Using the same example as Subsection 2.4.1 instead setting  $s_2 = \frac{(s_1 + s_3)}{2}$ , we search an optimal  $s_2$  within a range of  $(s_1 + 10, s_3 - 10)$ . Namely, the optimal  $\tau_1$ ,  $\tau_2$ , and  $s_2$  should be obtained simultaneously for this example. All other values of parameters remain the same.

Table 2.6 D-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$D\xi_{C_1}^{3(3)}$	152.65	247.44	155	1.13
$D\xi_{C_2}^{3(3)}$	148.33	226.85	155	1.09
$D\xi_{C_3}^{3(3)}$	152.93	247.44	155	1.19
$D\xi_{C_4}^{3(3)}$	143.25	211.74	155	1.12
$D\xi_{C_5}^{3(3)}$	179.40	245.78	155	1.08
$D\xi_{C_6}^{3(3)}$	161.43	211.03	155	1.11

Table 2.7 A-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$A\xi_{C_1}^{3(3)}$	155.44	247.47	155	1.13
$A\xi_{C_2}^{3(3)}$	148.03	226.84	155	1.09
$A\xi_{C_3}^{3(3)}$	155.42	247.47	155	1.19
$A\xi_{C_4}^{3(3)}$	142.41	211.73	155	1.11
$A\xi_{C_5}^{3(3)}$	155.21	247.47	155	1.09
$A\xi_{C_6}^{3(3)}$	142.43	211.73	155	1.11

Table 2.8 Q-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$Q\xi_{C_1}^{3(3)}$	156.19	247.47	155	1.09
$Q\xi_{C_2}^{3(3)}$	149.81	226.86	155	1.07
$Q\xi_{C_3}^{3(3)}$	156.71	247.47	155	1.14
$Q\xi_{C_4}^{3(3)}$	144.20	211.75	155	1.09
$Q\xi_{C_5}^{3(3)}$	156.45	247.47	155	1.06
$Q\xi_{C_6}^{3(3)}$	143.98	211.75	155	1.09

The optimal  $\tau_1$ ,  $\tau_2$ , and  $s_2$  for three-step-stress ALT design are listed in Tables 2.6, 2.7 and 2.8. Table 2.6 shows that the D-optimal three-step-stress designs have gained at a minimum of 8%, maximum of 19% respect to two-step-stress designs. In Table 2.7, A-optimal three-step-stress designs have gained at a minimum of 9%, maximum of 19% respect to two-step-stress designs. In Table 2.8, Q-optimal three-step-stress designs have gained at a minimum of 6%, maximum of 14% respect to two-step-stress designs.

## 2.5 Discussion

We have provided the optimal designs for an ALT when the baseline hazard function is a simple linear function. Under different constraints of minimum number of failures at each stress level, we optimally allocate stress-changing times and middle stress level so that the each loss function can be minimized. Six



constrained optimal designs are constructed under each of these three criteria in this chapter. The relative efficiencies between each of two designs are displayed in the Table 2.9.

Table 2.9 The relative efficiency

	$eff(3(1), 2)$			$eff(3(3), 2)$			$eff(3(3), 3(1))$		
	$D\xi$	$A\xi$	$Q\xi$	$D\xi$	$A\xi$	$Q\xi$	$D\xi$	$A\xi$	$Q\xi$
$C_1$	1.06	1.06	1.04	1.13	1.13	1.09	1.07	1.07	1.05
$C_2$	1.04	1.03	1.03	1.09	1.09	1.07	1.05	1.06	1.04
$C_3$	1.10	1.09	1.08	1.19	1.19	1.14	1.09	1.09	1.06
$C_4$	1.06	1.06	1.05	1.12	1.11	1.09	1.06	1.05	1.04
$C_5$	1.04	1.04	1.03	1.08	1.09	1.06	1.04	1.05	1.03
$C_6$	1.06	1.06	1.05	1.11	1.11	1.09	1.05	1.05	1.04
average	1.06	1.06	1.05	1.12	1.12	1.09	1.06	1.06	1.04

### 2.5.1 On the results of optimal designs when the middle stress level is fixed

From Tables 2.3, 2.4 and 2.5, we see that the optimal stress-changing times  $\tau_1$  and  $\tau_2$  are the same under three criteria. But the results are varied under different constraints. The optimal stress-changing times  $\tau_1$  and  $\tau_2$  increase as the  $MNF_1$  and  $MNF_2$  increase. On the other hand, the lower  $MNF_3$ , the higher  $\tau_2$  can obtain. Meanwhile, we have found that under each constraint, the D-, A- and Q-optimal designs are the same. Therefore, if the experimenter hasn't have fixed  $MNF_i$  in their mind, we would recommend to use  $D\xi_{C_3}^{3(1)}$ , which has maximum efficiency.

### 2.5.2 On the results of optimal designs when the middle stress level is not fixed

From Tables 2.6, 2.7 and 2.8, the optimal stress-changing times  $\tau_1$  and  $\tau_2$  are similar under three criteria except when the constraint plan is  $C_5$ . Under constraint plan  $C_5$ , the D-optimal stress-changing times are (179.40, 245.78), which is different from the A- and Q-optimal stress-changing times. We can see that all the optimal middle stress levels are all equal to 155, which is the lower boundary of  $s_2$ . Thus, we point out that middle stress level should be kept as close to the lower bound of  $s_2$  as possible. In addition, we obtain the optimal constraint plan,  $C_3$ , which has maximum efficiency. Thus, if the experimenter hasn't have special constraints in mind, we would recommend  $D\xi_{C_3}^{3(3)}$  or  $A\xi_{C_3}^{3(3)}$  for parameter estimation and  $Q\xi_{C_3}^{3(3)}$  for hazard rate prediction.

### 2.5.3 General remarks

The three-step-stress ALT designs have helped to reduce the loss functions. From the Table 2.9, the optimal three-step-stress designs have gained at a minimum of 3%, maximum of 10% for hazard rate prediction respect to two-step-stress designs. The average gain was 5.4% by optimal designing a three-step-stress ALT. Another method to help minimize the loss function would be simultaneously found the stress-changing times and middle stress level. By optimizing the middle stress, the minimum efficiency respect to two-step-stress designs gain of 3% has increased to 6%, maximum of 10% has increased to 19% (average of 5.4% increased to an average of 11%). Besides, we compute the average efficiency gain of optimal designs when the middle stress level is not fixed respective to the optimal designs when fixed the middle stress level, which is 5.5%.

Thus, we would recommend to use three-step-stress ALT designs when the baseline hazard function in a PH model is assumed a simple a simple linear form.

## Chapter 3

# Optimal designs when baseline hazard function is a quadratic function

### 3.1 Introduction

In this chapter, we extend the work of Jiao (2001), which has only found the optimal designs for PH-based simple step-stress ALT when baseline hazard function is a quadratic function, and investigate the optimal designs of multiple step-stress ALT.

### 3.2 Model assumptions

We have considered the PH-based ALT with the case where the baseline hazard function is a simple linear function in Chapter 2. In some practical situations, this simple model can be possibly underfitted. Therefore, in this chapter, we will focus on the PH model when the baseline hazard function is quadratic form. The PH model can be defined as

$$\lambda(t; s) = \lambda_0(t) \exp(\beta^T s), \quad (3.1)$$

where

$$\lambda_0(t) = \gamma_0 + \gamma_1 t + \gamma_2 t^2.$$

The cumulative hazard function is

$$\begin{aligned} \Lambda(t; s) &= \int_0^t \lambda(u) du = \int_0^t \lambda_0(u) \exp(\beta s) du \\ &= \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s). \end{aligned}$$

And the reliability function can be derived as

$$\begin{aligned} R(t; s) &= \exp(-\Lambda(t; s)) \\ &= \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s)\right). \end{aligned}$$

The cumulative distribution function can be expressed in terms of reliability function

$$\begin{aligned} F(t; s) &= 1 - R(t; s) \\ &= 1 - \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s)\right). \end{aligned}$$

Then we can derive the probability density function

$$\begin{aligned} f(t; s) &= F'(t; s) \\ &= (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s) \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s)\right). \end{aligned}$$

The log likelihood function of  $t$  is expressed as:

$$\begin{aligned} \ln L(t; \xi) &= I_3 \{I_2 \{I_1 [\ln f(t; s_1)] + (1 - I_1) [\ln f(x; s_2)]\} + (1 - I_2) [\ln f(y; s_3)]\} \\ &\quad + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 \{I_2 I_1 [\ln f(t; s_1)] + I_2 (1 - I_1) [\ln f(x; s_2)] + (1 - I_2) [\ln f(y; s_3)]\} \\ &\quad + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 I_2 I_1 [\ln f(t; s_1)] + I_3 I_2 (1 - I_1) [\ln f(x; s_2)] + I_3 (1 - I_2) [\ln f(y; s_3)] \\ &\quad + (1 - I_3) [-\Lambda(z; s_3)] \\ &= I_3 I_2 I_1 \left[ \ln(\gamma_0 + \gamma_1 t + \gamma_2 t^2) + \beta s_1 - \left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s_1) \right] \\ &\quad + I_3 I_2 (1 - I_1) \left[ \ln(\gamma_0 + \gamma_1 x + \gamma_2 x^2) + \beta s_2 - \left(\gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3\right) \exp(\beta s_2) \right] \\ &\quad + I_3 (1 - I_2) \left[ \ln(\gamma_0 + \gamma_1 y + \gamma_2 y^2) + \beta s_3 - \left(\gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3\right) \exp(\beta s_3) \right] \\ &\quad - (1 - I_3) \left[ \left(\gamma_0 z + \frac{\gamma_1}{2} z^2 + \frac{\gamma_2}{3} z^3\right) \exp(\beta s_3) \right], \end{aligned}$$

where  $x = a + t - \tau_1$ ,  $y = b + t - \tau_2$ ,  $z = b + c - \tau_2$  with  $a, b$  defined as in Chapter 2.

The first partial derivatives with respect to the model parameters are:

$$\begin{aligned}
\frac{\partial \ln L}{\partial \gamma_0} &= \frac{I_3 I_2 I_1}{\lambda_0(t)} - I_3 I_2 I_1 t \exp(\beta s_1) + \frac{I_3 I_2 (1 - I_1)}{\lambda_0(x)} \\
&\quad - I_3 I_2 (1 - I_1) x \exp(\beta s_2) + \frac{I_3 (1 - I_2)}{\lambda_0(y)} \\
&\quad - I_3 (1 - I_2) y \exp(\beta s_3) - (1 - I_3) z \exp(\beta s_3), \\
\frac{\partial \ln L}{\partial \gamma_1} &= \frac{I_3 I_2 I_1 t}{\lambda_0(t)} - \frac{I_3 I_2 I_1 t^2}{2} \exp(\beta s_1) + \frac{I_3 I_2 (1 - I_1) x}{\lambda_0(x)} \\
&\quad - \frac{I_3 I_2 (1 - I_1) x^2}{2} \exp(\beta s_2) + \frac{I_3 (1 - I_2) y}{\lambda_0(y)} \\
&\quad - \frac{I_3 (1 - I_2) y^2}{2} \exp(\beta s_3) - \frac{(1 - I_3) z^2}{2} \exp(\beta s_3), \\
\frac{\partial \ln L}{\partial \gamma_2} &= \frac{I_3 I_2 I_1 t^2}{\lambda_0(t)} - \frac{I_3 I_2 I_1 t^3}{3} \exp(\beta s_1) + \frac{I_3 I_2 (1 - I_1) x^2}{\lambda_0(x)} \\
&\quad - \frac{I_3 I_2 (1 - I_1) x^3}{3} \exp(\beta s_2) + \frac{I_3 (1 - I_2) y^2}{\lambda_0(y)} \\
&\quad - \frac{I_3 (1 - I_2) y^3}{3} \exp(\beta s_3) - \frac{(1 - I_3) z^3}{3} \exp(\beta s_3), \\
\frac{\partial \ln L}{\partial \beta} &= I_3 I_2 I_1 s_1 - I_3 I_2 I_1 s_1 \Lambda(t; s_1) + I_3 I_2 (1 - I_1) s_2 - I_3 I_2 (1 - I_1) s_2 \Lambda(x; s_2) \\
&\quad + I_3 (1 - I_2) s_3 - I_3 (1 - I_2) s_3 \Lambda(y; s_3) - (1 - I_3) s_3 \Lambda(z; s_3).
\end{aligned}$$

The second partial derivatives with respect to the model parameters are:

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \gamma_0^2} &= -\frac{I_3 I_2 I_1}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1)}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2)}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_1^2} &= -\frac{I_3 I_2 I_1 t^2}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x^2}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y^2}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_2^2} &= -\frac{I_3 I_2 I_1 t^4}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x^4}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y^4}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} &= -\frac{I_3 I_2 I_1 t}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_2} &= -\frac{I_3 I_2 I_1 t^2}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x^2}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y^2}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \gamma_1 \gamma_2} &= -\frac{I_3 I_2 I_1 t^3}{\lambda_0^2(t)} - \frac{I_3 I_2 (1 - I_1) x^3}{\lambda_0^2(x)} - \frac{I_3 (1 - I_2) y^3}{\lambda_0^2(y)}, \\
\frac{\partial^2 \ln L}{\partial \beta^2} &= -I_3 I_2 I_1 s_1^2 \Lambda(t; s_1) - I_3 I_2 (1 - I_1) s_2^2 \Lambda(x; s_2) \\
&\quad - I_3 (1 - I_2) s_3^2 \Lambda(y; s_3) - (1 - I_3) s_3^2 \Lambda(z; s_3).
\end{aligned}$$

The elements of the Fisher's information matrix for a single  $t$  are the negative expectations of second

partial derivative equations. Each of them will be derived as below:

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} &= \int_0^{\tau_1} \frac{1}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{1}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{1}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{1}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} &= \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^2}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y^2}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_2^2} \right\} &= \int_0^{\tau_1} \frac{t^4}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^4}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^4}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^4}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^4}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y^4}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} &= \int_0^{\tau_1} \frac{t}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_2} \right\} &= \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^2}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y^2}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1 \gamma_2} \right\} &= \int_0^{\tau_1} \frac{t^3}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^3}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^3}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^3}{\lambda_0(t)} \exp(\beta s_1) R(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^3}{\lambda_0(x)} \exp(\beta s_2) R(x; s_2) dt \\
&\quad + \int_{\tau_2}^c \frac{y^3}{\lambda_0(y)} \exp(\beta s_3) R(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} &= \int_0^{\tau_1} s_1^2 \Lambda(t; s_1) f(t; s_1) dt + \int_{\tau_1}^{\tau_2} s_2^2 \Lambda(x; s_2) f(x; s_2) dt \\
&\quad + \int_{\tau_2}^c s_3^2 \Lambda(y; s_3) f(y; s_3) dt + \int_c^{\infty} s_3^2 \Lambda(z; s_3) f(z; s_3) dt.
\end{aligned}$$

The correlations between the stress coefficient  $\beta$  and baseline parameters  $\gamma_0, \gamma_1$  and  $\gamma_2$  are equal to zero since in the PH model the baseline hazard rate is assumed to be independent of the covariates.

The Fisher's information matrix of the full sample with size  $n$  can be obtained as

$$F = n \begin{bmatrix} E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_2} \right\} & 0 \\ E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1 \gamma_2} \right\} & 0 \\ E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_2} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1 \gamma_2} \right\} & E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_2^2} \right\} & 0 \\ 0 & 0 & 0 & E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} \end{bmatrix}.$$

The covariance matrix of the maximum likelihood estimators (MLE) of  $\gamma_0, \gamma_1, \gamma_2, \beta$  is the inverse matrix of the Fisher's information matrix. It is:

$$\Sigma = F^{-1} = \begin{bmatrix} Var(\widehat{\gamma}_0) & Cov(\widehat{\gamma}_0, \widehat{\gamma}_1) & Cov(\widehat{\gamma}_0, \widehat{\gamma}_2) & 0 \\ Cov(\widehat{\gamma}_0, \widehat{\gamma}_1) & Var(\widehat{\gamma}_1) & Cov(\widehat{\gamma}_1, \widehat{\gamma}_2) & 0 \\ Cov(\widehat{\gamma}_0, \widehat{\gamma}_2) & Cov(\widehat{\gamma}_1, \widehat{\gamma}_2) & Var(\widehat{\gamma}_2) & 0 \\ 0 & 0 & 0 & Var(\widehat{\beta}) \end{bmatrix}$$

Denoting, the non-zero elements of Fisher's information as:

$$\begin{aligned}
F_0 &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0^2} \right\} = \int_0^{\tau_1} \frac{1}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{1}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{1}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{1}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{1}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3 \right) \exp(\beta s_3) \right] dt,
\end{aligned}$$

$$\begin{aligned}
F_1 &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\} = \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^2}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{t^2}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{t^2}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3 \right) \exp(\beta s_3) \right] dt,
\end{aligned}$$

$$\begin{aligned}
F_2 &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_2^2} \right\} = \int_0^{\tau_1} \frac{t^4}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^4}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^4}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^4}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{t^4}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{t^4}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3 \right) \exp(\beta s_3) \right] dt,
\end{aligned}$$

$$\begin{aligned}
F_{01} &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_1} \right\} = \int_0^{\tau_1} \frac{t}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{t}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{t}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3 \right) \exp(\beta s_3) \right] dt,
\end{aligned}$$

$$\begin{aligned}
F_{02} &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_0 \gamma_2} \right\} = \int_0^{\tau_1} \frac{t^2}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^2}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^2}{\lambda_0^2(y)} f(y; s_3) dt \\
&= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1^2} \right\},
\end{aligned}$$

$$\begin{aligned}
F_{12} &= E \left\{ -\frac{\partial^2 \ln L}{\partial \gamma_1 \gamma_2} \right\} = \int_0^{\tau_1} \frac{t^3}{\lambda_0^2(t)} f(t; s_1) dt + \int_{\tau_1}^{\tau_2} \frac{x^3}{\lambda_0^2(x)} f(x; s_2) dt + \int_{\tau_2}^c \frac{y^3}{\lambda_0^2(y)} f(y; s_3) dt \\
&= \int_0^{\tau_1} \frac{t^3}{\lambda_0(t)} \exp(\beta s_1) \exp \left[ -\left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_1) \right] dt \\
&\quad + \int_{\tau_1}^{\tau_2} \frac{t^3}{\lambda_0(x)} \exp(\beta s_2) \exp \left[ -\left( \gamma_0 x + \frac{\gamma_1}{2} x^2 + \frac{\gamma_2}{3} x^3 \right) \exp(\beta s_2) \right] dt \\
&\quad + \int_{\tau_2}^c \frac{t^3}{\lambda_0(y)} \exp(\beta s_3) \exp \left[ -\left( \gamma_0 y + \frac{\gamma_1}{2} y^2 + \frac{\gamma_2}{3} y^3 \right) \exp(\beta s_3) \right] dt,
\end{aligned}$$



$$\begin{aligned}
F_\beta &= E \left\{ -\frac{\partial^2 \ln L}{\partial \beta^2} \right\} = \int_0^{\tau_1} s_1^2 \Lambda(t; s_1) f(t; s_1) dt + \int_{\tau_1}^{\tau_2} s_2^2 \Lambda(x; s_2) f(x; s_2) dt \\
&\quad + \int_{\tau_2}^c s_3^2 \Lambda(y; s_3) f(y; s_3) dt + \int_c^\infty s_3^2 \Lambda(z; s_3) f(z; s_3) dt.
\end{aligned}$$

### 3.3 Loss functions

#### 3.3.1 The loss function under D-optimality

The first criterion we considered is D-optimality, which seeks to minimize the determinant of covariance matrix of the model parameter estimators. The loss function under D-optimality is

$$\begin{aligned}
L_D &= \det \begin{bmatrix} \text{Var}(\widehat{\gamma}_0) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_2) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Var}(\widehat{\gamma}_1) & \text{Cov}(\widehat{\gamma}_1, \widehat{\gamma}_2) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_2) & \text{Cov}(\widehat{\gamma}_1, \widehat{\gamma}_2) & \text{Var}(\widehat{\gamma}_2) & 0 \\ 0 & 0 & 0 & \text{Var}(\widehat{\beta}) \end{bmatrix} \\
&= \det \left( \begin{bmatrix} nF_0 & nF_{01} & nF_{02} & 0 \\ nF_{01} & nF_1 & nF_{12} & 0 \\ nF_{02} & nF_{12} & nF_2 & 0 \\ 0 & 0 & 0 & nF_\beta \end{bmatrix}^{-1} \right) \\
&= \frac{1}{n^4 (F_0 F_1 F_2 - F_0 F_{12}^2 - F_{01}^2 F_2 + 2F_{01} F_{02} F_{12} - F_{02}^2 F_1) F_\beta}.
\end{aligned} \tag{3.2}$$

#### 3.3.2 The loss function under A-optimality

The second criterion considered is A-optimality, which minimizes the trace of covariance matrix of the model parameter estimators. The loss function under A-optimality is used as

$$\begin{aligned}
L_A &= \text{tr} \begin{bmatrix} \text{Var}(\widehat{\gamma}_0) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_2) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_1) & \text{Var}(\widehat{\gamma}_1) & \text{Cov}(\widehat{\gamma}_1, \widehat{\gamma}_2) & 0 \\ \text{Cov}(\widehat{\gamma}_0, \widehat{\gamma}_2) & \text{Cov}(\widehat{\gamma}_1, \widehat{\gamma}_2) & \text{Var}(\widehat{\gamma}_2) & 0 \\ 0 & 0 & 0 & \text{Var}(\widehat{\beta}) \end{bmatrix} \\
&= \text{tr} \left( \begin{bmatrix} nF_0 & nF_{01} & nF_{02} & 0 \\ nF_{01} & nF_1 & nF_{12} & 0 \\ nF_{02} & nF_{12} & nF_2 & 0 \\ 0 & 0 & 0 & nF_\beta \end{bmatrix}^{-1} \right) \\
&= \frac{F_\beta F_0 F_1 + F_\beta F_0 F_2 - F_\beta F_{01}^2 - F_\beta F_{02}^2 + F_\beta F_1 F_2 - F_\beta F_{12}^2 \\
&\quad + F_0 F_1 F_2 - F_0 F_{12}^2 - F_{01}^2 F_2 + 2F_{01} F_{02} F_{12} - F_{02}^2 F_1}{n (F_0 F_1 F_2 - F_0 F_{12}^2 - F_{01}^2 F_2 + 2F_{01} F_{02} F_{12} - F_{02}^2 F_1) F_\beta}.
\end{aligned} \tag{3.3}$$

### 3.3.3 The loss function under Q-optimality

The third criterion we considered is the Q-optimality. We search for an optimal design that minimizes the AVAR of the hazard rate estimator over a specific period of time,  $T$ , at a normal design stress level,  $s_D$ . The loss function under Q-optimality can be derived as

$$\begin{aligned}
L_Q &= \int_0^T \text{Var} \left[ \widehat{\lambda}(t; s_D) \right] dt \\
&= \int_0^T \text{Var} \left[ (\widehat{\gamma}_0 + \widehat{\gamma}_1 t + \widehat{\gamma}_2 t^2) \exp(\widehat{\beta} s_D) \right] dt \\
&= \int_0^T \left\{ \left[ \begin{array}{cccc} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_2} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{array} \right] F^{-1} \left[ \begin{array}{cccc} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_2} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{array} \right]^T \right\}_{\widehat{\gamma}_0=\gamma_0, \widehat{\gamma}_1=\gamma_1, \widehat{\gamma}_2=\gamma_2, \widehat{\beta}=\beta} dt \\
&= \int_0^T \left\{ \begin{array}{c} \left[ \begin{array}{cccc} \exp(\beta s_D) & t \exp(\beta s_D) & t^2 \exp(\beta s_D) & s_D (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_D) \end{array} \right] \\ \left[ \begin{array}{cccc} nF_0 & nF_{01} & nF_{02} & 0 \\ nF_{01} & nF_1 & nF_{12} & 0 \\ nF_{02} & nF_{12} & nF_2 & 0 \\ 0 & 0 & 0 & nF_\beta \end{array} \right]^{-1} \\ \left[ \begin{array}{c} \exp(\beta s_D) \\ t \exp(\beta s_D) \\ t^2 \exp(\beta s_D) \\ s_D (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_D) \end{array} \right] \end{array} \right\} dt
\end{aligned} \tag{3.4}$$

where  $T$  is the specific period of time.

## 3.4 Optimal designs

The optimal decision variables  $(\tau_1, \tau_2, s_2)$  are chosen by minimizing the loss function (3.2), (3.3) or (3.4) with the given constraints. As the same as the constraints defined in Section 2.4, we have three constraints (2.9), (2.10) and (2.11). The corresponding designs  $^D\xi$ ,  $^A\xi$  and  $^Q\xi$  can be described as (2.12), (2.13) and (2.14).

### 3.4.1 Optimal designs when the middle stress level is fixed

We revisit the example presented in Subsection 2.4.1, however, the fitting model is a PH model with a quadratic baseline hazard function instead. Jiao (2001) provided the initial values for the model parameters, which are  $\gamma_0 = 0.0001$ ,  $\gamma_1 = 0.5$ ,  $\gamma_2 = 0$ ,  $\beta = -3800$ . All other values of the parameters in example remain the same. Thus, the accelerated stress levels remain as  $s_1 = 145^\circ C = \frac{1}{418.16K}$ ,  $s_3 = 250^\circ C = \frac{1}{523.16K}$  and  $s_2 = \frac{145+250}{2}^\circ C = 197.5^\circ C = \frac{1}{470.66K}$ . The resulting two-step-stress and three-step-stress optimal designs under D-, A-, and Q-optimality criteria are displayed in Tables 3.1, 3.2 and 3.3 respectively.

Table 3.1 D-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$D_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$D_{\xi_{C_{12}}}^2$	183	0.93
$D_{\xi_{C_2}}^{3(1)}$	177.96	209.16			0.89
$D_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$D_{\xi_{C_{34}}}^2$	156	1.14
$D_{\xi_{C_4}}^{3(1)}$	162.05	194.83			1.00
$D_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$D_{\xi_{C_5}}^2$	201	0.94
$D_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$D_{\xi_{C_6}}^2$	156	1.00

Table 3.2 A-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$A_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$A_{\xi_{C_{12}}}^2$	183	1.06
$A_{\xi_{C_2}}^{3(1)}$	177.96	209.16			1.03
$A_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$A_{\xi_{C_{34}}}^2$	156	1.09
$A_{\xi_{C_4}}^{3(1)}$	162.05	194.83			1.06
$A_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$A_{\xi_{C_5}}^2$	201	1.04
$A_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$A_{\xi_{C_6}}^2$	156	1.06

Table 3.3 Q-optimal stress-changing times

3-step-stress ALT			2-step-stress ALT		
Optimal design	$\tau_1$	$\tau_2$	Optimal design	$\tau_1$	$eff(3(1), 2)$
$Q_{\xi_{C_1}}^{3(1)}$	178.15	223.57	$Q_{\xi_{C_{12}}}^2$	183	0.88
$Q_{\xi_{C_2}}^{3(1)}$	177.96	209.16			0.86
$Q_{\xi_{C_3}}^{3(1)}$	162.85	219.44	$Q_{\xi_{C_{34}}}^2$	156	1.04
$Q_{\xi_{C_4}}^{3(1)}$	162.05	194.83			0.94
$Q_{\xi_{C_5}}^{3(1)}$	198.02	227.45	$Q_{\xi_{C_5}}^2$	201	0.91
$Q_{\xi_{C_6}}^{3(1)}$	162.05	194.83	$Q_{\xi_{C_6}}^2$	156	0.94

Designing a three-step-stress model only helped to reduce the A-loss function when we assume  $s_2$  is  $197.5^\circ C$ . Thus, we try to replace  $s_2$  by  $\frac{1}{2} \left( \frac{1}{418.16K} + \frac{1}{523.16K} \right) = 191.65^\circ C$ , which is the average of  $s_1$  and  $s_3$  after transforming the measurement units from Celsius to Kelvin which is often used in Arrhenius models. All other values of parameters remain the same as the example above.

Table 3.4 D-optimal stress-changing times

3-step-stress ALT			
Optimal design	$\tau_1$	$\tau_2$	$eff(3(2), 2)$
$D_{\xi_{C_1}}^{3(2)}$	178.15	225.93	0.98
$D_{\xi_{C_2}}^{3(2)}$	177.70	210.74	0.91
$D_{\xi_{C_3}}^{3(2)}$	156.68	220.66	1.17
$D_{\xi_{C_4}}^{3(2)}$	161.04	194.94	1.01
$D_{\xi_{C_5}}^{3(2)}$	180.74	226.26	0.78
$D_{\xi_{C_6}}^{3(2)}$	161.04	194.94	1.01

Table 3.5 Q-optimal stress-changing times

3-step-stress ALT			
Optimal design	$\tau_1$	$\tau_2$	$eff(3(2), 2)$
$Q_{\xi_{C_1}}^{3(2)}$	178.15	225.93	0.92
$Q_{\xi_{C_2}}^{3(2)}$	177.96	210.79	0.88
$Q_{\xi_{C_3}}^{3(2)}$	162.85	222.39	1.10
$Q_{\xi_{C_4}}^{3(2)}$	162.04	196.53	0.96
$Q_{\xi_{C_5}}^{3(2)}$	198.02	228.98	0.93
$Q_{\xi_{C_6}}^{3(2)}$	162.04	196.53	0.96

Tables 3.4 and 3.5 show that the efficiencies are not increase much by changing the middle stress level from 197.5 to 191.65 and the resulting stress-changing times are similar as the results in Tables 3.1 and 3.3.

### 3.4.2 Optimal designs when the middle stress level is not fixed

The results in Subsection 3.4.1 seem being more than two-step-stress ALT not efficient as we expected. Thus, using the same example as Subsection 3.4.1 instead setting  $s_2 = \frac{(s_1+s_3)}{2}$ , we search an optimal  $s_2$  within a range of  $(s_1 + 10, s_3 - 10)$ . Namely, the optimal  $\tau_1$ ,  $\tau_2$ , and  $s_2$  should be obtained simultaneously for this example. All other values of parameters remain the same.

Table 3.6 D-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$D_{\xi_{C_1}}^{3(3)}$	178.05	245.70	155	1.91
$D_{\xi_{C_2}}^{3(3)}$	175.60	220.01	155	1.42
$D_{\xi_{C_3}}^{3(3)}$	145.65	247.10	155	2.80
$D_{\xi_{C_4}}^{3(3)}$	154.65	210.94	155	1.79
$D_{\xi_{C_5}}^{3(3)}$	179.37	245.61	155	1.47
$D_{\xi_{C_6}}^{3(3)}$	154.65	210.94	155	1.79

Table 3.7 A-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$A_{\xi_{C_1}}^{3(3)}$	155.20	247.47	155	1.13
$A_{\xi_{C_2}}^{3(3)}$	147.99	226.84	155	1.09
$A_{\xi_{C_3}}^{3(3)}$	155.13	247.47	155	1.19
$A_{\xi_{C_4}}^{3(3)}$	142.41	211.73	155	1.11
$A_{\xi_{C_5}}^{3(3)}$	155.18	247.47	155	1.09
$A_{\xi_{C_6}}^{3(3)}$	142.39	211.73	155	1.11

Table 3.8 Q-optimal stress-changing times

3-step-stress ALT				
Optimal design	$\tau_1$	$\tau_2$	$s_2$	$eff(3(3), 2)$
$Q_{\xi_{C_1}^{3(3)}}$	178.15	245.96	155	1.71
$Q_{\xi_{C_2}^{3(3)}}$	177.96	224.53	155	1.39
$Q_{\xi_{C_3}^{3(3)}}$	148.51	247.29	155	2.36
$Q_{\xi_{C_4}^{3(3)}}$	162.04	210.97	155	1.62
$Q_{\xi_{C_5}^{3(3)}}$	198.03	241.97	155	1.37
$Q_{\xi_{C_6}^{3(3)}}$	162.04	210.97	155	1.62

### 3.5 Discussion

We derived the optimal design with baseline hazard function being a quadratic function. The optimal stress-changing times and middle stress level are chosen by minimizing the loss function. Six constrained optimal designs are constructed under three different criteria in this chapter. The relative efficiencies between each of two designs are displayed in the Table 3.9.

Table 3.9 The relative efficiency

	$eff(3(1), 2)$			$eff(3(2), 2)$		$eff(3(2), 3(1))$		$eff(3(3), 2)$			$eff(3(3), 3(1))$		
	$D_{\xi}$	$A_{\xi}$	$Q_{\xi}$	$D_{\xi}$	$Q_{\xi}$	$D_{\xi}$	$Q_{\xi}$	$D_{\xi}$	$A_{\xi}$	$Q_{\xi}$	$D_{\xi}$	$A_{\xi}$	$Q_{\xi}$
$C_1$	0.93	1.06	0.88	0.98	0.92	1.05	1.05	1.91	1.13	1.71	2.05	1.07	1.94
$C_2$	0.89	1.03	0.86	0.91	0.88	1.02	1.02	1.42	1.09	1.39	1.30	1.06	1.62
$C_3$	1.14	1.09	1.04	1.17	1.10	1.03	1.06	2.80	1.19	2.36	2.46	1.09	2.27
$C_4$	1.00	1.06	0.94	1.01	0.96	1.01	1.02	1.79	1.11	1.62	1.79	1.05	1.72
$C_5$	0.94	1.04	0.91	0.78	0.93	0.83	1.02	1.47	1.09	1.37	1.56	1.05	1.51
$C_6$	1.00	1.06	0.94	1.01	0.96	1.01	1.02	1.79	1.11	1.62	1.79	1.05	1.72
average	0.98	1.06	0.93	0.98	0.96	0.99	1.03	1.86	1.12	1.68	1.83	1.06	1.80

#### 3.5.1 On the results of optimal designs when the middle stress level is fixed

In Subsection 3.4.1, the three-step-stress designs with only  $\tau_1$ , and  $\tau_2$  being optimized have helped to reduce the total variances in estimation. From the table above, the optimal three-step-stress designs with  $s_2$  being fixed at  $197.5^{\circ}C$  have gained only for  $C_3$  at 14% under D-optimality criterion, and 4% under Q-optimality criterion respect to two-step-stress designs. Compare to the case of using  $s_2$  fixed at  $197.5^{\circ}C$ , when we replace  $s_2$  with  $191.65^{\circ}C$ , the D- and Q-optimal stress-changing times are not changing by much. For  $C_3$ , the efficiency respect to two-step-stress designs gain has increased to 17% under D-optimality, and increased to 10% under Q-optimality. It does appear that the lower the middle stress level, the higher efficiency. The optimal-stress-changing times  $\tau_1$  and  $\tau_2$  are the same under three criteria. Different constraint plan results in totally different results. Under three different criteria,  $C_3$  is the only constraint plan that the efficiency always greater than 1.

### 3.5.2 On the results of optimal designs when the middle stress level is not fixed

The three-step-stress ALT design with optimal middle stress level has helped to reduce the loss function for all cases. From Table 3.9, we have gained at a minimum of 9%, maximum of 180% respect to two-step-stress designs. Compare to the case of using  $s_2$  fixed at  $197.5^\circ C$ , when we replace  $s_2$  with optimal middle stress level, the average gain was high to 56%. The results are varied under three criteria in this case, but all the optimal middle stress levels equal to the lower bound of  $s_2$ . For each criterion, the most efficient case occurs when the constraint plan is  $C_3$ . Therefore, if the experimenter hasn't have special constraints in mind, we would recommend to use  ${}^D\xi_{C_3}^{3(3)}$  and  ${}^Q\xi_{C_3}^{3(3)}$ .

### 3.5.3 General remarks

The three-step-stress ALT designs with average middle stress level have only helped to reduce the total variances in estimation. Thus, we against to use average middle stress level when the PH-based three-step-stress ALT with the quadratic baseline hazard. Another method to help minimize the loss function would be simultaneously found the stress-changing times and middle stress level. By optimizing the middle stress, the maximum efficiency respect to two-step-stress designs gain of 14% has increased to 180%, average gain was high to 55%. Therefore, we strongly recommend to use our resulting optimal middle stress level.

## Chapter 4

# Robust designs for step-stress ALT when the baseline hazard function in a PH model is possibly misspecified

### 4.1 Introduction

Classical optimal designs are optimal only if the model assumed is exactly correct. Due to the nature of prediction made from ALT experimental data, attained under the stress levels higher than the normal design condition, extrapolation is encountered. In such case, the assumed model can not be tested. For possible imprecision in an assumed PH model, the method of constructing robust designs is needed.

### 4.2 Asymptotic distribution of MLEs with misspecification in the assumed baseline hazard function

It is assumed that Model (3.1) with a quadratic baseline hazard function, denoted by  $M_T$ , is the true model. However, Model (2.1) with a linear baseline hazard function is the fitted model, denoted by  $M_F$ . The maximum likelihood method has been used for estimating the model parameters. Define  $\boldsymbol{\alpha} = [\kappa_0, \kappa_1, \kappa_2, \alpha]^T$  and  $\boldsymbol{\beta} = [\gamma_0, \gamma_1, \beta]^T$ . Let  $\ell(\boldsymbol{\alpha}; \xi)$  be the log-likelihood function under  $M_T$ , and  $\ell(\boldsymbol{\beta}; \xi)$  be the log-likelihood function under  $M_F$ , both with the same design,  $\xi$ . The expected log-likelihood ratio under  $M_T$  over  $M_F$  is

$$I(\boldsymbol{\alpha} : \boldsymbol{\beta}) = E_{M_T} [\ell(\boldsymbol{\alpha}; \xi) - \ell(\boldsymbol{\beta}; \xi)]. \quad (4.1)$$

The likelihood function from observed lifetimes,  $t_i$ ,  $i = 1, \dots, n$  is

$$L(\boldsymbol{\beta}; \xi) = \prod_{i=1}^{n_1} f_F(t_i; s_1) \times \prod_{i=1}^{n_2} f_F(x_i; s_2) \times \prod_{i=1}^{n_3} f_F(y_i; s_3) \quad (4.2)$$

$$\times \prod_{i=1}^{n-n_1-n_2-n_3} (1 - F_F(z; s_3)),$$

where  $x = a+t-\tau_1$ ,  $y = b+t-\tau_2$ ,  $z = b+c-\tau_2$  with  $a, b$  defined as in Chapter 2,  $f_F$  is the probability density function (pdf) at failure time  $t$  and stress level  $s$  under  $M_F$ ,  $F_F$  is the cumulative distribution function (cdf) at failure time  $t$  and stress level  $s$  under  $M_F$ ; namely,

$$f_F(t; s) = (\gamma_0 + \gamma_1 t) \exp(\beta s) \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2\right) \exp(\beta s)\right),$$

$$F_F(t; s) = 1 - \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2\right) \exp(\beta s)\right).$$

The negative log-likelihood function is

$$-\ell(\boldsymbol{\beta}; \xi) = -n_1 \beta s_1 - \sum_{i=1}^{n_1} \left[ \ln(\gamma_0 + \gamma_1 t_i) - \left(\gamma_0 t_i + \frac{\gamma_1}{2} t_i^2\right) \exp(\beta s_1) \right] \quad (4.3)$$

$$-n_2 \beta s_2 - \sum_{i=1}^{n_2} \left[ \ln(\gamma_0 + \gamma_1 x_i) - \left(\gamma_0 x_i + \frac{\gamma_1}{2} x_i^2\right) \exp(\beta s_2) \right]$$

$$-n_3 \beta s_3 - \sum_{i=1}^{n_3} \left[ \ln(\gamma_0 + \gamma_1 y_i) - \left(\gamma_0 y_i + \frac{\gamma_1}{2} y_i^2\right) \exp(\beta s_3) \right]$$

$$+ (n - n_1 - n_2 - n_3) \left[ \left(\gamma_0 z + \frac{\gamma_1}{2} z^2\right) \exp(\beta s_3) \right].$$

Taking the derivative of (4.3) with respect to  $\gamma_0, \gamma_1$  and  $\beta$  yields

$$\frac{-\partial \ell(\boldsymbol{\beta}; \xi)}{\partial \gamma_0} = -\sum_{i=1}^{n_1} \left[ \frac{1}{(\gamma_0 + \gamma_1 t_i)} - t_i \exp(\beta s_1) \right] \quad (4.4)$$

$$-\sum_{i=1}^{n_2} \left[ \frac{1}{(\gamma_0 + \gamma_1 x_i)} - x_i \exp(\beta s_2) \right]$$

$$-\sum_{i=1}^{n_3} \left[ \frac{1}{(\gamma_0 + \gamma_1 y_i)} - y_i \exp(\beta s_3) \right]$$

$$+ (n - n_1 - n_2 - n_3) [z \exp(\beta s_3)],$$



$$\begin{aligned}
\frac{-\partial \ell(\boldsymbol{\beta}; \boldsymbol{\xi})}{\partial \gamma_1} &= -\sum_{i=1}^{n_1} \left[ \frac{t_i}{(\gamma_0 + \gamma_1 t_i)} - \frac{t_i^2}{2} \exp(\beta s_1) \right] \\
&\quad -\sum_{i=1}^{n_2} \left[ \frac{x_i}{(\gamma_0 + \gamma_1 x_i)} - \frac{x_i^2}{2} \exp(\beta s_2) \right] \\
&\quad -\sum_{i=1}^{n_3} \left[ \frac{y_i}{(\gamma_0 + \gamma_1 y_i)} - \frac{y_i^2}{2} \exp(\beta s_3) \right] \\
&\quad + (n - n_1 - n_2 - n_3) \left[ \frac{z^2}{2} \exp(\beta s_3) \right],
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\frac{-\partial \ell(\boldsymbol{\beta}; \boldsymbol{\xi})}{\partial \beta} &= -n_1 s_1 + \sum_{i=1}^{n_1} \left[ s_1 \left( \gamma_0 t_i + \frac{\gamma_1}{2} t_i^2 \right) \exp(\beta s_1) \right] \\
&\quad -n_2 s_2 + \sum_{i=1}^{n_2} \left[ s_2 \left( \gamma_0 x_i + \frac{\gamma_1}{2} x_i^2 \right) \exp(\beta s_2) \right] \\
&\quad -n_3 s_3 + \sum_{i=1}^{n_3} \left[ s_3 \left( \gamma_0 y_i + \frac{\gamma_1}{2} y_i^2 \right) \exp(\beta s_3) \right] \\
&\quad + (n - n_1 - n_2 - n_3) \left[ s_3 \left( \gamma_0 z + \frac{\gamma_1}{2} z^2 \right) \exp(\beta s_3) \right].
\end{aligned} \tag{4.6}$$

The expectation of (4.4), (4.5), and (4.6) are

$$\begin{aligned}
E\left(\frac{-\partial \ell(\boldsymbol{\beta}; \boldsymbol{\xi})}{\partial \gamma_0}\right) &\approx -E(n_1) \int_0^{\tau_1} \left[ \frac{1}{(\gamma_0 + \gamma_1 t)} - t \exp(\beta s_1) \right] f_T(t; s_1) dt \\
&\quad -E(n_2) \int_{\tau_1}^{\tau_2} \left[ \frac{1}{(\gamma_0 + \gamma_1 x)} - x \exp(\beta s_2) \right] f_T(x; s_2) dt \\
&\quad -E(n_3) \int_{\tau_2}^c \left[ \frac{1}{(\gamma_0 + \gamma_1 y)} - y \exp(\beta s_3) \right] f_T(y; s_3) dt \\
&\quad + (n - E(n_1) - E(n_2) - E(n_3)) (z \exp(\beta s_3)),
\end{aligned}$$

$$\begin{aligned}
E\left(\frac{-\partial \ell(\boldsymbol{\beta}; \boldsymbol{\xi})}{\partial \gamma_1}\right) &\approx -E(n_1) \int_0^{\tau_1} \left[ \frac{t}{(\gamma_0 + \gamma_1 t)} - \frac{t^2}{2} \exp(\beta s_1) \right] f_T(t; s_1) dt \\
&\quad -E(n_2) \int_{\tau_1}^{\tau_2} \left[ \frac{x}{(\gamma_0 + \gamma_1 x)} - \frac{x^2}{2} \exp(\beta s_2) \right] f_T(x; s_2) dt \\
&\quad -E(n_3) \int_{\tau_2}^c \left[ \frac{y}{(\gamma_0 + \gamma_1 y)} - \frac{y^2}{2} \exp(\beta s_3) \right] f_T(y; s_3) dt \\
&\quad + (n - E(n_1) - E(n_2) - E(n_3)) \frac{z^2}{2} \exp(\beta s_3),
\end{aligned}$$

$$\begin{aligned}
E\left(\frac{-\partial\ell(\boldsymbol{\beta}, \xi)}{\partial\boldsymbol{\beta}}\right) &\approx -E(n_1) s_1 + E(n_1) \int_0^{\tau_1} \left(s_1 \left(\gamma_0 t + \frac{\gamma_1}{2} t^2\right) \exp(\beta s_1)\right) f_T(t; s_1) dt \\
&\quad -E(n_2) s_2 + E(n_2) \int_{\tau_1}^{\tau_2} \left(s_2 \left(\gamma_0 x + \frac{\gamma_1}{2} x^2\right) \exp(\beta s_2)\right) f_T(x; s_2) dt \\
&\quad -E(n_3) s_3 + E(n_3) \int_{\tau_2}^c \left(s_3 \left(\gamma_0 y + \frac{\gamma_1}{2} y^2\right) \exp(\beta s_3)\right) f_T(y; s_3) dt \\
&\quad + (n - E(n_1) - E(n_2) - E(n_3)) \left(s_3 \left(\gamma_0 z + \frac{\gamma_1}{2} z^2\right) \exp(\beta s_3)\right).
\end{aligned}$$

Note that we approximate the expectation by its linear term in Taylor's expectations. Let  $\boldsymbol{\beta}^* = [\gamma_0^*, \gamma_1^*, \beta^*]^T$  be the value of  $\boldsymbol{\beta}$  that minimizes (4.1), which is the solutions of  $\gamma_0, \gamma_1, \beta$  by setting all equations in (4.4), (4.5), (4.6) equal to 0. Suppose that the data is collected under test design,  $\xi$ , with sample size  $n$ . The experimenter fits Model  $M_F$  to the data by using MLE method and let  $\widehat{\boldsymbol{\beta}}$  denote the quasi-MLE of  $\boldsymbol{\beta}$ . By Theorem 3.2 of White (1982),  $\sqrt{n}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)$  is asymptotically normal with mean  $\mathbf{0}$  and covariance matrix  $\mathbf{C}(\boldsymbol{\alpha}; \boldsymbol{\beta})$  which is defined as below.

$$\mathbf{C}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = [\mathbf{A}(\boldsymbol{\alpha}; \boldsymbol{\beta})]^{-1} \times \mathbf{B}(\boldsymbol{\alpha}; \boldsymbol{\beta}) \times [\mathbf{A}(\boldsymbol{\alpha}; \boldsymbol{\beta})]^{-1}, \text{ where}$$

$$\mathbf{A}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left[ -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} \right) \right], \text{ and} \quad (4.7)$$

$$\mathbf{B}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \boldsymbol{\beta}} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \boldsymbol{\beta}^T} \right) \right]. \quad (4.8)$$

We decompose Matrices (4.7) and (4.8) as the following blocks:

$$\mathbf{A}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad (4.9)$$

and

$$\mathbf{B}(\boldsymbol{\alpha}; \boldsymbol{\beta}) = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}. \quad (4.10)$$

The elements of (4.9) are

$$\begin{aligned}
A_{11} &= -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0^2} \right) \\
&\approx E(n_1) \int_0^{\tau_1} \frac{1}{(\lambda_0(t))^2} f_T(t; s_1) dt + E(n_2) \int_{\tau_1}^{\tau_2} \frac{1}{(\lambda_0(x))^2} f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c \frac{1}{(\lambda_0(y))^2} f_T(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
A_{22} &= -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1^2} \right) \\
&\approx E(n_1) \int_0^{\tau_1} \frac{t^2}{(\lambda_0(t))^2} f_T(t; s_1) dt + E(n_2) \int_{\tau_1}^{\tau_2} \frac{x^2}{(\lambda_0(x))^2} f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c \frac{y^2}{(\lambda_0(y))^2} f_T(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
A_{33} &= -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \beta^2} \right) \\
&\approx E(n_1) \int_0^{\tau_1} \left( s_1^2 \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right) f_T(t; s_1) dt \\
&\quad + E(n_2) \int_{\tau_1}^{\tau_2} \left( s_2^2 \left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right) f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c \left( s_3^2 \left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right) f_T(y; s_3) dt \\
&\quad + (n - E(n_1) - E(n_2) - E(n_3)) \left( s_3^2 \left( \gamma_0 z + \frac{\gamma_1}{2} z^2 \right) \exp(\beta s_3) \right),
\end{aligned}$$

$$\begin{aligned}
A_{12} &= A_{21} = -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0 \partial \gamma_1} \right) \\
&\approx E(n_1) \int_0^{\tau_1} \frac{t}{(\lambda_0(t))^2} f_T(t; s_1) dt + E(n_2) \int_{\tau_1}^{\tau_2} \frac{x}{(\lambda_0(x))^2} f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c \frac{y}{(\lambda_0(y))^2} f_T(y; s_3) dt,
\end{aligned}$$

$$\begin{aligned}
A_{13} &= A_{31} = -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0 \partial \beta} \right) \\
&\approx E(n_1) \int_0^{\tau_1} s_1 t \exp(\beta s_1) f_T(t; s_1) dt + E(n_2) \int_{\tau_1}^{\tau_2} s_2 x \exp(\beta s_2) f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c s_3 y \exp(\beta s_3) f_T(y; s_3) dt + (n - E(n_1) - E(n_2) - E(n_3)) s_3 z \exp(\beta s_3),
\end{aligned}$$

and

$$\begin{aligned}
A_{23} &= A_{32} = -E_{M_T} \left( \frac{\partial^2 \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1 \partial \beta} \right) \\
&\approx E(n_1) \int_0^{\tau_1} s_1 \frac{t^2}{2} \exp(\beta s_1) f_T(t; s_1) dt \\
&\quad + E(n_2) \int_{\tau_1}^{\tau_2} s_2 \frac{x^2}{2} \exp(\beta s_2) f_T(x; s_2) dt \\
&\quad + E(n_3) \int_{\tau_2}^c s_3 \frac{y^2}{2} \exp(\beta s_3) f_T(y; s_3) dt \\
&\quad + (n - E(n_1) - E(n_2) - E(n_3)) s_3 \frac{z^2}{2} \exp(\beta s_3),
\end{aligned}$$

where

$$\begin{aligned} f_T(t; s) &= (\kappa_0 + \kappa_1 t + \kappa_2 t^2) \exp(\alpha s) \exp\left(-\left(\kappa_0 t + \frac{\kappa_1}{2} t^2 + \frac{\kappa_2}{3} t^3\right) \exp(\alpha s)\right), \text{ and} \\ F_T(t; s) &= 1 - \exp\left(-\left(\kappa_0 t + \frac{\kappa_1}{2} t^2 + \frac{\kappa_2}{3} t^3\right) \exp(\alpha s)\right). \end{aligned}$$

The elements of (4.10) are

$$B_{11} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right) \right]^2 + \text{Var}_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right),$$

$$B_{22} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right) \right]^2 + \text{Var}_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right),$$

$$B_{33} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right) \right]^2 + \text{Var}_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right),$$

$$B_{12} = B_{21} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right\},$$

$$B_{13} = B_{31} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\},$$

and

$$B_{23} = B_{32} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\},$$

where

$$E(n_1) = n \int_0^{\tau_1} f_T(t; s_1) dt,$$

$$E(n_2) = (n - E(n_1)) \int_{\tau_1}^{\tau_2} f_T(x; s_2) dt, \text{ and}$$

$$E(n_3) = (n - E(n_1) - E(n_2)) \int_{\tau_2}^c f_T(y; s_3) dt.$$

We assume the observed lifetimes  $t_i$  are independent. We let  $E_{M_T}(t_i) = \varpi_1$ ,  $E_{M_T}(x_i) = \varpi_2$ ,  $E_{M_T}(y_i) = \varpi_3$ ,  $E_{M_T}(t_i^2) = \theta_1$ ,  $E_{M_T}(x_i^2) = \theta_2$ ,  $E_{M_T}(y_i^2) = \theta_3$ .

We first compute  $B_{11}$ :

$$B_{11} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right) \right]^2 + \text{Var}_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right),$$

where,

$$\begin{aligned}
E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0} \right) &\approx E(n_1) \int_0^{\tau_1} \left( \frac{1}{\lambda_0(t)} - t \exp(\beta s_1) \right) f_T(t; s_1) dt \\
&+ E(n_2) \int_{\tau_1}^{\tau_2} \left( \frac{1}{\lambda_0(x)} - x \exp(\beta s_2) \right) f_T(x; s_2) dt \\
&+ E(n_3) \int_{\tau_2}^c \left( \frac{1}{\lambda_0(y)} - y \exp(\beta s_3) \right) f_T(y; s_3) dt \\
&- (n - E(n_1) - E(n_2) - E(n_3)) (z \exp(\beta s_3)), \text{ and} \\
Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0} \right) &= \sum_{i=1}^{n_1} Var_{M_T} \left( \frac{1}{\lambda_0(t_i)} - t_i \exp(\beta s_1) \right) \\
&+ \sum_{i=1}^{n_2} Var_{M_T} \left( \frac{1}{\lambda_0(x_i)} - x_i \exp(\beta s_2) \right) \\
&+ \sum_{i=1}^{n_3} Var_{M_T} \left( \frac{1}{\lambda_0(y_i)} - y_i \exp(\beta s_3) \right).
\end{aligned}$$

In general, the first linear term in Taylor approximation for  $f(t)$ , can be used to obtain the 1st order approximation of  $Var[f(t)]$ :

$$Var[f(t)] \approx [f'(E(t))]^2 Var(t).$$

In our case,  $f(t_i) = \frac{1}{\lambda_0(t_i)} = \frac{1}{\gamma_0 + \gamma_1 t_i}$ , then

$$Var_{M_T} \left( \frac{1}{\gamma_0 + \gamma_1 t_i} \right) \approx \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 E_{M_T}(t_i))^4} Var_{M_T}(t_i),$$

where

$$Var_{M_T}(t_i) = E_{M_T}(t_i^2) - [E_{M_T}(t_i)]^2 = \theta_1 - \varpi_1^2.$$

Thus,

$$Var_{M_T} \left( \frac{1}{\gamma_0 + \gamma_1 t_i} \right) \approx \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_1)^4} (\theta_1 - \varpi_1^2).$$

Then we have

$$\begin{aligned}
Var_{M_T} \left( \frac{1}{\lambda_0(t_i)} - t_i \exp(\beta s_1) \right) &\approx Var_{M_T} \left( \frac{1}{\gamma_0 + \gamma_1 t_i} \right) + Var_{M_T}(t_i \exp(\beta s_1)) \\
&\approx \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_1)^4} (\theta_1 - \varpi_1^2) + \exp(2\beta s_1) (\theta_1 - \varpi_1^2) \\
&= (\theta_1 - \varpi_1^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_1)^4} + \exp(2\beta s_1) \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Var_{M_T} \left( \frac{1}{\lambda_0(x_i)} - x_i \exp(\beta s_2) \right) &\approx (\theta_2 - \varpi_2^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_2)^4} + \exp(2\beta s_2) \right), \text{ and} \\
Var_{M_T} \left( \frac{1}{\lambda_0(y_i)} - y_i \exp(\beta s_3) \right) &\approx (\theta_3 - \varpi_3^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_3)^4} + \exp(2\beta s_3) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned} Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right) &\approx E(n_1) (\theta_1 - \varpi_1^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_1)^4} + \exp(2\beta s_1) \right) \\ &+ E(n_2) (\theta_2 - \varpi_2^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_2)^4} + \exp(2\beta s_2) \right) \\ &+ E(n_3) (\theta_3 - \varpi_3^2) \left( \frac{\gamma_1^2}{(\gamma_0 + \gamma_1 \varpi_3)^4} + \exp(2\beta s_3) \right). \end{aligned}$$

Secondly, we approximate  $B_{22}$ :

$$B_{22} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right) \right]^2 + Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right),$$

where

$$\begin{aligned} E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right) &\approx E(n_1) \int_0^{\tau_1} \left( \frac{t}{\lambda_0(t)} - \frac{t^2}{2} \exp(\beta s_1) \right) f_T(t; s_1) dt \\ &+ E(n_2) \int_{\tau_1}^{\tau_2} \left( \frac{x}{\lambda_0(x)} - \frac{x^2}{2} \exp(\beta s_2) \right) f_T(x; s_2) dt \\ &+ E(n_3) \int_{\tau_2}^c \left( \frac{y}{\lambda_0(y)} - \frac{y^2}{2} \exp(\beta s_3) \right) f_T(y; s_3) dt \\ &- (n - E(n_1) - E(n_2) - E(n_3)) \left( \frac{z^2}{2} \exp(\beta s_3) \right), \\ Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right) &= \sum_{i=1}^{n_1} Var_{M_T} \left( \frac{t_i}{\lambda_0(t_i)} - \frac{t_i^2}{2} \exp(\beta s_1) \right) \\ &+ \sum_{i=1}^{n_2} Var_{M_T} \left( \frac{x_i}{\lambda_0(x_i)} - \frac{x_i^2}{2} \exp(\beta s_2) \right) \\ &+ \sum_{i=1}^{n_3} Var_{M_T} \left( \frac{y_i}{\lambda_0(y_i)} - \frac{y_i^2}{2} \exp(\beta s_3) \right) \end{aligned}$$

Using the first order Taylor approximation, we obtain

$$Var_{M_T} \left( \frac{t_i}{\lambda_0(t_i)} \right) = Var_{M_T} \left( \frac{t_i}{\gamma_0 + \gamma_1 t_i} \right) \approx \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_1} + \frac{\varpi_1 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_1)^2} \right)^2 (\theta_1 - \varpi_1^2).$$

Thus,

$$\begin{aligned}
& Var_{M_T} \left( \frac{t_i}{\lambda_0(t_i)} - \frac{t_i^2}{2} \exp(\beta s_1) \right) \\
& \approx Var_{M_T} \left( \frac{t_i}{\lambda_0(t_i)} \right) + Var_{M_T} \left( \frac{t_i^2}{2} \exp(\beta s_1) \right) \\
& \approx \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_1} + \frac{\varpi_1 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_1)^2} \right)^2 (\theta_1 - \varpi_1^2) + \exp(2\beta s_1) \varpi_1^2 (\theta_1 - \varpi_1^2) \\
& = (\theta_1 - \varpi_1^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_1} + \frac{\varpi_1 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_1)^2} \right)^2 + \exp(2\beta s_1) \varpi_1^2 \right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Var_{M_T} \left( \frac{x_i}{\lambda_0(x_i)} - \frac{x_i^2}{2} \exp(\beta s_2) \right) & \approx (\theta_2 - \varpi_2^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_2} + \frac{\varpi_2 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_2)^2} \right)^2 + \exp(2\beta s_2) \varpi_2^2 \right], \\
Var_{M_T} \left( \frac{y_i}{\lambda_0(y_i)} - \frac{y_i^2}{2} \exp(\beta s_3) \right) & \approx (\theta_3 - \varpi_3^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_3} + \frac{\varpi_3 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_3)^2} \right)^2 + \exp(2\beta s_3) \varpi_3^2 \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_1} \right) & \approx E(n_1) (\theta_1 - \varpi_1^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_1} + \frac{\varpi_1 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_1)^2} \right)^2 + \exp(2\beta s_1) \varpi_1^2 \right] \\
& + E(n_2) (\theta_2 - \varpi_2^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_2} + \frac{\varpi_2 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_2)^2} \right)^2 + \exp(2\beta s_2) \varpi_2^2 \right] \\
& + E(n_3) (\theta_3 - \varpi_3^2) \left[ \left( \frac{1}{\gamma_0 + \gamma_1 \varpi_3} + \frac{\varpi_3 \gamma_1}{(\gamma_0 + \gamma_1 \varpi_3)^2} \right)^2 + \exp(2\beta s_3) \varpi_3^2 \right].
\end{aligned}$$

Thirdly, we approximate  $B_{33}$ :

$$B_{33} = E_{M_T} \left\{ \left[ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right]^2 \right\} = \left[ E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right) \right]^2 + Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right),$$

where

$$\begin{aligned}
E_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right) &\approx E(n_1) s_1 - E(n_1) \int_0^{\tau_1} \left( s_1 \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 \right) \exp(\beta s_1) \right) f_T(t; s_1) dt \\
&\quad + E(n_2) s_2 - E(n_2) \int_{\tau_1}^{\tau_2} \left( s_2 \left( \gamma_0 x + \frac{\gamma_1}{2} x^2 \right) \exp(\beta s_2) \right) f_T(x; s_2) dt \\
&\quad + E(n_3) s_3 - E(n_3) \int_{\tau_2}^c \left( s_3 \left( \gamma_0 y + \frac{\gamma_1}{2} y^2 \right) \exp(\beta s_3) \right) f_T(y; s_3) dt \\
&\quad - (n - E(n_1) - E(n_2) - E(n_3)) \left( s_3 \left( \gamma_0 z + \frac{\gamma_1}{2} z^2 \right) \exp(\beta s_3) \right), \\
Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right) &= Var_{M_T}(n_1 s_1) + \sum_{i=1}^{n_1} Var_{M_T} \left( s_1 \left( \gamma_0 t_i + \frac{\gamma_1}{2} t_i^2 \right) \exp(\beta s_1) \right) \\
&\quad + Var_{M_T}(n_2 s_2) + \sum_{i=1}^{n_2} Var_{M_T} \left( s_2 \left( \gamma_0 x_i + \frac{\gamma_1}{2} x_i^2 \right) \exp(\beta s_2) \right) \\
&\quad + Var_{M_T}(n_3 s_3) + \sum_{i=1}^{n_3} Var_{M_T} \left( s_3 \left( \gamma_0 y_i + \frac{\gamma_1}{2} y_i^2 \right) \exp(\beta s_3) \right).
\end{aligned}$$

Using the first order Taylor approximation, we obtain

$$\begin{aligned}
Var_{M_T} \left( s_1 \left( \gamma_0 t_i + \frac{\gamma_1}{2} t_i^2 \right) \exp(\beta s_1) \right) &= s_1^2 \exp(2\beta s_1) Var_{M_T} \left( \gamma_0 t_i + \frac{\gamma_1}{2} t_i^2 \right) \\
&\approx s_1^2 \exp(2\beta s_1) (\gamma_0^2 + \gamma_1^2 \varpi_1^2) (\theta_1 - \varpi_1^2).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
Var_{M_T} \left( s_2 \left( \gamma_0 x_i + \frac{\gamma_1}{2} x_i^2 \right) \exp(\beta s_2) \right) &\approx s_2^2 \exp(2\beta s_2) (\gamma_0^2 + \gamma_1^2 \varpi_2^2) (\theta_2 - \varpi_2^2), \text{ and} \\
Var_{M_T} \left( s_3 \left( \gamma_0 y_i + \frac{\gamma_1}{2} y_i^2 \right) \exp(\beta s_3) \right) &\approx s_3^2 \exp(2\beta s_3) (\gamma_0^2 + \gamma_1^2 \varpi_3^2) (\theta_3 - \varpi_3^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var_{M_T} \left( \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \beta} \right) &\approx Var_{M_T}(n_1 s_1) + E(n_1) s_1^2 \exp(2\beta s_1) (\gamma_0^2 + \gamma_1^2 \varpi_1^2) (\theta_1 - \varpi_1^2) \\
&\quad + Var_{M_T}(n_2 s_2) + E(n_2) s_2^2 \exp(2\beta s_2) (\gamma_0^2 + \gamma_1^2 \varpi_2^2) (\theta_2 - \varpi_2^2) \\
&\quad + Var_{M_T}(n_3 s_3) + E(n_3) s_3^2 \exp(2\beta s_3) (\gamma_0^2 + \gamma_1^2 \varpi_3^2) (\theta_3 - \varpi_3^2).
\end{aligned}$$

For the cross product terms, we have

$$\begin{aligned}
B_{12} &= B_{21} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_1} \right\} \\
&= E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_1} \right\} + Cov_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0}, \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_1} \right\} \\
&\approx E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_0} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \boldsymbol{\xi})}{\partial \gamma_1} \right\},
\end{aligned}$$



$$\begin{aligned}
B_{13} &= B_{31} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} \\
&= E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} + Cov_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} \\
&\approx E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_0} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
B_{23} &= B_{32} = E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} \\
&= E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} + Cov_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \cdot \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\} \\
&\approx E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \gamma_1} \right\} E_{M_T} \left\{ \frac{\partial \ell(\boldsymbol{\beta}, \xi)}{\partial \beta} \right\}.
\end{aligned}$$

### 4.3 Optimality criteria

In this chapter, we consider two quantities of interest: (1) the hazard rate over a given time period at the normal design stress level; and (2) the reliability at a given time and the normal design stress level.

#### 4.3.1 Hazard function over time $T$

In this section, we determine the optimal stress-changing times  $\tau_1$  and  $\tau_2$  in order to minimize the asymptotic squared bias (ABIAS<sup>2</sup>), asymptotic variance (AVAR), and asymptotic mean squared error (AMSE) of average of the MLE of a hazard function over a specific period of time  $T$ , under normal design condition,  $s_D$ . For a given that, the MLE estimator of the hazard rate at normal design stress level  $s_D$  can be obtained by:

$$\widehat{\lambda}_{M_F}(t; s_D) = (\widehat{\gamma}_0 + \widehat{\gamma}_1 t) \exp(\widehat{\beta} s_D). \quad (4.11)$$

The ABIAS of the average of (4.11) over  $T$  is

$$\begin{aligned}
\text{ABIAS} \left[ \int_0^T \left( \widehat{\lambda}_{M_F}(t; s_D) | M_T \right) dt \right] &= \int_0^T \left( E_{M_T} \left[ \widehat{\lambda}_{M_F}(t; s_D) \right] - \lambda_{M_T}(t; s_D) \right) dt \\
&= \int_0^T \left( E_{M_T} \left[ (\widehat{\gamma}_0 + \widehat{\gamma}_1 t) \exp(\widehat{\beta} s_D) \right] \right. \\
&\quad \left. - (\kappa_0 + \kappa_1 t + \kappa_2 t^2) \exp(\alpha s_D) \right) dt \\
&\approx \int_0^T \left( \begin{array}{c} (\gamma_0^* + \gamma_1^* t) \exp(\beta^* s_D) \\ - (\kappa_0 + \kappa_1 t + \kappa_2 t^2) \exp(\alpha s_D) \end{array} \right) dt,
\end{aligned}$$

the AVAR of the average of (4.11) over  $T$  is

$$\begin{aligned}
&\text{AVAR} \left[ \int_0^T \left( \widehat{\lambda}_{M_F}(t; s_D) | M_T \right) dt \right] \\
&= \int_0^T \left[ \begin{array}{ccc} \frac{\partial \widehat{\lambda}}{\partial \gamma_0} & \frac{\partial \widehat{\lambda}}{\partial \gamma_1} & \frac{\partial \widehat{\lambda}}{\partial \beta} \end{array} \right] \Big|_{\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*} C(\boldsymbol{\alpha} : \boldsymbol{\beta} = \boldsymbol{\beta}^*) \left[ \begin{array}{ccc} \frac{\partial \widehat{\lambda}}{\partial \gamma_0} & \frac{\partial \widehat{\lambda}}{\partial \gamma_1} & \frac{\partial \widehat{\lambda}}{\partial \beta} \end{array} \right]^T \Big|_{\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^*} dt, \text{ and}
\end{aligned}$$

the AMSE of the average of (4.11) over  $T$  is

$$\begin{aligned} & \text{AMSE} \left[ \int_0^T \left( \widehat{\lambda}_{M_F}(t; s_D) | M_T \right) dt \right] \\ &= \left( \text{ABIAS} \left[ \int_0^T \left( \widehat{\lambda}_{M_F}(t; s_D) | M_T \right) dt \right] \right)^2 + \text{AVAR} \left[ \int_0^T \left( \widehat{\lambda}_{M_F}(t; s_D) | M_T \right) dt \right]. \end{aligned}$$

### 4.3.2 Reliability at time $t$

In this section, we construct the optimal designs for the stress-changing times  $\tau_1$  and  $\tau_2$  so that ABIAS<sup>2</sup>, AVAR, and AMSE of the MLE of the reliability, at a predetermined time  $t$  under normal design conditions can be minimized. According to invariance property of MLE, the MLE of the reliability at time  $t$  under normal design stress level  $s_D$  can be obtained by

$$\widehat{R}_{M_F}(t; s_D) = \exp \left[ - \left( \widehat{\gamma}_0 t + \frac{\widehat{\gamma}_1}{2} t^2 \right) \exp \left( \widehat{\beta} s_D \right) \right].$$

In order to compute the ABIAS<sup>2</sup>, AVAR and AMSE in a convenient way, we transform the reliability estimator into a simpler function of  $\widehat{\gamma}_0, \widehat{\gamma}_1$  and  $\widehat{\beta}$ , which is:

$$\widehat{N}_{M_F}(t; s_D) = \log \left( -\log \left[ \widehat{R}_{M_F}(t; s_D) \right] \right) = \log \left( \widehat{\gamma}_0 t + \frac{\widehat{\gamma}_1}{2} t^2 \right) + \widehat{\beta} s_D \quad (4.12)$$

The ABIAS of (4.12) is

$$\begin{aligned} \text{ABIAS} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right] &= E_{M_T} \left[ \widehat{N}_{M_F}(t; s_D) \right] - N_{M_T}(t; s_D) \\ &= E_{M_T} \left[ \log \left( \widehat{\gamma}_0 t + \frac{\widehat{\gamma}_1}{2} t^2 \right) + \widehat{\beta} s_D \right] \\ &\quad - \left[ \log \left( \kappa_0 t + \frac{\kappa_1}{2} t^2 + \frac{\kappa_2}{3} t^3 \right) + (\alpha s_D) \right] \\ &= \log \left( \gamma_0^* t + \frac{\gamma_1^*}{2} t^2 \right) + (\beta^* s_D) \\ &\quad - \log \left( \kappa_0 t + \frac{\kappa_1}{2} t^2 + \frac{\kappa_2}{3} t^3 \right) - (\alpha s_D). \end{aligned}$$

The ABIAS<sup>2</sup> of (4.12) is

$$\left( \text{ABIAS} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right] \right)^2,$$

the AVAR of (4.12) is

$$\begin{aligned} & \text{AVAR} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right] \\ &= \left[ \begin{array}{ccc} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{array} \right] \Big|_{\widehat{\beta}=\beta^*} C(\boldsymbol{\alpha} : \boldsymbol{\beta} = \boldsymbol{\beta}^*) \left[ \begin{array}{ccc} \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_0} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\gamma}_1} & \frac{\partial \widehat{\lambda}}{\partial \widehat{\beta}} \end{array} \right]^T \Big|_{\widehat{\beta}=\beta^*}, \text{ and} \end{aligned}$$

the AMSE of (4.12) is

$$\begin{aligned} & \text{AMSE} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right] \\ &= \left( \text{ABIAS} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right] \right)^2 + \text{AVAR} \left[ \widehat{N}_{M_F}(t; s_D) | M_T \right]. \end{aligned}$$

## 4.4 Robust designs

Under the constraints of minimum number of failures at each stress level, we optimally allocate stress-changing times and middle stress level so that the ABIAS<sup>2</sup>, AVAR and AMSE of (4.11) and (4.12) can be minimized, respectively. Namely, six different optimization criteria, listed in Section 4.3, are used in order to obtain the optimal stress-changing times and optimal middle stress level. As the same as the constraints defined in Section 2.4, we have three constraints (2.9), (2.10) and (2.11).

We denote  $B\xi_{C_k}^3$  as the optimal design obtained in benchmark under the given constraint  $C_k$ , where  $k = 1, \dots, 5$ . And  $R\xi_{C_k}^{3(i)}$  is the robust design obtained for three-step-stress ALT under the given constraint  $C_k$ , where  $k = 1, \dots, 5$ ;  $i = 1, 2, 3, 4$ ,  $i$  referring to the different quantity of interest and situation of  $s_2$  ( $i = 1$  when the quantity of interest is the hazard rate and  $s_2$  is fixed at  $197.5^\circ C$ ,  $i = 2$  when the quantity of interest is the hazard rate and  $s_2$  is unfixed,  $i = 3$  when the quantity of interest is the transformed reliability and  $s_2$  is fixed at  $197.5^\circ C$ , and  $i = 4$  when the quantity of interest is the transformed reliability and  $s_2$  is unfixed).

Besides, we also define the efficiencies of  $R\xi_{C_k}^{3(i)}$  relative to  $B\xi_{C_k}^3$  in terms of ABIAS<sup>2</sup>, AVAR and AMSE are

$$\begin{aligned} \text{eff}_B(3(i), 3) &= \frac{\text{The value of ABIAS}^2 \text{ under } B\xi_{C_k}^3}{\text{The value of ABIAS}^2 \text{ under } R\xi_{C_k}^{3(i)}}, \\ \text{eff}_V(3(i), 3) &= \frac{\text{The value of AVAR under } B\xi_{C_k}^3}{\text{The value of AVAR under } R\xi_{C_k}^{3(i)}}, \\ \text{eff}_M(3(i), 3) &= \frac{\text{The value of AMSE under } B\xi_{C_k}^3}{\text{The value of AMSE under } R\xi_{C_k}^{3(i)}}, \end{aligned}$$

and the efficiencies of  $R\xi_{C_k}^{3(i)}$  relative to  $R\xi_{C_k}^{3(j)}$  in terms of ABIAS<sup>2</sup>, AVAR and AMSE are

$$\begin{aligned} \text{eff}_B(3(i), 3(j)) &= \frac{\text{The value of ABIAS}^2 \text{ under } R\xi_{C_k}^{3(j)}}{\text{The value of ABIAS}^2 \text{ under } R\xi_{C_k}^{3(i)}}, \\ \text{eff}_V(3(i), 3(j)) &= \frac{\text{The value of AVAR under } R\xi_{C_k}^{3(j)}}{\text{The value of AVAR under } R\xi_{C_k}^{3(i)}}, \\ \text{eff}_M(3(i), 3(j)) &= \frac{\text{The value of AMSE under } R\xi_{C_k}^{3(j)}}{\text{The value of AMSE under } R\xi_{C_k}^{3(i)}}, \end{aligned}$$

where  $i = 1, 2, 3, 4$  and  $j = 1, 2, 3, 4$ .

### 4.4.1 Benchmark

We revisit the example presented in Subsection 3.4.1. The initial values for the model parameters  $\kappa_0 = 0.0001$ ,  $\kappa_1 = 0.5$ ,  $\kappa_2 = 0.0015$ ,  $\alpha = -3800$  are applied. All other values of the parameters in example remain the same. The optimal criteria are to minimize the ABIAS<sup>2</sup>, AVAR and AMSE of the MLE of hazard function over a specific period of time  $T$  under normal design conditions.

We have found  $(\gamma_0^*, \gamma_1^*)$  by minimizing the distance between quadratic baseline hazard function and linear baseline hazard function over the whole testing period,

$$(\gamma_0^*, \gamma_1^*) = \underset{\text{over } (\gamma_0, \gamma_1)}{\arg \min} \int_0^{300} [(\kappa_0 + \kappa_1 t + \kappa_2 t^2) - (\gamma_0 + \gamma_1 t)] dt. \quad (4.13)$$

After solving (4.13), we get

$$\begin{aligned} (\gamma_0^*, \gamma_1^*) &= \begin{pmatrix} \widehat{\gamma}_0 & \widehat{\gamma}_1 \end{pmatrix} \\ &= \begin{pmatrix} 0.7947 & 0.7947 \end{pmatrix} \end{aligned}$$

We have obtained the optimal stress-changing times by minimizing each optimization criterion when fixed  $s_2$  at 197.5 and under five different constraints. The results are displayed in Table 4.1.

Table 4.1 The optimal stress-changing times

Design	$\tau_1$	$\tau_2$
$B_{\xi C_1}^3$	160	201
$B_{\xi C_2}^3$	161	189
$B_{\xi C_3}^3$	148	198
$B_{\xi C_4}^3$	147	177
$B_{\xi C_5}^3$	177	203

The parameters of linear baseline have been found in order to minimize the distance between quadratic baseline hazard function and linear baseline hazard function over the whole testing period. Thus, the results listed in Table 4.1 will be used as a benchmark.

### 4.4.2 Robust designs when the quantity of interest is the hazard rate and the middle stress level is fixed

For this example, we are trying to find the MLE of  $\beta^* = [\gamma_0, \gamma_1, \beta]^T$ . We set the derivatives in (4.4), (4.5), and (4.6) to be zero, and solve for  $\gamma_0, \gamma_1, \beta$ . This is a constrained nonlinear problem. We solve it in an iterative way. First, we fix  $\beta$  at  $-3800$ , and search for  $\gamma_0^*$  and  $\gamma_1^*$  in positive ranges of  $\gamma_0$  and  $\gamma_1$  within their 95% confidence interval in order to maximize the log-likelihood function. Second, with  $\gamma_0^*$  and  $\gamma_1^*$  obtained from the first step, we update our estimate of  $\beta$ . By iterating these two steps, we obtain  $\beta^* = [\gamma_0^*, \gamma_1^*, \beta^*]^T$ . Finally, we can obtain the optimal  $\tau_1$  and  $\tau_2$  based on the most updated estimates for  $\beta^*$ . Our designs, the final estimate  $\beta^*$ , the efficiencies in terms of ABIAS<sup>2</sup>, AVAR and AMSE are presented in Tables 4.2-4.6. We

note that the maximums of efficiencies in terms of ABIAS<sup>2</sup>, AVAR and AMSE are marked with a "\*" in the following tables.

Table 4.2 Global searching results for Design  $R_{\xi_{C_1}^{3(1)}}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
90	131	0	0.04	-3730	0.9829	1.7375	0.9829
100	141	0	0.04	-3730	0.9829	1.4218	0.9829
110	151	0	0.04	-3730	0.9829	1.2263	0.9829
120	161	0	0.05	-3730	0.9832	1.1394	0.9832
130	171	0	0.06	-3730	0.9834	1.1622	0.9834
140	181	0	0.07	-3730	0.9836	1.3048	0.9836
150	191	0	0.09	-3730	0.9841	1.5997	0.9841
160	201	0	0.1	-3730	*0.9843	*2.0758	*0.9843

Table 4.3 Global searching results for Design  $R_{\xi_{C_2}^{3(1)}}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
91	119	0	0.04	-3730	0.9829	*2.9840	0.9829
101	129	0	0.04	-3730	0.9829	2.2759	0.9829
111	139	0	0.04	-3730	0.9829	1.8029	0.9829
121	149	0	0.05	-3730	0.9832	1.5358	0.9832
131	159	0	0.06	-3730	0.9834	1.4554	0.9834
141	169	0	0.07	-3730	0.9836	1.5526	0.9836
151	179	0	0.08	-3730	0.9838	1.8416	0.9838
161	189	0	0.11	-3730	*0.9845	2.3898	*0.9845

Table 4.4 Global searching results for Design  $R_{\xi_{C_3}^{3(1)}}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
78	128	0	0.04	-3730	0.9829	*3.6900	0.9829
88	138	0	0.04	-3730	0.9829	3.1390	0.9829
98	148	0	0.04	-3730	0.9829	2.7484	0.9829
108	158	0	0.05	-3730	0.9832	2.5004	0.9832
118	168	0	0.05	-3730	0.9832	2.4037	0.9832
128	178	0	0.06	-3730	0.9834	2.4718	0.9834
138	188	0	0.07	-3730	0.9836	2.7414	0.9836
148	198	0	0.09	-3730	*0.9841	3.2853	*0.9841

Table 4.5 Global searching results for Design  $R_{\xi_{C_4}}^{3(1)}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
77	107	0	0.04	-3730	0.9829	*12.4564	0.9829
87	117	0	0.04	-3730	0.9829	9.4775	0.9829
97	127	0	0.04	-3730	0.9829	7.2185	0.9829
107	137	0	0.04	-3730	0.9829	5.6636	0.9829
117	147	0	0.05	-3730	0.9832	4.7254	0.9832
127	157	0	0.05	-3730	0.9832	4.3456	0.9832
137	167	0	0.06	-3730	0.9834	4.4814	0.9834
147	177	0	0.08	-3730	*0.9838	5.1836	*0.9838

Table 4.6 Global searching results for Design  $R_{\xi_{C_5}}^{3(1)}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
87	113	0	0.04	-3730	0.9829	1.5123	0.9829
97	123	0	0.04	-3730	0.9829	1.1428	0.9829
107	133	0	0.04	-3730	0.9829	0.8812	0.9829
117	143	0	0.05	-3730	0.9832	0.7164	0.9832
127	153	0	0.05	-3730	0.9832	0.6393	0.9832
137	163	0	0.06	-3730	0.9834	0.6401	0.9834
147	173	0	0.08	-3730	0.9838	0.7231	0.9838
157	183	0	0.1	-3730	0.9843	0.8998	0.9843
167	193	0	0.12	-3730	0.9847	1.1931	0.9847
177	203	0	0.15	-3730	*0.9854	*1.6514	*0.9854

From tables above, we have found out that the maximum efficiencies in terms of ABIAS<sup>2</sup>, AVAR and AMSE occur when first and second stress-changing times both often appear at upper bound of the by given constraints except the efficiency in terms of AVAR when the constraints are  $C_2$ ,  $C_3$  and  $C_4$ . For the constraint plans,  $C_2$ ,  $C_3$  and  $C_4$ , the efficiency in terms of AVAR occurs when first and second stress-changing times are at the lower bound of the design constraint.

#### 4.4.3 Robust designs when the quantity of interest is the hazard rate and the middle stress level is not fixed

We continue using the example in Subsection 4.4.2. However, this time we remove the restriction on  $s_2$  and search for  $s_2$  within a range of (155, 240). Namely, we search for the optimal  $\tau_1$ ,  $\tau_2$  and  $s_2$  simultaneously under test plan  $C_1$ . The optimal criteria are still considered as to minimize the ABIAS<sup>2</sup>, AVAR, and AMSE of the MLE of hazard function over  $T = 10$  years under normal design conditions. All other values of parameters remain the same. We adopt a sequential design approach this time. The algorithm for choosing the optimal sequential stress-changing times and middle stress level can be described as follows:

- (1). With the initial value of  $\tau_1^{(0)}, \tau_2^{(0)}$  and  $s_2^{(0)}$ , find the initial estimates of hazard rate parameters  $\beta$ . Call these initial estimates  $\beta^{(0)}$ .
- (2). Using these initial estimates, evaluate the  $ABIAS^2$ ,  $AVAR$  and  $AMSE$ .
- (3). Remove the restriction on  $MNF_i$  and choose a new middle stress level  $s_2^*$  by minimizing the  $AMSE$  with respect to  $s_2$ .
- (4). With  $s_2^*$ , compute the new estimates of  $\beta^*$  by minimizing the  $AMSE$  with respect to  $\beta$ .
- (5). Update the parameter estimates to  $\beta^*$ , choose the stress-changing times  $\tau_1^*$  and  $\tau_2^*$  by minimizing the  $AMSE$  with the constraint  $C_1$ .
- (6). Update the stress-changing time estimates to  $\tau_1^*$  and  $\tau_2^*$ , then the next middle stress level  $s_2^*$  based on the new set of estimates can be derived.
- (7). Iterate the previous steps, until either  $\beta^*$  or  $\tau_1^*, \tau_2^*$  stays unchanged.

We apply the sequential approach to the example, the result is shown in the table below with initial values of  $\tau_1^{(0)} = 160, \tau_2^{(0)} = 201, \beta^{(0)} = [0, 0.1, -3730]^T, s_2^{(0)} = 197.5, T = 10 \text{ years} = 87600 \text{ hours}$ .

Table 4.7 The efficiencies of Design  $R_{\xi_{C_1}}^{3(2)}$  relative to  $R_{\xi_{C_1}}^{3(1)}$

step	$s_2$	$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$ $\widehat{\gamma}_1$ $\widehat{\beta}$	$eff_B(3(2), 3(1))$	$eff_V(3(2), 3(1))$	$eff_M(3(2), 3(1))$	Expected failure numbers
1	240	160	201	0 0.1 -3730	1.0000	1.0463	1.0000	123,25,5
2	240	160	201	0 0.1 -3730	1.0000	1.0463	1.0000	123,25,5
3	240	158	187	0 0.1 -3730	1.0000	0.9147	1.0000	120,20,10
4	155	158	187	0 0.1 -3730	1.0000	1.0073	1.0000	123,11,19
5	155	158	187	0 0.11 -3730	1.0003	1.0135	1.0002	123,11,19
6	155	160	218	0 0.11 -3730	1.0003	1.0350	1.0002	120,20,10

From this Table 4.7, we observe that the efficiencies in terms of  $ABIAS^2$  and  $AMSE$  are not change much as the design changes. We have gained by 0.03% and 0.02% in terms of  $ABIAS^2$  and  $AMSE$ . However, the efficiency in terms of  $AVAR$  is 3.5%.

#### 4.4.4 Robust designs when the quantity of interest is the transformed reliability and the middle stress level is fixed

We continue the example in Subsection 4.4.3. Assuming the quantity of experimenter's interest is the reliability at the end of warranty time under normal design conditions. We are interested in designing optimal stress-changing times so that the estimation of such reliability can be the most accurate. Therefore, our optimal criterion now is changed to minimize the  $ABIAS^2$ ,  $AVAR$ , and  $AMSE$  of the MLE of transformed reliability at a predetermined time, say one year, under normal design conditions. We focus on constraint plan  $C_1$ . Our resulting stress-changing times, the final estimates of model parameters, and the efficiencies relative to the first resulting design in terms of  $ABIAS^2$ ,  $AVAR$  and  $AMSE$  are listed in Table 4.8.

Table 4.8 Global searching results for Design  $R_{\xi_{C_1}}^{3(3)}$

$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{\beta}$	$eff_B$	$eff_V$	$eff_M$
90	131	0	0.04	-3730	1.0000	1.0000	1.0000
100	141	0	0.04	-3730	1.0000	0.8183	0.8185
110	151	0	0.04	-3730	1.0000	0.7057	0.7060
120	161	0	0.05	-3730	1.2253	1.0246	1.0247
130	171	0	0.06	-3730	1.4712	1.5049	1.5049
140	181	0	0.07	-3730	1.1840	1.5280	1.5275
150	191	0	0.09	-3730	1.6146	3.0969	3.0937
160	201	0	0.1	-3730	*1.8682	*4.9613	*4.9520

The figures of  $ABIAS^2$ ,  $AVAR$ , and  $AMSE$  of the MLE of reliability at one year under constraint plan  $C_1$  versus  $(\tau_1, \tau_2)$  are presented in Figures 4.1, 4.2, and 4.3 respectively.



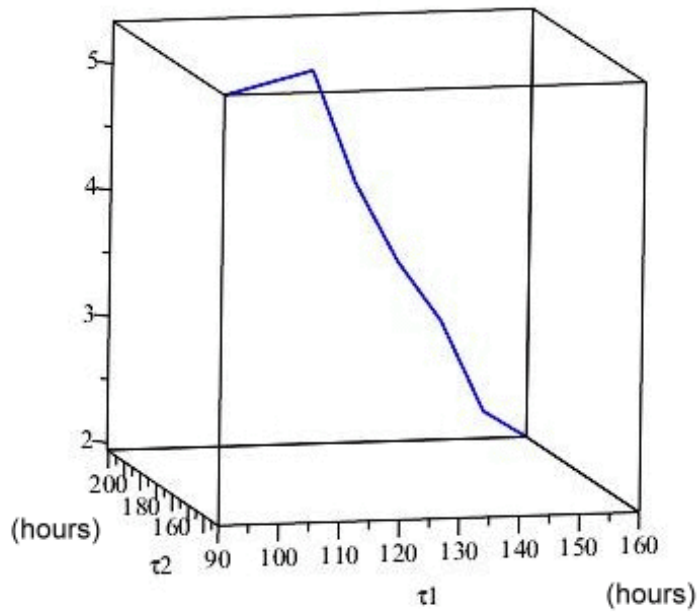


Figure 4.1: ABIAS<sup>2</sup> of the MLE of reliability at one year

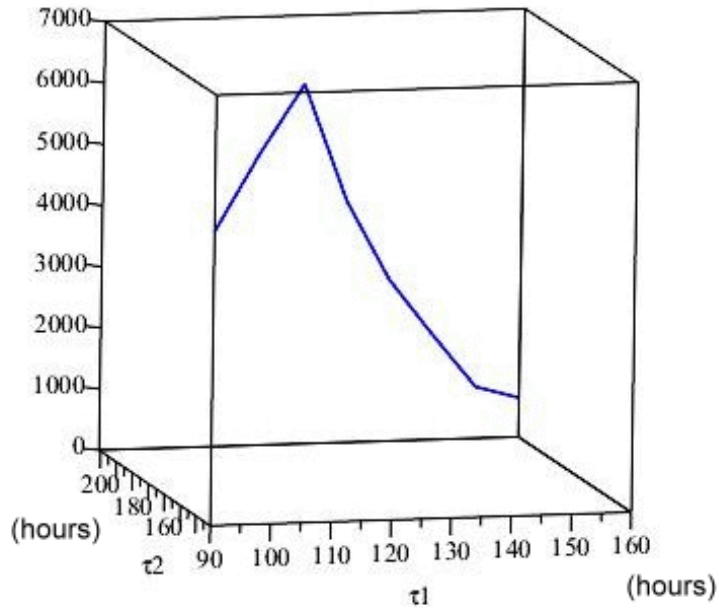


Figure 4.2: AVAR of the MLE of reliability at one year

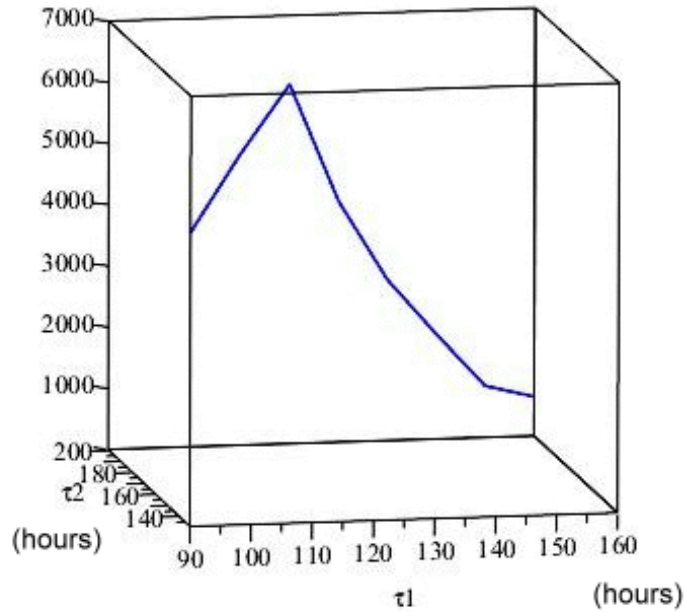


Figure 4.3: AMSE of the MLE of reliability at one year

From the figures above, we indicate that the  $ABIAS^2$ ,  $AVAR$  and  $AMSE$  are all minimized when the stress-changing times are around  $(160, 201)$ .

#### 4.4.5 Robust designs when the quantity of interest is the transformed reliability and the middle stress level is not fixed

We continue the example in Subsection 4.4.3 with both the values of parameters and the algorithm being unchanged. However, the optimal criteria are changed to minimize the  $ABIAS^2$ ,  $AVAR$ , and  $AMSE$  of the MLE of a transformed reliability at a predetermined time ( $t = 1 \text{ year}$ ) under normal design conditions. We focus on constraint plan  $C_1$ . The computation results are shown in the Table 4.9, with initial values of  $\tau_1^{(0)} = 160$ ,  $\tau_2^{(0)} = 201$ ,  $\beta^{(0)} = [0, 0.1, -3730]^T$ ,  $s_2^{(0)} = 197.5$ ,  $t = 1 \text{ year} = 8760 \text{ hours}$ .

Table 4.9 The efficiencies of Design  $R_{\xi_{C_1}}^{3(4)}$  relative to  $R_{\xi_{C_1}}^{3(3)}$

step	$s_2$	$\tau_1$	$\tau_2$	$\widehat{\gamma}_0$ $\widehat{\gamma}_1$ $\widehat{\beta}$	$eff_B(3(4), 3(3))$	$eff_V(3(4), 3(3))$	$eff_M(3(4), 3(3))$	Expected failure numbers
1	240	160	201	0 0.1 -3730	1.0000	1.0464	1.0462	123,25,5
2	240	160	201	0 0.1 -3730	1.0000	1.0464	1.0462	123,25,5
3	240	158	187	0 0.1 -3730	1.0000	0.9147	0.9160	120,20,10
4	155	158	187	0 0.1 -3730	1.0000	1.0073	1.0073	123,11,19
5	155	158	187	0 0.11 -3730	1.1523	1.2264	1.2261	123,11,19
6	155	160	218	0 0.11 -3730	1.1523	1.2524	1.2520	120,20,10

From this Table 4.9, we have gained efficiency by 15%, 25% and 25% in terms of ABIAS<sup>2</sup>, AVAR and AMSE respectively.

## 4.5 Discussion

In this chapter, robust designs are derived against possible departure from the linear baseline hazard function model. Optimal stress-changing times are obtained by minimizing the ABIAS<sup>2</sup>, AVAR and AMSE of each of the two quantities (hazard rate and transformed reliability). An iteration algorithm is described for sequentially determining the optimal stress-changing times and middle stress level.

### 4.5.1 On the results of robust designs when the quantity of interest is the hazard rate

Using a practical example, five constrained optimal designs have constructed for demonstrating our proposed approach. We note that the  $\widehat{\gamma}_0$  and  $\widehat{\beta}$  remain relatively unchanged constant as the stress-changing times vary for the example discussed. The value of  $\widehat{\gamma}_1$  is increasing as the stress-changing times increase, and the higher the value of  $\widehat{\gamma}_1$  the smaller the ABIAS<sup>2</sup>. The efficiencies of  $R_{\xi_{C_i}}^{3(1)}$  relative to  $R_{\xi_{C_i}}^{3(3)}$  in terms of ABIAS<sup>2</sup>, AVAR and AMSE are listed in Table 4.10.

Table 4.10 The efficiency of  $R\xi_{C_i}^{3(1)}$  relative to  $B\xi_{C_i}^3$

	$eff_B(3(1), 3)$	$eff_V(3(1), 3)$	$eff_M(3(1), 3)$
$C_1$	0.9843	2.0758	0.9843
$C_2$	0.9845	2.9840	0.9845
$C_3$	0.9841	3.6900	0.9841
$C_4$	0.9838	12.4564	0.9838
$C_5$	0.9854	1.6514	0.9854

In terms of AVAR, the average efficiency of our robust designs is as 4.57 times as that of their corresponding benchmarks. The minimum gain is 65% and some of our designs can be at maximum of 11.46 times more efficient than the benchmark. Furthermore, we compute the efficiencies of optimal design with optimal  $s_2$ ,  $R\xi_{C_1}^{3(2)}$ , relative to  $B\xi_{C_1}^3$  and the optimal design with  $s_2$  been fixed at  $197.5^\circ C$ ,  $R\xi_{C_1}^{3(1)}$ . The relative efficiencies in terms of ABIAS<sup>2</sup>, AVAR and AMSE are displayed in Table 4.11.

Table 4.11 The relative efficiency of  $R\xi_{C_1}^{3(2)}$

	$eff_B$	$eff_V$	$eff_M$
$R\xi_{C_1}^{3(2)}$ relative to $B\xi_{C_1}^3$	0.9845	2.1485	0.9845
$R\xi_{C_1}^{3(2)}$ relative to $R\xi_{C_1}^{3(1)}$	1.0003	1.0350	1.0002

Simultaneously finding optimal stress-changing times and optimal middle stress level has helped in reducing the AVAR for three-step-stress ALT. We have gained 115% in terms of AVAR respect to  $B\xi_{C_1}^3$ , however, the ABIAS<sup>2</sup> and AMSE seen not being corrected. Compare to  $R\xi_{C_1}^{3(1)}$ , we have gained efficiency by 0.03%, 3.5% and 0.02% in terms of ABIAS<sup>2</sup>, AVAR and AMSE respectively.

#### 4.5.2 On the results of robust designs when the quantity of interest is the transformed reliability

In Subsections 4.4.4 and 4.4.5, we have observed that the obtained optimal stress-changing times when the constraint plan is  $C_1$  are the same as the results in Subsections 4.4.2 and 4.4.3. The  $\widehat{\gamma}_0$  and  $\widehat{\beta}$  keep the same when the stress-changing times vary. The value of  $\widehat{\gamma}_1$  is increasing as the stress-changing times increase, and the higher value of  $\widehat{\gamma}_1$  resulted in smaller ABIAS<sup>2</sup>. When the quantity of interest is the transformed reliability at one year, the method of finding the stress-changing time and middle stress level simultaneously can help reduce the AVAR, ABIAS<sup>2</sup> and AMSE. From Table 4.9, we indicate that the efficiency gain can be as high as 25% for both AVAR and AMSE and 15% for ABIAS<sup>2</sup>.

#### 4.5.3 General remarks

Robust designs for three-step-stress ALT that minimize the ABIAS<sup>2</sup>, AVAR and AMSE of (4.11) and (4.12) have been derived respectively. These robust designs protect possible departure from the linear baseline hazard function assumption. Among all the estimators of the model parameters,  $\widehat{\gamma}_1$  is the only estimator changed as the stress-changing times vary. A higher value of  $\widehat{\gamma}_1$  results in a smaller ABIAS<sup>2</sup> and AMSE since AMSE is dominated by the ABIAS<sup>2</sup> term. Similar to Chapters 2 and 3, determining the middle stress level and stress-changing times simultaneously has helped to reduce the ABIAS<sup>2</sup>, AVAR, and AMSE of both

quantities interested. We also point out that the middle stress level should be kept as close to the lower bound of  $s_2$  as possible for this example.

# Chapter 5

## Simulation and comparison study

### 5.1 Introduction

In order to demonstrate the performance of resulting designs obtained in Chapters 2, 3 and 4, we carry out a simulation study. We provide a procedure to simulate data and to examine and compare the performances of our designs obtained from previous chapters.

### 5.2 Our simulation procedure

The following four steps describe our simulation procedure:

1. By given the initial values of true model parameters with the stress levels and stress-changing times of an optimal design we constructed previously, we can use Binomial distribution to simulate the data. We divide the time interval before the first stress-changing time  $\tau_1$ ,  $[0, \tau_1]$  into  $m_1$  subintervals:  $(t_{10}, t_{11}]$ ,  $(t_{11}, t_{12}]$ , ...,  $(t_{1, m_1-1}, t_{1 m_1}]$ , where  $0 = t_{10} < t_{11} < \dots < t_{1 m_1} = \tau_1$ . Let  $X_i$  be the failure number over the  $i$ th subinterval  $(t_{1, i-1}, t_{1 i}]$ . We assume that the distribution of  $X_i$  is a binomial distribution. Namely,

$$X_i \sim Bin(n - X_{i-1}, p_{1i}), i = 1, \dots, m_1,$$

where  $X_0 = 0$  and  $p_{1i}$  is the failure rate within the  $i$ th interval  $(t_{1, i-1}, t_{1 i}]$  under the low stress level,  $s_1$ . Then,  $p_{1i}$  can be derived as follows:

$$\begin{aligned} p_{1i} &= P\{t_{1, i-1} < t \leq t_{1 i} | s_1\} \\ &= \int_{t_{1, i-1}}^{t_{1 i}} (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_1) \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s_1)\right) dt. \end{aligned}$$

2. Similarly to Step 1, we divide the second the period  $(\tau_1, \tau_2]$  into  $m_2$  subintervals:  $(t_{20}, t_{21}]$ ,  $(t_{21}, t_{22}]$ , ...,  $(t_{2, m_2-1}, t_{2 m_2}]$ , where  $0 = t_{20} < t_{21} < \dots < t_{2 m_2} = \tau_2 - \tau_1$ . Let  $Y_i$  be the failure number over the  $i$ th interval  $(t_{2, i-1}, t_{2 i}]$ . The distribution of  $Y_i$  can be defined as

$$Y_i \sim Bin\left(\left(n - \sum_{i=1}^{m_1} X_i\right) - Y_{i-1}, p_{2i}\right), i = 1, \dots, m_2,$$

where  $Y_0 = 0$  and  $p_{2i}$  is the failure rate within the  $i$ th interval  $(t_{2,i-1}, t_{2i}]$  under the middle stress level,  $s_2$ . Then,  $p_{2i}$  can be derived as follow

$$\begin{aligned} p_{2i} &= P\{a + t_{2,i-1} < t \leq a + t_{2i} | s_2\} \\ &= \int_{a+t_{2,i-1}}^{a+t_{2i}} (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_2) \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s_2)\right) dt, \end{aligned}$$

where  $a = F_2^{-1}[F_1(\tau_1)]$ .

3. Now, we divide the third the period  $(\tau_2, c]$  into  $m_3$  subintervals:  $(t_{30}, t_{31}], (t_{31}, t_{32}], \dots, (t_{3,m_3-1}, t_{3m_3}]$ , where  $0 = t_{30} < t_{31} < \dots < t_{3m_3} = c - \tau_2$ . Let  $Z_i$  be the failure number over the  $i$ th interval  $(t_{3,i-1}, t_{3i}]$ . The distribution of  $Z_i$  can be defined as

$$Z_i \sim Bin\left(\left(n - \sum_{i=1}^{m_1} X_i - \sum_{i=1}^{m_2} Y_i\right) - Z_{i-1}, p_{3i}\right), i = 1, \dots, m_3,$$

where  $Z_0 = 0$  and  $p_{3i}$  is the failure rate within the  $i$ th interval  $(t_{3,i-1}, t_{3i}]$  under the high stress level,  $s_3$ . Then,  $p_{3i}$  can be derived as

$$\begin{aligned} p_{3i} &= P\{b + t_{3,i-1} < t \leq b + t_{3i} | s_3\} \\ &= \int_{b+t_{3,i-1}}^{b+t_{3i}} (\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_3) \exp\left(-\left(\gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3\right) \exp(\beta s_3)\right) dt, \end{aligned}$$

where  $b = F_3^{-1}[F_2(a + \tau_2 - \tau_1)]$ .

For simplify, we keep the length of all subinterval all equal to  $q$  hours. According to (4.2), the log-likelihood function can be expressed as

$$\begin{aligned} \ell(\boldsymbol{\beta}; \boldsymbol{\xi}) &= \left(\sum_{i=1}^{m_1} X_i\right) \beta s_1 + \sum_{i=1}^{m_1} \left[X_i \left(\ln(\gamma_0 + \gamma_1(qi)) - \left(\gamma_0(qi) + \frac{\gamma_1}{2}(qi)^2\right) \exp(\beta s_1)\right)\right] \\ &+ \left(\sum_{i=1}^{m_2} Y_i\right) \beta s_2 + \sum_{i=1}^{m_2} \left[Y_i \left(\ln(\gamma_0 + \gamma_1(a + qi)) - \left(\gamma_0(a + qi) + \frac{\gamma_1}{2}(a + qi)^2\right) \exp(\beta s_2)\right)\right] \\ &+ \left(\sum_{i=1}^{m_3} Z_i\right) \beta s_3 + \sum_{i=1}^{m_3} \left[Z_i \left(\ln(\gamma_0 + \gamma_1(b + qi)) - \left(\gamma_0(b + qi) + \frac{\gamma_1}{2}(b + qi)^2\right) \exp(\beta s_3)\right)\right] \\ &- \left(n - \sum_{i=1}^{m_1} X_i - \sum_{i=1}^{m_2} Y_i - \sum_{i=1}^{m_3} Z_i\right) \left[\left(\gamma_0(b + c - \tau_2) + \frac{\gamma_1}{2}(b + c - \tau_2)^2\right) \exp(\beta s_3)\right]. \end{aligned} \quad (5.1)$$

We carry out this simulation runs. For each run, we can compute the maximum likelihood estimates of the model parameters by maximizing (5.1). Then, use these  $r$  estimates to calculate the simulated squared bias (SBIAS<sup>2</sup>), simulated variance (SVAR), and simulated mean squared error (SMSE) of each parameter estimator. We define SBIAS<sup>2</sup>, SVAR and SMSE for  $\hat{\theta}$  are:

$$SBIAS^2(\hat{\theta}) = \left(\frac{\sum_{j=1}^r \hat{\theta}_j}{r} - \theta_0\right)^2,$$

$$\text{SVAR}(\widehat{\theta}) = \frac{1}{r-1} \sum_{j=1}^r \left( \widehat{\theta}_j - \frac{1}{r} \sum_{j=1}^r \widehat{\theta}_j \right)^2, \text{ and}$$

$$\text{SMSE}(\widehat{\theta}) = \text{SBIAS}^2(\widehat{\theta}) + \text{SVAR}(\widehat{\theta}),$$

where  $\theta_0$  is the given true parameter value,  $\widehat{\theta}_j$  is the estimate of  $\theta_0$  from the  $j$ th run. Then, we can have SBIAS<sup>2</sup>, SVAR and SMSE for  $\widehat{\gamma}_0, \widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\beta}$ , respectively. We also compute the SBIAS<sup>2</sup>, SVAR and SMSE of the MLE of hazard rate over  $T$  under normal design conditions. The true hazard function over  $T$  under normal design stress level  $s_D$  can be expressed as

$$\begin{aligned} g_1(\boldsymbol{\alpha}) &= \int_0^T \lambda(t; s_D) dt \\ &= \int_0^T ((\gamma_0 + \gamma_1 t + \gamma_2 t^2) \exp(\beta s_D)) dt, \end{aligned} \quad (5.2)$$

with  $\boldsymbol{\alpha} = [\gamma_0, \gamma_1, \gamma_2, \beta]^T$ .

The MLE of (5.2) at a time  $t$  under normal design stress level  $s_D$  is

$$\widehat{g_1(\boldsymbol{\alpha})} = \int_0^T \left( (\widehat{\gamma}_0 + \widehat{\gamma}_1 t + \widehat{\gamma}_2 t^2) \exp(\widehat{\beta} s_D) \right) dt, \quad (5.3)$$

with  $\widehat{\boldsymbol{\alpha}} = [\widehat{\gamma}_0, \widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\beta}]^T$ .

Besides, we compute the SBIAS<sup>2</sup>, SVAR, and SMSE of the MLE of transformed reliability at a pre-determined time  $t$  under normal design conditions. The true reliability function at time  $t$  under normal design stress  $s_D$  is

$$R(t; s_D) = \exp \left[ - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_D) \right]$$

In order to compute the SBIAS<sup>2</sup>, SVAR and SMSE in a convenient way, we transform the reliability into a simple function of  $\gamma_0, \gamma_1, \gamma_2$  and  $\beta$ . We denote such function as

$$\begin{aligned} g_2(\boldsymbol{\alpha}) &= \log(-\log[R(t; s_D)]) \\ &= \log \left( -\log \left( \exp \left[ - \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) \exp(\beta s_D) \right] \right) \right) \\ &= \log \left( \gamma_0 t + \frac{\gamma_1}{2} t^2 + \frac{\gamma_2}{3} t^3 \right) + \beta s_D. \end{aligned} \quad (5.4)$$

The MLE of (5.4) is

$$\widehat{g_2(\boldsymbol{\alpha})} = \log \left( \widehat{\gamma}_0 t + \frac{\widehat{\gamma}_1}{2} t^2 + \frac{\widehat{\gamma}_2}{3} t^3 \right) + \widehat{\beta} s_D. \quad (5.5)$$

We carry out this simulation procedure for a few optimal designs that we have constructed in Chapters 2, 3 and 4. (Both  $\gamma_2$  and  $\widehat{\gamma}_2$  are 0 In Chapter 2 while in Chapter 4 the value of  $\widehat{\gamma}_2$  is 0 when using (5.2), (5.3), (5.4), and (5.5) )



The SBIAS<sup>2</sup>, SVAR and SMSE of  $g_i$ ,  $i = 1, 2$  are

$$\begin{aligned} \text{SBIAS}^2 \left( \widehat{g_i(\boldsymbol{\theta})} \right) &= \left( \frac{\sum_{j=1}^r \widehat{g_i(\boldsymbol{\theta}_j)}}{r} - g_i^{True}(\boldsymbol{\theta}) \right)^2, \\ \text{SVAR} \left( \widehat{g_i(\boldsymbol{\theta})} \right) &= \frac{1}{r-1} \sum_{j=1}^r \left( \widehat{g_i(\boldsymbol{\theta}_j)} - \frac{1}{r} \sum_{j=1}^r \widehat{g_i(\boldsymbol{\theta}_j)} \right)^2 \\ &= \frac{1}{r-1} \left( \sum_{j=1}^r \widehat{g_i^2(\boldsymbol{\theta}_j)} - \frac{\left( \sum_{j=1}^r \widehat{g_i(\boldsymbol{\theta}_j)} \right)^2}{r} \right), \\ \text{SMSE} \left( \widehat{g_i(\boldsymbol{\theta})} \right) &= \text{SBIAS}^2 \left( \widehat{g_i(\boldsymbol{\theta})} \right) + \text{SVAR} \left( \widehat{g_i(\boldsymbol{\theta})} \right). \end{aligned}$$

## 5.3 Simulation using resulting designs

### 5.3.1 Performance of the optimal designs obtained in Chapter 2

Let the length of all subintervals equal to 5 hours. We demonstrate the performance of our resulting designs for the constraint plan being  $C_1$  which we obtained in Chapter 2. From Subsection 2.4.1,  $\xi_M^2 = [\tau_1 = 178.15, \tau_2 = 223.57, s_2 = 197.5]$ , is an optimal design with fixed middle stress at  $s_2 = 197.5^\circ C$ . After we update the  $s_2$  from average to optimal middle stress level, we get new D-, A-, and Q-optimal designs for  $C_1$ . Since the resulting D-, A- and Q-optimal designs are similar, we only present the simulation result for our resulting Q-optimal design. The Q-optimal design we obtained in Subsection 2.4.2 is  $\xi_Q^2 = [\tau_1 = 156.19, \tau_2 = 247.47, s_2 = 155]$ . And we use the same initial values as Subsection 2.4.2 for the model parameters,  $\gamma_0 = 0.0001$ ,  $\gamma_1 = 0.5$  and  $\beta = -3800$ . Table 5.1 lists the efficiencies of  $\xi_Q^2$  relative to  $\xi_M^2$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE of  $\widehat{\gamma}_0, \widehat{\gamma}_1, \widehat{\beta}$ , (5.3) and (5.5) and Table 5.2 displays the standard error.

Table 5.1 The efficiencies of  $\xi_Q^2$  relative to  $\xi_M^2$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE

	$\widehat{\beta}$	$\widehat{\gamma}_0$	$\widehat{\gamma}_1$	$\widehat{g}_1$	$\widehat{g}_2$
$eff_B(Q, M)$	0.9824	0.9803	1.0000	1.0175	0.9986
$eff_V(Q, M)$	2.2207	24508114.61	2.0038	1.7657	2.1449
$eff_M(Q, M)$	0.9838	0.9851	1.0000	1.0345	1.0016

Table 5.2 Standard error

Quantity	Design	standard error
$\beta$	$\xi_M^2$	5.6826
	$\xi_Q^2$	3.8133
$\gamma_0$	$\xi_M^2$	$2.2120 \cdot 10^{-7}$
	$\xi_Q^2$	$4.4681 \cdot 10^{-11}$
$\gamma_1$	$\xi_M^2$	$1.6330 \cdot 10^{-6}$
	$\xi_Q^2$	$1.1536 \cdot 10^{-6}$
$g_1$	$\xi_M^2$	$1.5137 \cdot 10^3$
	$\xi_Q^2$	$1.1392 \cdot 10^3$
$g_2$	$\xi_M^2$	$6.6858 \cdot 10^{-3}$
	$\xi_Q^2$	$4.5651 \cdot 10^{-3}$

Note: Reviewers suggest that we list the 95% confidence interval. However, the simulation data have not been recorded. Thus, we display the standard error instead.

### 5.3.2 Performance of the optimal designs obtained in Chapter 3

Let the length of all subintervals equal to 5 hours. We demonstrate the performance of our resulting designs which we obtained in Chapter 3. From Subsection 3.4.1,  $\xi_M^3 = [\tau_1 = 178.15, \tau_2 = 223.57, s_2 = 197.5]$ , is an optimal design with fixed middle stress at  $s_2 = 197.5^\circ C$ . After we update the  $s_2$  from average to optimal middle stress level, we get the updated D-, A- and Q-optimal designs. The D-optimal design and Q-optimal design are similar. Thus, we only present the simulation results for our resulting Q- and A-optimal designs in this example. The updated Q- and A-optimal designs are  $\xi_Q^3 = [\tau_1 = 178.15, \tau_2 = 245.96, s_2 = 155]$  and  $\xi_A^3 = [\tau_1 = 155.20, \tau_2 = 247.47, s_2 = 155]$ . As Chapter 3, the initial value for model parameter are  $\gamma_0 = 0.0001, \gamma_1 = 0.5, \gamma_2 = 0, \beta = -3800$ . Tables 5.3 and 5.4 display the efficiencies of  $\xi_Q^3$  and  $\xi_A^3$  relative to  $\xi_M^3$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE. The standard errors are listed in Table 5.5.

Table 5.3 The efficiencies of  $\xi_Q^3$  relative to  $\xi_M^3$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE

	$\hat{\beta}$	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{g}_1$	$\hat{g}_2$
$eff_B(Q, M)$	1.0271	31.4599	1.0004	199.0405	280.6279	4.3146
$eff_V(Q, M)$	0.7898	6.8388	0.0149	309.2855	3.8197	0.2415
$eff_M(Q, M)$	1.0258	9.8674	1.0004	29.3142	51.1622	3.7968

Table 5.4 The efficiencies of  $\xi_A^3$  relative to  $\xi_M^3$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE

	$\hat{\beta}$	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{g}_1$	$\hat{g}_2$
$eff_B(A, M)$	1.0098	21.2561	1.0004	504.6733	313.2065	4.6529
$eff_V(A, M)$	0.97136	11.4924	0.0277	420.4643	2.8329	0.2631
$eff_M(A, M)$	1.0097	14.0173	1.0004	42.7565	40.2554	4.0999

Table 5.5 Standard error

Quantity	Design	standard error
$\beta$	$\xi_M^3$	7.6834
	$\xi_Q^3$	8.6454
	$\xi_A^3$	7.7959
$\gamma_0$	$\xi_M^3$	$1.2834 \cdot 10^{-5}$
	$\xi_Q^3$	$4.9077 \cdot 10^{-6}$
	$\xi_A^3$	$3.7859 \cdot 10^{-6}$
$\gamma_1$	$\xi_M^3$	$5.8977 \cdot 10^{-7}$
	$\xi_Q^3$	$4.8381 \cdot 10^{-6}$
	$\xi_A^3$	$3.5458 \cdot 10^{-6}$
$\gamma_2$	$\xi_M^3$	$3.1505 \cdot 10^{-7}$
	$\xi_Q^3$	$1.7914 \cdot 10^{-8}$
	$\xi_A^3$	$1.5365 \cdot 10^{-8}$
$g_1$	$\xi_M^3$	$2.2084 \cdot 10^6$
	$\xi_Q^3$	$1.1300 \cdot 10^6$
	$\xi_A^3$	$1.3121 \cdot 10^6$
$g_2$	$\xi_M^3$	$2.1147 \cdot 10^{-2}$
	$\xi_Q^3$	$4.3036 \cdot 10^{-2}$
	$\xi_A^3$	$4.1230 \cdot 10^{-2}$

### 5.3.3 Performance of the robust designs obtained in Chapter 4

Let the length of all subintervals equal to 5 hours. We demonstrate the performance of our resulting robust designs which we obtained in Chapter 4. The optimal design with fixed middle stress,  $\xi_M^4 = [\tau_1 = 160, \tau_2 = 201, s_2 = 197.5]$ , is obtained in Chapter 4 and the design,  $\xi_G = [\tau_1 = 160, \tau_2 = 218, s_2 = 155]$ , is the obtained robust design when  $s_2$  is not fixed. The optimal criterion is to minimize the SBIAS<sup>2</sup>, SVAR and SMSE. Suppose from previous experience, the initial parameters are  $\gamma_0 = 0.0001$ ,  $\gamma_1 = 0.5$  and  $\beta = -3800$ . We choose the initial value of  $\gamma_2$  within the 95% confidence interval.

$$\begin{aligned}
 95\% \text{ CI} & : \widehat{\gamma}_2 \pm Z_{\frac{0.05}{2}} \sqrt{Var(\widehat{\gamma}_2)} \\
 & = [0 \pm 0.0015]
 \end{aligned}$$

However, the value should provide a reasonable baseline hazard function and reliability function under  $s_D$ . Thus, the reasonable value of  $\gamma_2$  should be within  $(0, 0.0015]$ . The efficiencies of  $\xi_G$  relative to  $\xi_M^4$  in terms of SBIAS<sup>2</sup>, SVAR and SMSE are displayed in Tables 5.6, and 5.7.

Table 5.6 The efficiencies of  $\xi_G$  relative to  $\xi_M^4$   
when the quantity of interest is hazard rate,  $g_1$ .

$\gamma_2$	$eff_B(G, M)$	$eff_V(G, M)$	$eff_M(G, M)$
0.00015	0.8262	1.0595	0.9759
0.0003	1.0205	0.9309	1.0183
0.00045	1.0093	1.5160	1.0120
0.0006	1.0127	0.9331	1.0125
0.00075	0.9894	0.9329	0.9893
0.0009	1.0016	1.1942	1.0018
0.00105	1.0029	0.9752	1.0028
0.0012	1.0052	1.3337	1.0054
0.00135	0.9978	1.2210	0.9979
0.0015	1.0063	1.0273	1.0063
average	0.9872	1.1124	1.0022

We have gained efficiency by 11% and 0.22% for SVAR and SMSE.

Table 5.7 The efficiencies of  $\xi_G$  relative to  $\xi_M^4$   
when the quantity of interest is transformed reliability,  $g_2$ .

$\gamma_2$	$eff_B(G, M)$	$eff_V(G, M)$	$eff_M(G, M)$
0.00015	1.0131	0.9383	1.0123
0.0003	0.9815	1.0131	0.9823
0.00045	0.9652	1.5070	0.9810
0.0006	0.9484	0.8996	0.9449
0.00075	1.1013	0.8145	1.0641
0.0009	0.9564	1.3765	1.0248
0.00105	0.9419	0.8611	0.9036
0.0012	0.6281	1.6702	1.2759
0.00135	0.2672	1.1972	1.1943
0.0015	1.9854	0.9779	1.1027
average	0.9789	1.1255	1.0486

We indicate that the efficiency gain can be more than 12% for SVAR, and 5% for SMSE.

## 5.4 Comparison and discussion

### 5.4.1 On the resulting optimal designs for Model (2.1)

We have obtained all the SBIAS<sup>2</sup>, SVAR and SMSE of MLE for  $\gamma_0, \gamma_1, \beta, g_1$  and  $g_2$  based on two different designs,  $\xi_M^2$  and  $\xi_Q^2$ . The efficiencies of  $\xi_Q^2$  relative to  $\xi_M^2$  in terms of SBIAS<sup>2</sup>, SVAR, and SMSE are displayed in the Table 5.1.

In Chapter 2, the optimal designs are obtained by minimizing the asymptotic variance of the  $\hat{g}_1$ . As we expect, when  $\xi_Q^2$  was adopted, we have gained 77% in terms of the SVAR( $\hat{g}_1$ ) compare to  $\xi_M^2$ . The SVAR

of MLE of the model parameters are all very much reduced, the efficiencies of  $\xi_Q^2$  relative to  $\xi_M^2$  in terms of  $\text{SVAR}(\widehat{\gamma}_0)$ ,  $\text{SVAR}(\widehat{\gamma}_1)$ , and  $\text{SVAR}(\widehat{\beta})$  we have gained are all higher than 100%. Moreover,  $\text{SBIAS}^2$  and  $\text{SMSE}$  of  $\widehat{g}_1$  as well as  $\text{SVAR}$  and  $\text{SMSE}$  of  $\widehat{g}_2$  (although they are not used as the optimal criteria in Chapter 2) are also being reduced. All the results indicate that  $\xi_Q^2$  outperforms the design  $\xi_M^2$ .

#### 5.4.2 On the resulting optimal designs for Model (3.1)

In Subsection 5.3.2, we have obtained  $\text{SVAR}$  of MLE of the model parameters  $\gamma_0$ ,  $\gamma_1$ ,  $\gamma_2$  and  $\beta$  and  $\text{SVAR}$  of MLE of  $g_1$  and  $g_2$  when each of the three optimal designs:  $\xi_M^3$ ,  $\xi_Q^3$  and  $\xi_A^3$  are adopted. The efficiencies of  $\xi_Q^3$  and  $\xi_A^3$  relative to  $\xi_M^3$  in terms of  $\text{SBIAS}^2$ ,  $\text{SVAR}$ , and  $\text{SMSE}$  are listed in the Tables 5.3 and 5.4.

These two tables display that optimally setting the middle stress level and stress-changing time simultaneously,  $\xi_Q^3$  and  $\xi_A^3$  can reduce the  $\text{SVAR}(\widehat{g}_1)$  by 282% and 183% compare to  $\xi_M^3$ . We note that not all of  $\text{SVAR}$ s of MLE of model parameters have been reduced by using  $\xi_Q^3$  or  $\xi_A^3$ . Only  $\text{SVAR}(\widehat{\gamma}_0)$  and  $\text{SVAR}(\widehat{\gamma}_2)$  have significantly dropped among all  $\text{SVAR}$ s. Moreover, we point out that the  $\text{SBIAS}^2$  and  $\text{SMSE}$  of  $\widehat{\gamma}_0$ ,  $\widehat{\gamma}_2$ ,  $\widehat{\beta}$ ,  $\widehat{g}_1$  and  $\widehat{g}_2$  (although they are not used as the optimal criteria in Chapter 3) are all reduced by adopting the either  $\xi_Q^3$  or  $\xi_A^3$ . The efficiencies of  $\xi_Q^3$  and  $\xi_A^3$  in terms of  $\text{SMSE}(\widehat{g}_1)$  are as high as 51.1622 and 40.2554 compared to  $\xi_M^3$ . This indicates that the performance of  $\xi_Q^3$  and  $\xi_A^3$  update designs are about 50 or 40 times better than  $\xi_M^3$  when experimenter is interested in estimating a hazard rate.

#### 5.4.3 On the robust designs constructed in Chapter 4

Simulation study in Subsection 5.5.3 indicates that the performance of the design is sensitive to the initial value of  $\gamma_2$ . The efficiencies of  $\xi_G$  relative to  $\xi_M^4$  in terms of average of  $\text{SBIAS}^2$ ,  $\text{SVAR}$ , and  $\text{SMSE}$  are listed in the Tables 5.6 and 5.7.

Table 5.6 shows that when the quantity of interest is hazard rate, the average efficiency gain in terms of  $\text{SMSE}$  is only around 0.2%, however, the average efficiency gain in terms of  $\text{SVAR}$  is more than 11%. Furthermore, the average efficiencies gain in terms of  $\text{SVAR}$  and  $\text{SMSE}$  are 12.55% and 4.86% when the quantity of interest is transformed reliability. These results show that the design  $\xi_G$  outperforms the  $\xi_M^4$ . However, the designs seems not protecting the estimation bias. This may be the reason why we loss the efficiency of  $\text{SBIAS}^2$  by 1.28% in Table 5.6 and 2.11% in Table 5.7.

#### 5.4.4 General remarks

For a three-step-stress ALT, the practitioner would naturally set the average of high and low stress as the middle stress level. From the results of simulation, we can see that the designs with optimal stress levels result in much lower  $\text{SBIAS}^2$ ,  $\text{SVAR}$  and  $\text{SMSE}$ . We recommend designing a three-step-stress model with an optimal middle stress level and optimal stress-changing times when a hazard rate is interested. We also suggest that under constraints on minimum number of failure of each stress level the middle stress level should be kept as close to the low stress level as possible. On the other hand, when a reliability estimation is required, the inaccuracy of the assume baseline hazard function can cause unavoidable bias. The proposed design methods seem not helping much in reducing such bias. However, the proposed designs performance very well when the model assumed is correct.

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# Appendix I: Notation

- $F$  - Fisher's information matrix  
 $n$  - total number of test units in experiment  
 $s_D$  - normal design stress level (stress under normal operating conditions)  
 $s_1$  - low accelerated stress  
 $s_2$  - middle accelerated stress  
 $s_3$  - high accelerated stress  
 $n_i$  - number of failures under  $s_i$   
 $\tau_1$  - first stress-changing time  
 $\tau_2$  - second stress-changing time  
 $c$  - censoring time  
 $\beta$  - parameters of PH model when baseline hazard function is a linear form  
 $\alpha$  - parameters of PH model when baseline hazard function is a quadratic form  
 $T$  - prespecified period of time over which the estimate is of interest  
 $\lambda(t; s)$  - hazard function of failure time at stress level  $s$   
 $\lambda_0(t)$  - the baseline hazard function  
 $\Lambda(t; s)$  - cumulative hazard function of failure time at stress level  $s$   
 $R(t; s)$  - reliability function of failure time at stress level  $s$   
 $F(t; s)$  - cumulative distribution function of failure time at stress level  $s$   
 $f(t; s)$  - probability density function of failure time at stress level  $s$   
 $L_D$  - the determinant of covariance matrix of the model parameter estimators  
 $L_A$  - the trace of covariance matrix of the model parameter estimators  
 $L_Q$  - the asymptotic variance of the hazard rate estimator over a period of time  $T$  at a design stress level  $s_D$   
 ${}^D\xi_{C_i}$  - D-optimal design when the constraint plan is  $C_i$   
 ${}^A\xi_{C_i}$  - A-optimal design when the constraint plan is  $C_i$   
 ${}^Q\xi_{C_i}$  - Q-optimal design when the constraint plan is  $C_i$   
 $MNF_i$  - the minimum number of failures at  $s_i$  stress level  
 $C_i$  -  $i$ th constraint plan for the  $MNF_1$ ,  $MNF_2$  and  $MNF_3$   
 $M_T$  - the model with quadratic baseline hazard function  
 $M_F$  - the model with linear baseline hazard function  
 $\lambda_{M_F}(t; s)$  - MLE of the hazard rate at a time  $t$  under stress  $s$



$R_{M_F}(t; s)$  - MLE of the reliability at time  $t$  under stress  $s$

$N_{M_F}(t; s)$  - the transform of  $R_{M_F}(t; s)$

ABIAS<sup>2</sup> - asymptotic squared bias

AVAR - asymptotic variance

AMSE - asymptotic mean squared error

SBIAS<sup>2</sup> - simulated squared bias

SVAR - simulated variance

SMSE - simulated mean squared error

$r$  - total number of simulations

$g_1$  - the hazard rate over a time period  $T$  under normal design stress level  $s_D$

$g_2$  - the transformed reliability at time  $t$  under normal design stress level  $s_D$

$\xi_M^i$  - the optimal design obtained in Chapter  $i$  when the middle stress level is fixed and constraint plan is  $C_1$

$\xi_Q^i$  - the Q-optimal design obtained in Chapter  $i$  when the middle stress level is unfixed and constraint plan is  $C_1$

$\xi_A^i$  - the A-optimal design obtained in Chapter  $i$  when the middle stress level is unfixed and constraint plan is  $C_1$

$\xi_G$  - the robust design obtained in Chapter 4 when the middle stress level is fixed and constraint plan is  $C_1$

# Appendix II: Programming Codes

## A: Codes for subsection 2.4.1

%This code is for computing the optimal stress-changing times  $\tau_1$  and  $\tau_2$ .

code for Matlabs

t1 = 145; t3 = 250; t2 =(t1+t3)\*(1/2);

s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 =1/(273.16+t2);

gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;

c = 300; n = 200;

a=@(tau) fsolve(@(a)-exp(-(gamma0.\*tau(1)+(1/2)\*gamma1.\*tau(1).^2)\*exp(beta.\*s1))+exp(-(gamma0.\*a+(1/2)\*gamma1.\*a.^2)\*exp(beta.\*s2)), 10)

b=@(tau) fsolve(@(b)-exp(-(gamma0.\*(a(tau(1))+tau(2)-tau(1))+(1/2).\*gamma1.\*(a(tau(1))+tau(2)-tau(1)).^2).\*exp(beta\*s2))+exp(-(gamma0.\*b+(1/2).\*gamma1.\*b.^2)\*exp(beta.\*s3)), 10)

c1=(1./2)\*gamma1.\*exp(beta.\*s1);

c2=(1./2)\*gamma1.\*exp(beta.\*s2);

c3=(1./2)\*gamma1.\*exp(beta.\*s3);

f1=@(x) exp(-c1.\*x.^2)\*1./x; f2=@(x) exp(-c2.\*x.^2)\*1./x; f3=@(x) exp(-c3.\*x.^2)\*1./x;

f4=@(x) exp(-c1.\*x.^2); f5=@(x) exp(-c2.\*x.^2); f6=@(x) exp(-c3.\*x.^2);

f7=@(x) exp(-c1.\*x.^2).\*x; f8=@(x) exp(-c2.\*x.^2).\*x; f9=@(x) exp(-c3.\*x.^2).\*x;

f10=@(x) exp(-c1.\*x.^2).\*x.^2; f11=@(x) exp(-c2.\*x.^2).\*x.^2; f12=@(x) exp(-c3.\*x.^2).\*x.^2;

f13=@(x) exp(-c1.\*x.^2).\*x.^3; f14=@(x) exp(-c2.\*x.^2).\*x.^3; f15=@(x) exp(-c3.\*x.^2).\*x.^3;

f16=@(x,tau) (s1.^2.\*(gamma0.\*x+(1/2).\*gamma1.\*x.^2+(1/3).\*gamma2.\*x.^3).\*exp(beta.\*s1).\*(gamma2.\*x.^2+gamma1.\*x+gamma0).\*exp(beta.\*s1)).\*exp(-(gamma0.\*x+(1/2).\*gamma1.\*x.^2+(1/3).\*gamma2.\*x.^3).\*exp(beta.\*s1))

f17=@(x,tau) (s2.^2.\*(gamma0.\*(a(tau(1))+x-tau(1))+(1/2).\*gamma1.\*(a(tau(1))+x-tau(1)).^2+(1/3).\*gamma2.\*(a(tau(1))+x-tau(1)).^3).\*exp(beta.\*s2).\*(gamma0+gamma1.\*(a(tau(1))+x-tau(1)).^2+gamma2.\*(a(tau(1))+x-tau(1)).^2).\*exp(beta.\*s2)).\*exp(-(gamma0.\*(a(tau(1))+x-tau(1))+(1/2).\*gamma1.\*(a(tau(1))+x-tau(1)).^2+(1/3).\*gamma2.\*(a(tau(1))+x-tau(1)).^3).\*exp(beta.\*s2))

f18=@(x,tau) (s3.^2.\*(gamma0.\*(b([tau(1) tau(2)])+x-tau(2))+(1/2).\*gamma1.\*(b([tau(1) tau(2)])+x-tau(2)).^2+(1/3).\*gamma2.\*(b([tau(1) tau(2)])+x-tau(2)).^3).\*exp(beta.\*s3).\*(gamma0+gamma1.\*(b([tau(1) tau(2)])+x-tau(2))+gamma2.\*(b([tau(1) tau(2)])+x-tau(2)).^2).\*exp(beta.\*s3)).\*exp(-(gamma0.\*(b([tau(1) tau(2)])+x-tau(2))+(1/2).\*gamma1.\*(b([tau(1) tau(2)])+x-tau(2)).^2+(1/3).\*gamma2.\*(b([tau(1) tau(2)])+x-tau(2)).^3).\*exp(beta.\*s3))

```

x-tau(2)).^2+(1/3).*gamma2.*(b([tau(1) tau(2)]+x-tau(2)).^3).*exp(beta.*s3))
A1=@(a,b) quad(f1,a,b); A2=@(a,b) quad(f2,a,b); A3=@(a,b) quad(f3,a,b);
B1=@(a,b) quad(f4,a,b); B2=@(a,b) quad(f5,a,b); B3=@(a,b) quad(f6,a,b);
C1=@(a,b) quad(f7,a,b); C2=@(a,b) quad(f8,a,b); C3=@(a,b) quad(f9,a,b);
D1=@(a,b) quad(f10,a,b); D2=@(a,b) quad(f11,a,b); D3=@(a,b) quad(f12,a,b);
E1=@(a,b) quad(f13,a,b); E2=@(a,b) quad(f14,a,b); E3=@(a,b) quad(f15,a,b);
Be1=@(tau,a,b) quadgk(@(x) f16(x,tau),a,b)
Be2=@(tau,a,b) quadgk(@(x) f17(x,tau),a,b)
Be3=@(tau,a,b) quadgk(@(x) f18(x,tau),a,b)
M0=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*A1(gamma0/
gamma1, gamma0/gamma1+tau(1))./gamma1+exp(beta.*s2)*exp(gamma0.^2.*
exp(beta.*s2)/(2.*gamma1))*A2(gamma0/gamma1+a(tau(1)), gamma0/gamma1+
tau(2)-tau(1)+a(tau(1)))./gamma1+exp(beta.*s3)*exp(gamma0.^2.*exp(beta.*
s3)/(2.*gamma1))*A3(gamma0/gamma1+b([tau(1) tau(2)]), gamma0/
gamma1+c-tau(2)+b([tau(1) tau(2)]))./gamma1
M1=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*(gamma0.^2.*
A1(gamma0/gamma1, gamma0/gamma1+tau(1))./gamma1.^2...
-2.*gamma0.*B1(gamma0/gamma1, gamma0/gamma1+tau(1))./gamma1+C1(gamma0/
gamma1, gamma0/gamma1+tau(1))./gamma1+exp(beta.*s2)*exp(gamma0.^2.*
exp(beta.*s2)/(2.*gamma1))*(gamma0.^2.*A2(gamma0/
gamma1+a(tau(1)),gamma0/gamma1+tau(2)-tau(1)+a(tau(1)))./gamma1.^2
-2.*gamma0.*B2(gamma0/gamma1+a(tau(1)),gamma0/gamma1+tau(2)-tau(1)+a(tau(1)))./
gamma1+C2(gamma0/gamma1+a(tau(1)),gamma0/gamma1+tau(2)-tau(1)+a(tau(1)))./gamma1
+exp(beta.*s3)*exp(gamma0.^2.*exp(beta.*s3)/(2.*gamma1))*(gamma0.^2.*A3(gamma0/
gamma1+b([tau(1) tau(2)]),gamma0/gamma1+c-tau(2)+b([tau(1) tau(2)]))./gamma1.^2
-2.*gamma0.*B3(gamma0/gamma1+b([tau(1) tau(2)]),gamma0/gamma1+c-tau(2)+
b([tau(1) tau(2)]))./gamma1+C3(gamma0/gamma1+b([tau(1) tau(2)]),gamma0/gamma1+
c-tau(2)+b([tau(1) tau(2)]))./gamma1
M01=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*
(B1(gamma0/gamma1,gamma0/gamma1+tau(1))-gamma0.*A1(gamma0/
gamma1,gamma0/gamma1+tau(1))./gamma1)./gamma1+exp(beta.*s2)*
exp(gamma0.^2.*exp(beta.*s2)/(2.*gamma1))*(B2(gamma0/gamma1+a(tau(1)),
gamma0/gamma1+tau(2)-tau(1)+a(tau(1)))-gamma0.*A2(gamma0/gamma1+a(tau
(1)),gamma0/gamma1+tau(2)-tau(1)+a(tau(1)))./gamma1)./gamma1+exp(beta.*
s3)*exp(gamma0.^2.*exp(beta.*s3)/(2.*gamma1))*(B3(gamma0/gamma1+b([tau(1) tau(2)]),gamma0/
gamma1+c-tau(2)+b([tau(1) tau(2)]))-gamma0.*A3(gamma0/gamma1+
b([tau(1) tau(2)]),gamma0/gamma1+c-tau(2)+b([tau(1) tau(2)]))./gamma1)./gamma1
Mb=@(tau)Be1(tau,0,tau(1))+Be2(tau,tau(1),tau(2))+Be3(tau,tau(2),c)+Be3(tau,c,inf)
Dloss=@(tau)1/((M0(tau).*M1(tau)-M01(tau).^2).*n.^3*Mb(tau))
Aloss=@(tau)M1(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2))+M0(tau)./(n.*(M0(tau).*
M1(tau)-M01(tau).^2))+1./(n.*Mb(tau))
t=87600

```

```

Qloss=@(tau)exp(beta.*s).*(exp(beta.*s).*M1(tau).*t./(n.*(M0(tau).*M1(tau)-M01(tau).^2))-(1/2).*
t.^2.*exp(beta.*s).*M01(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2)))...
+exp(beta.*s).*((1/3).*exp(beta.*s).*M0(tau).*t.^3./(n.*(M0(tau).*M1(tau)-M01(tau).^2))-(1/2).*t.^2.*
exp(beta.*s).*M01(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2)))...
+(1/3).*(gamma1.*t+gamma0).^3.*(exp(beta.*s)).^2.*s.^2./(n.*Mb(tau).*gamma1)
A=[1 -1]
B=[0]
tau0=[115 201]
lb=[0;0];
ub=[inf;inf];
%This code is the constraint function
function [c, ceq] = confun(tau)
t1 = 145; t3 = 250; t2 = 191.6536;
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 = 1/(273.16+t2);
gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;
c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2)*exp(beta.*s1))+
exp(-(gamma0.*a+(1/2)*gamma1.*a.^2)*exp(beta.*s2)), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau(1))+tau(2)-tau(1))+(1/2).*gamma1.*(a(tau(1))+
tau(2)-tau(1)).^2).*exp(beta.*s2))+exp(-(gamma0.*b+(1/2).*gamma1.*b.^2)*exp(beta.*s3)), 10)
c1=(1./2)*gamma1.*exp(beta.*s1);
c2=(1./2)*gamma1.*exp(beta.*s2);
c3=(1./2)*gamma1.*exp(beta.*s3);
lambda0=@(t)gamma0+gamma1.*t;
f19=@(t)(lambda0(t).*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s1)))
f20=@(t)(lambda0(t).*exp(beta.*s2).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s2)))
f21=@(t)(lambda0(t).*exp(beta.*s3).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s3)))
W1=@(a,b)quad(f19,a,b);W2=@(a,b)quad(f20,a,b);W3=@(a,b)quad(f21,a,b);
h1=@(tau)W1(0,tau(1))
h2=@(tau)W2(a(tau(1)),a(tau(1))+tau(2)-tau(1))
h3=@(tau)W3(b([tau(1) tau(2)]),c)
c(1)=30-n.*h1(tau)
c(2)=30-(n-n.*h1(tau)).*h2(tau)
c(3)=10-(n-n.*h1(tau)-(n-n.*h1(tau)).*h2(tau)).*h3(tau)
ceq = [];
end

```

## B: Codes for subsection 2.4.2

%This code is for computing the optimal stress-changing times  $\tau_1$ ,  $\tau_2$  and the optimal middle stress level.

```

code for Matlabs
t1 = 145; t3 = 250; t2 =@(tau)tau(3);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 =@(tau) 1/(273.16+t2(tau));
gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;
c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2)*exp(beta.*s1))+
exp(-(gamma0.*a+(1/2)*gamma1.*a.^2)*exp(beta.*s2(tau))), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau)+tau(2)-tau(1))+(1/2).*gamma1.*(a(tau)+tau(2)-tau(1))^2).*
exp(beta.*s2(tau)))+exp(-(gamma0.*b+(1/2).*gamma1.*b.^2)*exp(beta.*s3)), 100)
c1=(1./2)*gamma1.*exp(beta.*s1);
c2=@(tau)(1./2)*gamma1.*exp(beta.*s2(tau))
c3=(1./2)*gamma1.*exp(beta.*s3);
f1=@(x) exp(-c1.*x.^2)*1./x; f2=@(x,tau) exp(-c2(tau).*x.^2)*1./x; f3=@(x) exp(-c3.*x.^2)*1./x;
f4=@(x) exp(-c1.*x.^2); f5=@(x,tau) exp(-c2(tau).*x.^2); f6=@(x) exp(-c3.*x.^2);
f7=@(x) exp(-c1.*x.^2).*x; f8=@(x,tau)exp(-c2(tau).*x.^2).*x; f9=@(x) exp(-c3.*x.^2).*x;
f10=@(x) exp(-c1.*x.^2).*x.^2; f11=@(x,tau) exp(-c2(tau).*x.^2).*x.^2; f12=@(x) exp(-c3.*x.^2).*x.^2;
f13=@(x) exp(-c1.*x.^2).*x.^3; f14=@(x,tau) exp(-c2(tau).*x.^2).*x.^3; f15=@(x) exp(-c3.*x.^2).*x.^3;
f16=@(x,tau) (s1.^2.*(gamma0.*x+(1/2).*gamma1.*x.^2+(1/3).*gamma2.*x.^3).*exp(beta.*s1).
*(gamma2.*x.^2+gamma1.*x+gamma0).*exp(beta.*s1)).*exp(-(gamma0.*x+(1/2).*gamma1.*x.^2
+(1/3).*gamma2.*x.^3).*exp(beta.*s1))
f17=@(x,tau) (s2(tau).^2.*(gamma0.*(a(tau)+x-tau(1))+(1/2).*gamma1.*(a(tau)+x-tau(1)).^2+(1/3).*
gamma2.*(a(tau)+x-tau(1)).^3).*exp(beta.*s2(tau)).*(gamma0+gamma1.*(a(tau)+x-tau(1))...
+gamma2.*(a(tau)+x-tau(1)).^2).*exp(beta.*s2(tau))).*exp(-(gamma0.*(a(tau)+x-tau(1))+(1/2).*
gamma1.*(a(tau)+x-tau(1)).^2+(1/3).*gamma2.*(a(tau)+x-tau(1)).^3).*exp(beta.*s2(tau)))
f18=@(x,tau) (s3.^2.*(gamma0.*(b(tau)+x-tau(2))+(1/2).*gamma1.*(b(tau)+x-tau(2)).^2+(1/3).*
gamma2.*(b(tau)+x-tau(2)).^3).*exp(beta.*s3)).*(gamma0+gamma1.*(b(tau)+x-tau(2))...
+gamma2.*(b(tau)+x-tau(2)).^2).*exp(beta.*s3)).*exp(-(gamma0.*(b(tau)+x-tau(2))+(1/2).*gamma1.*
(b(tau)+x-tau(2)).^2+(1/3).*gamma2.*(b(tau)+x-tau(2)).^3).*exp(beta.*s3))
A1=@(a,b) quad(f1,a,b); A2=@(a,b,tau) quadgk(@(x)f2(x,tau),a,b); A3=@(a,b) quad(f3,a,b);
B1=@(a,b) quad(f4,a,b); B2=@(a,b,tau) quadgk(@(x)f5(x,tau),a,b); B3=@(a,b) quad(f6,a,b);
C1=@(a,b) quad(f7,a,b); C2=@(a,b,tau) quadgk(@(x)f8(x,tau),a,b); C3=@(a,b) quad(f9,a,b);
D1=@(a,b) quad(f10,a,b); D2=@(a,b,tau) quadgk(@(x)f11(x,tau),a,b); D3=@(a,b) quad(f12,a,b);
E1=@(a,b) quad(f13,a,b); E2=@(a,b,tau) quadgk(@(x)f14(x,tau),a,b); E3=@(a,b) quad(f15,a,b);
Be1=@(tau,a,b) quadgk(@(x) f16(x,tau),a,b)
Be2=@(tau,a,b) quadgk(@(x) f17(x,tau),a,b)
Be3=@(tau,a,b) quadgk(@(x) f18(x,tau),a,b)
M0=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*
A1(gamma0/gamma1, gamma0/gamma1+tau(1))./gamma1...
+exp(beta.*s2(tau))*exp(gamma0.^2.*exp(beta.*s2(tau))/(2.*gamma1))*A2(gamma0/

```

```

gamma1+a(tau), gamma0/gamma1+tau(2)-tau(1)+a(tau),tau)./gamma1...
+exp(beta.*s3)*exp(gamma0.^2.*exp(beta.*s3)/(2.*gamma1))*A3(gamma0/gamma1+
b(tau), gamma0/gamma1+c-tau(2)+b(tau))./gamma1
M1=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*(gamma0.^2.*
A1(gamma0/gamma1, gamma0/gamma1+tau(1))./gamma1.^2....
-2.*gamma0.*B1(gamma0/gamma1, gamma0/gamma1+tau(1))./gamma1+C1(gamma0/
gamma1, gamma0/gamma1+tau(1))./gamma1...
+exp(beta.*s2(tau))*exp(gamma0.^2.*exp(beta.*s2(tau))/(2.*gamma1))*(gamma0.^2.*
A2(gamma0/gamma1+a(tau),gamma0/gamma1+tau(2)-tau(1)+a(tau),tau)./gamma1.^2....
-2.*gamma0.*B2(gamma0/gamma1+a(tau),gamma0/gamma1+tau(2)-tau(1)+a(tau),tau)./gamma1+
C2(gamma0/gamma1+a(tau),gamma0/gamma1+tau(2)-tau(1)+a(tau),tau))./gamma1...
+exp(beta.*s3)*exp(gamma0.^2.*exp(beta.*s3)/(2.*gamma1))*(gamma0.^2.*A3(gamma0/
gamma1+b(tau),gamma0/gamma1+c-tau(2)+b(tau))./gamma1.^2....
-2.*gamma0.*B3(gamma0/gamma1+b(tau),gamma0/gamma1+c-tau(2)+b(tau))./gamma1+
C3(gamma0/gamma1+b(tau),gamma0/gamma1+c-tau(2)+b(tau))./gamma1
M01=@(tau)exp(beta.*s1)*exp(gamma0.^2.*exp(beta.*s1)/(2.*gamma1))*(B1(gamma0/gamma1,
gamma0/gamma1+tau(1))-gamma0.*A1(gamma0/gamma1,gamma0/gamma1+tau(1))./gamma1)./gamma1...
+exp(beta.*s2(tau))*exp(gamma0.^2.*exp(beta.*s2(tau))/(2.*gamma1))*(B2(gamma0/gamma1+a(tau),
gamma0/gamma1+tau(2)-tau(1)+a(tau),tau)...
-gamma0.*A2(gamma0/gamma1+a(tau),gamma0/gamma1+tau(2)-tau(1)+a(tau),tau)./gamma1)./gamma1...
+exp(beta.*s3)*exp(gamma0.^2.*exp(beta.*s3)/(2.*gamma1))*(B3(gamma0/gamma1+b(tau),gamma0/
gamma1+c-tau(2)+b(tau))...
-gamma0.*A3(gamma0/gamma1+b(tau),gamma0/gamma1+c-tau(2)+b(tau))./gamma1)./gamma1
Mb=@(tau)Be1(tau,0,tau(1))+Be2(tau,tau(1),tau(2))+Be3(tau,tau(2),c)+Be3(tau,c,inf)
Dloss=@(tau)1/((M0(tau).*M1(tau)-M01(tau).^2).^n.^3*Mb(tau))
Aloss=@(tau)M1(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2))+M0(tau)./(n.*(M0(tau).*
M1(tau)-M01(tau).^2))+1./(n.*Mb(tau))
t=87600
Qloss=@(tau)exp(beta.*s).*(exp(beta.*s).*M1(tau).*t./((n.*(M0(tau).*M1(tau)-M01(tau).^2))-
(1/2).*t.^2.*exp(beta.*s).*M01(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2)))...
+exp(beta.*s).*((1/3).*exp(beta.*s).*M0(tau).*t.^3./((n.*(M0(tau).*M1(tau)-M01(tau).^2))-
(1/2).*t.^2.*exp(beta.*s).*M01(tau)./(n.*(M0(tau).*M1(tau)-M01(tau).^2)))...
+(1/3).*(gamma1.*t+gamma0).^3.*(exp(beta.*s)).^2.*s.^2./((n.*Mb(tau).*gamma1)
A=[1 -1 0]
B=[0]
tau0=[180 250 160]
lb=[0;0;155];
ub=[inf;inf;240];
%This code is the constraint function
function [c, ceq] = confunthree(tau)
t1 = 145; t3 = 250; t2 =@(tau)tau(3);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 =@(tau) 1/(273.16+

```

```

t2(tau)); gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;
c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2)*exp(beta.*s1))+
exp(-(gamma0.*a+(1/2)*gamma1.*a.^2)*exp(beta.*s2(tau))), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau)+tau(2)-tau(1))+(1/2).*gamma1.*(a(tau)+
tau(2)-tau(1))^2).*exp(beta.*s2(tau)))+exp(-(gamma0.*b+(1/2).*gamma1.*b.^2)*exp(beta.*s3)), 100)
c1=(1./2)*gamma1.*exp(beta.*s1);
c2=@(tau)(1./2)*gamma1.*exp(beta.*s2(tau));
c3=(1./2)*gamma1.*exp(beta.*s3);
lambda0=@(t)gamma0+gamma1.*t+gamma2.*t.^2;
f19=@(t)(lambda0(t).*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s1)))
f20=@(t,tau)(lambda0(t).*exp(beta.*s2(tau)).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s2(tau))))
f21=@(t)(lambda0(t).*exp(beta.*s3).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s3)))
W1=@(a,b)quad(f19,a,b);W2=@(tau,a,b)quadgk(@(t)f20(t,tau),a,b)W3=@(a,b)quad(f21,a,b);
h1=@(tau)W1(0,tau(1))
h2=@(tau)W2(tau,a(tau),a(tau)+tau(2)-tau(1))
h3=@(tau)W3(b(tau),c)
c(1)=20-n.*h1(tau)
c(2)=10-(n-n.*h1(tau)).*h2(tau)
c(3)=10-(n-n.*h1(tau)-(n-n.*h1(tau)).*h2(tau)).*h3(tau)
ceq = [];
end

```

## C: Codes for subsection 3.4.1

%This code is for computing the optimal stress-changing times  $\tau_1$  and  $\tau_2$ .

```

code for Matlabs
t1 = 145; t3 = 250; t2 = (t1+t3)*(1/2);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 = 1/(273.16+t2);
gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;
c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2+(1/3).*gamma2.*tau(1).^3)*
exp(beta.*s1))+exp(-(gamma0.*a+(1/2)*gamma1.*a.^2+(1/3).*gamma2.*a.^3)*exp(beta.*s2)), 40)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau(1))+tau(2)-tau(1))+(1/2).*gamma1.*
(a(tau(1))+tau(2)-tau(1))^2+(1/3).*gamma2.*(a(tau(1))+tau(2)-tau(1))^3).*exp(beta.*s2))+exp(-
(gamma0.*b+(1/2).*gamma1.*b.^2+(1/3).*gamma2.*b.^3)*exp(beta.*s3)), 80)
f1=@(t) exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s1))./
(gamma2.*t.^2+gamma1.*t+gamma0);

```





```

f17=@(t,tau) (s2.^2.*(gamma0.*(a(tau)+t-tau(1))+(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+
(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*exp(beta.*s2)).*(gamma0+gamma1.*(a(tau)+t-tau(1))+
gamma2.*(a(tau)+t-tau(1)).^2).*exp(beta.*s2)).*exp(-(gamma0.*(a(tau)+t-tau(1))+(1/2).*
gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*exp(beta.*s2));
f18=@(t,tau) (s3.^2.*(gamma0.*(b(tau)+t-tau(2))+(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+
(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*s3)).*(gamma0+gamma1.*(b(tau)+t-tau(2))+
gamma2.*(b(tau)+t-tau(2)).^2).*exp(beta.*s3)).*exp(-(gamma0.*(b(tau)+t-tau(2))+(1/2).*
gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*s3));
A1=@(a,b) quad(f1,a,b); A2=@(tau,a,b) quadgk(@(t) f2(t,tau),a,b); A3=@(tau,a,b) quadgk(@(t) f3(t,tau),a,b);
B1=@(a,b) quad(f4,a,b); B2=@(tau,a,b) quadgk(@(t) f5(t,tau),a,b); B3=@(tau,a,b) quadgk(@(t) f6(t,tau),a,b);
C1=@(a,b) quad(f7,a,b); C2=@(tau,a,b) quadgk(@(t) f8(t,tau),a,b); C3=@(tau,a,b) quadgk(@(t) f9(t,tau),a,b);
D1=@(a,b) quad(f10,a,b); D2=@(tau,a,b) quadgk(@(t) f11(t,tau),a,b); D3=@(tau,a,b) quadgk(@(t) f12(t,tau),a,b);
E1=@(a,b) quad(f13,a,b); E2=@(tau,a,b) quadgk(@(t) f14(t,tau),a,b); E3=@(tau,a,b) quadgk(@(t) f15(t,tau),a,b);
G1=@(a,b) quad(f16,a,b); G2=@(tau,a,b) quadgk(@(t) f17(t,tau),a,b); G3=@(tau,a,b) quadgk(@(t) f18(t,tau),a,b);
%This code is the constraint function
function [c, ceq] = confun3(tau)
t1 = 145; t3 = 250; t2 = (t1+t3)*(1/2);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 = 1/(273.16+t2); gamma0 = 0.0001;
gamma1 = 0.5; gamma2 = 0; beta = -3800; c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2).*gamma1.*tau(1).^2+(1/3).*gamma2.*tau(1).^3)*
exp(beta.*s1))+exp(-(gamma0.*a+(1/2).*gamma1.*a.^2+(1/3).*gamma2.*a.^3).*exp(beta.*s2)), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau(1))+tau(2)-tau(1))+(1/2).*gamma1.*(a(tau(1))+tau(2)-
tau(1)).^2+(1/3).*gamma2.*(a(tau(1))+tau(2)-tau(1)).^3).*exp(beta.*s2))+exp(-(gamma0.*b+
(1/2).*gamma1.*b.^2+(1/3).*gamma2.*b.^3).*exp(beta.*s3)), 10)
c1=(1./2).*gamma1.*exp(beta.*s1);c2=(1./2).*gamma1.*exp(beta.*s2);c3=(1./2).*gamma1.*exp(beta.*s3);
lambda0=@(t)gamma0+gamma1.*t+gamma2.*t.^2;
f19=@(t)(lambda0(t).*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s1)))
f20=@(t)(lambda0(t).*exp(beta.*s2).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s2)))
f21=@(t)(lambda0(t).*exp(beta.*s3).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s3)))
W1=@(a,b)quad(f19,a,b);
W2=@(a,b)quad(f20,a,b);
W3=@(a,b)quad(f21,a,b);
h1=@(tau)W1(0,tau(1))
h2=@(tau)W2(a(tau(1)),a(tau(1))+tau(2)-tau(1))
h3=@(tau)W3(b([tau(1) tau(2)]),c)
c(1)=40-n.*h1(tau)
c(2)=20-(n-n.*h1(tau)).*h2(tau)
c(3)=10-(n-n.*h1(tau))-(n-n.*h1(tau)).*h2(tau)).*h3(tau)
ceq = [];
end

```

## D: Codes for subsection 3.4.2

%This code is for computing the optimal stress-changing times  $\tau_1$ ,  $\tau_2$  and the optimal middle stress level.

```

code for Matlabs
t1 = 145; t3 = 250; t2 =@(tau)tau(3);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 =@(tau) 1/(273.16+t2(tau));
gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800;
c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2)*exp(beta.*s1))+exp(-(gamma0.*
a+(1/2)*gamma1.*a.^2)*exp(beta.*s2(tau))), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau)+tau(2)-tau(1))+(1/2).*gamma1.*(a(tau)+tau(2)-tau(1))^2).*
exp(beta.*s2(tau)))+exp(-(gamma0.*b+(1/2).*gamma1.*b.^2)*exp(beta.*s3)), 100)
f1=@(t) exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s1))./
(gamma2.*t.^2+gamma1.*t+gamma0);
f2=@(t,tau) exp(beta.*s2(tau)).*exp(-(gamma0.*(a(tau)+t-tau(1))+(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+
(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*exp(beta.*s2(tau)))./(gamma0+gamma1.*(a(tau)+t-tau(1))+
gamma2.*(a(tau)+t-tau(1)).^2);
f3=@(t,tau) exp(beta.*s3).*exp(-(gamma0.*(b(tau)+t-tau(2))+(1/2).*gamma1.*(b(tau)+
t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*s3))./(gamma0+gamma1.*
(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2);
f4=@(t) t.^2.*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*
s1))./(gamma2.*t.^2+gamma1.*t+gamma0);
f5=@(t,tau) (a(tau)+t-tau(1)).^2.*exp(beta.*s2(tau)).*exp(-(gamma0.*(a(tau)+t-tau(1))+
(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*
exp(beta.*s2(tau)))./(gamma0+gamma1.*(a(tau)+t-tau(1))+gamma2.*(a(tau)+t-tau(1)).^2);
f6=@(t,tau) (b(tau)+t-tau(2)).^2.*exp(beta.*s3).*exp(-(gamma0.*(b(tau)+t-tau(2))+
(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*s3))./
(gamma0+gamma1.*(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2);
f7=@(t) t.^4.*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s1))./
(gamma2.*t.^2+gamma1.*t+gamma0);
f8=@(t,tau) (a(tau)+t-tau(1)).^4.*exp(beta.*s2(tau)).*exp(-(gamma0.*(a(tau)+t-tau(1))+
(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*exp(beta.*
s2(tau)))./(gamma0+gamma1.*(a(tau)+t-tau(1))+gamma2.*(a(tau)+t-tau(1)).^2);
f9=@(t,tau) (b(tau)+t-tau(2)).^4.*exp(beta.*s3).*exp(-(gamma0.*(b(tau)+t-tau(2))+
(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*
s3))./(gamma0+gamma1.*(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2);
f10=@(t) t.*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*
t.^3).*exp(beta.*s1))./(gamma2.*t.^2+gamma1.*t+gamma0);
f11=@(t,tau) (a(tau)+t-tau(1)).*exp(beta.*s2(tau)).*exp(-(gamma0.*(a(tau)+t-
tau(1))+(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*
exp(beta.*s2(tau)))./(gamma0+gamma1.*(a(tau)+t-tau(1))+gamma2.*(a(tau)+t-tau(1)).^2);
f12=@(t,tau) (b(tau)+t-tau(2)).*exp(beta.*s3).*exp(-(gamma0.*(b(tau)+t-tau(2))+

```

```

(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*
s3))./(gamma0+gamma1.*(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2);
f13=@(t) t.^3.*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*
t.^3).*exp(beta.*s1))./(gamma2.*t.^2+gamma1.*t+gamma0);
f14=@(t,tau) (a(tau)+t-tau(1)).^3.*exp(beta.*s2(tau)).*exp(-(gamma0.*(a(tau)+t-
tau(1))+(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*
exp(beta.*s2(tau)))./(gamma0+gamma1.*(a(tau)+t-tau(1))+gamma2.*(a(tau)+t-tau(1)).^2);
f15=@(t,tau) (b(tau)+t-tau(2)).^3.*exp(beta.*s3).*exp(-(gamma0.*(b(tau)+t-tau(2))+
(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*
s3))./(gamma0+gamma1.*(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2);
f16=@(t) (s1.^2.*(gamma0.*t+(1/2).*gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*
s1)).*(gamma2.*t.^2+gamma1.*t+gamma0).*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*
gamma1.*t.^2+(1/3).*gamma2.*t.^3).*exp(beta.*s1));
f17=@(t,tau) (s2(tau).^2.*(gamma0.*(a(tau)+t-tau(1))+(1/2).*gamma1.*(a(tau)+t-
tau(1)).^2+(1/3).*gamma2.*(a(tau)+t-tau(1)).^3).*exp(beta.*s2(tau))).*(gamma0+
gamma1.*(a(tau)+t-tau(1))+gamma2.*(a(tau)+t-tau(1)).^2).*exp(beta.*s2(tau))).*exp(-
(gamma0.*(a(tau)+t-tau(1))+(1/2).*gamma1.*(a(tau)+t-tau(1)).^2+(1/3).*gamma2.*
(a(tau)+t-tau(1)).^3).*exp(beta.*s2(tau)));
f18=@(t,tau) (s3.^2.*(gamma0.*(b(tau)+t-tau(2))+(1/2).*gamma1.*(b(tau)+t-
tau(2)).^2+(1/3).*gamma2.*(b(tau)+t-tau(2)).^3).*exp(beta.*s3)).*(gamma0+
gamma1.*(b(tau)+t-tau(2))+gamma2.*(b(tau)+t-tau(2)).^2).*exp(beta.*s3)).*exp(-(gamma0.*
(b(tau)+t-tau(2))+(1/2).*gamma1.*(b(tau)+t-tau(2)).^2+(1/3).*gamma2.*
(b(tau)+t-tau(2)).^3).*exp(beta.*s3));
A1=@(a,b) quad(f1,a,b); A2=@(tau,a,b) quadgk(@(t) f2(t,tau),a,b); A3=@(tau,a,b) quadgk(@(t) f3(t,tau),a,b);
B1=@(a,b) quad(f4,a,b); B2=@(tau,a,b) quadgk(@(t) f5(t,tau),a,b); B3=@(tau,a,b) quadgk(@(t) f6(t,tau),a,b);
C1=@(a,b) quad(f7,a,b); C2=@(tau,a,b) quadgk(@(t) f8(t,tau),a,b); C3=@(tau,a,b) quadgk(@(t) f9(t,tau),a,b);
D1=@(a,b) quad(f10,a,b); D2=@(tau,a,b) quadgk(@(t) f11(t,tau),a,b); D3=@(tau,a,b) quadgk(@(t) f12(t,tau),a,b);
E1=@(a,b) quad(f13,a,b); E2=@(tau,a,b) quadgk(@(t) f14(t,tau),a,b); E3=@(tau,a,b) quadgk(@(t) f15(t,tau),a,b);
G1=@(a,b) quad(f16,a,b); G2=@(tau,a,b) quadgk(@(t) f17(t,tau),a,b); G3=@(tau,a,b) quadgk(@(t) f18(t,tau),a,b);
M0=@(tau) A1(0,tau(1))+A2(tau,tau(1),tau(2))+A3(tau,tau(2),c);
M1=@(tau) B1(0,tau(1))+B2(tau,tau(1),tau(2))+B3(tau,tau(2),c);
M2=@(tau) C1(0,tau(1))+C2(tau,tau(1),tau(2))+C3(tau,tau(2),c);
M01=@(tau) D1(0,tau(1))+D2(tau,tau(1),tau(2))+D3(tau,tau(2),c);
M02=@(tau) B1(0,tau(1))+B2(tau,tau(1),tau(2))+B3(tau,tau(2),c);
M12=@(tau) E1(0,tau(1))+E2(tau,tau(1),tau(2))+E3(tau,tau(2),c);
Mb=@(tau) G1(0,tau(1))+G2(tau,tau(1),tau(2))+G3(tau,tau(2),c)+G3(tau,c,inf);
Dloss=@(tau) 1/(n.^4.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*M2(tau)+
2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)).*Mb(tau));
Aloss=@(tau) (M1(tau).*M2(tau)-M12(tau).^2)./(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*
M12(tau).^2-M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))
+(M0(tau).*M2(tau)-M02(tau).^2)./(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*
M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...

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+(M0(tau).*M1(tau)-M01(tau).^2)./(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*
M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))+1/(n.*Mb(tau))
T=87600
Qloss=@(tau)exp(beta.*s).*(exp(beta.*s).*(M1(tau).*M2(tau)-M12(tau).^2).*T./(n.*
(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*M2(tau)+2.*
M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...-(1/2).*T.^2.*exp(beta.*s).*
(M01(tau).*M2(tau)-M02(tau).*M12(tau))./(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-
M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
+(1/3).*T.^3.*exp(beta.*s).*(M01(tau).*M12(tau)-M02(tau).*M1(tau))./(n.*(M0(tau).*
M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*
M12(tau)-M02(tau).^2.*M1(tau)))...+exp(beta.*s).*(-1/4).*exp(beta.*s).*(M0(tau).*
M12(tau)-M01(tau).*M02(tau)).*T.^4./(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*
M12(tau).^2-M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
+(1/3).*exp(beta.*s).*(M0(tau).*M2(tau)-M02(tau).^2).*T.^3./(n.*(M0(tau).*
M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-M01(tau).^2.*
M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
-(1/2).*T.^2.*exp(beta.*s).*(M01(tau).*M2(tau)-M02(tau).*M12(tau))./(n.*
(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-
M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
+exp(beta.*s).*((1/5).*exp(beta.*s).*(M0(tau).*M1(tau)-M01(tau).^2).*T.^5./
(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-
M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
-(1/4).*exp(beta.*s).*(M0(tau).*M12(tau)-M01(tau).*M02(tau)).*T.^4./
(n.*(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-
M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
+(1/3).*T.^3.*exp(beta.*s).*(M01(tau).*M12(tau)-M02(tau).*M1(tau))./(n.*
(M0(tau).*M1(tau).*M2(tau)-M0(tau).*M12(tau).^2-
M01(tau).^2.*M2(tau)+2.*M01(tau).*M02(tau).*M12(tau)-M02(tau).^2.*M1(tau)))...
+(exp(beta.*s)).^2.*s.^2.*((1/5).*gamma2.^2.*T.^5+(1/2).*gamma1.*gamma2.*
T.^4+(1/3.*(2.*gamma0.*gamma2+gamma1.^2)).*T.^3+
gamma0.*gamma1.*T.^2+gamma0.^2.*T)/(n.*Mb(tau))
A=[1 -1 0]
B=[0]
tau0=[180 250 155]
lb=[0;0;155];
ub=[inf;inf;240];
%This code is the constraint function
function [c, ceq] = confun42(tau)
t1 = 145; t3 = 250; t2 =@(tau)tau(3);
s = 1/(273.16+50); s1 = 1/(273.16+t1); s3 = 1/(273.16+t3); s2 =@(tau) 1/(273.16+t2(tau));
gamma0 = 0.0001; gamma1 = 0.5; gamma2 = 0; beta = -3800; c = 300; n = 200;
a=@(tau) fsolve(@(a)-exp(-(gamma0.*tau(1)+(1/2)*gamma1.*tau(1).^2)*exp(beta.*s1))+

```

```

exp(-(gamma0.*a+(1/2)*gamma1.*a.^2)*exp(beta.*s2(tau))), 10)
b=@(tau) fsolve(@(b)-exp(-(gamma0.*(a(tau)+tau(2))-tau(1))+(1/2).*gamma1.*(a(tau)+tau(2)-
tau(1))^2).*exp(beta.*s2(tau)))+exp(-(gamma0.*b+(1/2).*gamma1.*b.^2)*exp(beta.*s3)), 100)
lambda0=@(t)gamma0+gamma1.*t+gamma2.*t.^2;
f19=@(t)(lambda0(t).*exp(beta.*s1).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s1)))
f20=@(t,tau)(lambda0(t).*exp(beta.*s2(tau)).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s2(tau))))
f21=@(t)(lambda0(t).*exp(beta.*s3).*exp(-(gamma0.*t+(1/2).*gamma1.*t.^2+
(1/3).*gamma2.*t.^3).*exp(beta.*s3)))
W1=@(a,b)quad(f19,a,b);W2=@(tau,a,b)quadgk(@(t)f20(t,tau),a,b);W3=@(a,b)quad(f21,a,b);
h1=@(tau)W1(0,tau(1))
h2=@(tau)W2(tau,a(tau),a(tau)+tau(2)-tau(1))
h3=@(tau)W3(b(tau),c)
% Nonlinear inequality constraints
c(1)=40-n.*h1(tau)
c(2)=10-(n-n.*h1(tau)).*h2(tau)
c(3)=10-(n-n.*h1(tau))-(n-n.*h1(tau)).*h2(tau).*h3(tau)
% Nonlinear equality constraints
ceq = [];
end

```

## E: Codes for the benchmark design

```

%This code is for computing the best parameter values for a true model but fitting a linear model.
code for Maple
with(Optimization);
Minimize(abs(int(gamma2*t^2+gamma1*t+gamma0-b*t-a, t = 0 .. 300)), assume = nonnegative);

%This code is for computing the ABIAS, AVAR and AMSE.
restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+197.5);
gamma0 := 0.1e-3; gamma1 := .5; gamma2 := 0.15e-2; beta := -3800;
gam0 := .7947; gam1 := .7947; theta := -3800;
lambda0 := gamma2*t^2+gamma1*t+gamma0;
'&lambda;t' := lambda0*exp(beta*s);
'&lambda;f' := gam1*t+gam0;
'&lambda;tf' := '&lambda;f'*exp(theta*s);
c := 300; n := 200; tau1 := 160; tau2 := 201;
Delta := solve(1-exp(-(gam0*tau1+(1/2)*gam1*tau1^2)*exp(theta*s1)) = 1-exp(-(gam0*a+(1/2)*
gam1*a^2)*exp(theta*s2)), a);

```

```

a := Delta[2];
Eta := solve(1-exp(-(gam0*(a+tau2-tau1)+(1/2)*gam1*(a+tau2-tau1)^2)*exp(theta*s2)) =
1-exp(-(gam0*b+(1/2)*gam1*b^2)*exp(theta*s3)), b);
b := Eta[2];
n1 := evalf(n*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = 0 .. tau1)));
n2 := evalf((n-n1)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s2)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s2)), t = a .. a+tau2-tau1)));
n3 := evalf((n-n1-n2)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s3)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s3)), t = b .. c)));
n4 := n-n1-n2-n3;
f1 := (gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*
gamma2*t^3)*exp(beta*s1));
f2 := (gamma0+gamma1*(a+t-tau1)+gamma2*(a+t-tau1)^2)*exp(beta*s2)*
exp(-(gamma0*(a+t-tau1)+(1/2)*gamma1*(a+t-tau1)^2+(1/3)*gamma2*(a+t-
tau1)^3)*exp(beta*s2));
f3 := (gamma0+gamma1*(b+t-tau2)+gamma2*(b+t-tau2)^2)*exp(beta*s3)*
exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*(b+t-tau2)^2+(1/3)*gamma2*(b+
t-tau2)^3)*exp(beta*s3));
A11 := n1*(int(f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int(f2/(gam0+
gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int(f3/(gam0+gam1*
(b+t-tau2))^2, t = tau2 .. c)));
A22 := n1*(int(t^2*f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int((a+
t-tau1)^2*f2/(gam0+gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int((b+t-
tau2)^2*f3/(gam0+gam1*(b+t-tau2))^2, t = tau2 .. c)));
A33 := evalf(n1*(int(s1^2*(gam0*t+(1/2)*gam1*t^2)*exp(theta*s1)*f1, t = 0 .. tau1))+
evalf(n2*(int(s2^2*(gam0*(a+t-tau1)+(1/2)*gam1*(a+t-tau1)^2)*exp(theta*s2)*
f2, t = tau1 .. tau2)))+evalf(n3*(int(s3^2*(gam0*(b+t-tau2)+(1/2)*gam1*(b+t-
tau2)^2)*exp(theta*s3)*f3, t = tau2 .. c)))+(n-n1-n2-n3)*s3^2*(gam0*(b+
c-tau2)+(1/2)*gam1*(b+c-tau2)^2)*exp(theta*s3)*(gamma0+gamma1*(b+
c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+
c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3));
A12 := n1*(int(t*f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int((a+t-tau1)*f2/
(gam0+gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int((b+t-tau2)*f3/
(gam0+gam1*(b+t-tau2))^2, t = tau2 .. c)));
A13 := evalf(n1*(int(s1*t*exp(theta*s1)*f1, t = 0 .. tau1))+evalf(n2*(int(s2*(a+
t-tau1)*exp(theta*s2)*f2, t = tau1 .. tau2)))+evalf(n3*(int(s3*(b+t-tau2)*exp(theta*
s3)*f3, t = tau2 .. c)))+(n-n1-n2-n3)*s3*(b+c-tau2)*exp(theta*s3)*(gamma0+
gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+c-
tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3));
A23 := evalf(n1*(int((1/2)*s1*t^2*exp(theta*s1)*f1, t = 0 .. tau1))+evalf(n2*(int((1/2)*
s2*(a+t-tau1)^2*exp(theta*s2)*f2, t = tau1 .. tau2)))+evalf(n3*(int((1/2)*s3*(b+t-

```

$\tau_2)^2 \exp(\theta s_3) f_3, t = \tau_2 \dots c)) + (1/2)^{(n-n_1-n_2-n_3)} s_3^3 (b+c-\tau_2)^2 \exp(\theta s_3) (\gamma_0 + \gamma_1 (b+c-\tau_2) + \gamma_2 (b+c-\tau_2)^2) \exp(\beta s_3) \exp(-(\gamma_0 (b+c-\tau_2) + (1/2) \gamma_1 (b+c-\tau_2)^2 + (1/3) \gamma_2 (b+c-\tau_2)^3) \exp(\beta s_3));$   
 $\text{'\&varpi;1' := int}(t^2 (\gamma_2 t^2 + \gamma_1 t + \gamma_0) \exp(\beta s_1) \exp(-(\gamma_0 t + (1/2) \gamma_1 t^2 + (1/3) \gamma_2 t^3) \exp(\beta s_1)), t = 0 \dots \tau_1);$   
 $\text{'\&varpi;2' := int}((a+t-\tau_1)^2 (\gamma_0 + \gamma_1 (a+t-\tau_1) + \gamma_2 (a+t-\tau_1)^2) \exp(\beta s_2) \exp(-(\gamma_0 (a+t-\tau_1) + (1/2) \gamma_1 (a+t-\tau_1)^2 + (1/3) \gamma_2 (a+t-\tau_1)^3) \exp(\beta s_2)), t = \tau_1 \dots \tau_2);$   
 $\text{'\&varpi;3' := int}((b+t-\tau_2)^2 (\gamma_0 + \gamma_1 (b+t-\tau_2) + \gamma_2 (b+t-\tau_2)^2) \exp(\beta s_3) \exp(-(\gamma_0 (b+t-\tau_2) + (1/2) \gamma_1 (b+t-\tau_2)^2 + (1/3) \gamma_2 (b+t-\tau_2)^3) \exp(\beta s_3)), t = \tau_2 \dots c);$   
 $\theta_1 := \text{int}(t^2 (\gamma_2 t^2 + \gamma_1 t + \gamma_0) \exp(\beta s_1) \exp(-(\gamma_0 t + (1/2) \gamma_1 t^2 + (1/3) \gamma_2 t^3) \exp(\beta s_1)), t = 0 \dots \tau_1);$   
 $\theta_2 := \text{int}((a+t-\tau_1)^2 (\gamma_0 + \gamma_1 (a+t-\tau_1) + \gamma_2 (a+t-\tau_1)^2) \exp(\beta s_2) \exp(-(\gamma_0 (a+t-\tau_1) + (1/2) \gamma_1 (a+t-\tau_1)^2 + (1/3) \gamma_2 (a+t-\tau_1)^3) \exp(\beta s_2)), t = \tau_1 \dots \tau_2);$   
 $\theta_3 := \text{int}((b+t-\tau_2)^2 (\gamma_0 + \gamma_1 (b+t-\tau_2) + \gamma_2 (b+t-\tau_2)^2) \exp(\beta s_3) \exp(-(\gamma_0 (b+t-\tau_2) + (1/2) \gamma_1 (b+t-\tau_2)^2 + (1/3) \gamma_2 (b+t-\tau_2)^3) \exp(\beta s_3)), t = \tau_2 \dots c);$   
 $B11 := \text{evalf}((n_1 \text{int}((1/(\gamma_1 t + \gamma_0) - t \exp(\theta s_1)) f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2 \text{int}((1/(\gamma_0 + \gamma_1 (a+t-\tau_1)) - (a+t-\tau_1) \exp(\theta s_2)) f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3 \text{int}((1/(\gamma_0 + \gamma_1 (b+t-\tau_2)) - (b+t-\tau_2) \exp(\theta s_3)) f_3, t = \tau_2 \dots c))) - (n-n_1-n_2-n_3) (1/(\gamma_0 + \gamma_1 (b+c-\tau_2)) - (b+c-\tau_2) \exp(\theta s_3)) (\gamma_0 + \gamma_1 (b+c-\tau_2) + \gamma_2 (b+c-\tau_2)^2) \exp(\beta s_3) \exp(-(\gamma_0 (b+c-\tau_2) + (1/2) \gamma_1 (b+c-\tau_2)^2 + (1/3) \gamma_2 (b+c-\tau_2)^3) \exp(\beta s_3)))^2 + n_1 (-\text{'\&varpi;1' }^2 + \theta_1) (\gamma_1^2 / (\gamma_1 \text{'\&varpi;1' } + \gamma_0)^4 + \exp(2 \theta s_1)) + n_2 (-\text{'\&varpi;2' }^2 + \theta_2) (\gamma_1^2 / (\gamma_1 \text{'\&varpi;2' } + \gamma_0)^4 + \exp(2 \theta s_2)) + n_3 (-\text{'\&varpi;3' }^2 + \theta_3) (\gamma_1^2 / (\gamma_1 \text{'\&varpi;3' } + \gamma_0)^4 + \exp(2 \theta s_3));$   
 $B22 := (n_1 \text{int}((t/(\gamma_1 t + \gamma_0) - (1/2) t^2 \exp(\theta s_1)) f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2 \text{int}(((a+t-\tau_1)/(\gamma_0 + \gamma_1 (a+t-\tau_1)) - (1/2) (a+t-\tau_1)^2 \exp(\theta s_2)) f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3 \text{int}(((b+t-\tau_2)/(\gamma_0 + \gamma_1 (b+t-\tau_2)) - (1/2) (b+t-\tau_2)^2 \exp(\theta s_3)) f_3, t = \tau_2 \dots c))) - (n-n_1-n_2-n_3) ((b+c-\tau_2)/(\gamma_0 + \gamma_1 (b+c-\tau_2)) - (1/2) (b+c-\tau_2)^2 \exp(\theta s_3)) (\gamma_0 + \gamma_1 (b+c-\tau_2) + \gamma_2 (b+c-\tau_2)^2) \exp(\beta s_3) \exp(-(\gamma_0 (b+c-\tau_2) + (1/2) \gamma_1 (b+c-\tau_2)^2 + (1/3) \gamma_2 (b+c-\tau_2)^3) \exp(\beta s_3)))^2 + n_1 (-\text{'\&varpi;1' }^2 + \theta_1) ((1/(\gamma_1 \text{'\&varpi;1' } + \gamma_0) + \text{'\&varpi;1' } \gamma_1 / (\gamma_1 \text{'\&varpi;1' } + \gamma_0)^2)^2 + \exp(2 \theta s_1) \text{'\&varpi;1' }^2) + n_2 (-\text{'\&varpi;2' }^2 + \theta_2) ((1/(\gamma_1 \text{'\&varpi;2' } + \gamma_0) + \text{'\&varpi;2' } \gamma_1 / (\gamma_1 \text{'\&varpi;2' } + \gamma_0)^2)^2 + \exp(2 \theta s_2) \text{'\&varpi;2' }^2) + n_3 (-\text{'\&varpi;3' }^2 + \theta_3) ((1/(\gamma_1 \text{'\&varpi;3' } + \gamma_0) + \text{'\&varpi;3' } \gamma_1 /$





```

exp(theta*s2)*f2, t = tau1 .. tau2))) + evalf(n3*s3-n3*(int(s3*(gam0*(b+t-tau2)+(1/2)*gam1*(b+t-
tau2)^2)*exp(theta*s3)*f3, t = tau2 .. c)))-(n-n1-n2-n3)*s3*(gam0*(b+c-tau2)+
(1/2)*gam1*c^2)*exp(theta*s3)*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*
exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3)))
T := 87600;
ABIAS := int((-gamma2*t^2+gam1*t-gamma1*t+gam0-gamma0)*exp(beta*s), t = 0 .. T);
A := Matrix(3, 3, {(1, 1) = A11, (1, 2) = A12, (1, 3) = A13, (2, 1) = A21, (2, 2) = A22, (2, 3) = A23,
(3, 1) = A31, (3, 2) = A32, (3, 3) = A33}); B := Matrix(3, 3, {(1, 1) = B11, (1, 2) = B12, (1, 3) = B13,
(2, 1) = B21, (2, 2) = B22, (2, 3) = B23, (3, 1) = B31, (3, 2) = B32, (3, 3) = B33}); C := 1/A.B.(1/A)
R := (gam0*tau1+(1/2)*gam1*tau1^2)*exp(theta*s)
AVAR := Vector[row](3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)}).C.Vector(
3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)})
AMSE := ABIAS^2+AVAR:

```

## F: Codes for subsections 4.4.2 and 4.4.3

```

%This code is for fitting linear model and getting MLEs of  $\gamma_0$ ,  $\gamma_1$  and  $\beta$  in Subsection 4.4.2;
and this code is also for Subsection 4.4.3 in each iteration.
%Global searching for  $\tau_1$  and  $\tau_2$  under constraint  $C_1$ 
code for maple
restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+197.5);
gamma0 := 0.1e-3; gamma1 := .5; gamma2 := 0.15e-2; beta := -3800;
gam0 := .7947; gam1 := .7947; theta := -3800;
lambda0 := gamma2*t^2+gamma1*t+gamma0;
'&lambda;t' := lambda0*exp(beta*s);
'&lambda;f' := gam1*t+gam0;
'&lambda;tf' := '&lambda;f'*exp(theta*s);
c := 300; n := 200; tau1 := 160; tau2 := 201;
Delta := solve(1-exp(-(gam0*tau1+(1/2)*gam1*tau1^2)*exp(theta*s1)) = 1-exp(-(gam0*a+(1/2)*
gam1*a^2)*exp(theta*s2)), a);
a := Delta[2];
Eta := solve(1-exp(-(gam0*(a+tau2-tau1)+(1/2)*gam1*(a+tau2-tau1)^2)*exp(theta*s2)) =
1-exp(-(gam0*b+(1/2)*gam1*b^2)*exp(theta*s3)), b);
b := Eta[2];
n1 := evalf(n*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = 0 .. tau1)));
n2 := evalf((n-n1)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s2)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s2)), t = a .. a+tau2-tau1)));
n3 := evalf((n-n1-n2)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s3)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s3)), t = b .. c)));

```

$n4 := n-n1-n2-n3;$   
 $f1 := (\text{gamma}2*t^2+\text{gamma}1*t+\text{gamma}0)*\exp(\text{beta}*s1)*\exp(-(\text{gamma}0*t+(1/2)*\text{gamma}1*t^2+(1/3)*\text{gamma}2*t^3)*\exp(\text{beta}*s1));$   
 $f2 := (\text{gamma}0+\text{gamma}1*(a+t-\text{tau}1)+\text{gamma}2*(a+t-\text{tau}1)^2)*\exp(\text{beta}*s2)*\exp(-(\text{gamma}0*(a+t-\text{tau}1)+(1/2)*\text{gamma}1*(a+t-\text{tau}1)^2+(1/3)*\text{gamma}2*(a+t-\text{tau}1)^3)*\exp(\text{beta}*s2));$   
 $f3 := (\text{gamma}0+\text{gamma}1*(b+t-\text{tau}2)+\text{gamma}2*(b+t-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+t-\text{tau}2)+(1/2)*\text{gamma}1*(b+t-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+t-\text{tau}2)^3)*\exp(\text{beta}*s3));$   
 $A11 := n1*(\text{int}(f1/(\text{gam}1*t+\text{gam}0)^2, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}(f2/(\text{gam}0+\text{gam}1*(a+t-\text{tau}1))^2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}(f3/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))^2, t = \text{tau}2 .. c)));$   
 $A22 := n1*(\text{int}(t^2*f1/(\text{gam}1*t+\text{gam}0)^2, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}((a+t-\text{tau}1)^2*f2/(\text{gam}0+\text{gam}1*(a+t-\text{tau}1))^2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}((b+t-\text{tau}2)^2*f3/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))^2, t = \text{tau}2 .. c)));$   
 $A33 := \text{evalf}(n1*(\text{int}(s1^2*(\text{gam}0*t+(1/2)*\text{gam}1*t^2)*\exp(\text{theta}*s1)*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}(s2^2*(\text{gam}0*(a+t-\text{tau}1)+(1/2)*\text{gam}1*(a+t-\text{tau}1)^2)*\exp(\text{theta}*s2)*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}(s3^2*(\text{gam}0*(b+t-\text{tau}2)+(1/2)*\text{gam}1*(b+t-\text{tau}2)^2)*\exp(\text{theta}*s3)*f3, t = \text{tau}2 .. c)))+(n-n1-n2-n3)*s3^2*(\text{gam}0*(b+c-\text{tau}2)+(1/2)*\text{gam}1*(b+c-\text{tau}2)^2)*\exp(\text{theta}*s3)*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3));$   
 $A12 := n1*(\text{int}(t*f1/(\text{gam}1*t+\text{gam}0)^2, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}((a+t-\text{tau}1)*f2/(\text{gam}0+\text{gam}1*(a+t-\text{tau}1))^2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}((b+t-\text{tau}2)*f3/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))^2, t = \text{tau}2 .. c)));$   
 $A13 := \text{evalf}(n1*(\text{int}(s1*t*\exp(\text{theta}*s1)*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}(s2*(a+t-\text{tau}1)*\exp(\text{theta}*s2)*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}(s3*(b+t-\text{tau}2)*\exp(\text{theta}*s3)*f3, t = \text{tau}2 .. c)))+(n-n1-n2-n3)*s3*(b+c-\text{tau}2)*\exp(\text{theta}*s3)*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3));$   
 $A23 := \text{evalf}(n1*(\text{int}((1/2)*s1*t^2*\exp(\text{theta}*s1)*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\text{int}((1/2)*s2*(a+t-\text{tau}1)^2*\exp(\text{theta}*s2)*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\text{int}((1/2)*s3*(b+t-\text{tau}2)^2*\exp(\text{theta}*s3)*f3, t = \text{tau}2 .. c)))+(1/2)*(n-n1-n2-n3)*s3*(b+c-\text{tau}2)^2*\exp(\text{theta}*s3)*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3));$   
 $\text{'\&varpi;1' := int}(t*(\text{gamma}2*t^2+\text{gamma}1*t+\text{gamma}0)*\exp(\text{beta}*s1)*\exp(-(\text{gamma}0*t+(1/2)*\text{gamma}1*t^2+(1/3)*\text{gamma}2*t^3)*\exp(\text{beta}*s1)), t = 0 .. \text{tau}1);$   
 $\text{'\&varpi;2' := int}((a+t-\text{tau}1)*(\text{gamma}0+\text{gamma}1*(a+t-\text{tau}1)+\text{gamma}2*(a+t-\text{tau}1)^2)*\exp(\text{beta}*s2)*\exp(-(\text{gamma}0*(a+t-\text{tau}1)+(1/2)*\text{gamma}1*(a+t-\text{tau}1)^2+(1/3)*\text{gamma}2*(a+t-\text{tau}1)^3)*\exp(\text{beta}*s2)), t = \text{tau}1 .. \text{tau}2);$

```

'&varpi;3' := int((b+t-tau2)*(gamma0+gamma1*(b+t-tau2)+gamma2*
(b+t-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*(b+
t-tau2)^2+(1/3)*gamma2*(b+t-tau2)^3)*exp(beta*s3)), t = tau2 .. c);
theta1 := int(t^2*(gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-
(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = 0 .. tau1);
theta2 := int((a+t-tau1)^2*(gamma0+gamma1*(a+t-tau1)+gamma2*
(a+t-tau1)^2)*exp(beta*s2)*exp(-(gamma0*(a+t-tau1)+(1/2)*gamma1*(a+t-
tau1)^2+(1/3)*gamma2*(a+t-tau1)^3)*exp(beta*s2)), t = tau1 .. tau2);
theta3 := int((b+t-tau2)^2*(gamma0+gamma1*(b+t-tau2)+gamma2*
(b+t-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*
(b+t-tau2)^2+(1/3)*gamma2*(b+t-tau2)^3)*exp(beta*s3)), t = tau2 .. c);
B11 := evalf((n1*(int((1/(gam1*t+gam0))-t*exp(theta*s1))*f1, t = 0 .. tau1))+
evalf(n2*(int((1/(gam0+gam1*(a+t-tau1)))-(a+t-tau1)*exp(theta*s2))*f2, t =
tau1 .. tau2)))+evalf(n3*(int((1/(gam0+gam1*(b+t-tau2)))-(b+t-tau2)*
exp(theta*s3))*f3, t = tau2 .. c)))-(n-n1-n2-n3)*(1/(gam0+gam1*(b+c-tau2))-
(b+c-tau2)*exp(theta*s3))*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+
c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*
(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3))^2+n1*
(-'&varpi;1'^2+theta1)*(gam1^2/(gam1*'&varpi;1'+gam0)^4+exp(2*theta*
s1))+n2*(-'&varpi;2'^2+theta2)*(gam1^2/(gam1*'&varpi;2'+gam0)^4+exp(2*
theta*s2))+n3*(-'&varpi;3'^2+theta3)*(gam1^2/(gam1*'&varpi;3'+gam0)^4+
exp(2*theta*s3));
B22 := (n1*(int((t/(gam1*t+gam0)-(1/2)*t^2*exp(theta*s1))*f1, t = 0 .. tau1))+
evalf(n2*(int(((a+t-tau1)/(gam0+gam1*(a+t-tau1)))-(1/2)*(a+t-tau1)^2*exp(theta*s2))*
f2, t = tau1 .. tau2)))+evalf(n3*(int(((b+t-tau2)/(gam0+gam1*(b+t-tau2)))-(1/2)*(b+t-tau2)^2*
exp(theta*s3))*f3, t = tau2 .. c)))-(n-n1-n2-n3)*((b+c-tau2)/(gam0+gam1*(b+c-tau2))-(1/2)*
(b+c-tau2)^2*exp(theta*s3))*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*
exp(beta*s3)*exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-
tau2)^3)*exp(beta*s3))^2+n1*(-'&varpi;1'^2+theta1)*((1/(gam1*'&varpi;1'+gam0)+'&varpi;1'*
gam1/(gam1*'&varpi;1'+gam0)^2)^2+exp(2*theta*s1)*'&varpi;1'^2)+n2*(-'&varpi;2'^2+theta2)*
((1/(gam1*'&varpi;2'+gam0)+'&varpi;2'*gam1/(gam1*'&varpi;2'+gam0)^2)^2+exp(2*theta*s2)*'
&varpi;2'^2)+n3*(-'&varpi;3'^2+theta3)*((1/(gam1*'&varpi;3'+gam0)+'&varpi;3'*gam1/
(gam1*'&varpi;3'+gam0)^2)^2+exp(2*theta*s3)*'&varpi;3'^2);
B33 := (n1*s1-n1*(int(s1*(gam0*t+(1/2)*gam1*t^2)*exp(theta*s1)*f1, t = 0 .. tau1))+
evalf(n2*s2-n2*(int(s2*(gam0*(a+t-tau1)+(1/2)*gam1*(a+t-tau1)^2)*exp(theta*s2)*
f2, t = tau1 .. tau2)))+evalf(n3*s3-n3*(int(s3*(gam0*(b+t-tau2)+(1/2)*gam1*(b+t-tau2)^2)*
exp(theta*s3)*f3, t = tau2 .. c)))-(n-n1-n2-n3)*s3*(gam0*(b+c-tau2)+(1/2)*gam1*c^2)*exp(theta*s3)*
(gamma0+gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*
(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3))^2+
n1*s1^2*exp(2*theta*s1)*(gam1*'&varpi;1'+gam0)^2*(-'&varpi;1'^2+theta1)+n2*s2^2*exp(2*
theta*s2)*(gam1*'&varpi;2'+gam0)^2*(-'&varpi;2'^2+theta2)+n3*s3^2*exp(2*theta*s3)*(gam1*'
&varpi;3'+gam0)^2*(-'&varpi;3'^2+theta3);

```

$$\begin{aligned}
& B12 := (n1*(\int((1/(\text{gam}1*t+\text{gam}0))-t*\exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\int((1/ \\
& (\text{gam}0+\text{gam}1*(a+t-\text{tau}1))-(a+t-\text{tau}1)*\exp(\text{theta}*s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3* \\
& (\int((1/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))-(b+t-\text{tau}2)*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))-(\text{n-n}1-\text{n}2-\text{n}3)* \\
& (1/(\text{gam}0+\text{gam}1*(b+c-\text{tau}2))-(b+c-\text{tau}2)*\exp(\text{theta}*s3))*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+ \\
& \text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+ \\
& (1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))* (n1*(\int((t/(\text{gam}1*t+\text{gam}0))- \\
& (1/2)*t^2*\exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\int(((a+t-\text{tau}1)/(\text{gam}0+\text{gam}1* \\
& (a+t-\text{tau}1))-(1/2)*(a+t-\text{tau}1)^2*\exp(\text{theta}*s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\int(((b+ \\
& t-\text{tau}2)/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))-(1/2)*(b+t-\text{tau}2)^2*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))- \\
& (\text{n-n}1-\text{n}2-\text{n}3)*((b+c-\text{tau}2)/(\text{gam}0+\text{gam}1*(b+c-\text{tau}2))-(1/2)*(b+c-\text{tau}2)^2*\exp(\text{theta}*s3))* \\
& (\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0* \\
& (b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))) \\
& B13 := (n1*(\int((1/(\text{gam}1*t+\text{gam}0))-t*\exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*(\int((1/(\text{gam}0+ \\
& \text{gam}1*(a+t-\text{tau}1))-(a+t-\text{tau}1)*\exp(\text{theta}*s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\int((1/(\text{gam}0+\text{gam}1* \\
& (b+t-\text{tau}2))-(b+t-\text{tau}2)*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))-(\text{n-n}1-\text{n}2-\text{n}3)*(1/(\text{gam}0+\text{gam}1* \\
& (b+c-\text{tau}2))-(b+c-\text{tau}2)*\exp(\text{theta}*s3))*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2* \\
& (b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1* \\
& (b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))* (n1*s1-n1*(\int(s1* \\
& (\text{gam}0*t+(1/2)*\text{gam}1*t^2)*\exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*s2-n2* \\
& (\int(s2*(\text{gam}0*(a+t-\text{tau}1)+(1/2)*\text{gam}1*(a+t-\text{tau}1)^2)*\exp(\text{theta}*s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3* \\
& s3-n3*(\int(s3*(\text{gam}0*(b+t-\text{tau}2)+(1/2)*\text{gam}1*(b+t-\text{tau}2)^2)*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))- \\
& (\text{n-n}1-\text{n}2-\text{n}3)*s3*(\text{gam}0*(b+c-\text{tau}2)+(1/2)*\text{gam}1*c^2)*\exp(\text{theta}*s3)*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+ \\
& \text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)* \\
& \text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))) \\
& B23 := (n1*(\int((t/(\text{gam}1*t+\text{gam}0))-(1/2)*t^2*\exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+ \\
& \text{evalf}(n2*(\int(((a+t-\text{tau}1)/(\text{gam}0+\text{gam}1*(a+t-\text{tau}1))-(1/2)*(a+t-\text{tau}1)^2*\exp(\text{theta}* \\
& s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*(\int(((b+t-\text{tau}2)/(\text{gam}0+\text{gam}1*(b+t-\text{tau}2))-(1/2)* \\
& (b+t-\text{tau}2)^2*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))-(\text{n-n}1-\text{n}2-\text{n}3)*((b+c-\text{tau}2)/(\text{gam}0+ \\
& \text{gam}1*(b+c-\text{tau}2))-(1/2)*(b+c-\text{tau}2)^2*\exp(\text{theta}*s3))*(\text{gamma}0+\text{gamma}1*(b+ \\
& c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)*\exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c- \\
& \text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))* (n1*s1-n1*(\int(s1*(\text{gam}0*t+(1/2)*\text{gam}1*t^2)* \\
& \exp(\text{theta}*s1))*f1, t = 0 .. \text{tau}1))+\text{evalf}(n2*s2-n2*(\int(s2*(\text{gam}0*(a+t-\text{tau}1)+(1/2)*\text{gam}1*(a+t-\text{tau}1)^2)* \\
& \exp(\text{theta}*s2))*f2, t = \text{tau}1 .. \text{tau}2)))+\text{evalf}(n3*s3-n3*(\int(s3*(\text{gam}0*(b+t-\text{tau}2)+(1/2)*\text{gam}1*(b+t- \\
& \text{tau}2)^2)*\exp(\text{theta}*s3))*f3, t = \text{tau}2 .. c)))-(\text{n-n}1-\text{n}2-\text{n}3)*s3*(\text{gam}0*(b+c-\text{tau}2)+ \\
& (1/2)*\text{gam}1*c^2)*\exp(\text{theta}*s3)*(\text{gamma}0+\text{gamma}1*(b+c-\text{tau}2)+\text{gamma}2*(b+c-\text{tau}2)^2)*\exp(\text{beta}*s3)* \\
& \exp(-(\text{gamma}0*(b+c-\text{tau}2)+(1/2)*\text{gamma}1*(b+c-\text{tau}2)^2+(1/3)*\text{gamma}2*(b+c-\text{tau}2)^3)*\exp(\text{beta}*s3))) \\
& T := 87600; \\
& ABIAS := \int(-\text{gamma}2*t^2+\text{gam}1*t-\text{gamma}1*t+\text{gam}0-\text{gamma}0)*\exp(\text{beta}*s), t = 0 .. T); \\
& A := \text{Matrix}(3, 3, \{(1, 1) = A11, (1, 2) = A12, (1, 3) = A13, (2, 1) = A21, (2, 2) = A22, (2, 3) = A23, \\
& (3, 1) = A31, (3, 2) = A32, (3, 3) = A33\}); B := \text{Matrix}(3, 3, \{(1, 1) = B11, (1, 2) = B12, (1, 3) = B13, \\
& (2, 1) = B21, (2, 2) = B22, (2, 3) = B23, (3, 1) = B31, (3, 2) = B32, (3, 3) = B33\}); C := 1/A.B.(1/A) \\
& R := (\text{gam}0*\text{tau}1+(1/2)*\text{gam}1*\text{tau}1^2)*\exp(\text{theta}*s)
\end{aligned}$$

```

AVAR := Vector[row](3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)}.C.Vector(
3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)})
AMSE := ABIAS^2+AVAR:

```

## G: Codes for subsections 4.4.4 and 4.4.5

%The code for graphing Figures 4.1, 4.2 and 4.3 in Subsection 4.4.4.

```

code for Maple
pointplot3d([[160, 201, (-1.3928)^2], [150, 191, (-1.4982)^2], [140, 181, (-1.7495)^2], [130, 171,
(-1.9037)^2], [120, 161, (-2.0860)^2], [110, 151, (-2.3091)^2], [100, 141, (-2.3091)^2], [90, 131,
(-2.3091)^2]], color = blue, symbol = box, connect = true, labels = ["&tau;1", "&tau;2", ""], title =
"&tau;1,&tau;2 vs ABIAS^2")
pointplot3d([[160, 201, 640.3515], [150, 191, 1025.8606], [140, 181, 2079.1538], [130, 171,
3176.9778], [120, 161, 4666.4500], [110, 151, 6774.7227], [100, 141, 5842.5740], [90, 131,
4781.0526]], color = blue, symbol = box, connect = true, labels = ["&tau;1", "&tau;2", ""], title =
"&tau;1,&tau;2 vs AVAR")
pointplot3d([[160, 201, 642.2914], [150, 191, 1028.1051], [140, 181, 2082.2146], [130, 171,
3180.6017], [120, 161, 4670.8012], [110, 151, 6780.0547], [100, 141, 5847.9060], [90, 131,
4786.3846]], color = blue, symbol = box, connect = true, labels = ["&tau;1", "&tau;2", ""], title =
"&tau;1,&tau;2 vs AMSE")

```

%The code for fitting linear model and getting MLEs of  $\gamma_0$ ,  $\gamma_1$  and  $\beta$  in Subsection 4.4.4;  
and this code is also for Subsection 4.4.5 in each iteration.

%Global searching for  $\tau_1$  and  $\tau_2$  under constraint  $C_1$

%The difference between codes in F and G is that we compute  $ABIAS^2$ , AVAR and AMSE  
for the estimated hazard rate in F (in Subsections 4.4.2 and 4.4.3), and for the estimated  
transformed reliability in G (in Subsections 4.4.4 and 4.4.5).

```

restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+197.5);
gamma0 := 0.1e-3; gamma1 := .5; gamma2 := 0.15e-2; beta := -3800;
gam0 := .7947; gam1 := .7947; theta := -3800;
lambda0 := gamma2*t^2+gamma1*t+gamma0;
'&lambda;t' := lambda0*exp(beta*s);
'&lambda;f' := gam1*t+gam0;
'&lambda;tf' := '&lambda;f'*exp(theta*s);
c := 300; n := 200; tau1 := 160; tau2 := 201;
Delta := solve(1-exp(-(gam0*tau1+(1/2)*gam1*tau1^2)*exp(theta*s1)) = 1-exp(-(gam0*a+(1/2)*
gam1*a^2)*exp(theta*s2)), a);
a := Delta[2];
Eta := solve(1-exp(-(gam0*(a+tau2-tau1)+(1/2)*gam1*(a+tau2-tau1)^2)*exp(theta*s2)) =
1-exp(-(gam0*b+(1/2)*gam1*b^2)*exp(theta*s3)), b);

```

```

b := Eta[2];
n1 := evalf(n*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = 0 .. tau1)));
n2 := evalf((n-n1)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s2)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s2)), t = a .. a+tau2-tau1)));
n3 := evalf((n-n1-n2)*(int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s3)*exp(-(gamma0*t+(1/2)*gamma1*
t^2+(1/3)*gamma2*t^3)*exp(beta*s3)), t = b .. c)));
n4 := n-n1-n2-n3;
f1 := (gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*
gamma2*t^3)*exp(beta*s1));
f2 := (gamma0+gamma1*(a+t-tau1)+gamma2*(a+t-tau1)^2)*exp(beta*s2)*
exp(-(gamma0*(a+t-tau1)+(1/2)*gamma1*(a+t-tau1)^2+(1/3)*gamma2*(a+t-
tau1)^3)*exp(beta*s2));
f3 := (gamma0+gamma1*(b+t-tau2)+gamma2*(b+t-tau2)^2)*exp(beta*s3)*
exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*(b+t-tau2)^2+(1/3)*gamma2*(b+
t-tau2)^3)*exp(beta*s3));
A11 := n1*(int(f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int(f2/(gam0+
gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int(f3/(gam0+gam1*
(b+t-tau2))^2, t = tau2 .. c)));
A22 := n1*(int(t^2*f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int((a+
t-tau1)^2*f2/(gam0+gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int((b+t-
tau2)^2*f3/(gam0+gam1*(b+t-tau2))^2, t = tau2 .. c)));
A33 := evalf(n1*(int(s1^2*(gam0*t+(1/2)*gam1*t^2)*exp(theta*s1)*f1, t = 0 .. tau1))+
evalf(n2*(int(s2^2*(gam0*(a+t-tau1)+(1/2)*gam1*(a+t-tau1)^2)*exp(theta*s2)*
f2, t = tau1 .. tau2)))+evalf(n3*(int(s3^2*(gam0*(b+t-tau2)+(1/2)*gam1*(b+t-
tau2)^2)*exp(theta*s3)*f3, t = tau2 .. c)))+(n-n1-n2-n3)*s3^2*(gam0*(b+
c-tau2)+(1/2)*gam1*(b+c-tau2)^2)*exp(theta*s3)*(gamma0+gamma1*(b+
c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+
c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3));
A12 := n1*(int(t*f1/(gam1*t+gam0)^2, t = 0 .. tau1))+evalf(n2*(int((a+t-tau1)*f2/
(gam0+gam1*(a+t-tau1))^2, t = tau1 .. tau2)))+evalf(n3*(int((b+t-tau2)*f3/
(gam0+gam1*(b+t-tau2))^2, t = tau2 .. c)));
A13 := evalf(n1*(int(s1*t*exp(theta*s1)*f1, t = 0 .. tau1))+evalf(n2*(int(s2*(a+
t-tau1)*exp(theta*s2)*f2, t = tau1 .. tau2)))+evalf(n3*(int(s3*(b+t-tau2)*exp(theta*
s3)*f3, t = tau2 .. c)))+(n-n1-n2-n3)*s3*(b+c-tau2)*exp(theta*s3)*(gamma0+
gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+c-
tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3));
A23 := evalf(n1*(int((1/2)*s1*t^2*exp(theta*s1)*f1, t = 0 .. tau1))+evalf(n2*(int((1/2)*
s2*(a+t-tau1)^2*exp(theta*s2)*f2, t = tau1 .. tau2)))+evalf(n3*(int((1/2)*s3*(b+t-
tau2)^2*exp(theta*s3)*f3, t = tau2 .. c)))+(1/2)*(n-n1-n2-n3)*s3*(b+c-tau2)^2*
exp(theta*s3)*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*
exp(beta*s3)*exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*

```

```

gamma2*(b+c-tau2)^3)*exp(beta*s3));
'&varpi;1' := int(t*(gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*
exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*
s1)), t = 0 .. tau1);
'&varpi;2' := int((a+t-tau1)*(gamma0+gamma1*(a+t-tau1)+gamma2*
(a+t-tau1)^2)*exp(beta*s2)*exp(-(gamma0*(a+t-tau1)+(1/2)*gamma1*
(a+t-tau1)^2+(1/3)*gamma2*(a+t-tau1)^3)*exp(beta*s2)), t = tau1 .. tau2);
'&varpi;3' := int((b+t-tau2)*(gamma0+gamma1*(b+t-tau2)+gamma2*
(b+t-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*(b+
t-tau2)^2+(1/3)*gamma2*(b+t-tau2)^3)*exp(beta*s3)), t = tau2 .. c);
theta1 := int(t^2*(gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-
(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = 0 .. tau1);
theta2 := int((a+t-tau1)^2*(gamma0+gamma1*(a+t-tau1)+gamma2*
(a+t-tau1)^2)*exp(beta*s2)*exp(-(gamma0*(a+t-tau1)+(1/2)*gamma1*(a+t-
tau1)^2+(1/3)*gamma2*(a+t-tau1)^3)*exp(beta*s2)), t = tau1 .. tau2);
theta3 := int((b+t-tau2)^2*(gamma0+gamma1*(b+t-tau2)+gamma2*
(b+t-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+t-tau2)+(1/2)*gamma1*
(b+t-tau2)^2+(1/3)*gamma2*(b+t-tau2)^3)*exp(beta*s3)), t = tau2 .. c);
B11 := evalf((n1*(int((1/(gam1*t+gam0))-t*exp(theta*s1))*f1, t = 0 .. tau1))+
evalf(n2*(int((1/(gam0+gam1*(a+t-tau1)))-(a+t-tau1)*exp(theta*s2))*f2, t =
tau1 .. tau2))) + evalf(n3*(int((1/(gam0+gam1*(b+t-tau2)))-(b+t-tau2)*
exp(theta*s3))*f3, t = tau2 .. c)))-(n-n1-n2-n3)*(1/(gam0+gam1*(b+c-tau2))-
(b+c-tau2)*exp(theta*s3))*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+
c-tau2)^2)*exp(beta*s3)*exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*
(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3))^2+n1*
(-'&varpi;1'^2+theta1)*(gam1^2/(gam1*'&varpi;1'+gam0)^4+exp(2*theta*
s1))+n2*(-'&varpi;2'^2+theta2)*(gam1^2/(gam1*'&varpi;2'+gam0)^4+exp(2*
theta*s2))+n3*(-'&varpi;3'^2+theta3)*(gam1^2/(gam1*'&varpi;3'+gam0)^4+
exp(2*theta*s3));
B22 := (n1*(int((t/(gam1*t+gam0))-(1/2)*t^2*exp(theta*s1))*f1, t = 0 .. tau1))+
evalf(n2*(int(((a+t-tau1)/(gam0+gam1*(a+t-tau1)))-(1/2)*(a+t-tau1)^2*exp(theta*s2))*
f2, t = tau1 .. tau2))) + evalf(n3*(int(((b+t-tau2)/(gam0+gam1*(b+t-tau2)))-(1/2)*(b+t-tau2)^2*
exp(theta*s3))*f3, t = tau2 .. c)))-(n-n1-n2-n3)*((b+c-tau2)/(gam0+gam1*(b+c-tau2))-(1/2)*
(b+c-tau2)^2*exp(theta*s3))*(gamma0+gamma1*(b+c-tau2)+gamma2*(b+c-tau2)^2)*
exp(beta*s3)*exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-
tau2)^3)*exp(beta*s3))^2+n1*(-'&varpi;1'^2+theta1)*((1/(gam1*'&varpi;1'+gam0))+'&varpi;1'*
gam1/(gam1*'&varpi;1'+gam0)^2)^2+exp(2*theta*s1)*'&varpi;1'^2+n2*(-'&varpi;2'^2+theta2)*
((1/(gam1*'&varpi;2'+gam0))+'&varpi;2'*gam1/(gam1*'&varpi;2'+gam0)^2)^2+exp(2*theta*s2)*'
&varpi;2'^2+n3*(-'&varpi;3'^2+theta3)*((1/(gam1*'&varpi;3'+gam0))+'&varpi;3'*gam1/
(gam1*'&varpi;3'+gam0)^2)^2+exp(2*theta*s3)*'&varpi;3'^2);
B33 := (n1*s1-n1*(int(s1*(gam0*t+(1/2)*gam1*t^2)*exp(theta*s1)*f1, t = 0 .. tau1))+
evalf(n2*s2-n2*(int(s2*(gam0*(a+t-tau1)+(1/2)*gam1*(a+t-tau1)^2)*exp(theta*s2)*

```

$$\begin{aligned}
& f2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^*s_3 - n_3^*(\int(s_3^*(\text{gam}_0^*(b+t-\tau_2) + (1/2)^*\text{gam}_1^*(b+t-\tau_2)^2)^* \\
& \exp(\theta^*s_3)^*f_3, t = \tau_2 \dots c)) - (n_1 - n_1 - n_2 - n_3)^*s_3^*(\text{gam}_0^*(b+c-\tau_2) + (1/2)^*\text{gam}_1^*c^2)^*\exp(\theta^*s_3)^* \\
& (\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^* \\
& (b+c-\tau_2) + (1/2)^*\text{gamma}_1^*(b+c-\tau_2)^2 + (1/3)^*\text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3)))^2 + \\
& n_1^*s_1^2^*\exp(2^*\theta^*s_1)^*(\text{gam}_1^*i^*\varphi_1 + \text{gam}_0)^2^*(-i^*\varphi_1^2 + \theta_1) + n_2^*s_2^2^*\exp(2^* \\
& \theta^*s_2)^*(\text{gam}_1^*i^*\varphi_2 + \text{gam}_0)^2^*(-i^*\varphi_2^2 + \theta_2) + n_3^*s_3^2^*\exp(2^*\theta^*s_3)^*(\text{gam}_1^*i^* \\
& \varphi_3 + \text{gam}_0)^2^*(-i^*\varphi_3^2 + \theta_3); \\
B12 := & (n_1^*(\int((1/(\text{gam}_1^*t + \text{gam}_0) - t^*\exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2^*(\int((1/ \\
& (\text{gam}_0 + \text{gam}_1^*(a+t-\tau_1)) - (a+t-\tau_1)^*\exp(\theta^*s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^* \\
& (\int((1/(\text{gam}_0 + \text{gam}_1^*(b+t-\tau_2)) - (b+t-\tau_2)^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - (n_1 - n_1 - n_2 - n_3)^* \\
& (1/(\text{gam}_0 + \text{gam}_1^*(b+c-\tau_2)) - (b+c-\tau_2)^*\exp(\theta^*s_3))^*(\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \\
& \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^*(b+c-\tau_2) + (1/2)^*\text{gamma}_1^*(b+c-\tau_2)^2 + \\
& (1/3)^*\text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3)))^*(n_1^*(\int((t/(\text{gam}_1^*t + \text{gam}_0) - \\
& (1/2)^*t^2^*\exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2^*(\int(((a+t-\tau_1)/(\text{gam}_0 + \text{gam}_1^* \\
& (a+t-\tau_1)) - (1/2)^*(a+t-\tau_1)^2^*\exp(\theta^*s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^*(\int(((b+t-\tau_2)/ \\
& (\text{gam}_0 + \text{gam}_1^*(b+t-\tau_2)) - (1/2)^*(b+t-\tau_2)^2^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - \\
& (n_1 - n_1 - n_2 - n_3)^*((b+c-\tau_2)/(\text{gam}_0 + \text{gam}_1^*(b+c-\tau_2)) - (1/2)^*(b+c-\tau_2)^2^*\exp(\theta^*s_3))^* \\
& (\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^* \\
& (b+c-\tau_2) + (1/2)^*\text{gamma}_1^*(b+c-\tau_2)^2 + (1/3)^*\text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3))) \\
B13 := & (n_1^*(\int((1/(\text{gam}_1^*t + \text{gam}_0) - t^*\exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2^*(\int((1/(\text{gam}_0 + \\
& \text{gam}_1^*(a+t-\tau_1)) - (a+t-\tau_1)^*\exp(\theta^*s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^*(\int((1/(\text{gam}_0 + \text{gam}_1^* \\
& (b+t-\tau_2)) - (b+t-\tau_2)^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - (n_1 - n_1 - n_2 - n_3)^*(1/(\text{gam}_0 + \text{gam}_1^* \\
& (b+c-\tau_2)) - (b+c-\tau_2)^*\exp(\theta^*s_3))^*(\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \text{gamma}_2^* \\
& (b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^*(b+c-\tau_2) + (1/2)^*\text{gamma}_1^* \\
& (b+c-\tau_2)^2 + (1/3)^*\text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3)))^*(n_1^*s_1 - n_1^*(\int(s_1^* \\
& (\text{gam}_0^*t + (1/2)^*\text{gam}_1^*t^2)^*\exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2^*s_2 - n_2^* \\
& (\int(s_2^*(\text{gam}_0^*(a+t-\tau_1) + (1/2)^*\text{gam}_1^*(a+t-\tau_1)^2)^*\exp(\theta^*s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^* \\
& s_3 - n_3^*(\int(s_3^*(\text{gam}_0^*(b+t-\tau_2) + (1/2)^*\text{gam}_1^*(b+t-\tau_2)^2)^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - \\
& (n_1 - n_1 - n_2 - n_3)^*s_3^*(\text{gam}_0^*(b+c-\tau_2) + (1/2)^*\text{gam}_1^*c^2)^*\exp(\theta^*s_3)^*(\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \\
& \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^*(b+c-\tau_2) + (1/2)^*\text{gamma}_1^*(b+c-\tau_2)^2 + (1/3)^* \\
& \text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3))) \\
B23 := & (n_1^*(\int((t/(\text{gam}_1^*t + \text{gam}_0) - (1/2)^*t^2^*\exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \\
& \text{evalf}(n_2^*(\int(((a+t-\tau_1)/(\text{gam}_0 + \text{gam}_1^*(a+t-\tau_1)) - (1/2)^*(a+t-\tau_1)^2^*\exp(\theta^* \\
& s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^*(\int(((b+t-\tau_2)/(\text{gam}_0 + \text{gam}_1^*(b+t-\tau_2)) - (1/2)^* \\
& (b+t-\tau_2)^2^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - (n_1 - n_1 - n_2 - n_3)^*((b+c-\tau_2)/(\text{gam}_0 + \\
& \text{gam}_1^*(b+c-\tau_2)) - (1/2)^*(b+c-\tau_2)^2^*\exp(\theta^*s_3))^*(\text{gamma}_0 + \text{gamma}_1^*(b+ \\
& c-\tau_2) + \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3)^*\exp(-(\text{gamma}_0^*(b+c-\tau_2) + (1/2)^*\text{gamma}_1^*(b+c- \\
& \tau_2)^2 + (1/3)^*\text{gamma}_2^*(b+c-\tau_2)^3)^*\exp(\beta^*s_3)))^*(n_1^*s_1 - n_1^*(\int(s_1^*(\text{gam}_0^*t + (1/2)^*\text{gam}_1^*t^2)^* \\
& \exp(\theta^*s_1))^*f_1, t = 0 \dots \tau_1)) + \text{evalf}(n_2^*s_2 - n_2^*(\int(s_2^*(\text{gam}_0^*(a+t-\tau_1) + (1/2)^*\text{gam}_1^*(a+t-\tau_1)^2)^* \\
& \exp(\theta^*s_2))^*f_2, t = \tau_1 \dots \tau_2)) + \text{evalf}(n_3^*s_3 - n_3^*(\int(s_3^*(\text{gam}_0^*(b+t-\tau_2) + (1/2)^*\text{gam}_1^*(b+t- \\
& \tau_2)^2)^*\exp(\theta^*s_3))^*f_3, t = \tau_2 \dots c)) - (n_1 - n_1 - n_2 - n_3)^*s_3^*(\text{gam}_0^*(b+c-\tau_2) + \\
& (1/2)^*\text{gam}_1^*c^2)^*\exp(\theta^*s_3)^*(\text{gamma}_0 + \text{gamma}_1^*(b+c-\tau_2) + \text{gamma}_2^*(b+c-\tau_2)^2)^*\exp(\beta^*s_3))^*
\end{aligned}$$



```

exp(-(gamma0*(b+c-tau2)+(1/2)*gamma1*(b+c-tau2)^2+(1/3)*gamma2*(b+c-tau2)^3)*exp(beta*s3)))
T := 87600;
ABIAS := int((-gamma2*t^2+gam1*t-gamma1*t+gam0-gamma0)*exp(beta*s), t = 0 .. T);
A := Matrix(3, 3, {(1, 1) = A11, (1, 2) = A12, (1, 3) = A13, (2, 1) = A21, (2, 2) = A22, (2, 3) = A23,
(3, 1) = A31, (3, 2) = A32, (3, 3) = A33}); B := Matrix(3, 3, {(1, 1) = B11, (1, 2) = B12, (1, 3) = B13,
(2, 1) = B21, (2, 2) = B22, (2, 3) = B23, (3, 1) = B31, (3, 2) = B32, (3, 3) = B33}); C := 1/A.B.(1/A)
R := ln(gam0*T+(1/2)*gam1*T^2)+theta*s;
AVAR := Vector[row](3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)}).C.Vector(
3, {(1) = diff(R, gam0), (2) = diff(R, gam1), (3) = diff(R, theta)})
AMSE := ABIAS^2+AVAR:

```

## H: Codes for subsection 5.3.1

```

%This code is for simulating the data for linear baseline function then finding MLEs of  $\gamma_0$ ,  $\gamma_1$  and  $\beta$ 
and computing the SBIAS2, SVAR and SMSE with fixed designs  $\xi_M^2$  and  $\xi_Q^2$  obtained in Chapter 2.
code for Maple
restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra); with(Optimization);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+155);
gamma0 := 0.1e-3; gamma1 := .5; beta := -3800;
c := 300; n := 200; tau1 := 178.15; tau2 := 247.44; u := 1000;
Delta := solve(1-exp(-(gamma0*tau1+(1/2)*gamma1*tau1^2)*exp(beta*s1)) = 1-exp(-(gamma0
*a+(1/2)*gamma1*a^2)*exp(beta*s2)), a);
a := Delta[2];
Eta := solve(1-exp(-(gamma0*(a+tau2-tau1)+(1/2)*gamma1*(a+tau2-tau1)^2)*exp(beta*
s2)) = 1-exp(-(gamma0*b+(1/2)*gamma1*b^2)*exp(beta*s3)), b);
b := Eta[2];
f1 := proc (L, T) options operator, arrow; int((gamma1*t+gamma0)*exp(beta*s1)*exp(-
(gamma0*t+(1/2)*gamma1*t^2)*exp(beta*s1)), t = L .. T) end proc;
f2 := proc (L, T) options operator, arrow; int((gamma1*t+gamma0)*exp(beta*s2)*exp(-
(gamma0*t+(1/2)*gamma1*t^2)*exp(beta*s2)), t = L .. T) end proc;
f3 := proc (L, T) options operator, arrow; int((gamma1*t+gamma0)*exp(beta*s3)*exp(-
(gamma0*t+(1/2)*gamma1*t^2)*exp(beta*s3)), t = L .. T) end proc;
for i to (1/5)*tau1-1 do for j to u do X[j, 1] := Statistics:-Sample(Binomial(200, f1(0, 5)), 1);
X[j, i+1] := Statistics:-Sample(Binomial(200-X[j, i][1], f1(5*i, 5*(i+1))), 1) end do end do;
for j to u do sumx[j] := add(X[j, i], i = 1 .. (1/5)*tau1) end do;
for i to 8 do for j to u do Y[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1], f2(a, a+5)), 1);
Y[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-Y[j, i][1], f2(a+5*i, a+5*(i+1))), 1)
end do end do;
for j to u do sumy[j] := add(Y[j, i], i = 1 .. 8) end do;
for i to 20 do for j to u do Z[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1],
f3(b, b+5)), 1); Z[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1]-Z[j, i][1],

```

```

f3(b+5*i, b+5*(i+1)), 1) end do end do;
for j to u do sumz[j] := add(Z[j, i], i = 1 .. 20) end do;
for j to u do logl[j] := -sumx[j][1]*bh*s1+add(X[j, i][1]*(ln(5*g1h*i+g0h)-(5*g0h*i+(1/2)*g1h*
(5*i)^2)*exp(-bh*s1)), i = 1 .. 32)-sumy[j][1]*bh*s2+add(Y[j, i][1]*(ln(g0h+g1h*(a+5*i))-(g0h*
(a+5*i)+(1/2)*g1h*(a+5*i)^2)*exp(-bh*s2)), i = 1 .. 8)-sumz[j][1]*bh*s3+add(Z[j, i][1]*(ln(g0h
+g1h*(b+5*i))-(g0h*(b+5*i)+(1/2)*g1h*(b+5*i)^2)*exp(-bh*s3)), i = 1 .. 20)-(200-sumx[j][1]-
sumy[j][1]-sumz[j][1])*(g0h*(b+c-tau2)+(1/2)*g1h*(b+c-tau2)^2)*exp(-bh*s3) end do;
for j to u do ma[j] := Maximize(logl[j], initialpoint = {bh = 3700, g0h = .1, g1h = 0.15e-2},
assume = nonnegative, iterationlimit = 400) end do;
aver := [add(ma[j][2][1], j = 1 .. u)/u, add(ma[j][2][2], j = 1 .. u)/u, add(ma[j][2][3], j = 1 .. u)/u];
av := eval([bh, g0h, g1h], aver);
for j to u do A[j] := eval([(bh-av[1])^2, (g0h-av[2])^2, (g1h-av[3])^2], ma[j][2]);
B[j] := eval([(bh-beta)^2, (g0h-gamma0)^2, (g1h-gamma1)^2], ma[j][2]) end do;
bias := [(-av[1]-beta)^2, (av[2]-gamma0)^2, (av[3]-gamma1)^2]
var := add(A[j], j = 1 .. u)/(u-1)
mse := add(B[j], j = 1 .. u)/u

```

## I: Codes for subsection 5.3.2

```

%This code is for simulating the data for quadratic baseline function then finding MLEs of  $\gamma_0, \gamma_1,$ 
 $\gamma_2$  and  $\beta$  and computing the SBIAS2, SVAR and SMSE with fixed designs  $\xi_M^3, \xi_Q^3$  and  $\xi_A^3$  obtained
in Chapter 3.
code for Maple
restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra); with(Optimization);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+197.5);
gamma0 := 0.1e-3; gamma1 := .5; gamma2 := 0; beta := -3800;
c := 300; n := 200; tau1 := 178.15; tau2 := 223.57; u := 1000;
Delta := solve(1-exp(-(gamma0*tau1+(1/2)*gamma1*tau1^2+(1/3)*gamma2*tau1^3)*exp(beta*s1)
s1)) = 1-exp(-(gamma0*a+(1/2)*gamma1*a^2+(1/3)*gamma2*a^3)*exp(beta*s2)), a);
a := Delta[2];
Eta := solve(1-exp(-(gamma0*(a+tau2-tau1)+(1/2)*gamma1*(a+tau2-tau1)^2+(1/3)*
gamma2*(a+tau2-tau1)^3)*exp(beta*s2)) = 1-exp(-(gamma0*b+(1/2)*gamma1*b^2+(1/3)*
gamma2*b^3)*exp(beta*s3)), b);
b := Eta[2];
f1 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*
exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = L .. T) end proc;
f2 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s2)*
exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s2)), t = L .. T) end proc;
f3 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s3)*
exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s3)), t = L .. T) end proc;
for i to (1/5)*tau1-1 do for j to u do X[j, 1] := Statistics:-Sample(Binomial(200, f1(0, 5)), 1);

```

```

X[j, i+1] := Statistics:-Sample(Binomial(200-X[j, i][1], f1(5*i, 5*(i+1))), 1) end do end do;
for j to u do sumx[j] := add(X[j, i], i = 1 .. (1/5)*tau1) end do;
for i to 9 do for j to u do Y[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1], f2(a, a+5)), 1);
Y[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-Y[j, i][1], f2(a+5*i, a+5*(i+1))), 1)
end do end do;
for j to u do sumy[j] := add(Y[j, i], i = 1 .. 9) end do;
for i to 15 do for j to u do Z[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1],
f3(b, b+5)), 1); Z[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1]-Z[j, i][1],
f3(b+5*i, b+5*(i+1))), 1) end do end do;
for j to u do sumz[j] := add(Z[j, i], i = 1 .. 15) end do;
for j to u do logl[j] := -sumx[j][1]*bh*s1+add(X[j, i][1]*(ln(g0h+5*g1h*i+(1/2)*g2h*(5*i)^2)-(5*
g0h*i+(1/2)*g1h*(5*i)^2+(1/3)*g2h*(5*i)^3)*exp(-bh*s1)), i = 1 .. 35)-sumy[j][1]*bh*s2+
add(Y[j, i][1]*(ln(g0h+g1h*(a+5*i)+(1/2)*g2h*(a+5*i)^2)-(g0h*(a+5*i)+(1/2)*g1h*(a+5*i)^2+
(1/3)*g2h*(a+5*i)^3)*exp(-bh*s2)), i = 1 .. 9)-sumz[j][1]*bh*s3+add(Z[j, i][1]*
(ln(g0h+g1h*(b+5*i)+(1/2)*g2h*(b+5*i)^2)-(g0h*(b+5*i)+(1/2)*g1h*(b+5*i)^2+(1/3)*g2h*
(b+5*i)^3)*exp(-bh*s3)), i = 1 .. 15)-(200-sumx[j][1]-sumy[j][1]-sumz[j][1])*(g0h*(b+
c-tau2)+(1/2)*g1h*(b+c-tau2)^2)*exp(-bh*s3) end do;
for j to u do ma[j] := Maximize(logl[j], initialpoint = {bh = 3700, g0h = .1, g1h = 0.15e-2, g2h = 0},
assume = nonnegative, iterationlimit = 400) end do;
aver := [add(ma[j][2][1], j = 1 .. u)/u, add(ma[j][2][2], j = 1 .. u)/u, add(ma[j][2][3],
j = 1 .. u)/u, add(ma[j][2][4], j = 1 .. u)/u];
av := eval([bh, g0h, g1h, g2h], aver);
for j to u do A[j] := eval([(bh-av[1])^2, (g0h-av[2])^2, (g1h-av[3])^2, (g2h-av[4])^2], ma[j][2]);
B[j] := eval([(-bh-beta)^2, (g0h-gamma0)^2, (g1h-gamma1)^2, (g2h-gamma2)^2], ma[j][2]) end do
bias := [(-av[1]-beta)^2, (av[2]-gamma0)^2, (av[3]-gamma1)^2, (av[4]-gamma2)^2]
var := add(A[j], j = 1 .. u)/(u-1)
mse := add(B[j], j = 1 .. u)/u

```

## J: Codes for subsection 5.3.3

```

%This code is for simulating the data for a model with a quadratic baseline function then finding
MLEs of  $\gamma_0$ ,  $\gamma_1$  and  $\beta$  by fitting a model with a linear baseline function and computing the SBIAS2,
SVAR and SMSE with fixed designs  $\xi_M^4$  and  $\xi_G$  obtained in Chapter 4.
code for Maple
restart; with(linalg); with(stats); with(plots); with(Statistics); with(LinearAlgebra); with(Optimization);
s := 1/(273.16+50); s1 := 1/(273.16+145); s3 := 1/(273.16+250); s2 := 1/(273.16+197.5);
gamma0 := 0.1e-3; gamma1 := .5; gamma2 := 0.15e-2; beta := -3800;
lambda0 := gamma2*t^2+gamma1*t+gamma0;
‘&lambda;t’ := lambda0*exp(beta*s);
‘&lambda;f’ := gam1*t+gam0;
‘&lambda;tf’ := ‘&lambda;f’*exp(theta*s);

```

```

c := 300; n := 200; tau1 := 160; tau2 := 201; u := 1000;
Atrue := int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s), t = 0 .. 87600);
Btrue := eval(log(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)+beta*s, t = 8760);
Delta := solve(1-exp(-(gamma0*tau1+(1/2)*gamma1*tau1^2+(1/3)*gamma2*tau1^3)*exp(beta*s1)) = 1-exp(-(gamma0*a+(1/2)*gamma1*a^2+(1/3)*gamma2*a^3)*exp(beta*s2)), a);
a := Delta[1];
Eta := solve(1-exp(-(gamma0*(a+tau2-tau1)+(1/2)*gamma1*(a+tau2-tau1)^2+(1/3)*gamma2*(a+tau2-tau1)^3)*exp(beta*s2)) = 1-exp(-(gamma0*b+(1/2)*gamma1*b^2+(1/3)*gamma2*b^3)*exp(beta*s3)), b);
b := Eta[1];
f1 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s1)*exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s1)), t = L .. T) end proc;
f2 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s2)*exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s2)), t = L .. T) end proc;
f3 := proc (L, T) options operator, arrow; int((gamma2*t^2+gamma1*t+gamma0)*exp(beta*s3)*exp(-(gamma0*t+(1/2)*gamma1*t^2+(1/3)*gamma2*t^3)*exp(beta*s3)), t = L .. T) end proc;
for i to (1/5)*tau1-1 do for j to u do X[j, 1] := Statistics:-Sample(Binomial(200, f1(0, 5)), 1);
X[j, i+1] := Statistics:-Sample(Binomial(200-X[j, i], f1(5*i, 5*(i+1))), 1) end do end do;
for j to u do sumx[j] := add(X[j, i], i = 1 .. (1/5)*tau1) end do;
for i to 8 do for j to u do Y[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1], f2(a, a+5)), 1);
Y[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-Y[j, i], f2(a+5*i, a+5*(i+1))), 1) end do end do;
for j to u do sumy[j] := add(Y[j, i], i = 1 .. 8) end do;
for i to 20 do for j to u do Z[j, 1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1], f3(b, b+5)), 1);
Z[j, i+1] := Statistics:-Sample(Binomial(200-sumx[j][1]-sumy[j][1]-Z[j, i], f3(b+5*i, b+5*(i+1))), 1) end do end do;
for j to u do sumz[j] := add(Z[j, i], i = 1 .. 20) end do;
for j to u do logl[j] := -sumx[j][1]*bh*s1+add(X[j, i][1]*(ln(5*g1h*i+g0h)-(5*g0h*i+(1/2)*g1h*(5*i)^2)*exp(-bh*s1)), i = 1 .. 32)-sumy[j][1]*bh*s2+add(Y[j, i][1]*(ln(g0h+g1h*(a+5*i))-(g0h*(a+5*i)+(1/2)*g1h*(a+5*i)^2)*exp(-bh*s2)), i = 1 .. 8)-sumz[j][1]*bh*s3+add(Z[j, i][1]*(ln(g0h+g1h*(b+5*i))-(g0h*(b+5*i)+(1/2)*g1h*(b+5*i)^2)*exp(-bh*s3)), i = 1 .. 20)-(200-sumx[j][1]-sumy[j][1]-sumz[j][1])*(g0h*(b+c-tau2)+(1/2)*g1h*(b+c-tau2)^2)*exp(-bh*s3) end do;
for j to u do ma[j] := Maximize(logl[j], initialpoint = {bh = 3700, g0h = .1, g1h = .15}, assume = nonnegative, iterationlimit = 400) end do;
for j to u do A[j] := eval([(g1h*t+g0h)*exp(-bh*s)], ma[j][2]); M[j] := int(A[j][1], t = 0 .. 87600) end do;
'&Delta;A1' := add(M[i], i = 1 .. u);
'&Delta;A2' := add(M[i]^2, i = 1 .. u);
bias2A := ('&Delta;A1'/u-Atrue)^2;
varA := ('&Delta;A2'-'&Delta;A1'^2/u)/(u-1);
mseA := bias2A+varA;
for j to u do B[j] := eval([log(g0h*t+(1/2)*g1h*t^2)-bh*s], ma[j][2]); N[j] := eval(B[j][1], t = 8760)

```

```
end do;
'Delta;B1' := add(N[i], i = 1 .. u);
'Delta;B2' := add(N[i]^2, i = 1 .. u);
bias2B := ('Delta;B1'/u-Btrue)^2;
varB := ('Delta;B2'-'Delta;B1'^2/u)/(u-1);
mseB := bias2B+varB;
```