Backbone Colouring of Graphs

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Abstract

Consider an undirected graph G and a subgraph of G, H. A q-backbone k-colouring of (G, H) is a mapping $f : V(G) \rightarrow \{1, 2, ..., k\}$ such that G is properly coloured and for each edge of H, the colours of its endpoints differ by at least q. The minimum number k for which there is a backbone k-colouring of (G, H) is the backbone chromatic number, $BBC_q(G, H)$.

It has been proved that the backbone k-colouring of (G, T) is at most 4 if G is a connected C_4-free planar graph or non-bipartite C_5-free planar graph or C_j-free, $j \in \{6, 7, 8\}$ planar graph without adjacent triangles.

In this thesis we improve the results mentioned above and prove that any connected planar graph without adjacent triangles has a 2-backbone k-colouring with at most 4 colours by using a discharging method.

In the second part of this thesis we further improve these results by proving that for any graph G with $\chi(G) \geq 4$, $BBC_2(G, T) = \chi(G)$. In fact, we prove the stronger result that a backbone tree T in G exists, such that $\forall uv \in T, |f(u) - f(v)| = 2$ or $|f(u) - f(v)| \geq k - 2$, $k = \chi(G)$.

For the case that G is a planar graph, according to Four-Colour theorem, $\chi(G) = 4$; so, $BBC_2(G, T) = 4$.

Acknowledgement

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Contents

Chapter 1

Introduction

Graph Theory is the study of graphs, which are discrete structures consisting of nodes and links between nodes. More specifically, nodes of a graph, called vertices, are connected by edges. One application of graph theory is the study of social networks where vertices represent the users of the network and edges represent the connections between users.

Also, a typical application of graph colouring is the scheduling problem. In this problem we want to assign a set of jobs to time slots in which jobs with the same resources do not have conflict with each other.

1.1 Overview

The purpose of the first section of Chapter 2 is to familiarize the reader with commonly used notations in the study of graph theory or specifically in this thesis. While the second section of Chapter 2 is a summary of backbone colouring results that have been previously proved by earlier authors.

In Chapter 3, we prove that every connected planar graph without adjacent triangles has a backbone for which the backbone chromatic number is at most 4 by using discharging rules. In Chapter 4, we extend these results and prove that every graph G with $\chi(G) \geq 4$ has a backbone for which the backbone chromatic number is equal to $\chi(G)$.

Chapter 5 contains the concluding remarks and the discussions about possible future directions.

Chapter 2

Preliminaries and Notions

In this chapter we introduce graph theoretical definitions and terminology that are used in this thesis. For undefined definitions and notations, we use [12] as our reference.

2.1 Preliminaries

Graph G consists of the vertex set $V(G)$ and the edge set $E(G)$. Each edge has two endpoints which are *adjacent* to each other. Vertex v is *incident* to edge e if v is an endpoint of e. The order of G is the number of vertices, $|V(G)|$ and the size of G is the number of edges, $|E(G)|$.

The graph obtained from G by deleting the edge e, $e \in E(G)$ or the set of edges S are denoted by $G - e$ and $G - S$, respectively. Similarly, the graph obtained from G by deleting the vertex v, $v \in V(G)$ or the set of vertices S' and all incident edges are denoted by $G - v$ and $G - S'$, respectively.

The degree of a vertex v is the number of vertices adjacent to v and it is denoted by $d(v)$; a k-vertex is a vertex of degree k. A *leaf* is a vertex of graph G with degree 1. A k^+-vertex and k^--vertex is a vertex of degree at least k and at most k, respectively. Let $N - G(A)$ denotes the set of neighbours of vertices in A in graph G.

Since each edge has two endpoints, we can say that the sum of all degrees is exactly twice the number of edges,

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

If all vertices of graph G have degree k, then G is said to be a k-regular. A connected 2-regular graph is called a *cycle.* A cycle graph of n vertices is denoted by C_n. A *complete graph* is a graph whose vertices are all pairwise adjacent. The complete graph with n vertices is denoted by K_n. *Triangle* is a common name for K_3.

The minimum degree of all vertices of graph G is denoted by $\delta(G)$ and the maximum degree is denoted by $\Delta(G)$.

A *path* in a graph is a finite sequence of distinct edges which connect a sequence of vertices. A graph is connected if there is at least one path between any two vertices of graph; otherwise, it is a disconnected graph. A *tree* T is a connected graph in which there is exactly one path between any two vertices of it.

A *component* is a largest connected subgraph of a graph. In other words, if G is disconnected, it has more than one component. The number of components of the graph G is denoted by $comp(G)$. A *cut vertex* or *cut edge* is a vertex or edge of graph G, respectively, whose deletion increases the number of components of G.

Cut edges are commonly called *bridges*.

A graph $H(V,E)$ is a subgraph of $G(V,E)$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of G is a subgraph that includes all the vertices of G. A *spanning tree* is a spanning subgraph that is a tree.

A graph is called *planar* if it can be drawn in the plane in such a way that no two edges intersect. An embedding of a planar graph divides the plane into a number of connected regions, called faces which are bounded by the edges of the graph. Vertex v is *incident* to face f if v is the endpoint of an edge in the boundary of f. A k-face is a face f with k edges in the boundary. A k^+-face and k^--face is a face with at least k edges in its boundary and at most k edges in its boundary, respectively.

A k-colouring of graph G is a mapping $f : V \to \{1, 2, ..., k\}$. If $f(u) \neq f(v)$ for any two adjacent vertices u and v, f is a proper colouring. The smallest integer k such that G has a proper k-colouring is called a *chromatic number* of G and is denoted by $\chi(G)$.

For a graph G and its subgraph H, a *q-backbone k-colouring* of (G,H) is a mapping $f : V(G) \to \{1, 2, ..., k\}$ such that:

$$|f(u) - f(v)| \geq \begin{cases} q & \text{if } uv \in E(H) \\ 1 & \text{if } uv \in E(G) \setminus E(H) \end{cases}$$

The minimum number k for which there is a q-backbone k-colouring of (G,H) is the backbone chromatic number of (G,H), denoted by $BBC_q(G,H)$. Let u and v be two endpoints of an edge $e \in E(G)$ and f be a colouring of G. If $|f(u) - f(v)| \geq$

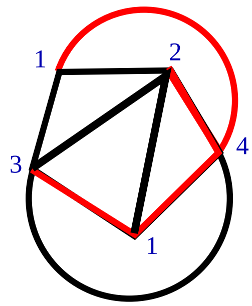


Figure 2.1: Example of backbone colouring.

q, then the edge e is called an f-edge. For the case that H is spanning tree T we called T the backbone of G.

A *circular q-backbone k-colouring* of (G, T) is a special case of q-backbone k-colouring of (G, H) which is a proper colouring of vertices with k colours such that $q \leq |f(u) - f(v)| \leq k - q$, for each edge $uv \in E(H)$.

The *circular q-backbone k-colouring chromatic number* of (G, T), denoted by $CBC_q(G, T)$, is the minimum k such that for any spanning trees of G there is a circular q-backbone k-colouring. We can say that if f is a circular q-backbone k-colouring of (G, T), then f is also a q-backbone k-colouring of (G, T).

For the case that $q = 2$ we simply say backbone k-colouring and use the notation $BBC(G, T)$. Similarly when $q = 2$ we say circular backbone k-colouring and use the notation $CBC(G, T)$.

In Figure 2.1 edges of graph G are shown by black colour and edges of spanning tree T are shown by red colour. Graph G has a backbone k-colouring with 4 colours. Similarly, G has a circular k-colouring with 5 colours.

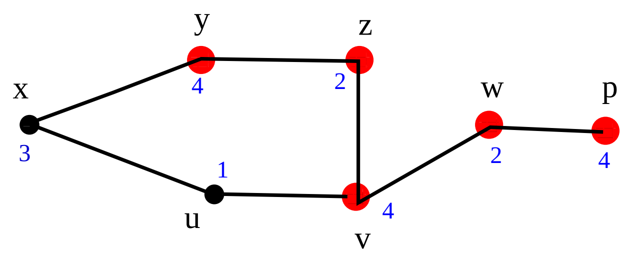


Figure 2.2: Example of (4,2)-Kempe chain.

Breadth First Search is one of the simplest algorithms for searching a graph G with a distinguished source vertex s. This algorithm explores the neighbours of s first, before moving to the next level neighbours by computing a shortest path from s to vertices reachable from s.

Consider a proper colouring of graph G with colour set S containing at least two distinct colours a and b. If v is a vertex of graph G with colour a then the *(a,b)-Kempe chain* of G containing v is the maximal connected subset of vertices of G, which contains v and whose vertices are all coloured either a or b.

We can use the (a,b)-Kempe chain for switching the colour of vertices in this subset without changing the colour of all vertices in G. By switching the colour of vertices in the subset we mean that for a vertex u in (a,b)-Kempe chain of v, $f'(u) = b$ if $f(u) = a$ and vice versa. We can observe that by recolouring vertices in this Kempe chain we did not change the proper colouring because adjacent vertices to any vertex of Kempe chain of V have other colouring rather than a or b; otherwise, they would be in the (a, b)-Kempe chain of v.

In Figure 2.2, the $(4, 2)$-Kempe chain of v is set $\{y, z, v, w, p\}$. If we switch the colour of vertices in the $(4, 2)$-Kempe chain of v, the new colour of vertices will be

$f'(y) = f'(v) = f'(p) = 2$ and $f'(z) = f'(w) = 4$.

2.2 Related Works

Backbone colouring was firstly motivated by the colouring problems related to frequency assignment and was introduced by Boersma et al. [2] in 2003. In addition to studying backbone colouring for (G, H), they studied $BBC(G, P)$ where P is a Hamiltonian Path, and for the special case of split graphs. Their main result was that for any connected graph G and spanning tree T, $BBC(G, T) \leq 2\chi(G) - 1$. This result together, with Four-Colour Theorem [1], implies that if G is a planar graph, then $BBC(G, T) \leq 7$. However, they conjecture that for any planar graph G, $BBC(G, T) \leq 6$ and by providing an example in which $BBC(G, T) = 6$, they showed that this upper bound, if true, is tight. In 2013, Campos et al. [3] proved this conjecture when T has diameter at most 4. In [2], the authors proved that if G is a split graph, $BBC(G, T) = \chi(G) + 2$ and if $\omega(G) \neq 3$ and G contains a Hamiltonian path, then for every Hamiltonian path P in G, $BBC(G, T) = \chi(G) + 2$. The general case of q-backbone k-colouring when $q \geq 2$, was first defined by Broersma et al. [4] in 2009. They consider the cases that backbone is either a collection of pairwise disjoint stars or a matching. According to their result, the minimum k for a star backbone S of G can differ by a multiplicative factor of at most $2 - \frac{1}{q}$ from $\chi(G)$.

In [9], Havet et al. showed that $BBC_q(G, H) \leq 3q + 1$ by Four-Colour Theorem [1] when G is planar and H is forest. They improve this upper bound by showing that

$BBC_q(G,H) \leq q+6$ and for the special case of $q=3$, $BBC_3(G,H) \leq 8$.
Another way of considering backbone colouring is to show that a spanning tree T of G exists, such that $BBC_q(G,T) \leq k$, instead of showing that this is true for any spanning tree T of G. In 2011, Bu and Zhang [5] proved that if G is a connected C_4-free planar graph, then there exists a spanning tree T of G such that $BBC(G,T) \leq 4$. In 2010, Zhang and Bu [6] proved that if G is a connected non-bipartite C_5-free planar graph, then there exists a spanning tree T of G such that $BBC(G,T) = 4$.
Also, in 2011, Bu and Li [7] proved that if G is connected C_6-free or C_7-free planar graphs without adjacent triangles, then there is a spanning tree T of G such that $BBC(G,T) \leq 4$. In 2012, Wang [8] showed that if G is a connected planar graph without C_8 and adjacent triangles, then there is a spanning tree T of G such that $BBC_2(G,T) \leq 4$. In 2015, Bu and Bao [11] showed that if G is a connected C_8-free or C_9-free planar graph without adjacent 4-cycles, then there is a spanning tree T of G such that $BBC(G,T) \leq 4$. The purpose of part one of my thesis is to extend these results by allowing graph G to have C_j where $j = 3, ..., n$. Also, in the second part of my thesis, we improve these results for any graph with $\chi(G) \geq 4$.
In 2014, Havet et al. [9] defined the concept of circular backbone colouring for the first time. They showed that even though $CBC_q(G,H) \leq q.\chi(G)$, this is not the best possible upper bound and they proved that $CBC_q(G,H) \leq 2q+4$. Also, they conjectured that $CBC_q(G,H) \leq 2q+3$.
In [10], Araujo et al. proved that if G is a planar graph without C_4 or C_5, then for any forest H that $H \subseteq G$, $CBC(G,H) \leq 7$. Also, they improve this upper bound

when H is a path forest. In other words, for the same class of graphs when the connected components of forest H are paths, $CBC_q(G,H) \leq 6$.

However, by using discharging method, recently we proved that if G is a connected C_j-free planar graph, $j \in 4,5,6,7$, then for every spanning tree T of G, $CBC(G,T) \leq 7$.

Chapter 3

Backbone Colouring

3.1 Upper bound on the backbone chromatic number of graphs

Observation 1. *Suppose that f and f' are two vertex colourings of G and $\forall v \in V(G)$, $f'(v) + f(v) = k + 1$. Then f is a backbone k-colouring of (G, T), if and only if, f' is a backbone k-colouring of (G, T). We call f the symmetric colouring of f' and vice versa.*

Proof. Assume that f is a backbone k-colouring of (G, T). According to the definition of backbone colouring:

If $uv \in E(T), |f(u) - f(v)| \geq 2, f'(u) = k + 1 - f(u)$ and $f'(v) = k + 1 - f(v)$. Thus, $|f'(u) - f'(v)| = |f(u) - f(v)| \geq 2$.

If $uv \in E(G) - E(T), |f(u) - f(v)| \geq 1$, then $|f'(u) - f'(v)| \geq 1$.

□

Theorem 1. *If G is a connected planar graph without adjacent triangles, then there exists a spanning tree T of G such that $BBC(G, T) \leq 4$.*

We will use proof by contradiction in the following two parts. First we assume for contradiction that there exists a planar graph G without adjacent triangles such that for a spanning tree T of G, $BBC(G, T) \geq 5$. Then we determine some properties for the graph G.
In the second part, we show that graph G does not exist using the discharging method. In other words, by assigning charges to each vertex and to each face of G, we carefully redistribute those charges such that the total charge of G is non-negative. However, by using Euler's formula we can show that initially the total charge of G was negative. This obvious contradiction states that the assumption of the existence of graph G was incorrect.

3.2 Properties of a minimum counterexample

Let $G(V, E)$ be a counterexample to Theorem 1 with minimum $|V|$. So, G is a connected planar graph without adjacent triangles and for every spanning tree T of G, $BBC(G, T) \geq 5$. Also, for every $G'(V', E')$ without adjacent triangles if $|V'| < |V|$, then there is a spanning tree T' such that $BBC(G', T') \leq 4$. We will use the color set $S = \{1, 2, 3, 4\}$.

Lemma 1. $\delta(G) \geq 4$.

Proof. We will show that G does not contain a leaf, 2-vertex and 3-vertex. These three properties are demonstrated separately below.

1. G has no leaf.

   Assume for a contradiction that G contains a leaf u adjacent to a vertex x. Let $G' = G - u$ with $|V'|$ number of vertices and $|E'|$ number of edges. Since $|V'| = |V| - 1$, by the minimality of G there is a spanning tree T' of G' such that (G', T') has a backbone 4-colouring. For any colouring of x we can find a proper colour for u. Without loss of generality, assume that $f(x) \in \{1, 2\}$ (set $\{3, 4\}$ is a symmetric colouring for $\{1, 2\}$). In this case, colour u with 4; so, ux is an f-edge. Let spanning tree $T = T' \cup \{ux\}$. Hence there is a spanning tree T of G such that (G, T) has a backbone 4-colouring; this is a contradiction.

2. G has no 2-vertex.

   Suppose to the contrary that G contains a 2-vertex u with two neighbours x, y. Let $G' = G - u$. Graph G' remains connected if u is not a cut vertex. Consider the cases of u being a cut vertex and u not being a cut vertex separately.

   (a) u is a cut vertex.

   In this case G' has two components C_1 and C_2. Without loss of generality, assume that $x \in C_1$ and $y \in C_2$. By the minimality of G, there is a spanning tree T_i of C_i such that (C_i, T_i) has a backbone 4-colouring for $i = 1, 2$. Without loss of generality, assume that $f(x) \in \{1, 2\}$ and $f(x) \leq f(y)$. If $\{f(x), f(y)\} = \{1, 2\}$, then colour u with 4. For the other cases, if $f(x) \neq f(y)$ consider the symmetric colouring of C_1 and colour u with 1. If $f(x) = f(y)$, colour u with 4; thus, ux and uy are f-edges. Suppose that $T = T_1 \cup T_2 \cup \{ux, uy\}$. So, T is a backbone of G such that $BBC(G, T) \leq 4$, contradiction.

(b) u is not a cut vertex.

Here G' is connected and $x, y \in G'$. by the minimality of G, there is a spanning tree T' of G' such that (G', T') has a backbone 4-colouring. Without loss of generality, assume that $f(x) \in \{1, 2\}$. For any colouring of x and y, let $f(u) = max(\{1, 2, 3, 4\} - \{f(x), f(y)\})$. One can easily observe that $max(|f(u) - f(x)|, |f(u) - f(y)|) \geq 2$. Without loss of generality, assume that $|f(u) - f(x)| \geq 2$; so, ux is an f-edge. Suppose that $T = T' \cup \{ux\}$. Thus, T is a backbone of G such that $BBC(G, T) \leq 4$, contradiction.

3. G has no 3-vertex.

Assume for a contradiction that G contains a 3-vertex u with neighbours x, y, z. Let set $A = \{x, y, z\}$. If u is a cut vertex, graph $G - u$ has more than one component which lead to the following three cases.

(a) u is not a cut vertex.

Here G' is connected and $x, y, z \in G'$. by the minimality of G, there is a spanning tree T' of G' such that $BBC(G', T') \leq 4$. Without loss of generality, assume that $f(x) \in \{1, 2\}$. For any colouring of x, y and z, let $f(u) = max(\{1, 2, 3, 4\} - \{f(x), f(y), f(z)\})$. One can show that $max(|f(u) - f(x)|, |f(u) - f(y)|, |f(u) - f(z)|) \geq 2$. Without loss of generality, assume that $|f(u) - f(y)| \geq 2$. Hence, uy is an f-edge and for a spanning tree $T = T' \cup \{uy\}$, $BBC(G,T) \leq 4$, contradiction.

(b) u is a cut vertex and $comp(G') = 2$.

Assume that C_1 and C_2 are two components of $G - uz$ and let $x, y \in C_1$ and $z \in C_2$. In graph $G - uz$ we identify x with z and we obtain graph G'; so, in G', $f(x) = f(z)$. By the minimality of G, there is a spanning tree T' of G' such that (G', T') has a backbone 4-colouring. Let T_i be $C_i \cap T'$ for $i = 1, 2$.

Without loss of generality, we assume that $f(x) = f(z) = \{1, 2\}$. If uz is an f-edge in G' then clearly there exists a spanning tree $T = T_1 \cup T_2 \cup \{uz\}$ such that $BBC(G,T)(G,T) \leq 4$, contradiction. If uz is not an f-edge, then regarding to the colouring of x, y, u we have the following three cases:

If $f(x) = f(z) = 1$ and $f(u) = 2$.

Consider the symmetric colouring of C_2 and $f'(x) = 4$. Hence, uz is an f-edge in G and $T = T_1 \cup T_2 \cup \{uz\}$, So for the spanning tree T in G, $BBC(G,T) \leq 4$, contradiction.

If $f(x) = f(z) = 2$ and $f(u) = 1$.

Consider the symmetric colouring of C_2 with $f'(x) = 3$. Hence, uz is an f-edge in G and $T = T_1 \cup T_2 \cup \{uz\}$, So for the spanning tree T in G, $BBC(G,T) \leq 4$, contradiction.

If $f(x) = f(z) = 2$ and $f(u) = 3$.

Since uy is an f-edge, $f(y) = 1$ and we can recolour u with 4. Hence, uz is an f-edge in G and there is a spanning tree $T = T_1 \cup T_2 \cup \{uz\}$ in (G, T) such that $BBC(G, T) \leq 4$, contradiction.

(c) u is a cut vertex and $comp(G') = 3$.

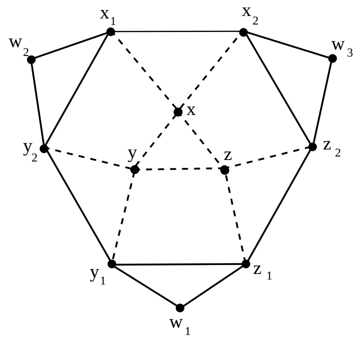
Assume that C_1, C_2 and C_3 are three components of G' and let $x \in C_1$, $y \in C_2$ and $z \in C_3$. By the minimality of G, there is a spanning tree T_i of C_i such that (C_i, T_i) has a backbone 4-colouring for $i = 1, 2, 3$. Without loss of generality, assume that $f(x) \leq f(y) \leq f(z)$. If any two vertices in set A have different colours, then $f(z) \in \{3, 4\}$. In this case consider the symmetric colouring of C_3 and let $f(u) = 4$.

If there exists two vertices v_1 and v_2 in A such that $f(v_1) = f(v_2)$, then similar to the case 2a we can find a proper colouring for u. Since ux, uy and uz are f-edges, then $T = T_1 \cup T_2 \cup T_3 \cup \{ ux, uy, uz \}$ is a spanning tree of G such that (G, T) has a backbone 4-colouring, contradiction.

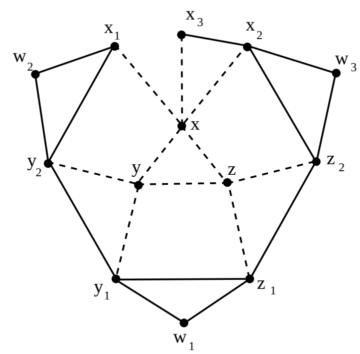
□

Lemma 2. *Graph G has at most one bridge.*

Proof. Assume otherwise and suppose that graph G has at least two bridges, e_1, e_2. We remove e_1 and G' is the remaining graph with two components, C_1 and C_2. Assume that $e_2 \in C_2$ with endpoints v_1 and v_2. According to Lemma 1, $\delta(G) \geq 4$ which states that $\delta(G') \geq 3$. Let $\{v_1, v_3, v_4\} \subseteq N'_G(v_2)$. Spanning tree T' of graph G' must contain e_2 because it does not lie on any cycle. So, assume that in G',



(a) bad $[4, 4, 4]$-triangle



(b) bad $[5, 4, 4]$-triangle

Figure 3.1: Examples of Lemma 3, bad triangles.

$f(v_1) = 2$, $f(v_3) = 1$ and $f(v_4) = 4$. Now $f(v_2) \notin \{1, 2, 3, 4\}$ and $BBC(G', T') \geq 5$, which is a contradiction. $\square$

A 4-face is weak if it is adjacent to four triangles. A $[4, 4, 4]$-triangle is bad if it is adjacent to three weak 4-faces. Similarly, a $[5, 4, 4]$-triangle is bad if it is adjacent to two weak 4-faces.

Lemma 3. *If x, y and z are three mutually adjacent 4-vertices in G, then x, y, z form a bad triangle, see Figure 3.1a.*

Proof. Suppose that G contains three mutually adjacent 4-vertices, x, y and z and let $G' = G - \{x, y, z\}$. By the minimality of G, there is a spanning tree T' of G' such that (G', T') has a backbone 4-colouring.

For now assume that G' is connected. At the end we show that if G' is not connected, we have the choice of expanding the spanning subgraph of G' to the connected one.

Let v_1 and v_2 be two neighbours of v, $v \in A = \{x, y, z\}$ which are in $B = N(A) - A$,

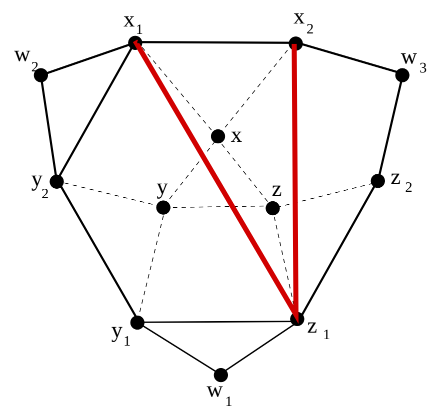


Figure 3.2: Example of graph G''.

i.e., $B = \{x_1, x_2, y_1, y_2, z_1, z_2\}$.

Without loss of generality, assume that $f(x_1) \leq f(x_2)$ and $f(y_1) \leq f(y_2)$ and $f(z_1) \leq f(z_2)$. For every $v \in A$ and $v_1, v_2 \in B$, list $L(v)$ is a set of available colours for vertex v which is $S - \{f(v_1), f(v_2)\}$.

For finding the spanning tree T we have the following two cases:

1. $L(x) = L(y) = L(z) = \{c_1, c_2\}$.

   In this case, we construct a graph $G''(V'', E'')$ by adding two new edges to $G - \{x, y, z\}$ which do not create any adjacent triangles in G'', see Figure 3.2. Here by adding z_1x_1 and z_1x_2 to $G - \{x, y, z\}$, G'' does not contain adjacent triangles. Since $f(z_1)$ is the same as $f(x_1)$, for every T'', $BBC(G'', T'') \geq 5$. By the minimality of G and the fact that $|V''| < |V|$, there is a spanning tree T'' of G'', $BBC(G'', T'') \leq 4$, which is a contradiction.

   The only challenging case is when by adding any two edges to G', where $BBC(G'', T'') \geq 5$, adjacent triangles are created, as in Figure 3.1a. Accord-

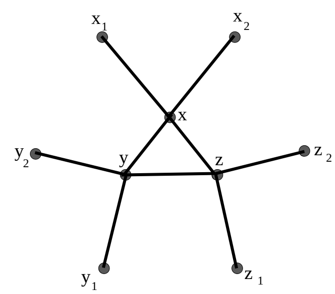


Figure 3.3: Three mutually adjacent 4-vertices.

ing to the definition this case is a bad $[4, 4, 4]$-triangle.

2. $\exists u, v \in A, L(u) - L(v) \neq \emptyset$ or $\forall u \in A, |L(u)| = 3$.

   If $\exists u, v \in A, L(u) - L(v) \neq \emptyset$.

   Without loss of generality, assume that $L(x) - L(y) \neq \emptyset$. In this case we provide a formula to assign proper colours to x, y and z in order to expand T' to T, spanning tree of graph G.

$$f(y) = min\Big(L(y) - L(x)\Big)$$

$$f(z) = \begin{cases} min\Big(L(z) - \{f(y)\}\Big) & : 1 \in L(z) \\ max\Big(L(z) - \{f(y)\}\Big) & : 1 \notin L(z) \end{cases}$$

$$f(x) = max\Big(L(x) - \{f(z)\}\Big)$$

   One can easily observe that if $\{1, 4\} \subseteq \{f(x), f(y), f(z)\}$, always we can find a subset of edges for expanding T' to spanning tree T. Without loss of generality, assume that $f(x) = 1$ and $f(y) = 4$. According to our assumption $|L(x)| = |L(y)| = 2$; so, for any colouring of x_1, x_2, y_1, y_2, edges xx_2 and yy_1 are always f-edges. Also, for any colouring of z, one of the edges xz or yz is always an f-edge; if $f(z) = 2$, yz is an f-edge and if $f(z) = 3$, xz is an f-edge.

If $\{1,4\} \nsubseteq \{f(x), f(y), f(z)\}$, the set of colours for x, y, z is either $\{2,3,4\}$ or its symmetric colouring $\{1,2,3\}$. Without loss of generality, assume that $f(x) = 2$, $f(y) = 4$ and $f(z) = 3$. Edges xy and yy_1 are f-edges. If $f(z_1) = 1$, then zz_1 is an f-edge. However, if $f(z_1) \neq 1$, it means that $1 \in L(z)$. Since $f(z) = 3$, $min\Big(L(z) - \{f(y)\}\Big) \neq 1$ which is true only if $f(y) = 1$; this contradicts the fact that $f(y) = 4$. So, in this case $f(z_1) = 1$.

If $\nexists u, v \in A, L(u) - L(v) \neq \emptyset$ and $\forall u \in A, |L(u)| = 3$.

In this case we provide a formula to assign proper colours to x, y and z in order to expand T' to T.

$$f(y) = min(L(y))$$
$$f(z) = max(L(z))$$
$$f(x) = \Big(L(x) - \{f(y)\}) - \{f(z)\}\Big)$$

Without loss of generality, assume that $\forall u \in A$, $L(u) = \{2,3,4\}$ or $L(u) = \{1,3,4\}$. If $\forall u \in A$, $L(u) = \{2,3,4\}$ then according to above formula colour y with 2, z with 4 and x with 3. So, edges yz, zz_1 and xx_1 are f- edges.

If $\forall u \in A$, $L(u) = \{1,3,4\}$ then according to above formula colour y with 1, z with 4 and x with 3. So, edges yz, zz_1 and xy are f- edges.

As a result, always we can find a subset of edges for expanding T' to spanning tree T such that $BBC(G,T) \leq 4$. This contradicts the minimality of G.

Now assume that G' is disconnected. Since G has at most one bridge, G' cannot have more that three components, C_1, C_2 and C_3 (three components for the case that x, y and z are cut vertices of G).
For finding a subset of edges to expand T', whether we can use the current colouring of vertices or the symmetric colouring of components with recolouring vertices. By the minimality of G, backbone colouring of each component is at most 4. If two edges in the set $\{xy, yz, xz\}$ are f-edges, then easily we can extend the spanning tree of each component to T. However, if just one of the edges in the set $\{xy, yz, xz\}$ is an f-edge and by considering the symmetric colouring of one component, we cannot find two f-edges in that set, then we have the following case:
$\exists u, v \in A, L(u) = L(v) = \{2, 3\}$.

Without loss of generality, assume that $L(y) = L(z) = \{2, 3\}$ and xy is an f-edge. In this case we consider a smaller graph $G_1 = G - xz$. Since yz is not an f-edge, for any spanning tree T_1 of G_1, $BBC(G_1, T_1) \geq 5$. This contradicts the minimality of G. So, this case does not exist.

For the case that $\forall u \in A, L(u) = \{2, 3\}$ or $\forall u \in A, L(u) = \{1, 4\}$, vertices x, y and z form a bad $[4, 4, 4]$-triangle. The proof can also be given with a similar argument to Case 1.

$\square$

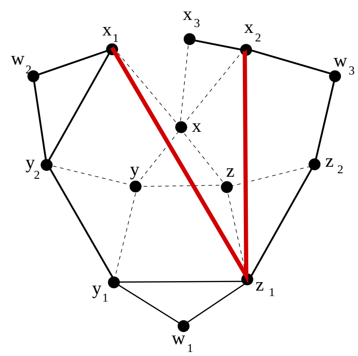


Figure 3.4: Example of graph G''.

Lemma 4. *If x, y and z are three mutually adjacent vertices with degrees $5, 4, 4$ in G, then x, y, z form a bad triangle, as shown in Figure 3.1b.*

Proof. Suppose that G contains three mutually adjacent vertices with degrees $4, 4, 5$ and without loss of generality, assume that $d(x) = 5$ and $d(y) = d(z) = 4$. Let $G' = G - \{x, y, z\}$ and for now assume that G' is connected. By the minimality of G, there is a spanning tree T' of G' such that $BBC(G', T') \leq 4$. Assume that $A = \{x, y, z\}$ and set $B' = N(A) - A = \{x_1, x_2, x_3, y_1, y_2, z_1, z_2\}$. For finding the spanning tree T, we consider the following two cases:

1. $L(y) = L(z)$ and $L(x) \subseteq L(y)$ and $|L(y)| = 2$.

   Without loss of generality, assume that $f(x_1) = f(z_1)$ and $f(x_2) = f(z_2)$. In this case, we construct a graph G'' by adding two new edges to $G - \{x, y, z\}$ which do not create any adjacent triangles in G'', Figure 3.4. Here by adding x_2z_1 and x_1z_1 to $G - \{x, y, z\}$, G'' does not contain adjacent triangles and for every spanning tree T'', $BBC(G'', T'') \geq 5$. Since, graph G'' is smaller

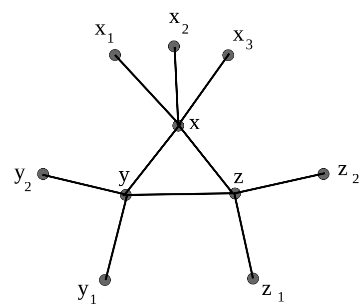


Figure 3.5: Three mutually adjacent vertices with degrees 5,4,4.

than G, by the minimality of G there exists a spanning tree T'' of G'' such that $BBC(G'', T'') \leq 4$, which is a contradiction.

The only challenging case is when by adding any two edges to $G - \{x, y, z\}$, where $BBC(G'', T'') \geq 5$, adjacent triangles in G'' are created, Figure 3.1. According to the definition, this case is a bad $[5, 4, 4]$-triangle.

2. $\exists u, v \in A, |L(u)| \leq |L(v)|, L(u) - L(v) \neq \emptyset$ or $\exists u \in A, |L(u)| = 3$.

   (a) If $\exists u, v \in A, |L(u)| \leq |L(v)|, L(u) - L(v) \neq \emptyset$ and $\forall u \in A, |L(u)| \leq 2$.

   If $\forall u \in A, |L(u)| = 2$, we can use the same formula as the Case 2 of Lemma 3 to assign proper colours to x, y and z. The proof can also be given with a similar argument to the Case 2 of Lemma 3.

   Now, if $|L(x)| = 1$, we can assign proper colours to x, y and z by using the following formula and expand T' to T.

$$f(x) = L(x)$$
$$f(y) = min\Big(L(y) - \{f(x)\}\Big)$$
$$f(z) = max\Big((L(z) - \{f(y)\}) - \{f(x)\}\Big)$$

One can easily observe that if $\{1,4\} \subseteq \{f(x), f(y), f(z)\}$, always we can find a subset of edges for expanding T' to T. Without loss of generality, assume that $f(x) = 1$ and $f(z) = 4$. As a result, xx_3 and zz_1 are f-edges. If $f(y) = 2$, yz will be an f-edge and if $f(y) = 3$, xy will be an f-edge.

If $\{1,4\} \nsubseteq \{f(x), f(y), f(z)\}$, also we can find a subset of edges for expanding T' to T. Without loss of generality, assume that $f(x) = 1$, $f(y) = 2$ and $f(z) = 3$. So, xx_3 and xz are f-edges. If at least one of the edges in the set $\{yy_1, yy_2, zz_1, zz_2\}$ is an f-edge easily we can expand spanning tree T' to T.

In this case, none of the edges in the set $\{yy_1, yy_2, zz_1, zz_2\}$ are f-edges, switch the name of y and z in the formula. For example, if $f(y_1) = 1$, $f(y_2) = 3$, $f(z_1) = 2$ and $f(z_2) = 4$, by switching y and z in the formula, $f(y) = 4$ and $f(z) = 1$. Hence, yy_1 and zz_2 are f-edges and we can expand T' to T.

(b) If $\exists u \in A, |L(u)| = 3$.

Without loss of generality, assume that $|L(x)| = 3$.

$$f(y) = min\Big(L(y)\Big)$$
$$f(z) = max\Big(L(z)\Big)$$
$$f(x) = max\Big((L(x) - \{f(y)\}) - \{f(z)\}\Big)$$

The proof can be given with a similar argument that is discussed in the previous case.

Now assume that G' is disconnected. Since G has at most one bridge, G' cannot

have more that four components, C_1, C_2, C_3 and C_4. Suppose that $x_i \in C_1$, $x_3 \in C_2$, $y_i \in C_3$ and $z_i \in C_4, i = 1, 2$ (four components for the case that x, y and z are cut vertices of G and one of the edges in set $\{xx_1, xx_2, xx_3\}$ is a bridge).
For finding a subset of edges to expand T', whether we can use the current colouring of vertices or the symmetric colouring of components with recolouring vertices. By the minimality of G, backbone colouring of each component is at most 4. If two edges in the set $\{xy, yz, xz\}$ are f-edges, then easily we can extend the spanning tree of each component to T. However, if just one of the edges in the set $\{xy, yz, xz\}$ is an f-edge and by considering the symmetric colouring of components, we cannot find two f-edges in that set, then we have the following case:
$\exists u, v \in A, L(u) \subseteq L(v) \subseteq \{2, 3\}$.

> Without loss of generality, assume that $L(x) \subseteq L(z) \subseteq \{2, 3\}$ and yz is an f-edge. In this case we consider a smaller graph $G_2 = G - xy$. Since xz is not an f-edge, for any spanning tree T_1 of G_1, $BBC(G_2, T_2) \geq 5$, but this contradicts our assumption that G is a minimum counterexample. So, this case does not exist.
>
> For the case that $\forall u \in A, L(u) \subseteq \{2, 3\}$ or $\forall u \in A, L(u) \subseteq \{1, 4\}$, vertices x, y and z form a bad triangle. The proof can also be given with a similar argument to Case 1.

$\square$

Corollary 1. *If a* $[4, 4, 4]$*-triangle is bad then in any colouring of* $G - \{x, y, z\}$, $L(y) = L(z) = \{p, q\}$ *and* $L(x) \subseteq L(y)$, $\{p, q\} \subset \{1, 2, 3, 4\}$.

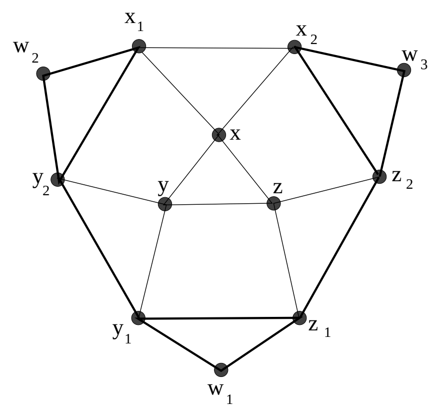


Figure 3.6: Example of sponsor vertices of bad $[4,4,4]$-triangle with degrees 4.

Corollary 2. *If a* $[5,4,4]$*-triangle is bad then in any colouring of* $G-\{x,y,z\}$*,* $L(y)=L(z)=\{p,q\}$ *and* $L(x)\subseteq L(y)$*,* $\{p,q\}\subset\{1,2,3,4\}$ *and* $p\neq q$*.*

3.3 Properties of bad $[4,4,4]$-triangle

We call vertices in set $B = \{x_1,x_2,y_1,y_2,z_1,z_2\}$ the *sponsor vertices* of the bad triangle. Vertices $w_i, i = 1,2,3$ in Figure 3.6 are the common neighbours of adjacent sponsor vertices. Let w_1 be the common neighbour of y_1 and z_1. According to Corollary 1, if $S-\{p,q\}=\{r,s\}$, $\{f(x_1),f(x_2)\}=\{f(y_1),f(y_2)\}= \{f(z_1),f(z_2)\}= \{r,s\}$.

Following lemmas will be frequently applied.

Lemma 5. *The degree of sponsor vertices is at least 5.*

Proof. First we prove it for $z_1, d(z_1) \geq 5$. Assume otherwise, that is, z_1 is a 4-vertex and $N(z_1) = \{z,z_2,y_1,w_1\}$, see Figure 3.6. Consider a colouring of $G-$

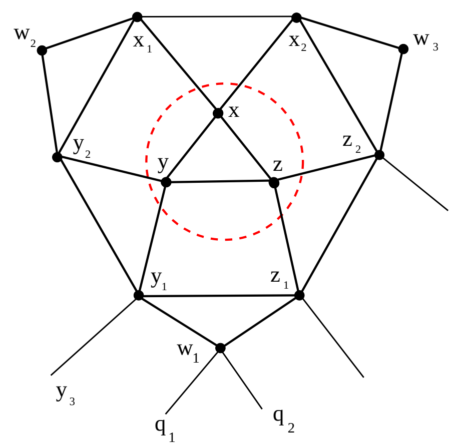


Figure 3.7: Example of vertices $w_i, i = 1, 2, 3$ with degrees 4.

$\{x, y, z\}$. Assume that $f(z_1) = r$. Since y_1 and z_1 are connected, $f(y_1) \neq f(z_1)$. Let $f(y_1) = s$ $(r \neq s)$. By Corollary 1, $f(z_2) \neq f(z_1)$ and $f(y_1) = f(z_2) = s$. So, z could be coloured with $f(w_1)$. As a result, all neighbours of z_1 are in two different colours, s and $f(z)$, and there are two colours for z_1, $S - \{s, f(z)\}$. This contradicts our assumption that for every colouring of G, the colour of sponsor vertices is r or s.

Second with the same argument we can prove that $d(v) \geq 5, v \in B$.

□

Lemma 6. *If two adjacent vertices in the set* $B = \{x_1, x_2, y_1, y_2, z_1, z_2\}$ *are* 5-*vertices, then their common vertex in the set* $\{w_1, w_2, w_3\}$ *(if there exists any) has degree at least 5.*

Proof. Assume for a contradiction that $d(y_1) = 5$ and $d(z_1) \geq 5$ and $d(w_1) = 4$. Consider a proper colouring of graph $G' = G - \{x, y, z\}$. According to Corollary

1, $\{f(y_1), f(y_2)\} = \{f(z_1), f(z_2)\} = \{r, s\}$. Without loss of generality assume that $f(z_2) = f(y_1) = r$ and $f(y_2) = f(z_1) = s$.

If we can recolour y_1 with colour p or q, then vertices x, y and z can be coloured with the colour set S for which $BBC(G,T) \leq 4$.

According to figure 3.7, $N'_G(y_1) = \{y_2, z_1, y_3, w_1\}$ and $\{z_3, z_4\} \in N'_G(z_1)$. Since, $f(z_1) = f(y_2) = s$ and the only available colour for y_1 is r, then $\{f(y_3), f(w_1)\} = \{p, q\}$. Similarly, since $f(z_2) = f(y_1) = r$ and the only available colour for z_1 is s, then $\{f(w_1), f(z_3)\} = \{p, q\}$.

If $\{f(q_1), f(q_2)\} = \{r, s\}$, then we can recolour y_1 with $\{p, q\} - \{f(y_3)\}$ because graph G does not have adjacent triangles which means that w_1 is not connected to y_3, contradiction.

Now if $r \notin \{f(q_1), f(q_2)\}$ or $s \notin \{f(q_1), f(q_2)\}$, then we can recolour w_1 with r or s, $f'(w_1) \in \{r, s\}$. Hence, y_1 or z_1 can be coloured with $f(w_1)$ which is p or q, contradiction.

□

Lemma 7. *If the sponsor vertex sponsored more than one bad triangle, then it should be a* 6^+*-vertex.*

Proof. Suppose for a contradiction that z_1 is a 5-vertex and it sponsors at least two bad triangle. Then it should be adjacent to more than three triangles. In this case two of these triangles are adjacent, the this contradicts our choice of G.

□

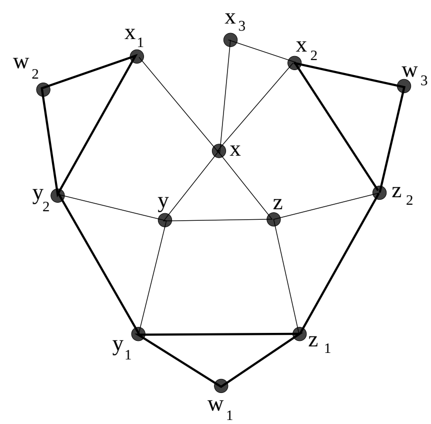


Figure 3.8: Example of sponsor Vertices of bad $[5, 4, 4]$-triangle with degree 4.

3.4 Properties of a bad $[5, 4, 4]$-triangle

According to Corollary 2, if $S - \{p, q\} = \{r, s\}$, $\{f(y_1), f(y_2)\} = \{f(z_1), f(z_2)\} = \{r, s\}$ and $\{r, s\} \subseteq \{f(x_1), f(x_2), f(x_3)\}$. Without loss of generality, assume that $f(x_1) = r$ and $f(x_2) = s$.

Lemma 8. *The degree of vertices z_1, z_2, y_1 and y_2 is at least 5.*

Proof. First we prove it for $z_1, d(z_1) \geq 5$. Assume otherwise, that is, z_1 is a 4-vertex where $N(z_1) = \{z, z_2, y_1, w_1\}$, as in Figure 3.8. Assume that $f(z_1) = s$. Since y_1 and z_1 are connected, $f(y_1) \neq f(z_1)$ and $f(y_1) = r$; also, $f(z_2) \neq f(z_1)$ and $f(y_1) = f(z_2) = r$ and $f(z)$ could be the same colour as $f(w_1)$. As a result, all neighbours of z_1 are in two different colours, r and $f(z)$, and there are two available colours for z_1 in the set $S - \{f(z)\}$. This contradicts our assumption that for every colouring of G, the colour of sponsor vertices is r or s.
Second with the same argument we can prove that $d(z_2) \geq 5$, $d(y_1) \geq 5$ and $d(y_2) \geq 5$.

□

Lemma 9. *The degree of w_1 is at least 5.*

The proof is similar to the proof of Lemma 2.

3.5 The discharging method

In this section we will complete our proof by using discharging rules.

Since G is a planar graph, then; according to Euler's formula, G has the characteristic that $|V(G)| - |E(G)| + |F(G)| = 2$. Hence, by using the fact that $\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$, we have the following result:
For every connected planar graph,

$$
\begin{aligned}
&\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) \\
&= 2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)| \\
&= -4(|V(G)| - |E(G)| + |F(G)|) = -8.
\end{aligned}
$$

We define an initial charge function of $\omega(x) = d(x) - 4$ where $x \in V(G) \cup F(G)$. Then we design appropriate discharging rules and redistribute charges accordingly. After charges are distributed according to discharging rules, a new charge function ω' is produced. When discharging is in progress, the sum of all charges is fixed. On the other hand,

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = \sum_{x \in V(G) \cup E(G)} \omega'(x).$$

We denote the amount of charges transformed from x to y by $\tau(x \to y)$ where $x, y \in V(G) \cup F(G)$. Our discharging rules are as follows:

Discharging rules:

R1 Every 5^+-vertex gives $\frac{1}{3}$ charge to each $[5^+, 5^+, 5^+]$-triangle.

R2 Every 5^+-vertex gives $\frac{1}{2}$ charge to each $[4, 5^+, 5^+]$-triangle.

R3 A sponsor vertex gives its residual charges to a bad triangle.

3.6 Proof of Theorem 1

For each $x \in V(G) \cup F(G)$ we will calculate $\omega'(x)$ and we will show that $\omega'(x) \geq 0$.

Proof.

- If $x \in V(G)$, according to Lemma 1, $\delta(G) \geq 4$,

$$\omega'(x) = d(x) - 4 \geq 0.$$

- If f^* is a [4,4,4]-triangle, $f^* \in F(G)$.

  According to Lemma 3, $[4, 4, 4]$-triangles in G are always bad, Figure 3.6. Let $f^* = [xyz]$.

All 3-faces of graph G have an initial charge of -1. By Lemma 5 and Lemma 2, $d(v) \geq 5$, $v \in \{w_1, y_1, z_1, z_2\}$. So, by using Rules R1 and R2 we have the following charge distributions:

$\tau(v \to f) = \frac{1}{3}, v \in \{w_1, y_1, z_1\}, f = w_1 y_1 z_1$

$\tau(v \to f) = \frac{1}{2}, v \in \{z_1, z_2\}, f = z_1 z_2 z$

Thus, $\omega'(z_1) = \frac{1}{6}$. With the same argument we can show that the new charge of all sponsor vertices of $[xyz]$ is $\frac{1}{6}$. Since bad triangle has six sponsor vertices, by using Rule R3 we have the following result:

$\omega'[xyz] = 3 - 4 + 6.\frac{1}{6} = 0$

If sponsor vertices are 6^+-vertices, they are adjacent to at most three triangles. Let z_1 be a 6-vertex, then $\omega'(z_1) = 6 - 4 - \frac{1}{3} - \frac{1}{2} = \frac{7}{6}$. So, z_1 can give at least $\frac{1}{6}$ charge to f^*.

- If f^* is a [5,4,4]-triangle, $f^* \in F(G)$.

According to Lemma 4, $[5, 4, 4]$-triangle in G are always bad, Figure 3.6. Let $f^* = [xyz]$.

By Lemma 8 and Lemma 9, $d(v) \geq 5$, $v \in \{w_1, y_1, y_2, z_1, z_2\}$. Also, x is a 5-vertex and $\omega(x) = 1$. So, by using Rules R1 and R2, we have the following charge distributions:

$\tau(v \to f) = \frac{1}{3}, v \in \{w_1, y_1, z_1\}, f = w_1 y_1 z_1$

$\tau(v \to f) = \frac{1}{2}$, where $v \in \{z_1, z_2\}, f = z_1 z_2 z$ or $v \in \{y_1, y_2\}, f = y_1 y_2 y$

$\tau(x \to f^*) = \frac{1}{2}$

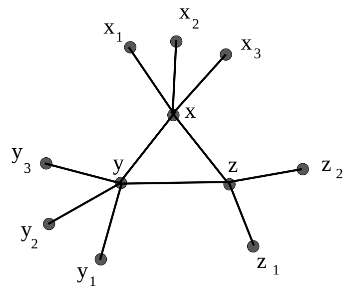


Figure 3.9: Example of [4,5,5]-triangle.

By Rule R3, $\omega'(y_1) = \omega'(z_1) = \omega'(z_2) = \frac{1}{6}$, that is $\omega'(f^*) = 3 - 4 + 3.\frac{1}{6} + \frac{1}{2} \geq 0$.

If vertex $v, v \in \{y_1, y_2, z_1, z_2, w_1\}$ is 6^+-vertex, it is adjacent to at most three triangles. Then $\omega'(v) = 6 - 4 - \frac{1}{3} - \frac{1}{2} = \frac{7}{6}$ and v can give at least $\frac{1}{6}$ charge to f^*.

- If f^* is a $[4^+, 5^+, 5^+]$-triangle, $f^* \in F(G)$.

  Let $f^* = [xyz]$, Figure 3.9. One can easily observe that, for a k-vertex v in graph G, v is adjacent to at most $\lfloor\frac{k}{2}\rfloor$ triangles. Let y and z be two 5^+-vertices of $[xyz]$. So, $\omega(v) = k - 4, v \in \{y, z\}$. By Rule R4,

$$\tau(v \to f^*) = \frac{k-4}{\lfloor\frac{k}{2}\rfloor}, v \in \{y, z\}.$$

  If $k = 5$, v is adjacent to at most two triangle. So, $\tau(v \to f^*) \geq \frac{1}{2}$.

  If $k \geq 6$, $\tau(v \to f^*) = \frac{k-4}{\lfloor\frac{k}{2}\rfloor} \geq 2 - \frac{8}{k} \geq \frac{4}{6}$.

  As a result, $\tau(v \to f^*) \geq \frac{1}{2}$, where $v \in \{y, z\}$; $\omega'(f^*) \geq 3 - 4 + 2.\frac{1}{2} \geq 0$.

- If f^* is a 4^+-face, $f^* \in F(G)$,

$$\omega'(f^*) = d(f^*) - 4 \geq 0.$$

Thus, we proved that $\omega'(x) \geq 0$ for $x \in V(G) \cup F(G)$ which leads the following contradiction,

$$0 \leq \sum_{x \in V(G) \cup E(G)} \omega'(x) = -8.$$

This contradiction shows that the assumption of existence of a minimum counterexample G was incorrect. As a result, we have succeeded in proving that backbone k-colouring of all connected planar graphs without adjacent triangles is at most 4.

$\square$

Chapter 4

Algorithm for Finding a Backbone

In this chapter we will extend the result obtained in Chapter 3 to all classes of graphs. In Chapter 3, as we coloured all vertices of graph G properly, we made the backbone of a graph. However, in this chapter, after colouring all vertices of the graph properly, we will find a backbone tree with the algorithm to be described below.

4.1 Existence of the backbone of every graph G with $\chi(G) \geq 4$

Theorem 2. *For every graph G with the chromatic number $\chi(G) = k$, there is a spanning tree T and a proper k-colouring f such that $\forall uv \in E(T),\ |f(u) - f(v)| = 1$.*

Proof. Consider a proper colouring of G with a colour function $f : V(G) \to S, |S| = \chi(G)$. An edge uv of the graph G is called *good* if $|f(u) - f(v)| = 1$. We also define an edge uv of the graph G as being *bad* if $|f(u) - f(v)| \geq 2$. Let x be an arbitrary vertex in G. Let T be the Breadth First Search tree with source x.

If all edges of T are good, simply we consider T as a backbone of G. Otherwise assume that T has q number of bad edges. Assume that set B is the set of all bad edges of T, $|B| = q$. Let e be the first bad edge in T with endpoints u and v where $f(u) = r$ and $f(v) = s$, such that $|r - s| \geq 2$.

Let C_x and C_v be two components of $T - B$ where $x, u \in C_x$ and $v \in C_v$. In other words, C_x is the maximal subgraph of T that contains x and no other bad edges. Our goal is to expand C_x step by step in a way that no other bad edges are left in C_x.

If there exists one good edge in G between C_x and C_v, simply we add that edge to $T - e$. So, it can be seen that the number of vertices in C_x is increased by at least 1. But if there does not exist any good edge between C_x and C_v in G, we have the following two cases:

Case I. $|r - s| = 2$.

> Without loss of generality, assume that $s - r = 2$. In this case we consider the $(s, s-1)$-Kempe chain of v and switch the colour of vertices in the Kempe chain. So $f'(v) = s - 1 = r + 1$ and uv is a good edge; as a result, we add uv to $T - B$ and it is clear that the number of vertices in C_x is increased by at least 1.
>
> It is not possible that the $(s, s-1)$-Kempe chain of v has any intersection with C_x. Assume otherwise and let y be an intersection of $(s, s-1)$-Kempe chain of v and C_x. Let z be a neighbour of y in C_v and, also, in $(s, s-1)$-

Kempe chain of v. Since both y and z are in $(s, s-1)$-Kempe chain of v and they are adjacent to each other, $|f(y) - f(z)| = 1$ and yz will be a good edge. But we assumed that there are not any good edges between C_x and C_v, contradiction.

Case II. $|r - s| \geq 3$.

Let $|r - s| = t$ and without loss of generality, assume that $r = s + t$. We consider the following algorithm for recolouring vertices.

Let i be a counter from 1 to $t - 1$. Consider $(s + i - 1, s + i)$-Kempe chain of v and switch the colour of vertices in the Kempe chain; then add 1 to the counter i. Do this step until $i = t - 1$. So, $f'(v) = s + t - 1 = r - 1$ and uv is a good edge; then, we add uv to $T - B$. Also, it is clear that the number of vertices in C_x is increased by at least 1.

Until here we replace the first bad edge by another good edge and we will repeat this step until there are not any more bad edges in T. As a result, $\forall uv \in E(T), |f(u) - f(v)| = 1$.

□

Note: The reason that we did not consider $(s, r - 1)$ directly is that it might have an intersection with C_x.

For example, consider Figure 4.1. In this figure, black edges are edges of graph G and edges of the spanning tree T are shown by red colour. Let uv be a first bad edge in T. Since there are no other good edges between C_x and C_v, we should

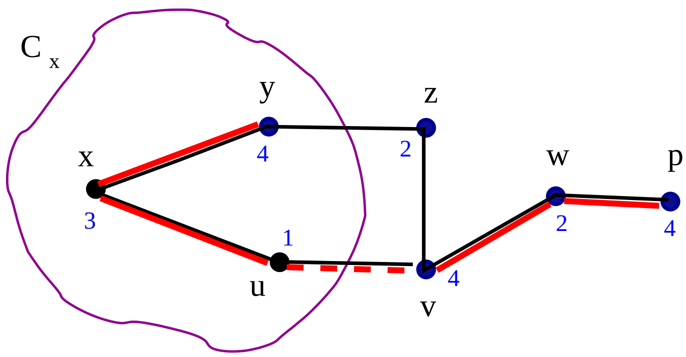


Figure 4.1: Example of $(4, 2)$-Kempe chain.

consider the Kempe chain of v. If we consider $(4, 2)$-Kempe chain of v directly, the subset of vertices in the Kempe chain will be $\{y, z, v, w, p\}$. These vertices are shown by blue colour in Figure 4.1. By switching colours of $(4, 2)$-Kempe chain, $f'(v) = 2$ and $f'(y) = 2$. But we were not allowed to change the colour of any vertices in C_x.

Corollary 3. *For every graph G if $\chi(G) \geq 4$, then there is a spanning tree T such that $BBC(G, T) = \chi(G)$.*

Proof. Assign vertices of G into k, $k = \chi(G)$ colour classes and put these sets in an increasing order. Next, we add edges of T in between the sets. According to Theorem 2, there are edges just between two consecutive pair of colour classes. Next step is to change the permutation of colour classes in the following order:

$$1, 2, 3, ..., k \rightarrow 2, 4, ..., 1, 3, ...$$

It is clear that the colour class k will be in the middle of permutation or at the end, corresponding to k being even or odd respectively.

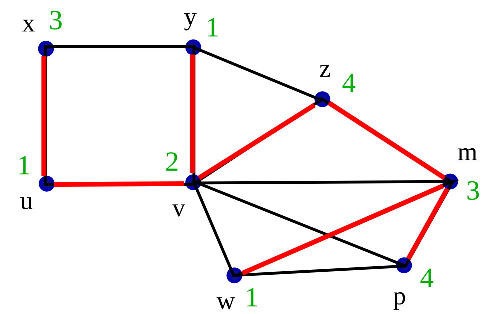


Figure 4.2: Example of finding a backbone of graph G with an arbitrary spanning tree T.

Hence, $\forall uv \in T, |f(u) - f(v)| \geq 2$ and all edges of T are f-edges. So, we find a spanning tree T of G such that $BBC(G,T) = \chi(G)$. Also, we prove the stronger result that $\forall uv \in T, |f(u) - f(v)| = 2$ or $|f(u) - f(v)| \geq k - 2$.

□

4.2 Example of finding a backbone of graph

Here is an example of using this algorithm for finding the backbone tree of graph G, Figure 4.2. In this figure black edges are edges of graph G. Let x be an arbitrary vertex in G and T be the breadth-first tree with source x. Edges of the spanning tree T are shown by red colour. Also, green numbers are the vertex colouring of G.

Let xu is the first bad edge in T. Set of all bad edges in T is $B = \{xu, vz, mw\}$. Graph $T - \ xu$ has two components, C_x, C_v and there are not any good edges between C_x and C_v. So, we consider the $Kempe(1,2)$ of u in G and switch the colour of vertices. According to Figure 4.2, $Kempe(1,2) = \{u, v, y, w\}$ and after

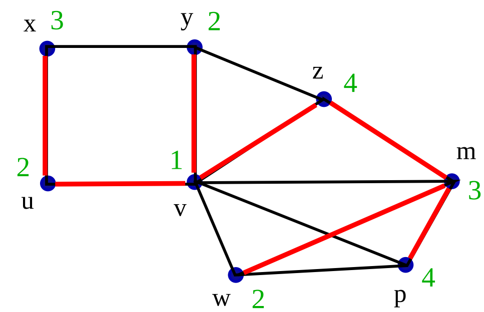


Figure 4.3: Switching the colour of vertices in the $Kempe(1,2)$ of u in G.

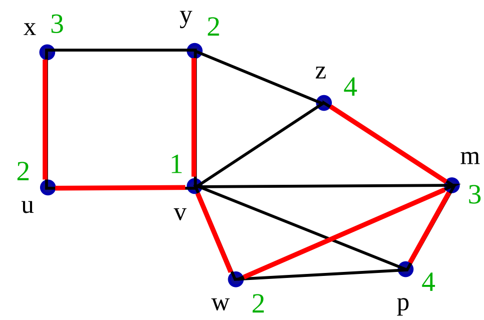


Figure 4.4: Replacing the first bad edge of T by another good edge from G.

switching colours, $f'(u) = 2, f'(y) = f'(w) = f'(v) = 1$ as we can see in Figure 4.3. Now xu is a good edge and we keep it in T.

Next step is to find a first bad edge of T in Figure 4.3 which is vz. So, $B = \{vz\}$. Components C_x and C_v are two components of $T - B$. In graph G, there exists a good edge vw between C_x and C_v and simply we add vw to $T - B$.

By using the Breadth First Search algorithm we can see that there are not any more bad edges in T. So, we find the spanning tree T such that all edges are good edges, as in Figure 4.4.

After that, we assign vertices of G into the 4 colour classes and put these sets in

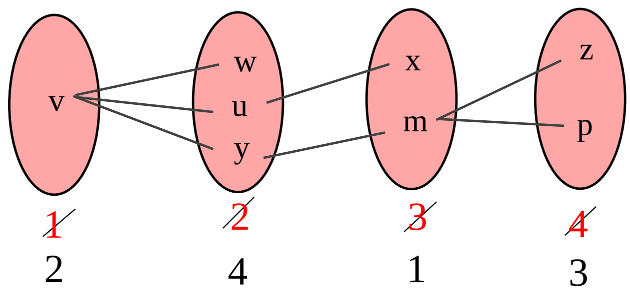


Figure 4.5: Recolouring each colour class of G.

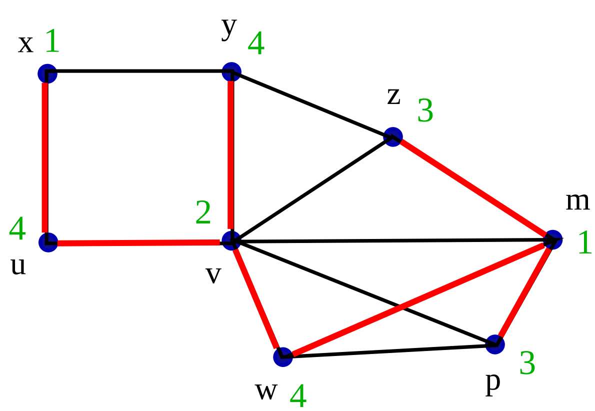


Figure 4.6: The spanning tree T is the backbone of graph G.

an increasing order, as in Figure 4.5. Now we should recolour each colour class of G in the following order: $1 \to 2, 2 \to 4, 3 \to 1, 4 \to 3$. Hence, according to the proper colouring of G in Figure 4.6, all red edges of graph are f-edges. As a result, spanning tree T is the backbone of G and $BBC(G,T) = 4$.

Chapter 5

Conclusion

In this research it has been proved that for every connected planar graph without adjacent triangles, backbone k-colouring is at most 4. Also, we proved the stronger result that for every graph G with $\chi(G) \geq 4$, a spanning tree T of G exists, such that $BBC(G, T) = \chi(G)$.

In future work, the following questions could be answered:

- Is it possible to prove that for every connected planar graph and every spanning tree T of G, the circular backbone k-colouring of (G, T) is at most 7?
- Can we find any tight upper bound for the circular q-backbone k-colouring chromatic number of planar graphs?
- Is it possible to find any tight upper bound for the q-backbone k-colouring chromatic number of all classes of graphs?

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