
Edge-choosability of Planar Graphs

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Abstract

According to the List Colouring Conjecture, if G is a multigraph then $\chi'(G) = \chi'_l(G)$. In this thesis, we discuss a relaxed version of this conjecture that every simple graph G is edge- $(\Delta + 1)$ -choosable as by Vizing's Theorem $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. We prove that if G is a planar graph without 7-cycles with $\Delta(G) \neq 5, 6$ or without adjacent 4-cycles with $\Delta(G) \neq 5$, or with no 3-cycles adjacent to 5-cycles, then G is edge- $(\Delta + 1)$ -choosable.

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Chapter 1

Introduction

A *graph* G is an ordered pair $(V(G), E(G))$ consisting of $V(G)$, a set of vertices and $E(G)$ a set of unordered pair of vertices of G (not necessarily distinct). If e is an edge joining two vertices u and v , then we denote the edge e by uv and the vertices u and v are called the *end vertices* of e . *Multiple edges* (also called parallel edges) are two or more edges with the same end vertices. If a graph G has multiple edges, then it is called a *multigraph*. A *simple* graph G has no multiple edges.

In this thesis, we consider *simple planar* graphs. A graph G is called *planar*, if it can be embedded in the plane such that its edges intersect only at their end vertices. Two edges are adjacent if they have one common end vertex. This vertex is called *incident* to both edges. The degree of a vertex v in a graph G , denoted by $d(v)$, is the number of edges in G which are incident to v . We denote the minimum and maximum degrees of the vertices of G by

$\delta(G)$ and $\Delta(G)$, respectively.

An *edge-colouring* of graph G is an assignment of colours to all edges of graph G such that adjacent edges are not assigned the same colours. The *edge chromatic number* of graph G , denoted by $\chi'(G)$, is the minimum number of colours needed to achieve an edge-colouring of graph G . Vizing's theorem states

Theorem 1. *For a simple graph G , $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.*

A generalized type of edge colouring is *list-edge-colouring* in which a list of colours is assigned to each edge of G and each edge should be coloured with an available colour on its list. The *edge choice number* of graph G , denoted by $\chi'_l(G)$, is the smallest integer k such that each list contains at least k colours for every edge in G and a list-edge-colouring is achievable.

1.1 History

The following conjecture has been investigated independently by Vizing, Gupta, Albers and Collins, and Bollobás and Harris [JT95, HC92], and is known as the *List Colouring Conjecture* which states

Conjecture 1. *For all multigraphs G , $\chi'(G) = \chi'_l(G)$.*

The conjecture has been proven for a few special cases, such as bipartite multigraphs [Gal95], complete graphs with odd number of vertices [HJ97], multicircuits [Woo99], outerplanar graphs [WL01a], and planar graphs with

$\Delta(G) \geq 12$ which can be embedded in a surface of non-negative characteristic [BKW97]. In 1976, Vizing proposed a weaker conjecture [Kos92].

Conjecture 2. *Every simple graph G is edge- $(\Delta + 1)$ -choosable.*

This conjecture is a relaxed version of the List Colouring conjecture which states for every simple graph G , the upper bound of edge choice number of graph G is equal to the upper bound of the edge chromatic number of graph G as by Vizing's Theorem $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Harris showed if G is a graph with $\Delta(G) \geq 3$, then $\chi'_l(G) \leq 2\Delta(G) - 2$. This implies that Conjecture 2 is true for graphs with $\Delta(G) = 3$. Juvan, Mohar, and Škrekovski [JMŠ99] proved that the Vizing's conjecture holds for all graphs G with $\Delta(G) = 4$. Conjecture 2 has also been proven for other special cases such as complete graphs [HJ97], graphs with girth at least $8\Delta(G)(\ln\Delta(G) + 1.1)$ [Kos92], and planar graphs with $\Delta(G) \geq 9$ [CH10]. In March 2013, Bonamy improved the lower bound of 9 and proved Vizing's Conjecture holds for planar graphs with $\Delta(G) \geq 8$.

Vizing's Conjecture has been studied extensively. Most of the known results are for planar graphs. We provide a list of planar graphs that hold Vizing's Conjecture below.

[Wang et al. 2001] without 6-cycles with $\Delta \neq 5$ [WL01b]

[Wang and Lih 2002] without 5-cycles [WL02]

[Zhang and Wu 2004] triangle-free [ZW04]

[Zhang and Wu 2004] without 4-cycles with $\Delta \neq 5$ [ZW04]

[Wang 2005] without 4-cycles with $\Delta = 5$ or without 6-cycles with $\Delta = 5$ [Wan05]

[Shen et al. 2008] without 4-cycles [SZHZ08]

[Cranston 2009] without adjacent triangle with $\Delta \neq 5$ [Cra09]

[Hou et al. 2009] without adjacent triangles with $\Delta \neq 5$ [HLC09]

[Hou et al. 2009] without 7-cycles with $\Delta \neq 5, 6$ [HLC09]

[Ma et al. 2011] without intersecting 4-cycles with $\Delta \neq 5$ [MWCZ11]

1.2 Overview

In this thesis, we apply the discharging method to show that Vizing's Conjecture holds for 3 more cases. We correct some errors discovered in the proof of edge-choosability of planar graphs without 7-cycles [HLC09]. Ma et al. proved planar graphs G without intersecting 4-cycles with $\Delta(G) \neq 5$ are edge- $(\Delta + 1)$ -choosable [MWCZ11]. We improve this result by proving Vizing's conjecture holds for planar graph G without adjacent 4-cycles with $\Delta(G) \neq 5$. We will also prove that this conjecture holds for planar graphs without 3-cycles adjacent to 5-cycles.

We have definitions and notations in Chapter 2 which will aid the reader in understanding the language used in this thesis. In Chapter 3, we discuss

past results and explain the tools that were used commonly in their proofs. We introduce tree structure which is used in later chapters. We also prove the following theorem by fixing the errors in its proof.

Theorem 2. *Every planar graph G without 7-cycles is edge- k -choosable, where $k = \max\{8, \Delta(G) + 1\}$.*

In Chapter 4, we prove the following theorems by using tree structure.

Theorem 3. *Every planar graph G without adjacent 4-cycles is edge- k -choosable, where $k = \max\{7, \Delta(G) + 1\}$.*

Theorem 4. *Every planar graph G without 3-cycles adjacent to 5-cycles is edge- k -choosable, where $k = \max\{6, \Delta(G) + 1\}$.*

In Chapter 5, we will explain the results we discuss in Chapter 3 and Chapter 4. We also go through future questions that might be asked by someone regarding edge-choosability of planar graphs without adjacent 4-cycles with $\Delta(G) = 5$ and planar graphs without 7-cycles with $\Delta(G) = 5, 6$.

Chapter 2

Preliminaries and Notions

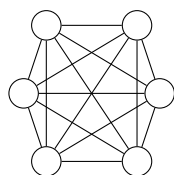
In this chapter, we define terminology used in this thesis and will also discuss in depth the special graphs mentioned in Section 1.1 that hold the List Colouring Conjecture or Vizing's Conjecture.

2.1 Special Families of Graphs

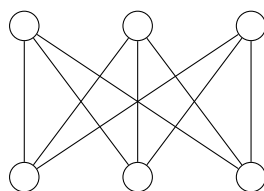
We mentioned certain types of graphs in the previous chapter. We define these graphs here.

A *complete graph* is a simple graph in which any two vertices are adjacent as shown in Figure 2.1. A complete graph with n vertices is denoted by K_n .

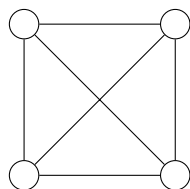
A *bipartite graph* is a graph whose vertex set can be divided into two disjoint subsets A and B such that every edge has one end vertex in A and one end vertex in B . If a bipartite graph is simple and every vertex in A is joined to every vertex in B , then G is called a *complete bipartite graph*. A

Figure 2.1: Complete Graph K_6 .

complete bipartite graph with sets of vertices A and B such that $|A| = m$ and $|B| = n$, is denoted by $K_{m,n}$.

Figure 2.2: Complete Bipartite Graph $K_{3,3}$.

Graph G is *planar* if it can be embedded in the plane such that its edges do not intersect except at the end vertices. For example, Figure 2.3 is a planar graph, since it can be embedded in the plane which is shown in Figure 2.5. However, the graph shown in Figure 2.4 is not a planar graph.

Figure 2.3: Planar graph K_4

If G is a planar graph, then any plane drawing of G divides the plane

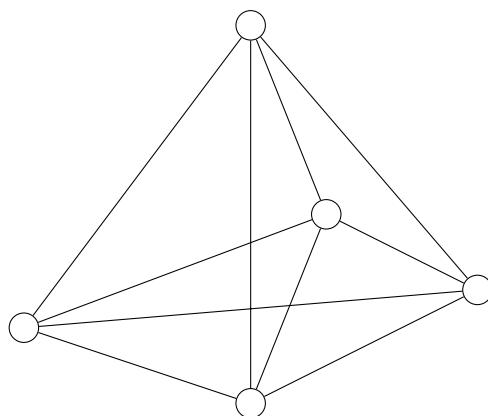
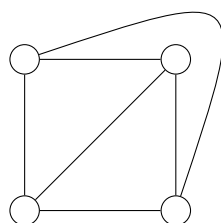


Figure 2.4: Non-planar graph.

Figure 2.5: Embedded planar graphs K_4 .

into regions, called *faces*. One of these faces is unbounded, and is called the *infinite face*. The set of faces of graph G is denoted by $F(G)$. An *outerplanar* graph is a planar graph which can be embedded in such a way that all of the vertices belong to the unbounded face of graph G . For example K_4 is a planar graph but it is not an outerplanar graph. However, Figure 2.6 is outerplanar.

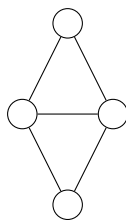


Figure 2.6: Outerplanar graph.

2.2 Vertex Degrees

The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$ or shortly $d(v)$, is the number of vertices adjacent to v . A vertex v is called a k -vertex if $d(v) = k$ or k^+ -vertex if $d(v) \geq k$. The maximum degree of a graph G , denoted by $\Delta(G)$, and the minimum degree of graph G , denoted by $\delta(G)$, are the largest vertex degree and smallest vertex degree in graph G , respectively. For example in Figure 2.7, the maximum degree $\Delta(G) = 5$ and minimum degree $\delta(G) = 3$.

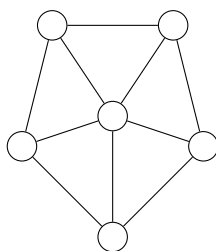


Figure 2.7: The maximum degree is 5 and minimum degree is 3.

The *degree of a face* $f \in F(G)$ denoted by $d(f)$, is the number of edges on the boundary of f and each cut edge is counted twice. A face $f \in F(G)$ is called a k -face or a k^+ -face if $d(f) = k$ or $d(f) \geq k$, respectively.

We use $[d(v_1), d(v_2), \dots, d(v_n)]$ to represent a face $f \in F(G)$ with vertices v_1, v_2, \dots, v_n of the degrees $d(v_1), d(v_2), \dots, d(v_n)$, that lie on the boundary of f in clockwise order. We also use $[v_1, v_2, \dots, v_n]$ to represent face $f \in F(G)$ as well.

A vertex v is *incident* to a face f if v lies on the boundary of face f .

For example, v is incident to a 5-face in Figure 2.8. Let $m_k(v)$ denote

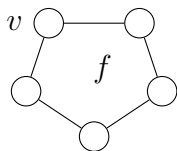


Figure 2.8: Vertex v incident to 5-face f .

the number of k -faces incident to v for all $v \in V(G)$ and $n_k(f)$ denote the number of k -faces adjacent to face f for all $f \in F(G)$.

For a face $f \in F(G)$, let $\delta(f)$ denote the minimum degree of vertices incident to f . For example in Figure 2.9, $\delta(f) = 2$.

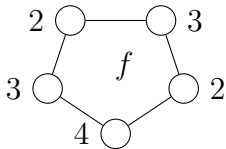


Figure 2.9: The degree of vertices are shown on vertices by numbers.

2.3 Circuits, Paths and Cycles

A $u - v$ *path* of length k is a sequence of k vertices starting at u and ending at v , where consecutive vertices in the sequence are adjacent in graph G such that each vertex is used at most once. A path of length k is denoted by P_k . A *circuit* is a sequence of vertices starting at u and ending at the same vertex, where consecutive vertices in the sequence are adjacent in graph G such that each edge is used at most once. A *cycle* of length k is a closed path of length k where the first and last vertices are the same. It is denoted by C_k . An *even cycle* or an *odd cycle* is a cycle of length k where K is an even or odd integer, respectively. A *multicircuit* is a multigraph whose underlying simple graph is a circuit. If for each two vertices u, v in graph G , there is a path between u and v then we call graph G a *connected* graph. A *tree* is a connected graph without a cycle.

Two cycles are *intersecting* if they have an edge or a vertex in common. Two cycles are *adjacent* if they have an edge in common. We discuss graphs excluding adjacent 4-cycles in Section 4.1 and 3-cycles adjacent to 5-cycles in Section 4.2.

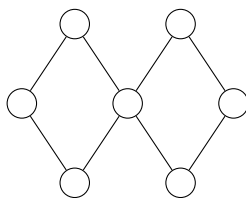
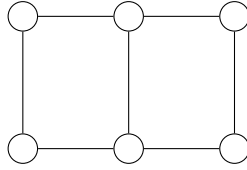


Figure 2.10: Two intersecting C_4 .

Figure 2.11: Two adjacent C_4 .

The minimum length of a cycle in a graph G is the *girth* of graph G .

2.4 Edge Colouring and List-edge-colouring

An *edge- k -colouring* of a graph $G = (V, E)$ is a mapping $c : E \rightarrow \{1, 2, \dots, k\}$, where $\{1, 2, \dots, k\}$ is a set of colours. An edge- k -colouring is *proper* if adjacent edges' colours are different. A graph G is *edge- k -colourable* if it has a proper edge- k -colouring. The *edge chromatic number*, denoted by $\chi'(G)$, is the smallest integer k such that G is edge- k -colourable. For example, for all integer n , $\chi'(C_{2n}) = 2$ and $\chi'(C_{2n+1}) = 3$.

Proof. Consider an even cycle $C_{2n} = [v_1, v_2, \dots, v_{2n-1}, v_{2n}]$ and two distinct colours α and β . We colour the edges $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$ with α and the edges $v_2v_3, v_4v_5, \dots, v_{2n}v_1$ with β . This is an edge-2-colouring of C_{2n} . We need at least 2 colours to colour two adjacent edges of C_{2n} and we showed an edge-2-colouring of C_{2n} , so $\chi'(C_{2n}) = 2$.

Also assume an odd cycle $C_{2n+1} = [v_1, v_2, \dots, v_{2n-1}, v_{2n}, v_{2n+1}]$ and three distinct colours α, β and γ . We colour the edges $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$ with α as well as the edges $v_2v_3, v_4v_5, \dots, v_{2n}v_{2n+1}$ with β . Then, for the edge

$v_{2n+1}v_1$, we have to colour this edge with the third colour γ . Since its adjacent edges $v_1v_2, v_{2n}v_{2n+1}$ have been already coloured with α and β , respectively. This is an edge-3-colouring of C_{2n+1} . It has been proved that we need at least 3 colours to colour all edges in C_{2n+1} and we showed an edge-3-colouring of C_{2n+1} , so $\chi'(C_{2n+1}) = 3$.

□

A generalized type of edge colouring is *list-edge-colouring* that lists the colours $L(e)$ that are assigned to each edge of G . A Graph G is *edge- L -colourable* if it has proper edge colouring such as ψ that $\psi(e) \in L(e)$ for all edges $e \in E(G)$. Graph G is *edge- k -choosable* if G is edge- L -colourable for every edge assignment L satisfying $|L(e)| \geq k$ for all edges $e \in E(G)$. The *edge choice number*, denoted by $\chi'_l(G)$, is the smallest integer k such that G is edge- k -choosable. It is clear that $\chi'_l(G) \geq \chi'(G)$, since we have to colour each edge of graph G with a colour from the list assigned to that edge. For example, for all integer n , $\chi'_l(C_{2n}) = 2$. We prove this in following theorem.

Theorem 5. *All even cycles C_{2n} are edge-2-choosable [Gal95].*

Proof. Consider an even cycle $C_{2n} = [v_1, v_2, \dots, v_{2n-1}, v_{2n}]$. We discuss two different cases to colour cycle C .

Case 1. There are two adjacent edges such that their assigned list of colours are not the same. Without loss of generality, assume that $v_1v_2, v_{2n}v_1$ are those edges with different assigned lists. Since $L(v_1v_2) \neq L(v_{2n}v_1)$, then there is one colour such that $\alpha \in L(v_1v_2)$ and $\alpha \notin L(v_{2n}v_1)$. Let us colour

the edge v_1v_2 with α . Then colour $v_2v_3, v_3v_4, \dots, v_{2n-1}v_{2n}$, successively. We claim there is still one colour available on $L(v_{2n}v_1)$ for colouring this edge, since $\alpha \notin L(v_{2n}v_1)$.

Case 2. All edges have the same assigned list of colours $L(e) = \{\alpha, \beta\}$. Then we colour the edges $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$ with α and colour the edges $v_2v_3, v_4v_5, \dots, v_{2n}v_1$ with β .

□

We will use Theorem 5 in following chapters.

Chapter 3

Review of Past Results

In this chapter, we review the methods used in past results concerned with proving Vizing's Conjecture for different cases of planar graphs. The two common techniques used in almost all proofs I have ever seen discussing edge-choosability of planar graphs, are the *minimal counterexample* and the *discharging* methods. We apply these two methods in our proofs as well. We briefly explain how these two methods work.

As you know, we are trying to prove Vizing's Conjecture holds for some certain planar graphs. For example, we are trying to prove Vizing's Conjecture holds for planar graphs X . We begin by assuming that the statement "Vizing's Conjecture holds for planar graphs X " is not true. Then there exists a planar graph X that does not hold Vizing's Conjecture. Since we are discussing finite graphs in this thesis, then there exists a minimal counterexample. It means if X is a minimal counterexample that does not hold Vizing's Conjecture

and Y is a non-empty subgraph of X , then $X - Y$ holds Vizing's Conjecture. The second method is the discharging method that was used to prove the Four Colour Theorem. This method is commonly used to prove that a certain class of planar graphs contains some subgraphs from a specified list. Then it is proven planar graph X contains subgraph Y by discharging method. We remove subgraph Y from graph X . Since X is a minimal counterexample, then $X - Y$ holds Vizing's Conjecture. We try to colour subgraph Y as well. Then we can say graph X holds Vizing's Conjecture which is a contradiction.

We provide you with a list of two common unavoidable configurations in all discussed planar graphs of past results below.

- (1) an edge uv such that $d(u) + d(v) \leq \Delta(G) + 2$,
- (2) an even cycle $(3, \Delta, \dots, 3, \Delta)$.

We exclude these configurations of a minimal counterexample as well as additional configurations called the *bad subgraphs* which are illustrated in Figure 3.1.

In 2011, Ma et al. [MWCZ11] used a structure which will play a central role in the discharging method used in our following proofs. Without using this structure, we are not able to prove theorems mentioned in Section 1.2. We call this structure *tree structure* which is explained in the following section.

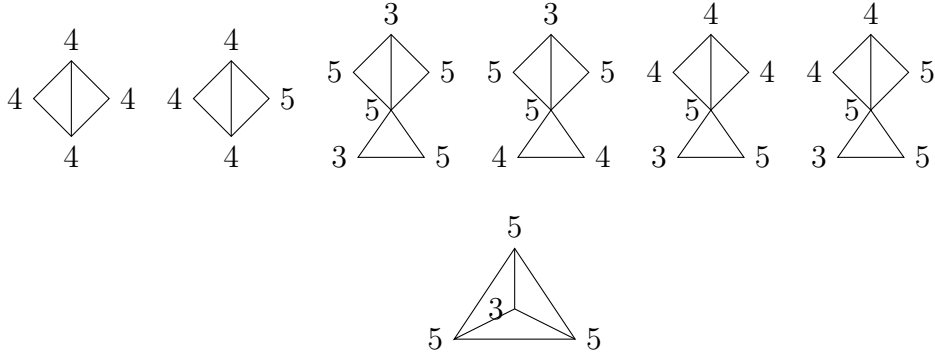


Figure 3.1: Bad subgraphs, the degree of the vertices are shown by numbers.

3.1 Tree Structure

If G is a planar graph including neither an edge uv with $d(u) + d(v) \leq \Delta(G) + 2$ nor an even cycle $(3, \Delta, \dots, 3, \Delta)$, then for each 3-vertex of graph G two adjacent Δ -vertices are considered in graph G such that two 3-vertices do not share a common Δ -vertex. We prove this in Lemma 1 as follows.

Lemma 1. *Let G be a planar graph that excludes the following configurations:*

- (1) *an edge uv that $d(u) + d(v) \leq \Delta(G) + 2$,*
- (2) *an even cycle $(3, \Delta, \dots, 3, \Delta)$.*

Let G' be the subgraph of G induced by the edges incident to 3-vertices. We claim G' contains a bipartite subgraph $G'' = (V_1'', V_2'', E(G''))$, such that for each vertex $v \in V_1''$, $d_{G''}(v) = 2$ and for each vertex $v \in V_2''$, $d_{G''}(v) = 1$. Then for each edge $uv \in E(G'')$, $d_G(u) = 3$ and $d_G(v) = \Delta(G)$, v is called a Δ -master of u and u is called a 3-dependent of v .

We emphasize that each Δ -vertex is Δ -master of at most one 3-dependent vertex.

Proof. According to the assumption, G' is the subgraph of G induced by the edges incident to 3-vertices and it does not include an edge uv that $d(u) + d(v) \leq \Delta(G) + 2$. Then for each edge $uv \in G'$, $d(u) + d(v) \geq \Delta(G) + 3$. Therefore, each edge in subgraph G' is incident to a 3-vertex and a Δ -vertex of graph G . Next we prove subgraph G' contains a bipartite subgraph $G'' = (V_1'', V_2'', E(G''))$, such that for each vertex $v \in V_1''$, $d_{G''}(v) = 2$ and for each vertex $v \in V_2''$, $d_{G''}(v) = 1$.

We claim subgraph G' has no odd cycles. By contradiction, we assume that G' has an odd cycle. Then this odd cycle contains an edge uv such that either $d(u) = d(v) = 3$ or $d(u) = d(v) = \Delta(G)$ which is a contradiction, since each edge of subgraph G' is incident to a 3-vertex and a Δ -vertex of graph G . By assumption, subgraph G' has no even cycles as well. Thus, G' is a tree. Take one of the 3-vertices as its root and let G' be a rooted tree. Each 3-vertex in this rooted tree is adjacent to two Δ -vertices as its children and one Δ -vertex as its parent. Let us consider each 3-vertex except the root vertex in G' and its children with their incident edges as well as the root vertex and its two arbitrary children with their incident edges as subgraph G'' , the set of all 3-vertices as V_1'' , and the set of all Δ -vertices as V_2'' . It is easy to see each vertex $v \in V_1''$, $d_{G''}(v) = 2$ and each vertex $v \in V_2''$, $d_{G''}(v) = 1$. Also for each edge $uv \in E(G'')$, $d_G(u) = 3$ and $d_G(v) = \Delta(G)$. That are called 3-dependent of v and Δ -master of u , respectively. We claim that each

Δ -vertex is Δ -master of at most one 3-dependent vertex. Otherwise, there are two 3-dependent vertices sharing a common Δ -master. Consider the cycle incident to these vertices. Therefore, the length of this cycle is infinite or an even integer number which are both contradiction to our assumptions. Since graph G is considered a finite graph, it does not contain an even cycle $(3, \Delta, \dots, 3, \Delta)$ by assumption. \square

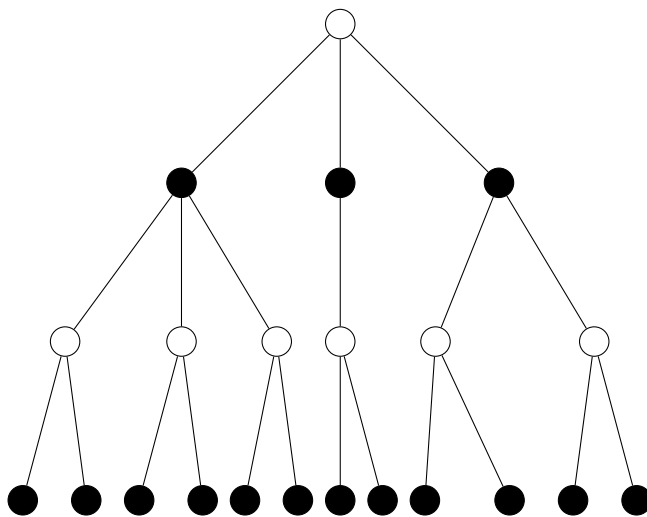


Figure 3.2: Tree structure, 3-vertices and Δ -vertices are shown by white nodes and black nodes, respectively.

Tree structure provides a good source of weight for 3-dependent vertices, since they have a shortage of weight and two Δ -master vertices take care of them.

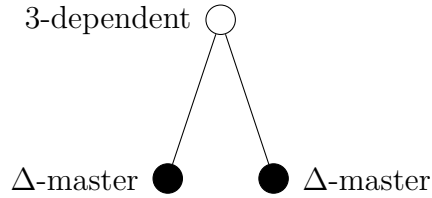


Figure 3.3: Each 3-dependent has two distinct Δ -masters.

3.2 Edge-choosability of Planar Graphs Without 7-cycles

Hou, Liu and Cai [HLC09] proved that every planar graph G without 7-cycles is edge- $(\Delta + 1)$ -choosable with $\Delta(G) \geq 7$. Hou et al. used the discharging method to prove this case. They ignored the case that graphs may have a non-simple 7-face as it is shown in Figure 3.4. Also they used observations which are hard to be verified. We found counterexamples for observations used in their proof. As their proof is not complete, we provide a new proof by using tree structure and two simple observations which are verified easily.

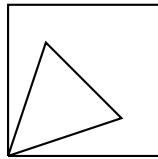


Figure 3.4: Non-simple 7-face.

Next we prove Theorem 2 of Chapter 1.2 which says planar graph G without 7-cycles are edge- k -choosable where $k = \max\{8, \Delta(G) + 1\}$.

Lemma 2. *Consider a subgraph H of graph G to be one of the following configurations:*

- (1) *an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$;*
- (2) *an even cycle $(3, \Delta, \dots, 3, \Delta)$.*

If $G - H$ is edge- k -choosable, then G is edge- k -choosable.

Proof. We consider two different cases to prove Lemma 2:

Case 1. Consider an edge uv is a subgraph H in G such that $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$. We remove the subgraph H from graph G , according to the assumption $G - H$ is edge- k -choosable. We know there are at most $k - 1 = \max\{7, \Delta(G)\}$ edges incident to u or v in $G - H$. As we know for each edge assignment L , $|L(uv)| \geq k$. Then there is at least one colour available on edge uv which is different from the colours of incident edges to u and v in $G - H$. So we can colour the edge uv with this colour. Then G is edge- k -choosable.

Case 2. Consider subgraph H is an even cycle $[v_1, v_2, \dots, v_{2n-1}, v_{2n}]$ such that $d(v_1) = d(v_3) = \dots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \dots = d(v_{2n}) = \Delta(G)$. For each edge assignment L of graph G , $|L(e)| \geq k \quad \forall e \in E(G)$. We remove subgraph H from graph G , according

to the assumption that $G - H$ is edge- k -choosable. Assume $G - H$ has an edge- L -colouring ϕ . Each edge of subgraph H is adjacent to $k - 2 = \max\{6, \Delta(G) - 1\}$ edges in $G - H$, so there are at least two colours available on each edge of H which are different from the colours of incident edges to vertices of subgraph H in $G - H$. We assign these available colours to edges of H by edge assignment L' . Then $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$ implies that $|L'(e)| \geq 2 \quad \forall e \in E(H)$. As it is proven in Theorem 5 of chapter 2, all even cycles are edge-2-choosable [Gal95]. Then G is edge- k -choosable.

□

Corollary 1. *If graph G is a minimal counterexample to Theorem 2, then it includes neither an edge uv that $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$ nor an even cycle $(3, \Delta, \dots, 3, \Delta)$ by Lemma 2.*

Properties of the minimal counterexample planar graphs without 7-cycle that are not edge- k -choosable where $k = \max\{8, \Delta(G) + 1\}$ are discussed below.

- (P1) Graph G does not have any simple 7-face.
- (P2) A k -vertex, where $k \geq 6$, is not incident with five continuous triangles.
- (P3) For each edge $uv \in E(G)$, $d(u) + d(v) \geq \max\{10, \Delta(G) + 3\}$.
- (P4) Each 3-dependent vertex has exactly two distinct Δ -master vertices by

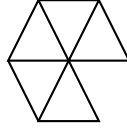


Figure 3.5: Five continuous triangles.

Lemma 1 [MWCZ11]. Also each Δ -vertex is Δ -master of at most one 3-dependent vertex.

Observation 1. Let v, u be two adjacent 5-vertices of G with $m_3(v) \geq 4$. Since graph G has no 7-cycles, then $m_3(u) \leq 3$.

Observation 2. Let v, u be two adjacent 5-vertices of G with $m_3(v) \geq 4$. If $m_3(u) = 3$, then u is not incident to three continuous triangles. It is incident to three 3-faces in the way shown in Figure 3.6 and it is incident to at most one 4-face.

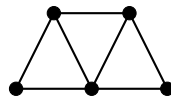
Figure 3.6: Vertex u with $m_3(u) = 3$.

Figure 3.7: Three continuous triangles.

We show that the minimal counterexample includes one of Lemma 2's configurations by using the discharging method.

Lemma 3. *Let G be a planar graph without 7-cycles. Then G has one of the following configurations:*

- (1) *an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$;*
- (2) *an even cycle $(3, \Delta, \dots, 3, \Delta)$.*

Proof. We prove this lemma by contradiction. Assume that G is a planar graph without 7-cycles such that for each edge $uv \in E(G)$, $d(u) + d(v) \geq \max\{10, \Delta + 3\}$ and it does not include an even cycle $(3, \Delta, \dots, 3, \Delta)$. Thus, we have $\delta(G) \geq 3$ by lacking configuration (1) of Lemma 3. We use the discharging method to prove this lemma. We first define initial weights for $V(G) \cup F(G)$ by $w(v) = 3d(v) - 8$ for each vertex of G and $w(f) = d(f) - 8$ for each face of G . Total sum of initial weight of vertices and faces is

$$w = \sum_{x \in V(G) \cup F(G)} w(x) = \sum_{x \in V(G)} (3d(v) - 8) + \sum_{x \in F(G)} (d(f) - 8) \quad (3.1)$$

By handshaking lemma we have:

$$\sum_{x \in V(G)} (3d(v) - 8) = 6|E(G)| - 8|V(G)|, \quad \sum_{x \in F(G)} (d(f) - 8) = 2|E(G)| - 8|F(G)| \quad (3.2)$$

According to Euler's Formula

$$w(x) = 8(|E(G)| - |V(G)| - |F(G)|) = -16 \quad (3.3)$$

We transfer weights from vertices to faces to make all new weights non-negative for $V(G) \cup F(G)$. Through this redistribution the sum of weights won't be changed. Thus, the non-negative total sum of new weights leads to a contradiction by the negative sum of initial weights according to Euler's formula. Hence the proof is complete.

We apply the following rules to transfer weight from vertices to faces and denote new weights $w^*(x)$ for $x \in V(G) \cup F(G)$:

(R1) From each Δ -master to its 3-dependent vertex, transfer 1.

(R2) From each 3-vertex to each of its incident faces f , transfer 1.

(R3) From each 4-vertex to each of its incident faces f , transfer 1.

(R4) From each 5-vertex to each of its incident 3-faces f , transfer

$$\left\{ \begin{array}{ll} (i) & 7/5 \quad \text{if } m_3(v) = 5; \\ (ii) & 3/2 \quad \text{if } m_3(v) = 4; \\ (iii) & 9/5 \quad \text{if } m_3(v) = 3; \\ (iv) & 2 \quad \text{Otherwise.} \end{array} \right.$$

(R5) From each 5-vertex to each of its incident faces f such that $4 \leq d(f) \leq 7$, transfer

$$\left\{ \begin{array}{ll} (i) & 1 \quad \text{if } d(f) = 4; \\ (ii) & 3/5 \quad \text{if } d(f) = 5; \\ (iii) & 1/3 \quad \text{if } 6 \leq d(f) \leq 7. \end{array} \right.$$

(R6) From each 6^+ -vertex to each of its incident face f , transfer:

$$\left\{ \begin{array}{ll} 2, & \text{if } d(f)=3; \\ 1, & \text{if } 4 \leq d(f) \leq 7. \end{array} \right.$$

Now we show that $w^*(x)$ is non-negative for all $x \in V(G) \cup F(G)$. Suppose that v is a k -vertex. Therefore

- If $k = 3$, by considering property (P4) it has two Δ -masters and receives weight from its Δ -masters. Then $w^*(v) \geq w(v) + 2 \times 1 - 3 \times 1 = 0$ by (R1) and (R2).
- If $k = 4$, $w^*(v) \geq w(v) - 4 \times 1 = 0$ by (R3).
- If $k = 5$, we have the following cases:
 - If $m_3(v) = 5$, according to (i) of (R4) $w^*(v) = w(v) - 5 \times 7/5 = 0$.
 - If $m_3(v) = 4$, according to (ii) of (R4) and (R5), $w^*(v) \geq w(v) - 4 \times 3/2 - 1 = 0$.
 - If $m_3(v) = 3$, according to (iii) of (R4) and (R5) and Observation 2, $w^*(v) \geq w(v) - 3 \times 9/5 - 1 - 3/5 = 0$.

- If $m_3(v) \leq 2$, according to (iv) of (R4) and (R5), $w^*(v) \geq w(v) - 2 \times 2 - 3 \times 1 = 0$.
- If $k = 6$, we consider the following cases:
 - If $m_3(v) = 4$, by (R6) $w^*(v) \geq w(v) - 4 \times 2 - 2 \times 1 = 0$.
 - If $k \geq 7$, by (R1) and (R6) $w^*(v) = w(v) - m_3(v) \times 2 - (7 - m_3(v)) - 1 \geq w(v) - 13 = 0$.

We show that w^* is non-negative for all faces as well. Suppose that f is a k -face.

- If $k = 3$, consider vertex v is incident to face f and gives minimum weight to this face.

Case1. If face f receives $7/5$ from vertex v , then by Observation 2 and (iii) of (R4) this face receives at least $9/5$ from other vertices. Thus $w^*(f) \geq w(f) + 7/5 + 2 \times 9/5 = 0$;

Case2. If face f receives $3/2$ from vertex v , then by Observation 2 and (iii) of (R4) this face receives at least $9/5$ from other vertices. Thus $w^*(f) \geq w(f) + 3/2 + 2 \times 9/5 > 0$;

Case3. Otherwise, $w^*(f) \geq w(f) + 3 \times 9/5 > 0$.

- If $k = 4$, then $w^*(f) = w(f) + 4 \times 1 = 0$.
- If $k = 5$, then $w^*(f) \geq w(f) + 5 \times 3/5 = 0$.

- If $k = 6$, then $w^*(f) \geq w(f) + 6 \times 1/3 = 0$.
- If $k = 7$, then $w^*(f) \geq w(f) + 7 \times 1/3 > 0$.

Therefore, $\sum_{x \in V(G) \cup F(G)} w^*(x) \geq 0$. However, $\sum_{x \in V(G) \cup F(G)} w(x) = -16$ by Euler's Formula which is a contradiction. Since we have transferred the weight between vertices to faces, total weight is not changed. We draw this conclusion that the minimal counterexample must contain either an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$ or an even cycle $(3, \Delta, \dots, 3, \Delta)$ which is a contradiction. Then planar graphs without 7-cycles with $\Delta(G) \geq 7$ are edge- $(\Delta + 1)$ -choosable.

□

Corollary 2. *It has been already proven that all graphs G with $\Delta(G) \leq 4$ are edge- $(\Delta + 1)$ -choosable. In this section, we proved all planar graphs G without 7-cycles with $\Delta(G) \geq 7$ are edge- $(\Delta + 1)$ -choosable. Then all planar graphs without 7-cycles with $\Delta(G) \neq 5, 6$ are edge- $(\Delta + 1)$ -choosable.*

Chapter 4

Our Results

In this chapter, we explain how Vizing's Conjecture holds for two different cases: planar graphs without adjacent 4-cycles and planar graphs without 3-cycles adjacent to 5-cycles. As it is mentioned in Chapter 3, Ma et al. [MWCZ11] proved all planar graphs G without intersecting 4-cycles with $\Delta(G) \neq 5$ are edge- $(\Delta + 1)$ -choosable. We improve this result by proving planar graphs G without adjacent 4-cycles with $\Delta(G) \neq 5$ are edge- $(\Delta + 1)$ -choosable in Section 4.1. We also prove Vizing's conjecture holds for planar graphs without 3-cycles adjacent to 5-cycles in Section 4.2.

4.1 Edge-choosability of Planar Graphs Without Adjacent 4-cycles

First we determine forbidden figures from planar graphs without adjacent 4-cycles as shown in the following figure. A pair of adjacent 4-cycles with two vertices and one edge in common and a pair of adjacent 4-cycles with three vertices and one edge in common are shown in Figure 4.1.

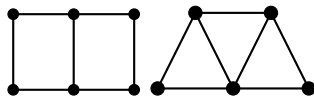


Figure 4.1: Two adjacent 4-Cycles.

Next We prove Theorem 3 of Chapter 1. It says if G is a planar graph without adjacent 4-cycles, then it is edge- k -choosable where $k = \max\{7, \Delta(G)+1\}$.

By contradiction, there exists a minimal counterexample G that is not edge- k -choosable. We prove G does not include certain subgraphs discussed in Lemma 4.

Lemma 4. *Consider a subgraph H of graph G to be one of the following configurations.*

(1) *an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$;*

(2) *an even cycle $(3, \Delta, \dots, 3, \Delta)$.*

If $G - H$ is edge- k -choosable, then G is edge- k -choosable.

Proof. We consider two different cases to prove Lemma 4:

Case 1. Consider an edge uv is a subgraph H in G such that $d(u)+d(v) \leq \max\{8, \Delta(G)+2\}$. We remove subgraph H from graph G and according to the assumption, $G - H$ is edge- k -choosable. We know there are at most $k - 1 = \max\{6, \Delta(G)\}$ edges incident to u and v in $G - H$. By definition, we know for each edge assignment L , $|L(uv)| \geq k$. Then there is at least one colour available on $L(uv)$ which is different from the colours of incident edges to u and v in $G - H$. Therefore, the edge uv can be coloured with this colour. Then graph G is edge- k -choosable.

Case 2. Consider an even cycle $[v_1, v_2, \dots, v_{2n-1}, v_{2n}]$ as subgraph H in G with $d(v_1) = d(v_3) = \dots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \dots = d(v_{2n}) = \Delta(G)$. For each edge assignment L of graph G , $|L(e)| \geq \Delta(G) + 1 \quad \forall e \in E(G)$. We remove subgraph H from graph G and according to the assumption, $G - H$ is edge- k -choosable. Assume $G - H$ has an edge- L -colouring ϕ . Each edge of subgraph H is adjacent to $k - 2 = \max\{5, \Delta(G) - 1\}$ edges in $G - H$ as each vertex is incident to two edges in the even cycle. Therefore, there are at least two colours available on each edge's assigned list of H . We assign these available colours to each edge of H by an edge assignment L' . Then $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. This implies $|L'(e)| \geq 2 \quad \forall e \in E(H)$. As it is proven in Theorem 5 of chapter 2, all even cycles are edge-2-choosable [Gal95]. Then graph G is edge- k -

choosable.

□

Corollary 3. *If graph G is a minimal counterexample to Theorem 3, then it includes neither an edge uv that $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$ nor an even cycle $(3, \Delta, \dots, 3, \Delta)$ by Lemma 4.*

The properties of the minimal counterexample planar graph G without adjacent 4-cycles which is not edge- k -choosable where $k = \max\{7, \Delta(G) + 1\}$, are discussed below.

- (P1) For any edge $uv \in E(G)$, $d(u) + d(v) \geq \max\{9, \Delta(G) + 3\}$. Then $\delta(G) \geq 3$ and each 3-vertex is adjacent to Δ -vertices where $\Delta(G) \geq 6$.
- (P2) Each 3-dependent vertex has distinct two Δ -master vertices. This property has been proven in Lemma 1 of chapter 3 [MWCZ11].

Next we prove this minimal counterexample includes either an edge uv with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$ or an even cycle $(3, \Delta, \dots, 3, \Delta)$ by the following lemma which is a contradiction.

Lemma 5. *Let graph G be a minimal counterexample to Theorem 3. Then graph G contains at least one of the following configurations:*

- (1) an edge $uv \in E(G)$ with $d(u) + d(v) \leq k + 1 = \max\{8, \Delta(G) + 2\}$;
- (2) an even cycle $(3, \Delta, \dots, 3, \Delta)$.

Proof. Considering the properties discussed, this minimal counterexample does not include neither an edge $uv \in E(G)$ with $d(u)+d(v) \leq \max\{8, \Delta(G)+2\}$ nor an even cycle $(3, \Delta, \dots, 3, \Delta)$. We use the discharging method to prove such a minimal counterexample with those properties does not exist. We define initial weight for vertices and faces of graph G . Next we transfer weights between vertices and faces according to the discharging rules. Let w denote the initial weight on $V(G) \cup F(G)$ by $w(v) = d(v) - 4$ for each vertex of G and $w(f) = d(f) - 4$ for each face of G . The total sum of the initial weight of the vertices and faces is -8 by Euler's formula:

$$w = \sum_{x \in V(G) \cup F(G)} w(x) = \sum_{x \in V(G)} (d(v)-4) + \sum_{x \in F(G)} (d(f)-4) = -4(|V|-|E|+|F|) = -8$$

We transfer weights between vertices and faces to make all new weights non-negative. We denote new weights by $w^*(x)$ for all $x \in V(G) \cup F(G)$. Through this redistribution, the sum of weights won't be changed. Thus, the non-negative total sum of new weights leads to a contradiction by the negative sum of initial weights according to Euler's formula. Hence the proof is complete.

$$0 \leq \sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) = -8$$

Considering Lemma 1 in chapter 3, each 3-dependent vertex has distinct two Δ -master vertices from which receive weight. Also, if the planar graph without adjacent 4-cycles does not include three continuous triangles, then

each k -vertex is incident to at most $\lfloor 2k/3 \rfloor$ 3-faces.

Now we redistribute weights between vertices and faces based on the following rules:

(R1) From each Δ -master to its 3-dependent, transfer $1/2$.

(R2) From each 5-vertex to each incident 3-face f , transfer

$$\begin{cases} (i) & 1/3 & \text{if } m_3(v) = 3; \\ (ii) & 1/2 & \text{if } m_3(v) \leq 2. \end{cases}$$

(R3) From each 6-vertex to each incident 3-face f , transfer

$$\begin{cases} (i) & 1/3 & \text{if } m_3(v) = 4; \\ (ii) & 1/2 & \text{if } m_3(v) \leq 3. \end{cases}$$

(R4) From each 7^+ -vertex to each incident 3-face f , transfer $1/2$.

(R5) From each 5^+ -face to each adjacent 3-face f , transfer $1/6$.

Now we show that $w^*(x)$ is non-negative for all $x \in V(G) \cup F(G)$. Suppose that v is a k -vertex. We discuss different cases below.

- If $k = 3$, by Lemma 1 of Chapter 3 it has two distinct Δ -master vertices. Then by (R1) $w^*(v) = w(v) + 2 \times 1/2 = 0$.
- If $k = 4$, it does not contribute in the discharging method and its weight remains fixed. Then $w^*(v) = w(v) = 0$.

- If $k = 5$, we know $m_3(v) \leq 3$. Then we have the following cases:
 - If $m_3(v) = 3$, vertex v gives $1/3$ to each incident 3-face by (i) of (R2). Thus, $w^*(v) = w(v) - 3 \times 1/3 = 0$.
 - If $m_3(v) \leq 2$, by (ii) of (R2) $w^*(v) \geq w(v) - 2 \times 1/2 = 0$.
- If $k = 6$, we know $m_3(v) \leq 4$ and it is Δ -master of at most one 3-dependent by Lemma 1 of Chapter 3. Then we have the following cases:
 - If $m_3(v) = 4$, then v gives $1/3$ to each incident 3-face. Thus $w^*(v) = w(v) - 4 \times 1/3 - 1/2 > 0$ by (i) of (R3) and (R1).
 - If $m_3(v) \leq 3$, then $w^*(v) \geq w(v) - 3 \times 1/2 - 1/2 = 0$ by (ii) of (R3) and (R1).
- If $k \geq 7$, we know $m_3(v) \leq \lfloor 2k/3 \rfloor$ and it is Δ -master of at most one 3-dependent by Lemma 1 of Chapter 3. $w^*(v) \geq w(v) - \lfloor 2k/3 \rfloor \times 1/2 - 1/2 > 0$ by (R1) and (R4).

We show that w^* is non-negative for all faces as well. Suppose that f is a k -face.

- If $k = 3$, then we have the following cases:
 - If f is adjacent to at least two 5^+ -faces, by (R5) and the fact that each triangle has at least two 5^+ -vertices and they are giving at least $1/3$ weight to their incident face f , $w^*(f) \geq w(f) + 2 \times 1/6 + 2 \times 1/3 = 0$;

- If f is adjacent to at most one 5^+ -face and $\delta(f) \leq 4$, then considering the fact that two incident 5^+ -vertices are giving $1/2$ to face f , $w^*(f) \geq w(f) + 2 \times 1/2 = 0$;
- Otherwise, f is adjacent to at most one 5^+ -face and $\delta(f) \geq 5$. Then $w^*(f) \geq w(f) + 3 \times 1/3 = 0$.
- If $k = 4$, it does not contribute in the discharging method and its weight remains fixed, $w^*(f) = w(f) = 0$.
- If $k \geq 5$, then $w^*(f) \geq w(f) - k \times 1/6 = 5/6 \times k - 4 > 0$.

Therefore, $\sum_{x \in V(G) \cup F(G)} w^*(x) \geq 0$. However, $\sum_{x \in V(G) \cup F(G)} w(x) = -8$ by Euler's Formula which is a contradiction. The proof is complete. \square

Then the minimal counterexample should include either an edge uv with $d(u) + d(v) \leq \max\{8, \Delta(G) + 2\}$ or an even cycle $(3, \Delta, \dots, 3, \Delta)$ which is contradiction to corollary 3. Therefore the minimal counterexample does not exist. We conclude all planar graphs without adjacent 4-cycles are edge- k -choosable where $k = \max\{7, \Delta(G) + 1\}$.

Corollary 4. *It has been already proven that all graphs G with $\Delta(G) \leq 4$ are edge- $(\Delta + 1)$ -choosable. In this section, we proved planar graphs G without adjacent 4-cycles with $\Delta(G) \geq 6$ are edge- $(\Delta + 1)$ -choosable. Then all planar graphs G without adjacent 4-cycles with $\Delta(G) \neq 5$ are edge- $(\Delta + 1)$ -choosable.*

4.2 Edge-choosability of Planar Graphs Without Adjacent 3- and 5-cycles

We determine forbidden figures from planar graphs without 3-cycles adjacent to 5-cycles as shown in the following figures. These figures show a 3-face adjacent to a 5-cycle as well as a 3-face adjacent to a 5-face.

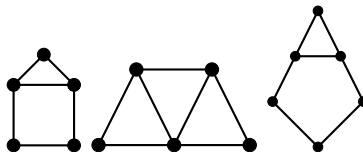


Figure 4.2: 3-Cycles adjacent to 5-Cycles.

In this section we prove Theorem 4. By contradiction, there exist a minimal counterexample G which is a planar graph without 3-cycles adjacent to 5-cycles and is not edge- k -choosable. We prove this minimal counterexample does not contain the certain subgraphs discussed in the following lemma.

Lemma 6. *Let graph G be a minimal counterexample to Theorem 4. Then it does not contain the following subgraphs:*

- (1) *an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{7, \Delta(G) + 2\}$;*
- (2) *an even cycle $(3, \Delta, \dots, 3, \Delta)$;*
- (3) *at least one of the bad subgraphs H in Figure 4.3.*

Proof. By contradiction, we assume the minimal counterexample G contains at least one of the above subgraphs. We discuss the following different cases.

4.2 Edge-choosability of Planar Graphs Without Adjacent 3- and 5-cycles 4

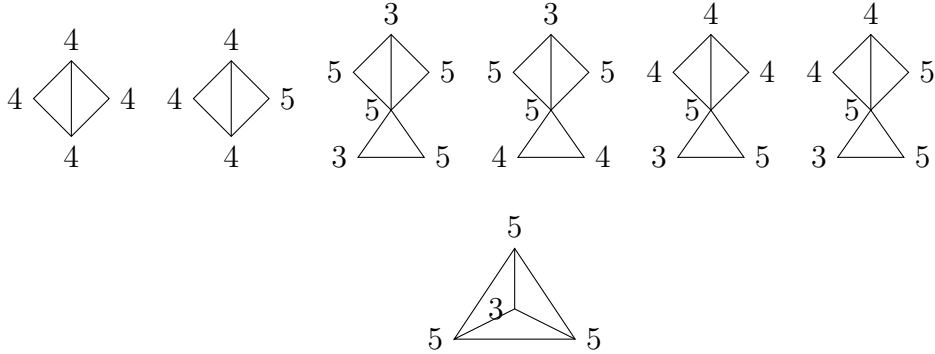


Figure 4.3: Bad subgraphs, the degree of the vertices are shown by numbers.

Case 1. Consider subgraph H is an edge uv that $d(u) + d(v) \leq \max\{7, \Delta(G) + 2\}$. We remove subgraph H from G . Since G is a minimal counterexample, then $G - H$ is edge- k -choosable. There are at most $k - 1 = \max\{5, \Delta(G)\}$ edges incident to u and v in $G - H$. According to the definition, for each edge assignment L , $|L(uv)| \geq k$. Then, there is at least one colour available on the list of colours assigned to the edge uv which is different from the colours of incident edges to u and v in $G - H$. Therefore, the edge uv could be coloured with this colour. Then G is edge- k -choosable which is a contradiction to our assumption.

Case 2. Consider graph G contains subgraph $H = [v_1, v_2, \dots, v_{2n-1}, v_{2n}]$ such that $d(v_1) = d(v_3) = \dots = d(v_{2n-1}) = 3$ and $d(v_2) = d(v_4) = \dots = d(v_{2n}) = \Delta(G)$. For each edge assignment L and each edge of graph G , $|L(e)| \geq k$. We remove subgraph H from graph G . Since G is a

4.2 Edge-choosability of Planar Graphs Without Adjacent 3- and 5-cycles 4

minimal counterexample, then $G - H$ is edge- k -choosable. Assume that $G - H$ has an edge-L-colouring ϕ . Since each edge in subgraph H is adjacent to $\max\{4, \Delta(G) - 1\}$ edges in $G - H$, there are at least two colours available on each edge of H . We assign these available colours to edges of H by edge assignment L' . Then $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$ which implies $|L'(e)| \geq 2 \quad \forall e \in E(H)$. As it is proven in Lemma 1 of Chapter 2, all even cycles are edge-2-choosable [Gal95]. Then G is edge- k -choosable which is a contradiction to our assumption.

Case 3. Consider graph G contains the following bad subgraph H with $d(v_1) = d(v_2) = d(v_4) = 4$, $d(v_3) = 5$. We remove subgraph H

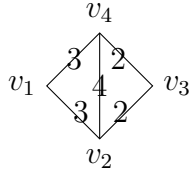


Figure 4.4: The number of colours available on each edge of bad subgraph H are shown on each edge.

from graph G . Since G is a minimal counterexample, then $G - H$ is edge- k -choosable. For each edge assignment L of graph G such that $|L(e)| \geq k \quad \forall e \in E(G)$, $G - H$ has an edge-L-colouring ϕ . For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is shown in Figure 4.4, $|L'(v_1v_2)| \geq 3$, $|L'(v_1v_4)| \geq 3$, $|L'(v_2v_3)| \geq 2$, $|L'(v_3v_4)| \geq 2$ and $|L'(v_2v_4)| \geq 4$. We

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consider three different subcases.

Subcase 3.1. If $\alpha \in L'(v_3v_4) \cap L'(v_1v_2)$, then we colour edges v_1v_2, v_3v_4 with α . Next we colour v_2v_3, v_2v_4, v_1v_4 , successively.

Subcase 3.2. If $L'(v_2v_4) = 4$ and $\alpha \in L'(v_3v_4) \cup L'(v_1v_2)$ and $\alpha \notin L'(v_2v_4)$, then we colour the edge v_3v_4 with α , next we colour $v_2v_3, v_1v_2, v_1v_4, v_2v_4$, successively.

Subcase 3.3. If $L'(v_2v_4) \geq 5$, we can colour the edges $v_3v_4, v_2v_3, v_1v_2, v_1v_4, v_4v_2$, successively.

Therefore, G is edge- k -choosable which is a contradiction to our assumption.

We have another similar bad subgraph H with $d(v_1) = d(v_2) = d(v_3) = 4 = d(v_4) = 4$ which is illustrated in Figure 4.5. If planar graph G contains this bad subgraph, the same argument as above is used to show G is edge- k -choosable.

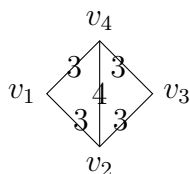


Figure 4.5: The number of colours available on each edge of bad subgraph H are shown on each edge.

We remove subgraph H from graph G . Since G is a minimal counterexample, $G - H$ is edge- k -choosable. For each edge assignment L

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of graph G such that $|L(e)| \geq k \quad \forall e \in E(G)$, $G - H$ has an edge-L-colouring ϕ . For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is illustrated in Figure 4.5, $|L'(v_1v_2)| \geq 3$, $|L'(v_1v_4)| \geq 3$, $|L'(v_2v_3)| \geq 3$, $|L'(v_3v_4)| \geq 3$ and $|L'(v_2v_4)| \geq 4$. We could prove planar graph G containing this bad subgraph H , is edge- k -choosable with the same argument discussed above for Case 3 which is a contradiction to our assumption.

Case 4. Consider graph G containing the following bad subgraph H with $d(v) = d(v_1) = d(v_3) = d(v_5) = 5$, $d(v_2) = d(v_4) = 3$. We remove

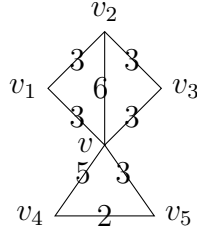


Figure 4.6: The number of colours available on each edge of bad subgraph H are shown on each edge.

subgraph H . Since G is a minimal counterexample, then $G - H$ is edge- k -choosable. Then for each edge assignment L of graph G that $|L(e)| \geq k \quad \forall e \in E(G)$, subgraph $G - H$ has an edge-L-colouring ϕ .

For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is shown in Figure 4.6, $|L'(vv_1)| \geq 3$, $|L'(vv_3)| \geq 3$, $|L'(vv_5)| \geq 3$, $|L'(v_1v_2)| \geq 3$,

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$|L'(v_2v_3)| \geq 3$, $|L'(v_4v_5)| \geq 2$, $|L'(vv_4)| \geq 5$ and $|L'(vv_2)| \geq 6$. We consider three different subcases as follows:

Subcase 4.1. If $\alpha \in L'(v_2v_3) \cap L'(vv_1)$, then we colour the edges vv_1, v_2v_3 with α . Next we colour the edges $vv_3, vv_5, v_4v_5, vv_4, vv_2, v_1v_2$, successively.

Subcase 4.2. If $L'(v_2v_3) \cap L'(vv_1) = \phi$ and $L'(v_1v_2) = 3$, there is a colour such as β in $L'(v_2v_3) \cup L'(vv_1)$ and $\beta \notin L'(v_1v_2)$. Without loss of generality we assume $\beta \in L'(vv_1)$ and we colour the edge vv_1 with β . Next we colour the edges $vv_3, vv_5, v_4v_5, vv_4, vv_2, v_2v_3, v_1v_2$, successively.

Subcase 4.3. If $L'(v_2v_3) \cap L'(vv_1) = \phi$ and $L'(v_1v_2) \geq 4$, we can colour the edges in this order $vv_1, vv_3, vv_5, v_4v_5, vv_4, v_2v_3, vv_2, v_1v_2$. Then G is edge- k -choosable which is a contradiction to our assumption.

Case 5. We consider graph G containing the following bad subgraph H with $d(v) = d(v_1) = d(v_3) = 5$, $d(v_2) = 3$ and $d(v_4) = d(v_5) = 4$.

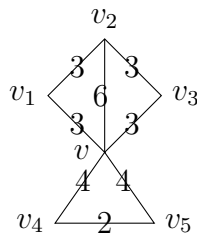


Figure 4.7: The number of colours available on each edge of bad subgraph H are shown on each edge.

We remove subgraph H . Since G is a minimal counterexample, $G - H$

is edge- k -choosable. For each edge assignment L of graph G such that $|L(e)| \geq k \quad \forall e \in E(G)$, $G - H$ has an edge- L -colouring ϕ . For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is illustrated in Figure 4.7, $|L'(vv_1)| \geq 3$, $|L'(vv_3)| \geq 3$, $|L'(v_1v_2)| \geq 3$, $|L'(v_2v_3)| \geq 3$, $|L'(v_4v_5)| \geq 2$, $|L'(vv_4)| \geq 4$, $|L'(vv_5)| \geq 4$ and $|L'(vv_2)| \geq 6$. We consider three different subcases as follows:

Subcase 5.1. If $\alpha \in L'(v_2v_3) \cap L'(vv_1)$, then we colour the edges vv_1 , v_2v_3 with α .

Next we colour the edge vv_4 with the colour which is not in the list of colours assigned to the edge v_4v_5 . Then colour edges $vv_3, vv_5, v_4v_5, vv_2, v_1v_2$, successively.

Subcase 5.2. If $L'(v_1v_2) = 3$ and $\beta \in L'(v_2v_3) \cup L'(vv_1)$ but $\beta \notin L'(v_1v_2)$. Without loss of generality we assume $\beta \in L'(vv_1)$ and $\beta \notin L'(v_1v_2)$. Next we colour the edge vv_4 with a colour which is not in the list of colours assigned to the edge v_4v_5 . Then colour edges $vv_3, vv_5, vv_4, v_4v_5, vv_2, v_2v_3, v_1v_2$, successively.

Subcase 5.3. If $L'(v_1v_2) \geq 4$, then colour edge vv_4 with a colour which is not in the list of colours assigned to the edge v_4v_5 . Then colour edges in this order $vv_1, vv_3, vv_5, v_4v_5, v_2v_3, vv_2, v_1v_2$.

In all of the above subcases we assumed $L'(v_4v_5) = 2$. It is obvious if $L'(v_4v_5) \geq 3$ we can still colour the edges in the same order. Then G is

edge- k -choosable which is a contradiction to our assumption.

Case 6. We consider graph G containing the following bad subgraph H with $d(v) = d(v_5) = d(v_3) = 5$, $d(v_4) = 3$ and $d(v_1) = d(v_2) = 4$.

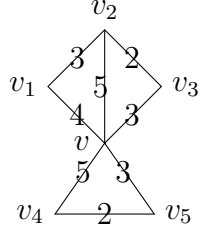


Figure 4.8: The number of colours available on each edge of bad subgraph H are shown on each edge.

We remove subgraph H . Since G is a minimal counterexample, then $G - H$ is edge- k -choosable. For each edge assignment L of graph G , $|L(e)| \geq k \quad \forall e \in E(G)$. Then $G - H$ has an edge- L -colouring ϕ . For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is shown in Figure 4.8, $|L'(vv_1)| \geq 4$, $|L'(vv_3)| \geq 3$, $|L'(v_4v_5)| \geq 2$, $|L'(v_1v_2)| \geq 3$, $|L'(vv_5)| \geq 3$, $|L'(v_2v_3)| \geq 2$ and $|L'(vv_2)| \geq 5$, $|L'(vv_4)| \geq 5$.

We consider three different subcases.

Subcase 6.1. If there is a colour α such that $\alpha \in L'(v_2v_3)$ and $\alpha \notin L'(v_1v_2)$, then we colour the edge v_2v_3 with α . Next we colour the edge vv_5 with the colour which is not in the list of colours assigned to the edge v_4v_5 . Then colour edges $vv_3, vv_1, vv_2, v_1v_2, vv_4, v_4v_5$, successively.

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Subcase 6.2. If there is a colour β such that $\beta \in L(v_1v_2)$ and $\beta \notin L(vv_1)$, then we colour the edge v_1v_2 with β . Next we colour the edge vv_5 with the colour which is not in the list of colour assigned to the edge v_4v_5 . Then colour the edges $v_2v_3, vv_3, vv_2, vv_1, vv_4, v_4v_5$, successively.

Subcase 6.3. Otherwise, $\gamma \in L'(v_2v_3) \cap L'(vv_1)$. Then we colour the edges v_2v_3, vv_1 with γ . Next we colour the edges $vv_5, v_4v_5, vv_3, vv_4, vv_2, v_1v_2$, successively.

In all of the above subcases we considered $L'(v_4v_5) = 2$. If $L'(v_4v_5) \geq 3$, it is obvious that we can colour all edges in the same order. Then G is edge- k -choosable which is a contradiction to our assumption.

We have a similar bad subgraph H with $d(v) = d(v_5) = 5, d(v_4) = 3$ and $d(v_1) = d(v_2) = d(v_3) = 4$ which is illustrated in Figure 4.9

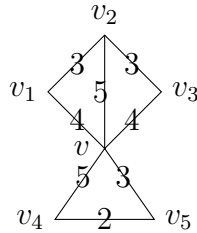


Figure 4.9: The number of colours available on each edge of bad subgraph H are shown on each edge.

If G contains bad subgraph H , we remove subgraph H . Since G is a minimal counterexample, $G - H$ is edge- k -choosable. As we know for each edge assignment L of graph G , $|L(e)| \geq k \quad \forall e \in E(G)$. Then $G - H$

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has an edge-L-colouring ϕ . For all edges of H , we assign a list of available colours $L'(e) = L(e) \setminus \{\phi(e') \mid e' \in G - H \text{ is adjacent to } e \text{ in } G\}$. As it is shown in Figure 4.9, $|L'(vv_1)| \geq 4$, $|L'(vv_3)| \geq 4$, $|L'(v_4v_5)| \geq 2$, $|L'(v_1v_2)| \geq 3$, $|L'(vv_5)| \geq 3$, $|L'(v_2v_3)| \geq 3$ and $|L'(vv_2)| \geq 5$, $|L'(vv_4)| \geq 5$. We can colour subgraph H with the same argument discussed above for Case 6. Then G is edge- k -choosable which is a contradiction to our assumption.

Case 7. We consider graph G containing the following bad subgraph H with $d(v) = 3$, $d(v_1) = d(v_2) = d(v_3) = 5$.

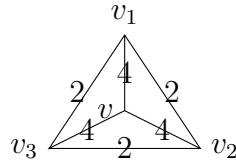


Figure 4.10: The number of colours available on each edge of bad subgraph H are shown on each edge.

According to the assumption that graph G is a minimal counterexample, then we can colour the edges v_1v_2, v_2v_3, v_1v_3 which is a triangle with lists of 2 colours assigned to each edge. Without loss of generality, we colour edges v_1v_2, v_2v_3, v_1v_3 with colours α, β, γ , respectively. Assume that we colour the edge vv_1 with x and colour the edge vv_2 with y . If we could colour the edge vv_3 , then the proof is complete. Otherwise, the list of colours assigned to the edge vv_3 is $\{\beta, \gamma, x, y\}$. If there are any available colours except $\{\alpha, \beta, \gamma, x, y\}$ on edges vv_1, vv_2 , then we

can colour one of these edges with that colour and one colour would release on the edge vv_3 . Otherwise, the list of assigned colours to the edges vv_1, vv_2 are $\{\alpha, \gamma, x, y\}, \{\alpha, \beta, x, y\}$, respectively.

Let L be an edge assignment of subgraph H and $\{i, j, k\} = \{1, 2, 3\}$. Let define each pair of edges vv_i, v_jv_k a pair of opposite edges. For all pairs of opposite edges vv_i, v_jv_k if $L(vv_i) \cap L(v_jv_k) \subset \{x, y\}$, then we could colour all edges of subgraph H properly. Otherwise, for each edge v_iv_j the set of assigned colours to this edge is $L(v_iv_j) \subset \{\alpha, \beta, \gamma\}$.

Without loss of generality, we assume that $L(vv_2) \cap L(v_1v_3) = \{\alpha\}$. If $\gamma \in L(v_1v_2)$ then we colour edges vv_2, v_1v_3 with α and colour edges $v_1v_2, v_2v_3, vv_1, vv_3$ with γ, β, x, y , respectively. Otherwise, $L(v_1v_2) = \{\alpha, \beta\}$ and we discuss two different subcases below.

Subcase 7.1. Assume that $L(v_2v_3) = \{\alpha, \beta\}$ and the list of colours assigned to the edges of subgraph H are shown in Figure 4.11. Then

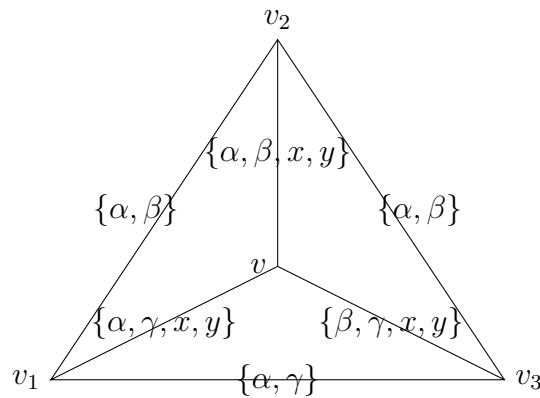


Figure 4.11: Assignment lists of colours to the edges of subgraph H .

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we colour the edges vv_3, v_1v_2 with β and colour edges $v_2v_3, v_1v_3, vv_1, vv_2$ with α, γ, y, x , respectively.

Subcase 7.2. Assume that $L(v_2v_3) = \{\beta, \gamma\}$ and the lists of colours assigned to each edge of subgraph H are shown in Figure 4.12. Then

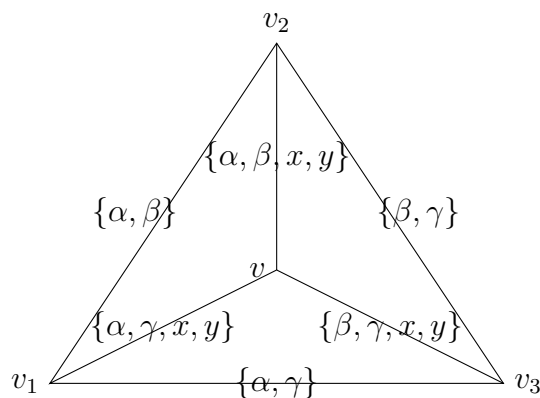


Figure 4.12: Assignment lists of colours to the edges of subgraph H .

we colour the edges vv_3, v_1v_2 with β and colour edges $v_2v_3, v_1v_3, vv_1, vv_2$ with γ, α, y, x , respectively. Then graph G is edge- k -choosable which is a contradiction to our assumption.

□

Next we prove the minimal counterexample G contains at least one of the subgraphs discussed in Lemma 6 which is a contradiction.

Lemma 7. *Let G be a minimal counterexample to Theorem 4, then it contains at least one of the following configurations:*

- (1) an edge $uv \in E(G)$ with $d(u) + d(v) \leq \max\{7, \Delta(G) + 2\}$;

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(2) an even cycle $(3, \Delta, \dots, 3, \Delta)$;

(3) at least one of the bad subgraphs in Figure 4.3.

Proof. We prove this lemma by contradiction. Consider the minimal counterexample G does not include the above configurations. Then for each edge $uv \in E(G)$, $d(u) + d(v) \geq \max\{8, \Delta + 3\}$. We use the discharging method to prove this lemma.

Let us define the initial weight for $V(G) \cup F(G)$ by $w(v) = d(v) - 4$ for each vertex of G and $w(f) = d(f) - 4$ for each face of G . Considering Lemma 1 of Chapter 3, each 3-dependent has exactly two distinct Δ -master vertices. Also, since planar graph G does not have any 3-cycles adjacent to 5-cycles it does not include three continuous triangles. Hence, each k -vertex is incident to at most $\lfloor 2k/3 \rfloor$ 3-faces.

Total sum of the initial weight of vertices and faces is -8 by Euler's formula:

$$w = \sum_{x \in V(G) \cup F(G)} w(x) = \sum_{x \in V(G)} (d(x) - 4) + \sum_{x \in F(G)} (d(x) - 4) = -4(|V| - |E| + |F|) = -8$$

We transfer weights between vertices and faces to make all new weights non-negative for all vertices and faces in graph G . We denote new weights by $w^*(x)$ for all $x \in V(G) \cup F(G)$. Through this redistribution, the sum of weights won't be changed. Thus, the non-negative total sum of new weights leads to a contradiction by the negative sum of initial weights according to Euler's formula. Hence the proof is complete.

$$0 \leq \sum_{x \in V \cup F} w^*(x) = \sum_{x \in V \cup F} w(x) = -8$$

Considering Lemma 1 in chapter 3, each 3-dependent vertex has distinct two Δ -master vertices from which receive weight. Also, since planar graphs without 3-cycles adjacent to 5-cycles do not include three continuous triangles, each k -vertex is incident to at most $\lfloor 2k/3 \rfloor$ 3-faces.

We define a 5-vertex v of graph G as a **special vertex** if it is incident to $(5, 4, 4)$ -face and two adjacent $(5, 5, 5)$ -face and $(5, 5, 3)$ -face which is illustrated in Figure 4.13.

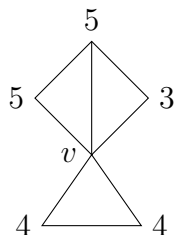


Figure 4.13: Special 5-vertex v incident to three 3-faces, number on each vertex shows the degrees of that vertex

Next we redistribute the weights between vertices and faces according to the following rules:

- (R1) From each Δ -master to its 3-dependent, transfer $1/2$.

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(R2) From each 5-vertex v to each incident 3-face f , transfer

$$\left\{ \begin{array}{ll} \text{(i)} & \frac{1}{m_3(v)} \quad \text{if } v \text{ is not a } \Delta\text{-master;} \\ \text{(ii)} & 1/9 \quad \text{if } f \text{ is a } (5, 5, 5)\text{-face;} \\ \text{(iii)} & 1/5 \quad \text{if } f \text{ is a } (5, 4, 4)\text{-face and } v \text{ is special vertex;} \\ \text{(iv)} & \frac{1}{2m_3(v)} \quad \text{Otherwise.} \end{array} \right.$$

(R3) From each 6^+ -vertex to each incident 3-face transfer $1/3$;

(R4) From each 6^+ -face to each adjacent 3-face f , transfer $w(f)/n_3(f)$.

Now we show that $w^*(x)$ is non-negative for all $x \in V(G) \cup F(G)$. Suppose that v is a k -vertex. Then

- If $k = 3$, it has two Δ -masters and it receives weight from them by (R1). Then $w^*(v) = w(v) + 2 \times 1/2 = 0$.
- If $k = 4$, it doesn't contribute in discharging rules and its weight remains fixed. Then $w^*(v) = w(v) = 0$.
- If $k = 5$, we know $m_3(v) \leq 3$. Then we have the following cases:
 - According to (R2) If v is not a Δ -master, then by (i) $w^*(v) \geq w(v) - m_3(v) \times 1/m_3(v) = 0$.
 - If v is a Δ -master and it is not a special vertex, then $w^*(v) \geq w(v) - 1/2 - m_3(v) \times 1/2m_3(v) = 0$ by (iv) of (R2) and (R1).

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- Otherwise, v is a Δ -master and it is a special vertex. Then $w^*(v) \geq w(v) - 1/2 - 1/5 - 1/9 - 1/6 > 0$ by (ii), (iii), (iv) of (R2) and (R1).
- If $k \geq 6$, we know $m_3(v) \leq \lfloor 2k/3 \rfloor$ and it is Δ -master of at most one 3-dependent by (P2) of Lemma 1. Then $w^*(v) \geq w(v) - 2k/3 \times 1/3 - 1/2 > 0$ by (R1) and (R3).

We show that w^* is non-negative for all faces as well. Suppose that f is a k -face. Then

- If $k = 3$, we have the following cases:
 - If f is not adjacent to any other 3-faces, then by (R4) $w^*(f) \geq w(f) + 3 \times 1/3 = 0$;
 - If f is adjacent to another 3-face f' , we know f is not a $(4, 4, 4)$ -face by excluding configuration (3) of Lemma 7. Then it should be $(4^+, 4^+, 5^+)$ or $(3, 5^+, 5^+)$. We consider the following cases:
 - Case 1. If f is a $(5^+, 5^+, 5^+)$ -face, it receives weight from adjacent faces by (R4) and at least $1/9$ from each incident 5^+ -vertex by (ii) of (R2). Then $w^*(f) \geq w(f) + 3 \times 1/9 + 2 \times 1/3 = 0$;
 - Case 2. If f is a $(4, 5^+, 5^+)$ -face or a $(3, 5^+, 5^+)$ -face, it receives weight from adjacent faces by (R4) and at least $1/6$ from each incident 5^+ -vertex by (iv) of (R2). Then $w^*(f) \geq w(f) + 2 \times 1/6 + 2 \times 1/3 = 0$;

Case 3. Let denote face f by $[u, v, w]$ such that $d(u) = d(w) = 4$ and $d(v) = 5$, then we have the following cases for vertex v :

- * If uw is a common edge between two faces f and f' and v is incident with at most two 3-faces, then f receives weight from the adjacent 6^+ -face by (R4) and at least $1/4$ from 5-vertex by (iv) of (R2). Then $w^*(f) \geq w(f) + 2 \times 2/5 + 1/4 > 0$;
- * If uw is a common edge between two faces f and f' and v is a special vertex, then by (R4) and (iii) of (R2) $w^*(f) \geq w(f) + 2 \times 2/5 + 1/5 = 0$;
- * If uw is a common edge between two faces f and f' and v is incident to two adjacent $(4, 5, 5), (3, 5, 5)$ -faces, then by (R4) and (iv) of (R2) $w^*(f) \geq w(f) + 2/5 + 1/2 + 1/6 > 0$;
- * If uw is a common edge between two faces f and f' and v is incident to two adjacent $(4^+, 4^+, 4^+)$ -faces, then by (R4) and (i) of (R2) $w^*(f) = w(f) + 1/3 + 2 \times 2/5 > 0$;
- * If uw is a common edge between two faces f and f' and $m_3(v) = 2$, then by (R4) and (iv) of (R2) $w^*(f) \geq w(f) + 2 \times 2/5 + 1/4 > 0$;
- * If uw is a common edge between two faces f and f' and $m_3(v) = 3$, then v is not adjacent to any 3-dependent due to the lacking of bad subgraphs. Therefore, according to (R4)

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and (i) of (R2) $w^*(f) \geq w(f) + 2/5 + 1/3 + 1/3 > 0$.

- If $k = 4, 5$, it does not contribute in the discharging method. Then $w^*(f) = w(f) = 0$.
- If $k \geq 6$, then by (R3) $w^*(f) \geq w(f) - n_3(f) \times w(f)/n_3(f) = 0$.

The proof of Lemma 7 is complete.

□

Therefore, the minimal counterexample G includes at least one of the configurations discussed in Lemma 7 which is a contradiction to Lemma 6. Then there does not exist a minimal counterexample. We conclude all planar graphs without 3-cycles adjacent to 5-cycles with $\Delta(G) \geq 5$ are edge- $(\Delta + 1)$ -choosable.

Corollary 5. *It has already been proven that all graphs G with $\Delta(G) \leq 4$ are edge- $(\Delta + 1)$ -choosable. In this section, we proved all planar graphs G without 3-cycles adjacent to 5-cycles with $\Delta(G) \geq 5$ are edge- $(\Delta + 1)$ -choosable. Then all planar graphs without 3-cycles adjacent to 5-cycles are edge- $(\Delta + 1)$ -choosable.*

Chapter 5

Conclusion

Proving Vizing's Conjecture in general has not been achieved yet. This conjecture has already been proven for some special cases. We improved some of these previous results by proving planar graphs G without 7-cycles with $\Delta(G) \neq 5, 6$, planar graphs G without adjacent 4-cycles with $\Delta(G) \neq 5$, and planar graphs without 3-cycles adjacent to 5-cycles are edge- $(\Delta + 1)$ -choosable. One might ask why we did research on these cases. We explain below why we studied those three cases.

As it is mentioned in Chapter 1, Vizing's Conjecture holds for planar graph G without adjacent triangles with $\Delta(G) \neq 5$. First we tried to improve this result by proving Vizing's Conjecture is true for planar graphs G without three continuous triangles with $\Delta(G) \neq 5$. But through the discharging method, we had a shortage of weight for 6-vertices and we could not provide this shortage with a good source of weight. Therefore, we proved Vizing's

Conjecture holds for weaker cases including planar graphs without adjacent 4-cycles or planar graphs without 3-cycles adjacent to 5-cycles which do not contain three continuous triangles. We also found an error in the proof of Vizing's Conjecture for the case of planar graphs without 7-cycles and proved this case with a different approach.

Therefore, there are two questions left for further research regarding edge-choosability of planar graphs. Is Vizing's Conjecture true for planar graphs without three continuous triangles? Also does Vizing's Conjecture hold for planar graphs G without adjacent 4-cycles with $\Delta(G) = 5$ or planar graphs without 7-cycles with $\Delta(G) = 5, 6$? These are important questions that are not resolved yet.

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