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Some Travelling Wave Solutions of KdV-Burgers Equation

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Abstract

In this paper we study the extended Tanh method to obtain some exact solutions of KdV-Burgers equation. The principle of the Tanh method has been explained and then apply to the nonlinear KdV-Burgers evolution equation. A finite power series in tanh is considered as an ansatz and the symbolic computational system is used to obtain solution of that nonlinear evolution equation. The obtained solutions are all travelling wave solutions.

Keywords: Travelling wave, Tanh method, KdV-Burgers equation, Travelling wave solutions

1 Introduction

Travelling waves are conspicuous in many linear equations and nonlinear modeling of waves, for example, sound wave, string wave, water wave etc. The nonlinear KdV(Korteweg-de Vries) equation is a famous model[2] of water wave with long wave and small but finite amplitude. This equation admits the permanent travelling wave solutions. Besides many nonlinear models of wave can be found in different physical problems which satisfy different travelling waves. Let us consider one dimensional linear wave equation, as a simple example, that admits travelling waves:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $c > 0$ denotes speed of the wave and u is a function of spatial variable x and time variable t usually describe amplitude of the wave. The equation has the general solution of the form

$$u(x,t) = f(x - ct) + g(x + ct) \tag{2}$$

The functions $f(x-ct)$ and $g(x+ct)$ describe the right and the left moving waves respectively with a constant speed $c>0$. The wave $f(x-ct)$ can be observed with a new system of co-ordinates (x', t') so that $x=0$ at $t=0$. At time t , the position x of the wave is ct , since c is a speed of the wave, i.e. $x=ct$. Then the new co-ordinate is $(x'=x+ct, t'=t)$. Its mean that the wave moves to the distance $(x+ct)$ at time t.

If we draw the solution $f(x-ct)$ with a constant speed $c=2$, we will get the following figure which shows the right moving wave at different values of t. The figure shows that at $t=0$ the spatial space $x=0$ and at $t=5$ and 10 the wave moves at right direction at a distance $x=10$ and 20 respectively.

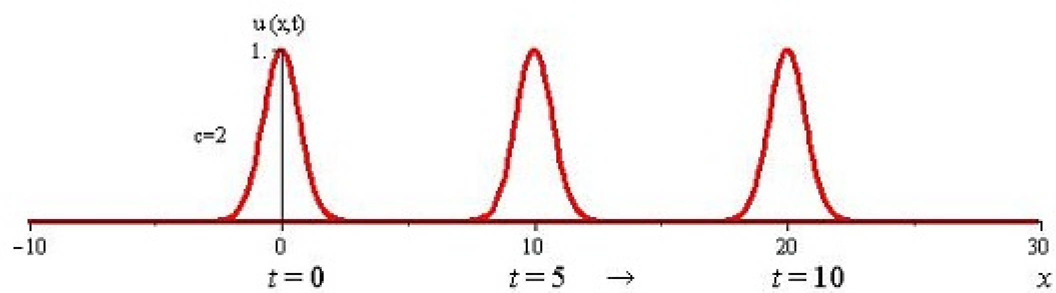

Fig.1: Right travelling wave at different values of t with constant speed $c=2$

Similarly, the solution $g(x+ct)$,with constant speed $c=2$, have the following left direction travelling wave which also moves at a distance $x=-10$ and -20 when $t=5$ and 10 respectively .

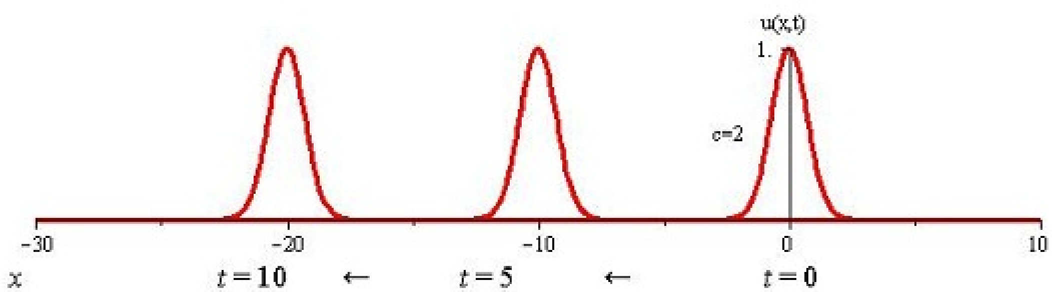

Fig.2: Left travelling wave at different values of t with constant speed $c=2$

In other words, a travelling wave is a wave in which the medium moves in the direction of propagation of the wave[9]. Such waves arise in the study of nonlinear differential equations where waves are represented by the form $u(x,t)=f(x-ct)$, where u is a disturbance moving in the negative or positive x direction if $c<0$ or $c>0$ or respectively. If the travelling solution $u(x,t)$

depends only on the difference between the two coordinates of partial differential equations, then the solution keeps its exact shape as it was originally. If the transition from the asymptotic state at $\xi = -\infty$ to the other asymptotic state at $\xi = \infty$ of a travelling wave is localized in ξ , where $\xi = x - ct$, then it is also called solitary wave.

In this paper our aim is to find some traveling wave solutions of KdV-Burgers equation using the extended Tanh method. The KdV-Burgers equation[6] is read of the form:

$$u_t + 2uu_x - \lambda u_{xx} + \beta u_{xxx} = 0, \tag{3}$$

where β and $\lambda > 0$ are constants and u is a function of spatial variable x and time variable t. The equation combines nonlinearity uu_x , linear dispersion u_{xxx} and dissipation u_{xx}. This is a well known nonlinear model of viscous elastic medium and is found in many physical phenomena.

2 Extended Tanh Method

The standard Tanh method was first developed in 1996 by Milfliet and Herman [11] where they used tanh as a new variable since all derivatives of a tanh are represented by tanh itself. The method explains that if we take a travelling wave variable $\xi = x - ct$ so that dependent variable $u(x,t) = u(\xi)$, then it converts any partial differential equation(PDE)

$$P(u(x,t), u_t, u_x, u_{xx}, u_{xxx}, ...) = 0 \tag{4}$$

to an ordinary differential equation(ODE)

$$F(u(\xi), \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, ...) = 0 \tag{5}$$

The ODE (5) can be integrated as many time as all terms contains derivatives and one can set integration constant to zero[11] since $u(\xi), \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, ... \to 0$ as $\xi \to \pm\infty$.

Suppose we introduce a transformation of independent variable ξ as

$$Y = \tanh(k\xi) \text{ and } \xi = x - ct \tag{6}$$

where, k is the number of waves and c is the speed of wave.
Then,

$$\frac{d}{d\xi} = k(1 - Y^2)\frac{d}{dY} \tag{7}$$

$$\frac{d^2}{d\xi^2} = k^2(1-Y^2)^2\frac{d^2}{dY^2} - 2k^2Y(1-Y^2)\frac{d}{dY} \tag{8}$$

$$\frac{d^3}{d\xi^3} = k^3(1-Y^2)^3\frac{d^3}{dY^3} - 6k^3Y(1-Y^2)^2\frac{d^2}{dY^2} - 2k^3(1-Y^2)(1-3Y^2)\frac{d}{dY} \tag{9}$$

and so on.

The tanh method admits the solution for PDE (4) of the form:

$$u(\xi) = \sum_{i=0}^{M} A_i Y^i + \sum_{j=1}^{M} B_j Y^{-j} \tag{10}$$

where M to be determined by

$$M + r = nM \tag{11}$$

as balancing the highest order (r) of linear terms with the highest order (n) of nonlinear term in the resulting ODE. Here A_i, B_j are real parameters. Now putting (10) into (5) and equating the coefficients of Y of the resulting system to zero, one have many algebraic equation with unknowns A_i, B_j and c . Solving these algebraic equations for the unknowns, we can find exact solutions of (4).

3 Tanh Method In KdV-Burgers Equation

We rewrite the KdV-Burgers equation (3) in the following form for our convenience:

$$u_t + (u^2)_x - \lambda u_{xx} + \beta u_{xxx} = 0, \tag{12}$$

Suppose that $\xi = x - ct$ be a moving co-ordinate with speed c, and $u(x,t) = u(\xi)$ represent the wave solutions. Then using these transformations to the PDE (3) we can find an ordinary differential equation (ODE).

The resultant ODE is

$$-c\frac{du}{d\xi} + \frac{d}{d\xi}(u^2) - \lambda\frac{d^2u}{d\xi^2} + \beta\frac{d^3u}{d\xi^3} = 0 \tag{13}$$

Integrating once we have

$$-cu(\xi) + u^2 - \lambda\frac{du}{d\xi} + \beta\frac{d^2u}{d\xi^2} = 0 \tag{14}$$

We set integration constant to zero as described before and since this is a nonlinear travelling wave equation. According to the Tanh method, from the equation (14), we have $M = 2$.

Let us consider the solution of KdV-Burgers equation is of the form:

$$u(\xi) = \sum_{i=0}^{2} A_i Y^i + \sum_{j=1}^{2} B_j Y^{-j}, \tag{15}$$

where $Y = \tanh(k\xi)$, k is the number of waves and $\xi = x - ct$.

Putting (15) in (14) and equating the coefficients of powers of Y to zero, we have a system of many algebraic equations, where $c, \lambda, \beta, k \neq 0$.

4 Solutions Set of Parameters

Solving, by MAPLE, the algebraic equations found in the section-3, there exist sixteen set solutions of the parameters. But for $A_0 = A_1 = A_2 = B_1 = B_2 = 0$ and $A_1 = A_2 = B_1 = B_2 = 0$ the KdV-Burgers equation has some trivial solutions and we ignore these solutions. The twelve non-trivial solution sets for coefficients of (15) are given below:
Set-1:

$$A_0 = -\frac{3\lambda^2}{50\beta}, A_1 = A_2 = 0, B_1 = -\frac{3\lambda^2}{25\beta}, B_2 = -\frac{3\lambda^2}{50\beta}, c = -\frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{10\beta}$$

Set-2:

$$A_0 = \frac{9\lambda^2}{50\beta}, A_1 = A_2 = 0, B_1 = -\frac{3\lambda^2}{25\beta}, B_2 = -\frac{3\lambda^2}{50\beta}, c = \frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{10\beta}$$

Set-3:

$$A_0 = -\frac{9\lambda^2}{50\beta}, A_1 = B_1 = -\frac{3\lambda^2}{50\beta}, A_2 = B_2 = -\frac{3\lambda^2}{200\beta}, c = -\frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{20\beta}$$

Set-4:

$$A_0 = \frac{3\lambda^2}{20\beta}, A_1 = B_1 = -\frac{3\lambda^2}{50\beta}, A_2 = B_2 = -\frac{3\lambda^2}{200\beta}, c = \frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{20\beta}$$

Set-5:

$$A_0 = -\frac{9\lambda^2}{100\beta}, A_1 = B_1 = \frac{3\lambda^2}{50\beta}, A_2 = B_2 = -\frac{3\lambda^2}{200\beta}, c = -\frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{20\beta}$$

Set-6:

$$A_0 = \frac{3\lambda^2}{20\beta}, A_1 = B_1 = \frac{3\lambda^2}{50\beta}, A_2 = B_2 = -\frac{3\lambda^2}{200\beta}, c = -\frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{20\beta}$$

Set-7:

$$A_0 = -\frac{3\lambda^2}{50\beta}, A_1 = A_2 = 0, B_1 = \frac{3\lambda^2}{25\beta}, B_2 = -\frac{3\lambda^2}{50\beta}, c = -\frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{10\beta}$$

Set-8:

$$A_0 = \frac{9\lambda^2}{50\beta}, A_1 = A_2 = 0, B_1 = \frac{3\lambda^2}{25\beta}, B_2 = -\frac{3\lambda^2}{50\beta}, c = \frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{10\beta}$$

Set-9:

$$A_0 = -\frac{3\lambda^2}{50\beta}, A_1 = \frac{3\lambda^2}{25\beta}, A_2 = -\frac{3\lambda^2}{50\beta}, B_1 = B_2 = 0, c = -\frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{10\beta}$$

Set-10:

$$A_0 = \frac{9\lambda^2}{50\beta}, A_1 = \frac{3\lambda^2}{25\beta}, A_2 = -\frac{3\lambda^2}{50\beta}, B_1 = B_2 = 0, c = \frac{6\lambda^2}{25\beta}, k = -\frac{\lambda}{10\beta}$$

Set-11:

$$A_0 = -\frac{3\lambda^2}{50\beta}, A_1 = -\frac{3\lambda^2}{25\beta}, A_2 = -\frac{3\lambda^2}{50\beta}, B_1 = B_2 = 0, c = -\frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{10\beta}$$

Set-12:

$$A_0 = \frac{9\lambda^2}{50\beta}, A_1 = -\frac{3\lambda^2}{25\beta}, A_2 = -\frac{3\lambda^2}{50\beta}, B_1 = B_2 = 0, c = \frac{6\lambda^2}{25\beta}, k = \frac{\lambda}{10\beta}$$

5 Some Exact Solutions

Each solution set of the parameters described above section gives an exact solution of KdV-Burgers equation. But set-1 and set-7 of section-3 give the same solution and similar cases arise for the pair of set 2 and 8, 3 and 5, 4 and 6, 9 and 11, 10 and 12. Thus we have 6 exact solutions of the KdV-Burgers equation which are as follows:
Solution-1:

$$u_1(x,t) = -\frac{3\lambda^2}{50\beta}\left[1 + 2\coth\left(\frac{\lambda}{10\beta}x + \frac{3\lambda^3}{125\beta^2}t\right) + \coth^2\left(\frac{\lambda}{10\beta}x + \frac{3\lambda^3}{125\beta^2}t\right)\right]$$

Solution-2:

$$u_2(x,t) = \frac{3\lambda^2}{50\beta}\left[3 - 2\coth\left(\frac{\lambda}{10\beta}x - \frac{3\lambda^3}{125\beta^2}t\right) - \coth^2\left(\frac{\lambda}{10\beta}x - \frac{3\lambda^3}{125\beta^2}t\right)\right]$$

Solution-3:

$$u_3(x,t) = -\frac{3\lambda^2}{200\beta}\Bigg[6 + 4\tanh\left(\frac{\lambda}{20\beta}x + \frac{3\lambda^3}{250\beta^2}t\right) + \tanh^2\left(\frac{\lambda}{20\beta}x + \frac{3\lambda^3}{250\beta^2}t\right) + 4\coth\left(\frac{\lambda}{20\beta}x + \frac{3\lambda^3}{250\beta^2}t\right) + \coth^2\left(\frac{\lambda}{20\beta}x + \frac{3\lambda^3}{250\beta^2}t\right)\Bigg]$$

Solution-4:

$$u_4(x,t) = \frac{3\lambda^2}{200\beta}\Bigg[10 - 4\tanh\left(\frac{\lambda}{20\beta}x - \frac{3\lambda^3}{250\beta^2}t\right) - \tanh^2\left(\frac{\lambda}{20\beta}x - \frac{3\lambda^3}{250\beta^2}t\right) - 4\coth\left(\frac{\lambda}{20\beta}x - \frac{3\lambda^3}{250\beta^2}t\right) - \coth^2\left(\frac{\lambda}{20\beta}x - \frac{3\lambda^3}{250\beta^2}t\right)\Bigg]$$

Solution-5:

$$u_5(x,t) = -\frac{3\lambda^2}{50\beta}\left[1 + 2\tanh\left(\frac{\lambda}{10\beta}x + \frac{3\lambda^3}{125\beta^2}t\right) + \tanh^2\left(\frac{\lambda}{10\beta}x + \frac{3\lambda^3}{125\beta^2}t\right)\right]$$

Solution-6:

$$u_6(x,t) = \frac{3\lambda^2}{50\beta}\left[3 - 2\tanh\left(\frac{\lambda}{10\beta}x - \frac{3\lambda^3}{125\beta^2}t\right) - \tanh^2\left(\frac{\lambda}{10\beta}x - \frac{3\lambda^3}{125\beta^2}t\right)\right]$$

All the six solutions are travelling wave solutions. Some of them(1,3,5) are left travelling and rest of them(2,4,6) are right travelling wave solutions.

6 Concluding Remark

Using extended Tanh method KdV-Burgers equation has six exact solutions which describe six different travelling waves with different speeds and frequencies. The main advantage of this method is that one can use MAPLE or Mathematica easily for computing tedious calculation. The Tanh method is a powerful tool for solving nonlinear partial differential equation. It is more reliable and efficient method to handle nonlinear partial differential equation.

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