Acyclic 5-Choosability of Planar Graphs Without Adjacent Short Cycles

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Abstract

The conjecture claiming that every planar graph is acyclic 5-choosable [Borodin et al., 2002] has been verified for several restricted classes of planar graphs. Recently, O. V. Borodin and A. O. Ivanova, [Journal of Graph Theory, 68(2), October 2011, 169-176], have shown that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$. We improve the above mentioned result and prove that every planar graph without an i-cycle adjacent to a j-cycle with $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 5$ if $i = 4$ is acyclically 5-choosable.

Acknowledgement

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Chapter 1

Introduction

Graph theory is the study of graphs, which are structures which model items and the relations existing between these items. A graph contains a set of vertices, and a set of edges. For example, one can think of cities as vertices on a map, and an edge; i.e., a line with vertices as its endpoints, would exist form one vertex to another vertex if there was a road connecting these two cities. As in real life applications, these edges can be *directed*, as in the case of a one way street, or *undirected*, such as a street in which traffic flows in both directions.

Let $G = (V, E)$ denote a *graph* with vertex set $V(G)$ and edge set $E(G)$. Two vertices u and v in $V(G)$ are called *adjacent* if they are endpoints of an edge in $E(G)$. This edge is denoted as uv. A *proper vertex colouring* of G is one in which no edge is monochromatic; i.e., adjacent vertices are assigned different colours.

A *k-colouring* of G is a partition $V(G) = V_1 \cup \ldots \cup V_k$ of the vertices of G into k pairwise disjoint sets. If G can be properly coloured with k colours then we say G is *k-colourable*. The chromatic number of G, denoted $\chi(G)$, is the least number of colours required to properly colour G; i.e., $\chi(G) = k$ if G can be properly coloured with k colours but not with $k-1$ colours.

A proper vertex colouring of a graph G is *acyclic* if G contains no bi-coloured cycle; i.e., no cycle is coloured with only two colours. If the fewest number of colours required to acyclically colour a graph G is k, then the acyclic chromatic number $\chi_a(G) = k$.

Consider a list assignment $L(v)$, where by a list of admissible colours is assigned to each vertex v of G. We say G is *L-list colourable* if there exists a proper vertex colouring of G with this list assignment $L(v)$ such that each v is coloured with one of its admissible colours, and the colouring is proper. If, regardless of how these lists are assigned to the vertices of G, there exists a proper L-list colouring for every list assignment with $|L(v)| \geq k$ for every vertex $v \in G$, then G is k-choosable. That is, if any one vertex is assigned a list of $k-1$ admissible colours, then there exists some list assignment in which a proper colouring is not possible. The list-chromatic number $\chi^l(G)$ is the smallest integer k such that $|L(v)| \geq k$ for every vertex $v \in G$ which produces such a colouring. If a graph G can be acyclically coloured for all list assignment with $|L(v)| \geq k$, then G is *acyclically k-choosable* and the acyclic list-chromatic number $\chi_a^l(G) = k$.

1.1 History

In 1973, Grünbaum [Grü73] defined a proper colouring of a graph to be acyclic if every cycle uses at least three colours and proved that every planar graph is acyclically 9-colourable (a *planar graph* is a graph which can be embedded in a plane such that two edges never cross). It was also, at this time, conjectured that every planar graph has an acyclic 5-colouring. He provided an example of a planar graph which is not acyclically 4-colourable (refer to Figure 1.1), making 5 the best possible bound. Borodin [Bor79] proved Grünbaums conjecture in 2002, improving earlier bounds of 8, 7, and 6, attributed to Mitchem [Mit74], Albertson and Berman [AB77], and Kostochka [Kos76], respectively. To further demonstrate that 5 is the best bound, there are bipartite 2-degenerate planar graphs that are not acyclically 4-colourable [KM76]. Note that a bipartite d-degenerate planar graph H is a planar graph such that every subgraph of H has a vertex of degree at most d and can be partitioned into two disjoint set of non-adjacent vertices.

Thomassen [Tho94] proved that every planar graph is 5-choosable. This is the best possible bound, as Voigt [Voi93] presented a non-4-choosable planar graph. Borodin [Bor79] conjectured

Conjecture 1. *Every planar graph is acyclically 5-choosable.*

A proof of this conjecture would improve the results of both Borodin [Bor79] and Thomassen [Tho94].

Borodin et al. [BFDFK$^+$02] proved that every planar graph is acyclically

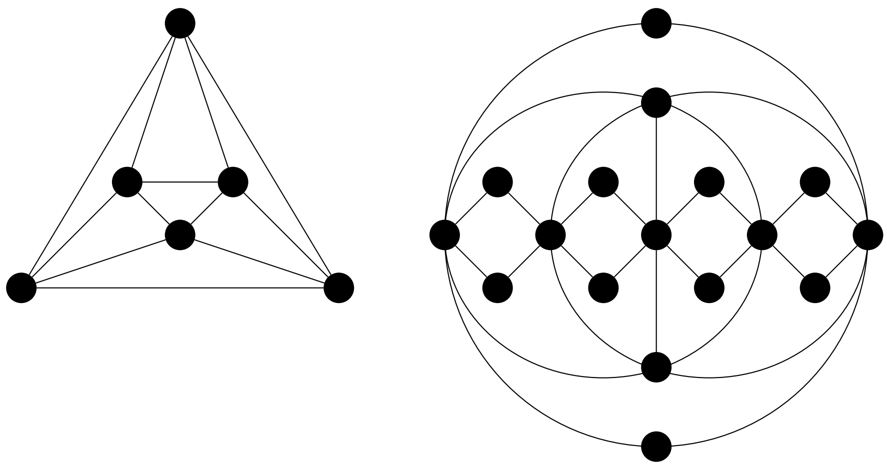

Figure 1.1: Grünbaum's example and Kostochka-Mel'nikov's example.

7-choosable. As of yet, proofs for the k-choosability, $3 \leq k \leq 6$ of planar graphs have been verified for restricted classes of planar graphs [BCIR10] [Bor09] [CR10] [Mon07] [WC09]. Work on the 5-choosability of planar graphs, such as those graphs

- with girth at least five [MOR06],
- without 4- and 5-cycles or without 4- and 6-cycles [MRW07],
- with neither 4-cycles nor chordal 6-cycles [ZX09],
- with neither 4-cycles nor two 3-cycles at distance less than 3 [CW08],
- and without 4-cycles and intersecting 3-cycles [CR12],

have recently been absorbed by Borodin and Ivanova [BI11] which proved that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$. The purpose of this thesis is to improve this last result by allowing the adjacency of 4- and 6-cycles which were previously forbidden.

Theorem 1. *Every planar graph without an i-cycle adjacent to a j-cycle with* $3 \leq j \leq 5$ *if* $i = 3, 4$ *is acyclically 5-choosable.*

The method of proof will be by contradiction, and shown in two parts using the discharging method. First we assume for contradiction that there exists a graph G with the necessary restrictions, i.e., contains no short cycles which have at least one edge in common, but cannot be acyclically 5-coloured with an arbitrary list assignment for $v \in G$ such that $|L(v)| \geq 5$. Then we determine some notion of the structure of G. This is achieved though finding reducible configurations that would lead to a contradiction in the assumption that G is not 5-choosable. Finally we assign charges to each vertex and to each face of G such that the total charge of G is negative. We carefully redistribute those charges and show that, without adding or removing charge, the total charge of G is non-negative. Hence a contradiction, meaning the assumption of the existence of such a graph G was incorrect.

1.2 Overview

The purpose of Chapter 2 is to help familiarize the reader with notation that is commonly used in the study of Graph Theory and to notation used specific to Colouring Problems and this thesis. Chapter 3 is a summary reducible configurations that has been previously proved by other authors who are examining the problem of planar graphs and there 5-choosability. Chapter 4 is comprised of three different parts. Firstly, the list of reducible configurations is expanded, as some new configurations are considered. Next, the proof of Theorem 1 is started by providing the reader a list of rules by which the discharging will follow. Lastly, an examination of the new charges carried by each face and each vertex of the minimum counterexample is calculated. Finally, this thesis will end in Chapter 5 with a look at the future of the answer to the question of the 5-choosability of planar graphs.

Chapter 2

Preliminaries and Notions

In this chapter, we will become familiar with the notation used in the next few chapters.

2.1 Vertices, Edges and Faces

A k-vertex, k^+-vertex and k^--vertex is a vertex of degree k, at least k and at most k, respectively. The degree of a vertex refers to the number of edges incident to it.

Similarly, we define a k-face, k^+-face, k^--face where a face f has exactly, at least, or at most k edges in its boundary, respectively.

A vertex v which is the endpoint of an edge in the boundary of a face is said to be *incident* to that face. A triangle is synonymous with a 3-face. It is a region enclosed by three successively adjacent vertices embedded in a plane. It has three boundary edges and, as expected, each triangle has three

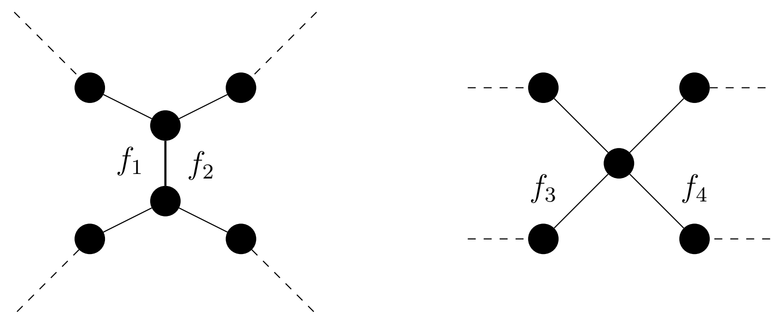

Figure 2.1: Two adjacent faces f_1 and f_2. Two intersecting faces f_3 and f_4.

incident vertices.

Usually, a face $f \in F(G)$; the set of faces of an embedded graph G, is written as $f = [u_1, u_2, ..., u_n]$ if $u_1, u_2, ..., u_n$ are the boundary vertices of f in a cyclic order.

Two cycles are *adjacent* if they have two consecutive adjacent vertices v_i and v_{i+1} in common as boundary vertices. Otherwise, for the purposes of this thesis, two cycles *intersect* if they share common vertices but are not adjacent. If two cycles are not adjacent in a graph G, then they are not adjacent faces in the embedding of G (see Fig.2.1).

2.2 Neighbourhoods

For a vertex v and an integer $i \geq 1$, let $t(v)$ denote the number of 3-faces incident to v and let $n_i(v)$ denote the number of i-vertices adjacent to v.

For a face $f \in F(G)$ and an integer $j \geq 2$, let $n_j(f)$ denote the number of j-vertices incident to f.

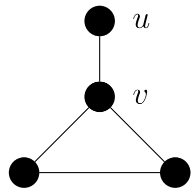

Figure 2.2: A pendant light 3-vertex v of u.

Let $N(v)$ denote the set of neighbours of a vertex v. In other words, $N(v)$ refers to the number of vertices adjacent to v.

A 3-face $f = [v_1v_2v_3]$ is called an (a_1, a_2, a_3)-face if the degree of the vertex v_i is a_i for $i = 1, 2, 3$. Similarly we define a 4-face $f = [v_1v_2v_3v_4]$ as being a (a_1, a_2, a_3, a_4)-face if the degree of the vertex v_i is a_i for $i = 1, 2, 3, 4$.

2.3 Pendant Light Vertices

A 3-vertex v is *pendant* if it is incident to a 3-face. If a vertex u is adjacent to a 3-vertex v such that the edge uv is not in the boundary of a 3-face, then the edge uv is called *light*. Hence, we call v a pendant light 3-vertex of u if uv is light and v is pendant (see Fig. 2.2).

If v is a pendant light 3-vertex of u which is incident to an (a_1, a_2, a_3)-face, then we call v a pendant light (a_1, a_2, a_3)-vertex of u. Let $p_3(u)$ denote the number of pendant light 3-vertices of a vertex u.

2.4 Path Colouring

A path is a set of vertices $v_1, \ldots, v_k$ and edges $e_1, \ldots, e_{k-1}$ which are in sequential order $v_1, e_1, \ldots, e_{k-1}, v_k$. Let α and β be any two distinct colours. An *alternating* $(\alpha,\ \beta)$*-path* in G is a path where by each vertex is coloured α or β in alternating order. A path $v_1, e_1, \ldots, e_{k-1}, v_k$ is called a cycle if $v_1 = v_k$.

Chapter 3

Reducible Configurations - Past Work

For all figures in this thesis, a vertex is represented by a solid point $\bullet$ when all of its incident edges are indicated; otherwise it is represented by a hollow point $\circ$. Letters which appear early in the alphabet, such as $a, b, c, \ldots$ will denote a colour which is assigned to a vertex. Indexed colours $c_1, c_2, \ldots$ are colours assigned to indexed vertices $v_1, v_2, \ldots$ and are distinct colours unless otherwise specified. Lastly, letters which appear late in the alphabet, such as $u, v, \ldots, y, z$ will denote vertices.

To show the acyclic 5-choosability of every planar graph which does not contain an i-cycle adjacent to a j-cycle for $i = 3, 4$ and $j = 3, 4, 5$, assume for contradiction that there exists a planar graph G which does not contain the above adjacent cycles, is not acyclically 5-choosable. This graph G is a minimum planar graph; i.e., it can be embedded in a plane without any of its edges crossing, and has the fewest number of vertices such that it cannot be acyclically *List* 5-coloured. Firstly, we assume G is a connected planar graph; that is, every two vertices in G belong to a path in G. As per the assumption that G is minimum, each component $C_1, \ldots, C_k$ of a graph which is not connected is acyclically 5-choosable. Hence G is acyclically 5-choosable which contradicts our choice of G. Next, we shall assume that G is simple. A *simple graph* is a graph with no loops or multiple edges. A *loop* is an edge uv such that u and v are the same vertex. Obviously, a loop cannot be properly coloured. *Multiple edges* are edges a set of two or more edges which share the same two endpoints. Obviously, every multiple cannot be acyclically coloured.

The following is a list of reducible configuration that are not in G. If any of these configurations where in G, then one or more vertices can be removed, the remaining graph can be acyclically *List* 5-coloured and this colouring can be extended to the vertices that were removed. Hence the graph G would be acyclically *List* 5-coloured.

Lemma 1. *As a consequence of G being a minimum counterexample, the following conditions hold:*

(C1) There are no 1*-vertices. [MRW07]*

(C2) No 2*-vertex is adjacent to a* 4^-*-vertex. [MRW07]*

(C3) Let v *be a* 3*-vertex.*

(C3.1) If v is adjacent to a 3-vertex, then v is not adjacent to any other 4^--vertex. [MRW07]

(C3.2) v has no pendant 3-vertex. [MRW07]

(C4) Let v be a 4-vertex, then v has no pendant 3-vertex. [BI11].

(C5) Let v be a 5-vertex.

(C5.1) v is adjacent to at most one 2-vertex. [MRW07]

(C5.2) If $n_2(v) = 1$, then v has no pendant 3-vertex. [MRW07]

(C5.3) If $n_2(v) = 1$ and v is incident to a 3-face, then $n_3(f) = 0$. [CW08]

(C5.4) If $n_2(v) = 0$, then $p_3(v) \leq 3$. [CR12]

(C5.5) If $n_2(v) = 0$ and $t(v) = 1$, then $p_3(v) \leq 2$. [CR12]

(C5.6) If v is incident to a $(5, 3, 4)$-face, then $p_3(v) \leq 1$. [CR12]

(C6) Let v be a 6-vertex.

(C6.1) v is adjacent to at most four 2-vertices. [MRW07]

(C6.2) If $n_2(v) = 4$, then v is not adjacent to any 3-vertex. [MRW07]

(C6.3) If $n_2(v) = 4$, then $t(v) = 0$. [CW08]

(C6.4) If $n_2(v) = 3$ and $t(v) = 1$, then $p_3(v) = 0$. [CR12]

(C6.5) *If* $n_2(v) = 2$*, then* $p_3(v) \leq 2$*. [CR12]*

(C6.6) *If* $n_2(v) = 2$*, and* $t(v) = 1$*, then* $p_3(v) \leq 1$*. [CR12]*

(C6.7) *If* $n_2(v) = 0$ *and* v *is incident to a* $(3, 3, 6)$*-face, then* $p_3(v) \leq 2$*. [CR12]*

(C6.8) *If* v *is incident to a* $(3, 4, 6)$*-face, then* $n_2(v) \leq 2$*. [CR12]*

(C7) *Let* v *be a* 7*-vertex.*

(C7.1) v *is adjacent to at most five* 2*-vertices. [MRW07]*

(C7.2) *If* $n_2(v) = 4$*, then* $n_3(v) \leq 2$*. [CW08]*

(C7.3) *If* $n_2(v) = 5$*, then* $n_3(v) = 0$ *and* $t(v) = 0$*. [CR12]*

(C7.4) *If* $n_2(v) = 4$ *and* $t(v) = 1$*, then* $p_3(v) = 0$*. [CR12]*

(C7.5) *If* $n_2(v) = 3$ *and* v *is incident to a* $(7, 3, 3)$*-face, then* $p_3(v) \leq 1$*. [CR12]*

(C8) *Let* v *be a* 8*-vertex.*

(C8.1) v *is adjacent to at most six 2-vertices. [CR12]*

(C8.2) *If* $t(v) = 1$*, then* $n_2(v) \leq 5$*. [CR12]*

(C9) *Let* f *be a* 3*-face* $[xyz]$ *with* $d(x) \leq d(y) \leq d(z)$*.*

(C9.1) $d(x) \neq 2$. *[MRW07]*

(C9.2) $[xyz]$ *does not satisfy* $d(x) = d(y) = 3$ *and* $d(z) \leq 5$. *[MRW07]*

(C9.3) $[xyz]$ *does not satisfy* $d(x) = 3$ *and* $d(y) = d(z) = 4$. *[MRW07]*

(C10) *There does not exist a* 5*-face* $[x_1x_2x_3x_4x_5]$ *such that* $d(x_1) = 2, d(x_3) = 3$, *and* $d(x_2) = 5$. *[MRW07]*

The following section is a proof of Lemma 1 and will demonstrate how the presence of these configurations would lead to a contradiction in the choice of G as a minimum counterexample.

3.1 Restrictions on a 1-vertex

Proof. ***(C1)*** Suppose that there exists a 1-vertex $v \in V(G)$ adjacent to the vertices u, as in Figure 3.1.

Figure 3.1: A 1-vertex in G.

By minimality of G, the graph $G - \{v\}$ admits acyclically list 5-colouring π. Extend this colouring of $G - \{v\}$ to G by letting $\pi(v)$ be a colour in $L(v)$ different than $\pi(u)$. Now G is acyclically 5-choosable, which contradicts the choice of G. Hence G does not contain a 1-vertex. $\square$

3.2 Restrictions on a 2-vertex

Proof. ***(C2)*** Suppose that there exists a 2-vertex $v \in V(G)$ adjacent to the vertices u and w with $d(w) \leq 4$ as in Figure 3.2.

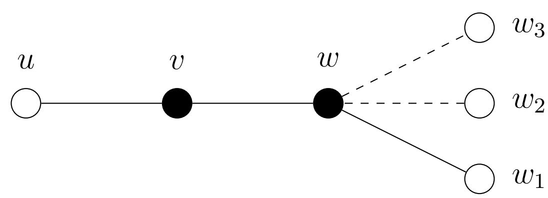

Figure 3.2: A 2-vertex in G with a neighbour having degree ≤ 4.

Let $w_1, \ldots, w_k$ be the neighbours of w that are different from v, where $k \leq 3$. By minimality of G, the graph $G - \{v\}$ admits an acyclic list 5-colouring π. Extend this colouring to G by considering the following:

If $\pi(u) \neq \pi(w)$, colour v with a colour in $L(v)$ different from $\pi(u)$ and $\pi(w)$. Otherwise, $\pi(u) = \pi(w)$. Colour v with a colour different from $\pi(w), \pi(w_1), \ldots, \pi(w_k)$. Now G is acyclically 5-choosable, which contradicts the choice of G. Hence G does not contain a 2-vertex adjacent to a 4^{-}-vertex. □

Note that if a 2-vertex v where incident to a 3-face in the graph G, then its two neighbours u and w would always have the property that $\pi(u) \neq \pi(w)$, since u is adjacent to w in $G - v$. Hence, there is always a proper acyclic 5-colouring if G regardless of the degrees of the neighbours of v. This configuration will be examined in Condition (C9.1).

3.3 Restrictions on a 3-vertex

Proof. ***(C3.1)*** Suppose that v is adjacent to a 3-vertex v_1, a vertex v_2 of degree at most 4, and a vertex v_3 as in Figure 3.3.

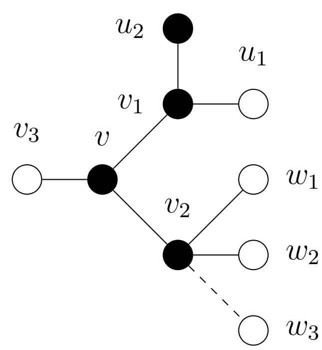

Figure 3.3: A 3-vertex v adjacent to a 3-vertex v_1 and to a vertex v_2 of degree at most 4.

Let u_1, u_2 be the neighbours of v_1 different from v, and $w_1, w_2, \ldots, w_k$, for $k \leq 3$, be the neighbours of v_2 different from v. Note that by Condition $(C2)$, $d(v_2) \neq 2$. Let π be an acyclic list 5-colouring of $G - \{v\}$. If v_1, v_2, v_3 have pairwise distinct colours, then colour v with a colour different from $\pi(v_1), \pi(v_2), \pi(v_3)$. If $\pi(v_1) = \pi(v_2) \neq \pi(v_3)$, colour v with a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_3),\ \pi(u_1), \pi(u_2)\}$. If $\pi(v_1) = \pi(v_3) \neq \pi(v_2)$, colour v with a colour different from $\pi(v_1)$, $\pi(v_2), \pi(u_1), \pi(u_2)$. It remains to consider the following two cases.

- Assume that $\pi(v_2) = \pi(v_3) \neq \pi(v_1)$. If there exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(w_1), \ldots, \pi(w_k)\}$, then colour v with c. Otherwise, it follows that $k = 3$ and w_1, w_2, w_3 have pairwise distinct colours. Recolour v_2 with a colour different from those of v_2, w_1, w_2, w_3 and reduce to the previous case.

- Assume that $\pi(v_1) = \pi(v_2) = \pi(v_3)$. If either $\pi(u_1) \neq \pi(u_2)$, or $\pi(w_1), \ldots, \pi(w_k)$ are pairwise distinct, recolour v_1 or v_2 to reduce to the previous case. So suppose that $\pi(u_1) = \pi(u_2)$ and, without loss of generality, assume $\pi(w_1) = \pi(w_2)$. In this case, colour v with a colour different from the colours assigned to $v_1, u_1, w_1, \ldots, w_k$.

The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

3.4 Restrictions on a pendant light 3-vertex

Proof. ***(C3.2)(C4)*** Let v be a pendant light 3-vertex of u, with $f = [vv_1v_2]$ being a 3-face. Suppose, for contradiction, that $d(u) \leq 4$. Let $u_1, \ldots, u_k, k \leq 3$ be the neighbours of u different from v as in Figure 3.4. Note that by Condition $(C2)$, $d(u) \neq 2$.

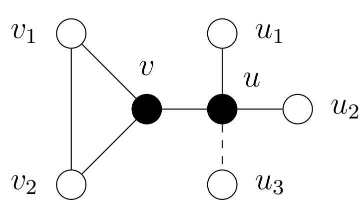

Figure 3.4: A 4-vertex v adjacent to a pendant light 3-vertex u.

Then by the minimality of G, $G - v$ admits an acyclic L-colouring π. If v_1, v_2 and u are coloured with pairwise distinct colours, then colour v with a colour different from the colours assigned to its neighbours. Otherwise, by the symmetry, suppose $\pi(v_1) = \pi(u)$. Colour v with a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(u_1), \ldots, \pi(u_k)\}$. If there is no such c, then $k = 3$, and $L(v) = \{\pi(v_1), \pi(v_2), \pi(u_1), \pi(u_2), \pi(u_3)\}$. If v cannot be acyclically coloured with $\pi(u_1), \pi(u_2)$ or $\pi(u_3)$, then recolour u with a colour c_u different from $\pi(u_1)$, $\pi(u_2), \pi(u_3)$ and $\pi(v_1)$. If $c_u \neq \pi(v_2)$ then colour v with a colour different from the colours assigned to its neighbours. Otherwise, $c_u = \pi(v_2)$. Then properly acyclically colour v with one of $\pi(u_1), \pi(u_2)$ or $\pi(u_3)$, as $G - v$ is planar and there does not exist a $(\pi(u_i), \pi(v_2))$-path from u_i to v_2 for some $1 \leq i \leq 3$. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

3.5 Restrictions on a 5-vertex

Proof. ***(C5.1)*** Suppose that G contains a 5-vertex v adjacent to two 2-vertices v_1, v_2 and other vertices v_3, v_4, v_5 as in Figure 3.5

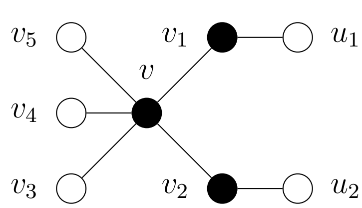

Figure 3.5: A 5-vertex v adjacent to two 2-vertices v_1, v_2.

For $i = \{1, 2\}$, let u_i be the neighbour of v_i different from v. By the minimality of G, $G - v_1$ has an acyclic list 5-colouring π. If $\pi(v) \neq \pi(u_1)$, colour v_1 with a colour different from the colours assigned to v and u_1. Otherwise, if v_1 can not be coloured acyclically, suppose that $\pi(v) = \pi(u_1) = \pi(u_2)$, and $\pi(v_i)$ for $i = 2, 3, 4, 5$ are pairwise distinct colours. $L(v) = \{\pi(v), \pi(v_2), \ldots, \pi(v_5)\}$ and there exists $(\pi(v), \pi(v_i))$-paths from u_1 to v_i for $i = 2, 3, 4, 5$. If $L(v) \neq L(v_1)$, recolour v with a colour in $L(v) \setminus L(v_1)$ and then give v_1 a proper colouring. If $L(v) = L(v_1)$, recolour v with $\pi(v_2)$, then colour v_1 with $\pi(v_3)$ and v_2 with a colour different from the colours assigned to v, u_2. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.2)*** Suppose that G contains a 5-vertex v adjacent to a 2-vertex v_1 with neighbour of v and u_1, a pendant light 3-vertex v_2 which is incident a 3-face $[v_2xy]$, and other vertices v_3, v_4, v_5 as in Figure 3.6.

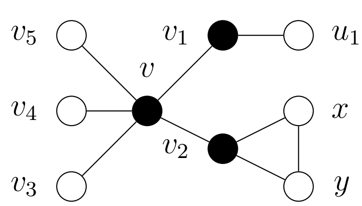

Figure 3.6: A 5-vertex v adjacent to a 2-vertex and a pendant light 3-vertex.

Let π be an acyclic list 5-colouring of $G - \{v_1\}$. If v_1 cannot be acyclically coloured by any of its assigned acceptable colours, assume that $|L(v)| = 5$, $\pi(v) = \pi(x) = \pi(u_1)$ and $\pi(v_i)$ for $i = 2, 3, 4, 5$ are pairwise distinct colours.

$L(v) = \{\pi(v), \pi(v_2), \ldots, \pi(v_5)\}$ and there exists $(\pi(v), \pi(v_i))$-paths from u_1 to v_i for $i = 2, 3, 4, 5$. If $L(v) \neq L(v_1)$, recolour v with a colour in $L(v) \setminus L(v_1)$ and then give v_1 a proper colouring. If $L(v) = L(v_1)$, recolour v with $\pi(v_2)$, then colour v_1 with π_3 and v_2 with a colour different from the colours assigned to v, x, y. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.3)*** Let $v_1, v_2, \ldots, v_5$ be the neighbours of a 5-vertex v with $d(v_1) = 2$ and $N(v_1) = v, u_1$. Assume that v is incident to a 3-face $f = [vv_2v_3]$ such that $n_3(f) \geq 1$. By Condition $(C9.2)$, $n_3(f) = 1$, say $d(v_2) = 3$. Let x_2 be the neighbour of v_2 different from v and v_3 as in Figure 3.7.

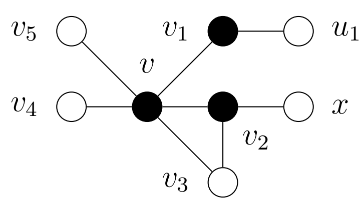

Figure 3.7: A 5-vertex v adjacent to a 2-vertex v_1 and a 3-face $[vv_2v_3]$ with $d(v_2) = 3$.

By the minimality of G, $G - \{v_1\}$ has an acyclic list 5-colouring π. If $\pi(u_1) \neq \pi(v)$,then let $\pi(v_1) = c_1 \in L(v_1) \setminus \{\pi(u_1), \pi(v)\}$. Otherwise, $\pi(u_1) = \pi(v)$. If there does not exist a colour $c_1 \in L(v_1)$ which acyclically colours G properly, then it is the case that $\pi(v) = \pi(u_1) = \pi(x_2)$. If $L(v) \neq L(v_1)$, recolour v with a colour in $L(v) \setminus L(v_1)$ and then $v1$ is properly coloured. If $L(v) = L(v_1)$, recolour v with $\pi(v_2)$ and colour v_1 with $\pi(v_3)$, then recolour

v_2 with a colour different from $\pi(x_2), \pi(v)$ and $\pi(v_3)$. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.4)*** Assume to the contrary that $p_3(v) \geq 4$ so that $v_1, \ldots, v_4$ are pendant light 3-vertices of v and v_5 is the other neighbour of v with $d(v_5) \geq 3$. Let x_i, y_i be the neighbours of v_i, different from v, for $i \leq 4$ as in Figure 3.8.

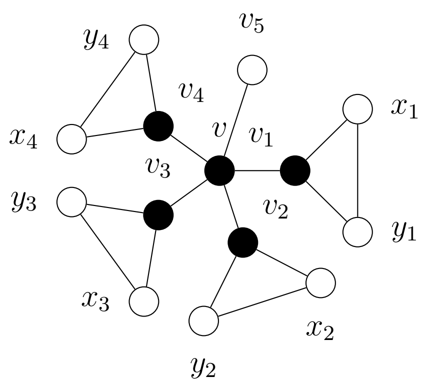

Figure 3.8: A 5-vertex v adjacent to a at least four pendant light 3-vertices v_1, v_2, v_3, v_4.

By the minimality of $G, G-\{v, v_1, \ldots, v_4\}$ admits an acyclic list 5-colouring π. Notice that $\pi(x_i) \neq \pi(y_i)$ for all $1 \leq i \leq 4$ since x_i and y_i are adjacent. Let $S = \{x_1, y_1, \ldots, x_4, y_4\}$. Since $|L(v) \setminus \{\pi(v_5)\}| \geq 4$ and $|S| = 8$, then there exists a colour $c \in L(v) \setminus \{\pi(v_5)\}$ which appears at most twice on the set S, say $\pi(x_1) = \pi(x_2) = c$. Then, colour v with c, v_1 with $c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(y_1)\}, v_2$ with $c_2 \in L(v_2) \setminus \{c, c_1, \pi(v_5), \pi(y_2)\}$, and v_i with a colour different from $c, \pi(x_i), \pi(y_i)$ for $i = 3, 4$. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.5)*** Let v_1, v_2, v_3, v_4, v_5 be neighbours of a 5-vertex v in clockwise order. Assume to the contrary that $[vv_1v_2]$ is a 3-face and v_3, v_4, v_5 are pendant light 3-vertices of v. Let $x_3, y_3, x_4, y_4, x_5, y_5$ be, respectively, neighbours of v_3, v_4, v_5 different from v as in Figure 3.9.

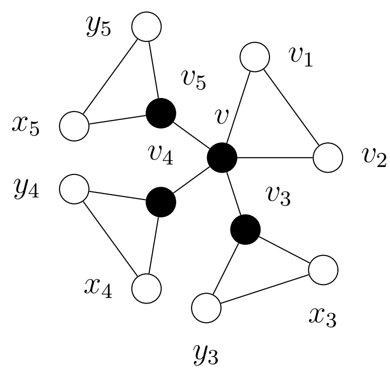

Figure 3.9: A 5-vertex v adjacent to at least three pendant light 3-vertices v_1, v_2, v_3, v_4.

Let $G' = G - \{v, v_3, v_4, v_5\}$. By the minimality of G, G' admits an acyclic list 5-colouring π. Notice that, $\pi(v_1) \neq \pi(v_2)$ and $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{3,4,5\}$. Let $S = \{x_3, x_4, x_5, y_3, y_4, y_5\}$. Notice also that $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$ and $|S| = 6$. Then there exists a colour in $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most twice on the set S. Consider two cases:

1. If there exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ which appears at most once on the set S, then assume, without loss of generality, that $\pi(x_3) = c$. Let $\pi(v) = c$, v_3 with $c_3 \in L(v_3) \setminus \{c, \pi(v_1), \pi(v_2), \pi(y_3)\}$, and finally colour v_i with a colour different from $c, \pi(x_i), \pi(y_i)$ for $i = 4, 5$.

2. Otherwise, each colour in $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appears exactly twice on the set S. Without loss of generality, assume that $\pi(x_3) = \pi(x_5), \pi(y_4) = \pi(y_5)$ and $\pi(y_3) = \pi(x_4)$.

   - If there is no alternating $(\pi(y_3), \pi(v_1))$-path in G' connecting y_3 and v_1, then colour v with $\pi(y_3)$, v_4 with $a \in L(v_4) \setminus \{\pi(v_1), \pi(v_2), \pi(y_4), \pi(y_3)\}$, v_3 with $b \in L(v_3) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3), a\}$, and finally colour v_5 with a colour distinct from $\pi(x_3), \pi(y_4), \pi(y_3)$.

   - If there is no alternating $(\pi(y_5), \pi(v_2))$-path in G' connecting y_5 and v_2, then colour v with $\pi(y_5)$, v_4 with $c \in L(v_4) \setminus \{\pi(v_1), \pi(v_2), \pi(y_4), \pi(y_3)\}$, v_5 with $d \in L(v_5) \setminus \{\pi(v_1), \pi(x_3), \pi(y_4)c\}$, and finally colour v_3 with a colour distinct from $\pi(x_3), \pi(y_4), \pi(y_3)$.

Since there cannot be both an alternating $(\pi(y_3), \pi(v_1))$-path connecting y_3 and v_1, and an alternating $(\pi(y_5), \pi(v_2))$-path connecting y_5 and v_2 due to the planarity of G, then the colouring of G' can be extended to G. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.6)*** Let v_1, v_2, v_3, v_4, v_5 be neighbours of a 5-vertex v in clockwise order. Assume that $[vv_1v_2]$ is an incident $(5, 3, 4)$-face of v, with $d(v_1) = 3$, $d(v_2) = 4$ and v_3, v_4 are pendant light 3-vertices of v. Let u be the neighbour of v_1 different from v and v_2, and let x_2 and y_2 be the neighbour of v_2 different from v and v_1 and let x_3, y_3, x_4, y_4 be, respectively, neighbours of v_3, v_4 different from v as in Figure 3.10.

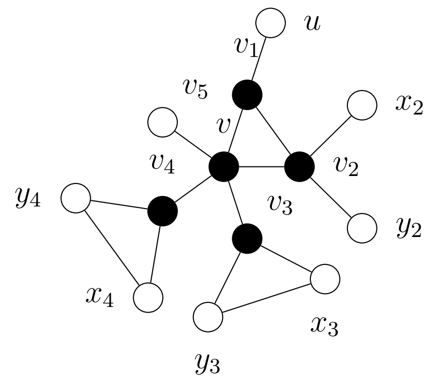

Figure 3.10: A 5-vertex v adjacent to a $(3, 4, 5)$-face and two pendant light 3-vertices.

Let $G' = G - \{v, v_1, v_3, v_4\}$. By the minimality of G, G' admits an acyclic list 5-colouring π . Let $S = \{u, x_3, y_3, x_4, y_4\}$, and consider the following two cases:

1. If $\pi(v_2) \neq \pi(v_5)$, then since $|L(v) \setminus \{\pi(v_2), \pi(v_5)\}| \geq 3$ and $|S| = 5$, there exists a colour $c \in L(v) \setminus \{\pi(v_2), \pi(v_5)\}$ which appears at most once on the set S. Let $\pi(v) = c$. If $\pi(u) = c$, then let $\pi(v_1) = c_1 \in L(v_1) \setminus \{\pi(v_2), \pi(v_5), c\}$, and then colour v_i with a colour different from $c, \pi(x_i), \pi(y_i)$ for $i = \{3, 4\}$. Otherwise assume, without loss of generality, that $\pi(x_3) = c$. Let $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_2), \pi(v_5), \pi(y_3)\}$, colour v_4 with a colour different from $c, \pi(x_4), \pi(y_4)$, then colour v_1 as follows:

   - If $\pi(u) \neq \pi(v_2)$, colour v_1 different from $c, \pi(u)$ and $\pi(v_2)$.
   - If $\pi(u) = \pi(v_2)$, colour v_1 different from $c, \pi(v_2)$, $\pi(x_2)$ and $\pi(y_2)$.

2. If $\pi(v_2) = \pi(v_5)$, consider the following:

    - If $\pi(x_2) = \pi(y_2)$, then there exists a colour in $L(v) \setminus \{\pi(v_2), \pi(x_2)\}$ which appears at most once on the set S. Then the proof can also be given with a similar argument to the previous case.

    - Otherwise, $x_2 \neq y_2$. Recolour v_2 with a colour different from $\pi(v_2), \pi(x_2), \pi(y_2)$ and then reduce the proof to the former case.

The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C5.7)*** Let v_1, v_2, v_3, v_4, v_5 be neighbours of a 5-vertex v in clockwise order. Assume to the contrary that $[vv_1v_2]$ is a $(5, 3, 5^+)$-face with $d(v_1) = 3$ and $d(v_2) \geq 5$. Let v_3, v_4 be pendant light $3-$vertices of v and let v_5 be a $3-$vertex. Let u be the neighbour of v_1 different from v and v_2 and let $N(v_i) = \{v, x_i, y_i\}$ for $i = 3, 4, 5$ as in Figure 3.11.

Consider the graph $G - \{v, v_3, v_4\}$. By the minimality of G, $G - \{v, v_3, v_4\}$ admits an acyclic $L-$ colouring π. Let $S = \{x_3, y_3, x_4, y_4\}$, $\alpha = |\{\pi(x_3), \pi(y_3), \pi(x_4), \pi(y_4)\}|$, and consider the following three cases.

- If $\pi(v_1), \pi(v_2), \pi(v_5)$ are pairwise distinct colours, then consider the following:

    ◇ If there exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5)\}$ which appears at most once on the set S, say $\pi(x_3) = c$, then let $\pi(v) = c$ and

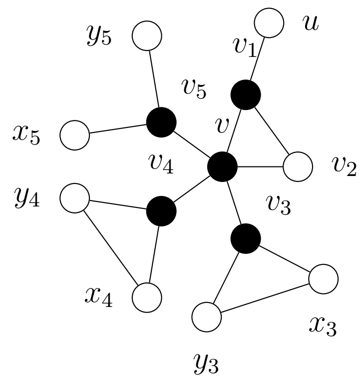

Figure 3.11: A 5-vertex v adjacent to a $(3,4,5)$-face and two pendant light 3-vertices.

$\pi(v_4) = c_4 \in L(v_4) \setminus \{c, \pi(x_4), \pi(y_4)\}$. Then let $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(y_3), \pi(v_1), \pi(v_2), \pi(v_5)\}$.

If there is no such colour c_3, then $L(v_3) = \{c, \pi(y_3), \pi(v_1), \pi(v_2), \pi(v_5)\}$. If v_3 cannot be acyclically coloured, then $\pi(u) = c$. So, let $\pi(v_3) = \pi(v_1)$ and recolour v_1 with $c_1 \in L(v_1) \setminus \{c, \pi(v_2), \pi(v_3), \pi(v_5)\}$.

- ◇ Now assume, that all colours in $L(v) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5)\}$ each appear twice in S, so assume without loss of generality that $\pi(x_3) = \pi(x_4)$ and $\pi(y_3) = \pi(y_4)$. If $\pi(u) \neq \pi(v_2)$, then colour v with $pi(v_1)$, recolour $a \in L(v_1) \setminus \{\pi(v_1), \pi(v_2), \pi(v_5), \pi(u)\}$, and colour v_i with a colour distinct from $\pi(v), \pi(x_3), \pi(y_3)$ for $i = \{3, 4\}$.

  Otherwise, suppose that $\pi(u) = \pi(v_2)$. If $\pi(x_5) = \pi(y_5)$, then let $\pi(v) = b \in L(v) \setminus \{\pi(x_3), \pi(y_3), \pi(x_5)\}$, $\pi(v_3) = c \in L(v_3) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3)\}$ and $\pi(v_4) = d \in L(v_4) \setminus \{\pi(v_2), \pi(x_3), \pi(y_3)\}$.

If $\pi(x_5) \neq \pi(y_5)$, colour v with $\pi(v_5)$, then recolour v_5 with a colour different from $\pi(v_2), \pi(v_5), \pi(x_5), \pi(y_5)$, and give a proper colouring for v_3 and v_4.

- Assume that $\pi(v_5) = \pi(v_1) \neq \pi(v_2)$. If $\pi(u) \neq \pi(v_2)$, recolour v_1 with a colour different from $\pi(v_1), \pi(v_2), \pi(u)$ and then colour the other vertices as in the previous case where $\pi(v_1), \pi(v_2), \pi(v_5)$ are pairwise distinct colours. Now suppose that $\pi(u) = \pi(v_2)$. There exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ which appears at most once on the set S, say $\pi(x_3) = c$. Colour v with c, v_3 with a colour in $L(v_3) \setminus \{\pi(v_1), \pi(v_2), c, \pi(y_3)\}$, and v_4 with a colour different from $c, \pi(x_4), \pi(y_4)$.

- Assume that $\pi(v_5) = \pi(v_2) \neq \pi(v_1)$. If $\pi(x_5) \neq \pi(y_5)$, then recolour v_5 with a colour different from $\pi(v_5), \pi(v_2), \pi(x_5), \pi(y_5)$ and thus reduce the proof to the previous case where $\pi(v_1), \pi(v_2), \pi(v_5)$ are pairwise distinct colours. Suppose now that $\pi(x_5) = \pi(y_5)$. If there exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(v_1), \pi(x_5)\}$ appearing at most once on the set S, say $\pi(x_3) = c$, then let $\pi(v) = c$, v_3 with a colour distinct from $\pi(v_1), \pi(v_5), c, \pi(y_3)$, and colour v_4 with a colour different from $c, \pi(x_4), \pi(y_4)$. Otherwise, assume, without loss of generality that $L(v) = \{1, 2, \pi(x_5), 4, 5\}$ and that $\pi(v_5) = \pi(v_2) = 1, \pi(v_1) = 2, \pi(x_3) = \pi(x_4) = 4$ and $\pi(y_3) = \pi(y_4) = 5$. If $\pi(u) \neq 1$, recolour v_1 with $a \in L(v_1) \setminus \{1, 2, \pi(u)\}$ and then reduce the proof to the previous case. Otherwise, $\pi(u) = 1$. Colour v with $4, v_3$ with a colour

$b \in L(v_3) \setminus \{1, 4, 5\}$ and v_4 with a colour in $L(v_4) \setminus \{1, 4, 5, b\}$.

The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

3.6 Restrictions on a 6-vertex

Proof. ***(C6.1)*** Suppose that G contains a 6-vertex v adjacent to five 2-vertices $v_1, \ldots, v_5$ and a vertex v_6 as in Figure 3.12.

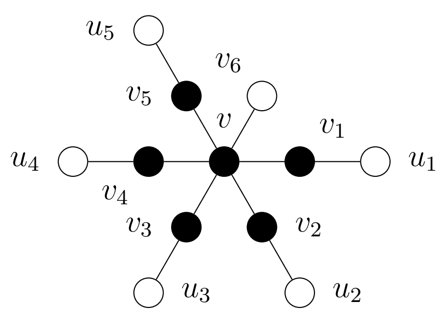

Figure 3.12: A 6-vertex adjacent to five 2-vertices.

Let u_i be the neighbour of v_i different from v for $i = 1, 2, \ldots, 5$. Let π be an acyclic list 5-colouring of $G - \{v, v_1, \ldots, v_5\}$. Then there exists q colour $c \in L(v) \setminus \{\pi(v_6)\}$ which appears at most once on the vertices $u_1, u_2, \ldots, u_5$. Without loss of generality, suppose that c appears on u_1. Colour v with c, v_1 with a colour different from u_1, v, v_6, and for each $i = 2, 3, 4, 5$ and colour v_i with a colour different v and u_i. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G.

□

Proof. ***(C6.2)*** Suppose that G contains a 6-vertex v adjacent to four 2-vertices v_1, v_2, v_3, v_4, a 3-vertex v_5, and a vertex v_6 as in Figure 3.13. Let

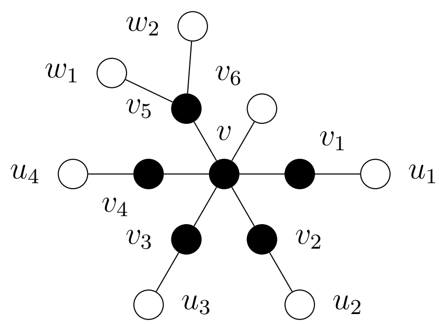

Figure 3.13: A 6-vertex adjacent to four 2-vertices and a 3-vertex.

u_i be the neighbour of v_i different from v for $i = 1, 2, 3, 4$, and let w_1, w_2 be the neighbours of v_5 different from v. Suppose that π is an acyclic list 5-colouring of $G - \{v, v_1, v_2, v_3, v_4\}$. If $\pi(v_5) \neq \pi(v_6)$, then there exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$ which appears at most once on the vertices u_1, u_2, u_3, u_4. Suppose that c appears on u_1 possibly. Colour v with c, v_1 with a colour different from the colours assigned to c, v_5, v_6, and v_i with a colour different from $c, \pi(u_i)$ for $i = 2, 3, 4$. If $\pi(v_5) = \pi(v_6)$, then if $\pi(w_1) \neq \pi(w_2)$, recolour v_5 with a colour different from the colours assigned to w_1, w_2, v_6, and then reduce the proof to the previous case. Suppose then that $\pi(w_1) = \pi(w_2)$. Again, there exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(w_1)\}$ which appears at most once on the vertices u_1, u_2, u_3, u_4. Without loss of generality, suppose $\pi(u_1) = c$. Colour v with c, v_1 with a colour different from the colours assigned to v, u_1, v_5, w_1, and v_i with a colour different from the colours assigned to v, u_i for $i = 2, 3, 4$. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. $\square$

Proof. ***(C6.3)*** Assume to the contrary that v_1, v_2, v_3, v_4 are 2-vertices and $[vv_5v_6]$ is a 3-face as in Figure 3.13.

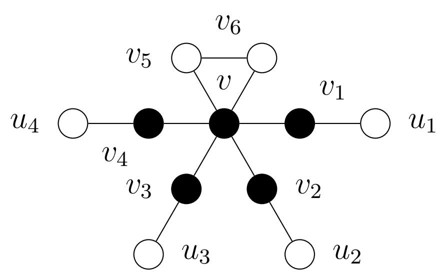

Figure 3.14: A 6-vertex adjacent to four 2-vertices and incident a 3-face.

Let π be an acyclic list 5-colouring of $G - \{v, v_1, v_2, v_3, v_4\}$. Obviously, $\pi(v_5) \neq \pi(v_6)$. There exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(v_6)\}$ appearing at most once on u_1, u_2, u_3, u_4. Suppose that c appears on u_1 possibly. Colour v with c, v_1 with a colour different from the colours assigned to v, u_1, v_5, v_6, and v_i with a colour different from the colours assigned to v, u_i for $i = 2, 3, 4$. The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. $\square$

Proof. ***(C6.5)*** Assume to the contrary that $[vv_1v_2]$ is an incident 3-face, v_3, v_4, v_5 are 2-vertices and v_6 is an pendant light 3-vertex of v as in Figure 3.15. Let u_1, u_2, u_3 be neighbours of v_1, v_2, v_3 different from v, let $[v_4vv_5]$ be a 3-face

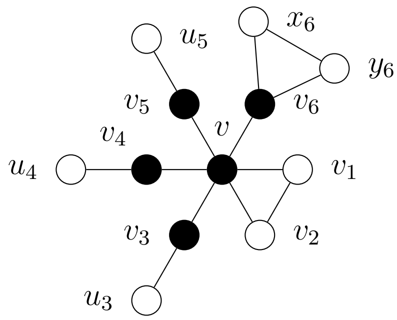

Figure 3.15: A 6-vertex with $n_2(v) = 3, t(v) = 1$ and $p_3(v) = 1$

and let x_6, y_6 be neighbours of v_6 different from v.

By minimality, $G - \{v, v_3, v_4, v_5, v_6\}$ has an acyclic 5-colouring π. Let $S = \{u_3, u_4, u_5, x_6, y_6\}$. Since $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$ and $|S| = 5$, there exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most once on the set S. First colour v with c and in order to colour the remaining uncoloured vertices, without loss of generality, consider the following two cases.

- If $\pi(u_3) = c$ then colour v_i with a colour different from $c, \pi(v_1), \pi(v_2), \pi(u_i)$ for $i = 3, 4, 5$, and v_6 with a colour different from $c, \pi(x_6), \pi(y_6)$.
- If $\pi(x_6) = c$, then colour v_i with a colour different from $c, \pi(u_i)$ for $i = 3, 4, 5$, and v_6 with a colour different from $c, \pi(v_1), \pi(v_2), \pi(y_6)$.

$\square$

Proof. ***(C6.5)*** Suppose to the contrary that v_1, v_2 are 2-vertices and v_3, v_4, v_5 are pendant light 3-vertices of v as in Figure 3.16.

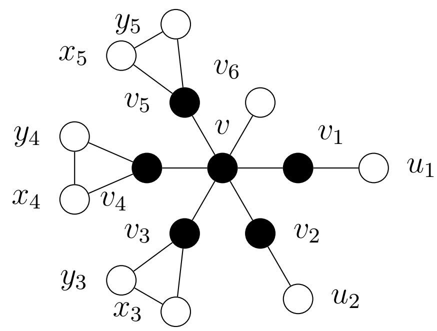

Figure 3.16: A 6-vertex adjacent to two 2-vertices and three pendant light 3-vertices.

By the minimality of $G, G-\{v, v_1, v_2, \ldots, v_5\}$ has an acyclic list 5-colouring π. It is obvious that $\pi(x_i) \neq \pi(y_i)$ for all $i = 3, 4, 5$. Let $S = \{u_1, u_2, x_3, y_3, x_4, y_4, x_5, y_5\}$. Since $|L(v) \setminus \{\pi(v_6)\}| \geq 4$ and $|S| = 8$, there exists a colour belonging to $L(v)\setminus\{\pi(v_6)\}$ appearing at most twice on the set S. First assume that there exists a colour $c \in L(v) \setminus \{\pi(v_6)\}$ which appears at most once on the set S. Colour v with c, v_i with a colour different from $c, \pi(v_6), \pi(u_i)$ for $i = 1, 2$, and v_j with a colour different from $c, \pi(v_6), \pi(x_j), \pi(y_j)$ for $j = 3, 4, 5$. Now assume, without loss of generality, that $L(v) = \{1, 2, 3, 4, 5\}$, $\pi(v_6) = 1$, and each colour belonging to $\{2, 3, 4, 5\}$ appears exactly twice on the set S. One can easily observe that there exist two vertices x and y, where $x, y \in S \setminus \{u_1, u_2\}$, such that $\pi(x) = \pi(y)$. Without loss of generality, assume that $\pi(x_3) = \pi(x4_4 = 2$. Colour v with $2, v_3$ with a colour $a \in L(v_3) \setminus \{1, 2, \pi(y_3)\}, v_4$ with a colour $b \in L(v_4) \setminus \{1, 2, a, \pi(y_4)\}$, v_i with a colour different from $2, \pi(u_i)$ for $i = 1, 2$, and finally colour v_6 with a colour different from $2, \pi(x_6), \pi(y_6)$. □

Proof. ***(C6.6)*** Assume to the contrary that $[vv_1v_2]$ is a incident 3-face, v_3, v_4 are 2-vertices and v_5, v_6 are pendant light 3-vertices of $v$as in Figure 3.17.

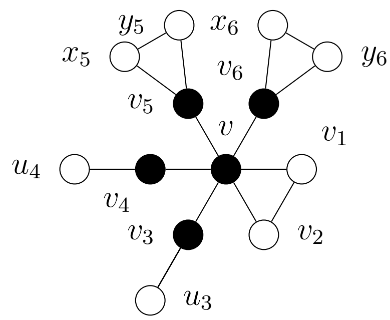

Figure 3.17: A 6-vertex adjacent to two 2-vertices, two pendant light 3-vertices and incident a 3-face.

By the minimality of $G, G - \{v, v_3, v_4, v_5, v_6\}$ admits an acyclic list 5-colouring π. Let $S = \{u_3, u_4, x_5, y_5, x_6,\ y_6\}$. It is easy to observe that $|L(v) \setminus \{\pi(v_1), \pi(v_2)\}| \geq 3$ and $|S| = 6$. Based on this fact, there exists a colour belonging to $L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most twice on the set S. First assume that there exists a colour $c \in L(v) \setminus \{\pi(v_1), \pi(v_2)\}$ appearing at most once on the set S. By symmetry, colour v with c. Then colour the remaining uncoloured vertices in the following way: If $\pi(u_3) = c$, colour v_3 with a colour different from $c, \pi(v_1), \pi(v_2)$, and then assign v_i with a colour different from the colours assigned to that of its neighbours for $i = 4, 5, 6$. If $\pi(x_5) = c$, colour v_5 with a colour different from $c, \pi(v_1), \pi(v_2), \pi(y_5)$, and then assign v_j with a colour different from the colours assigned to that of its neighbours for $j = 3, 4, 6$. Now, assume that $L(v) = \{1, 2, 3, 4, 5\}, \pi(v_1) = 1, \pi(v_2) = 2$ and each colour in $\{3, 4, 5\}$ appears exactly twice on the set S. If

$\pi(u_3) = \pi(u_4)$, say $\pi(u_3) = \pi(u_4) = 3$, then colour v with $3, v_3$ with a colour $a \in L(v_3) \setminus \{1, 2, 3\}, v_4$ with a colour $b \in L(v_4) \setminus \{1, 2, 3, a\}$, and finally colour v_i with a colour distinct from $3, \pi(x_i), \pi(y_i)$ for $i = 5, 6$. Otherwise, without loss of generality, suppose that $\pi(u_3) = \pi(x_5) = 3$. Then colour v with $3, v_5$ with $c \in L(v_5) \setminus \{1, 2, 3, \pi(y_5)\}, v_3$ with $d \in L(v_3) \setminus \{1, 2, 3, c\}$, and finally assign a proper colouring for v_4 and v_6 easily. □

Proof. ***(C6.7)*** Assume to the contrary that $[vv_1v_2]$ is a $(6, 3, 3)$-face, i.e. $d(v_1) = d(v_2) = 3$, and v_3, v_4, v_5 are pendant light 3-vertices of v as in Figure 3.18. Let $N(v_1) = \{u_1, v_2, v\}$ and $N(v_2) = \{u_2, v_1, v\}$. By the minimality

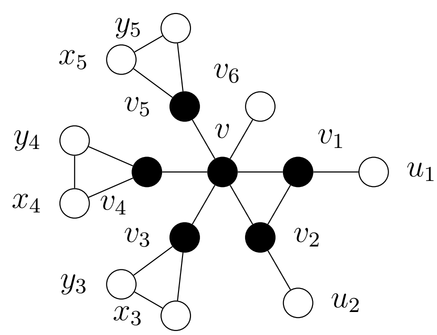

Figure 3.18: A 6-vertex adjacent to three pendant light 3-vertices and incident a $(3, 3, 6)$-face.

of $G, G - \{v, v_1, v_2, \ldots, v_5\}$ has an acyclic list 5-colouring π. Notice that $\pi(x_i) \neq \pi(y_i)$ for each $i \in \{3, 4, 5\}$. Let $S = \{u_1, u_2, x_3, y_3, x_4, y_4, x_5, y_5\}$. Since $|L(v) \setminus \{\pi(v_6)\}| \geq 4$ and $|S| = 8$, there exists a colour belonging to $L(v) \setminus \{\pi(v_6)\}$ appearing at most twice on the set S. If there exists a colour in $L(v) \setminus \{\pi(v_6)\}$ appearing at most once on S, the proof can also be given with a similar argument to the previous lemma **??**. Now assume, without loss of generality, that $L(v) = \{1, 2, 3, 4, 5\}, \pi(v_6) = 1$, and each colour in $\{2, 3, 4, 5\}$ appears exactly twice on the set S. It is easy to see that there exist two vertices $x, y \in \{x_3, y_3, x_4, y_4 x_5, y_5\}$ having the same colour, set $\pi(x_3) = \pi(x_4) = 2$. Colour v with $2, v_1$ with a colour a different from $2, \pi(u_1), \pi(u_2), v_2$ with a colour different from $a, 2, \pi(u_2), v_3$ with a colour $b \in L(v_3) \setminus \{1, 2, \pi(y_3)\}, v_4$ with a colour $c \in L(v_4) \setminus \{1, 2, b, \pi(y_4)\}$, and finally assign a proper colouring for v_5. $\square$

Proof. ***(C6.8)*** Suppose that $[vv_1v_2]$ is $(6, 3, 4)$-face such that $d(v_1) = 3$ and $d(v_2) = 4$. Let $N(v_1) = \{u_1, v_2, v\}$ and $N(v_2) = \{x_2, y_2, v_1, v\}$. Assume to the contrary that v_3 is a 2-vertex with a neighbour u_3 different from v and v_4, v_5, v_6 are pendant light 3-vertices of $v$as in Figure 3.19. By the

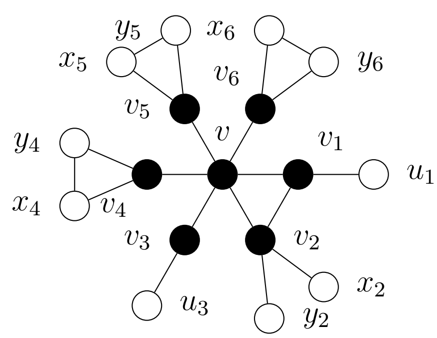

Figure 3.19: A 6-vertex adjacent to two 2-vertices and incident a $(3, 4, 6)$-face.

minimality of $G, G - \{v, v_1, v_3, v_4, v_5, v_6\}$ admits an acyclic list 5-colouring π. Let $S = \{u_1, u_3, x_4, y_4,\ x_5, y_5, x_6, y_6\}$. It is easy to see that there exists a colour belonging to $L(v) \setminus \{\pi(v_2)\}$ appearing at most twice on the set S, since

$|L(v) \setminus \{\pi(v_2)\}| \geq 4$ and $|S| = 8$. Consider the following two cases. First assume that there exists a colour $c \in L(v) \setminus \{\pi(v_2)\}$ which appears at most once on the set S. Colour v with c first, then colour v_3 with a colour different from $c, \pi(u_3), \pi(v_2)$, and v_i with a colour different from $c, \pi(v_2), \pi(x_i), \pi(y_i)$ for $i = 4, 5, 6$. Then colour v_1 in the following way: If $\pi(u_1) = \pi(v_2)$, then assign v_1 a colour in $L(v_1) \setminus \{c, \pi(v_2), \pi(x_2),$
$pi(y_2)\}$. Otherwise, assign a colour in $L(v_1) \setminus \{c, \pi(v_2), \pi(u_1)\}$ to v_1.

Now assume, without loss of generality., that $L(v) = \{1, 2, 3, 4, 5\}, \pi(v_2) = 1$, and each colour in $\{2, 3, 4, 5\}$ appears exactly twice on the set S. Then there exist two vertices x and y belonging to $\{x_4, y_4, x_5, y_5, x_6, y_6\}$ having the same colour. Without loss of generality., assume that $\pi(x_4) = \pi(x_5) = 2$. First colour v with $2, v_3$ with a colour different from $2, \pi(u_3), v_4$ with a colour $a \in L(v_4) \setminus \{1, 2, \pi(y_4)\}, v_5$ with a colour $b \in L(v_5) \setminus \{1, 2, a, \pi(y_5)\}, v_6$ with a colour different from $2, \pi(x_6), \pi(y_6)$, and finally colour v_1 in the following way: If $\pi(u_1) = \pi(v_2) = 1$, then assign v_1 with a colour in $L(v_1) \setminus \{1, 2, \pi(x_2), \pi(y_2)\}$. Otherwise, assign a colour in $L(v_1) \setminus \{1, 2, \pi(u_1)\}$ to v_1. □

3.7 Restrictions on a 7-vertex

Proof. ***(C7.1)*** Suppose that G contains a 7-vertex v adjacent to six 2-vertices $v_1, \ldots, v_6$ and one other vertex v_7 as in Figure 3.20

Let $G' = G - \{v, v_i\}$ for $1 \leq i \leq 6$. Then G' admits an acyclic list 5-colouring π by the minimality of G. There exists a colour $c \in L(v) \setminus \{\pi(v_7)\}$

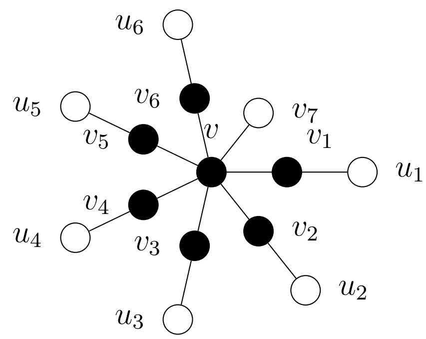

Figure 3.20: A 7-vertex adjacent to six 2-vertices

which appears at most once in u_i for $1 \leq i \leq 6$, so let $\pi(v) = c$. Assuming $\pi(u_1) = c$, let $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(u_i)\}$ for $2 \leq i \leq 6$ and let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_7)\}$. □

Proof. ***(C7.2)*** Assume to the contrary that the neighbours of v have degrees as follows: v_1, v_2, v_3, v_4 are 2-vertices and v_5, v_6, v_7 are 3-vertices. Let u_1, u_2, u_3, u_4 be, respectively, the neighbours of v_1, v_2, v_3, v_4 that are different from v. Let $x_5, y_5, x_6, y_6, x_7, y_7$ be, respectively, the neighbours of v_5, v_6, v_7 that are different from v as in Figure 3.21.

By the minimality of $G, G' = G - \{v, v_1, v_2, v_3, v_4\}$ has an acyclic list 5-colouring π. Let $\alpha = |\{\pi(v_5), \pi(v_6), \pi(v_7)\}|$ and let $S = \{u_1, u_2, u_3, u_4\}$. Consider the following possibilities:

1. $\alpha = 3$. If there exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(v_6), \pi(v_7)\}$ appearing at most once on S, say $\pi(u_1) = c$, then colour v with c, let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}$ and colour v_i with a colour

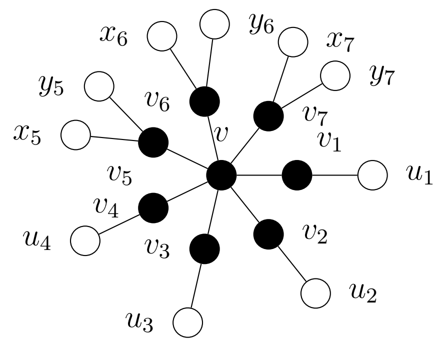

Figure 3.21: A 7-vertex adjacent to four 2-vertices and three 3-vertices.

different from c and $\pi(u_i)$ for $i = 2, 3, 4$.

Otherwise, colours c and c' each appear twice in the colouring of S, so assume that $\pi(v_1) = \pi(v_2) = c$. If $L(v_1) \neq L(v_2)$, then let $\pi(v) = c$, let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}$ and let $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, c_1, \pi(v_5), \pi(v_6), \pi(v_7)\}$. Finally colour v_3 and v_4 differently from there respective neighbours. If $L(v_1) = L(v_2)$ and a proper acyclic list 5-colouring G' cannot be extended to G when $\pi(v) = c$, then $|L(v_1) - \{c, \pi(v_5), \pi(v_6), \pi(v_7)\}| = 1$ and there exists a $(c, \pi(v_j))$-path from u_i to v for $i = 1, 2$ and $j = 5, 6, 7$. Then colour v with c', v_1 and v_2 with a colour different from c and c',then let $\pi(v_3) = c_3 \in L(v_3) \setminus \{c', \pi(v_5), \pi(v_6), \pi(v_7)\}$ and $\pi(v_4) = c_4 \in L(v_4) \setminus \{c', c_3, \pi(v_5), \pi(v_6), \pi(v_7)\}$. If there is no such c_4, then colour v_4 with one of $\pi(v_5), \pi(v_6), \pi(v_7)$ as there does not exist a $(c', \pi(v_j))$-path from u_4 to v for some $j = 5, 6, 7$ because G is planar.

2. $\alpha = 2$. Without loss of generality, assume that $\pi(v_5) = \pi(v_6)$. If

$\pi(x_5) \neq \pi(y_5)$ or $\pi(x_6) \neq \pi(y_6)$, recolour v_5 or v_6 to reduce to the previous case (1). Thus, suppose $\pi(x_5) = \pi(y_5)$ and $\pi(x_6) = \pi(y_6)$. There exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(v_7), \pi(x_5)\}$ appearing at most twice on S, say $\pi(u_1) = \pi(u_2) = c$. Colour v with c, v_1 with a colour c_1 different from $\{c, \pi(v_5), \pi(v_7)\}, v_2$ with a colour different from $\{c, c_1, \pi(v_5), \pi(v_7)\}$, and give a proper colouring for v_3 and v_4.

3. $\alpha = 1$. If there exists $j \in \{5, 6, 7\}$ such that $\pi(x_j) \neq \pi(y_j)$, then recolour v_j to reduce to the former case (2).
   Otherwise, $\pi(x_j) = \pi(y_j)$ for all $j \in \{5, 6, 7\}$. There exists a colour $c \in L(v) \setminus \{\pi(v_5), \pi(x_5), \pi(x_6)\}$ appearing at most twice on S, say $\pi(u_1) = \pi(u_2) = c$. Colour v with c, v_1 with a colour in $L(v_1) \setminus \{\pi(v_5), c\}, v_2$ with a colour different from $\{\pi(v_5), c, \pi(v_1)\}$, then properly colour v_3 and v_4.

The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

Proof. ***(C7.3)*** Let v be a 7-vertex with neighbours $v_1, \ldots, v_7$ such that $v_1, \ldots, v_5$ are 2-vertices having neighbours $u_1, \ldots, u_5$ different from v as in Figure 3.22. By the minimality of $G, G' = G - \{v, v_1, v_2, v_3, v_4, v_5\}$ has an acyclic list 5-colouring π. Consider the following two cases:

1. Assume for contradiction that $f = [vv_6v_7]$ is a 3-face in G. Then there exists a colour $c \in L(v) \setminus \{\pi(v_6), \pi(v_7)\}$ which appears at most once in the colouring of $u_1, \ldots, u_5$, say $\pi(u_1) = c$. Let $\pi(v) = c$, let

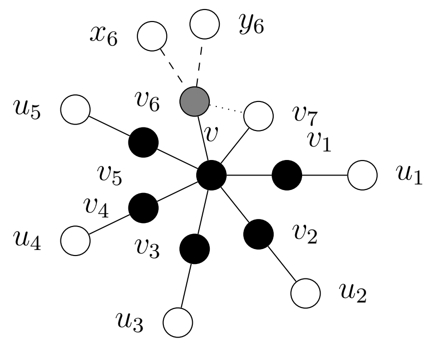

Figure 3.22: A 7-vertex adjacent to five 2-vertices and a 3-vertex or incident a 3-face (v_6 is gray as it is either a 3-vertex or incident to a 3-face).

$\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_6), \pi(v_7)\}$ and give a proper colouring to v_i which is different from c and $\pi(u_i)$ for $i = 2, 3, 4, 5$.

2. Assume for contradiction that $d(v_6) = 3$. If $\pi(v_6) \neq \pi(v_7)$, then the colouring of $v, v_1, v_2, v_3, v_4, v_5$ is as in case (1), so assume $\pi(v_6) = \pi(v_7)$. If the neighbours, x_6, y_6 of $\pi(v_6)$ different from v are not assigned the same colour; i.e., $\pi(x_6) \neq \pi(y_6)$, then recolour $\pi(v_6)$ with a colour in $L(v_6)$ which is different from $\pi(v_6), \pi(x_6), \pi(y_6)$ and colour $v, v_1, v_2, v_3, v_4, v_5$ as in case (1). Assume then that $\pi(x_6) = \pi(y_6)$. There exists a colour $c \in L(v) \setminus \{\pi(v_6), \pi(x_6)\}$ which appears at most once in the colouring of $u_1, \ldots, u_5$, say $\pi(u_1) = c$. Let $\pi(v) = c$, let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_6)\}$ and give a proper colouring to v_i which is different from c and $\pi(u_i)$ for $i = 2, 3, 4, 5$.

The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. $\square$

Proof. ***(C7.4)*** Suppose to the contrary that $v_1, \ldots, v_4$ are 2-vertices, $[vv_5v_6]$ is a 3-face, and v_7 is a pendant light 3-vertex of v as in Figure 3.23. By the minimality of G, $G' = G - \{v, v_1, v_2, v_3, v_4, v_7\}$ admits an acyclic list 5-colouring π. Let $S = \{u_1, u_2, u_3, u_4, x_7, y_7\}$, the neighbours of v_1, v_2, v_3, v_4, v_7 respectively.

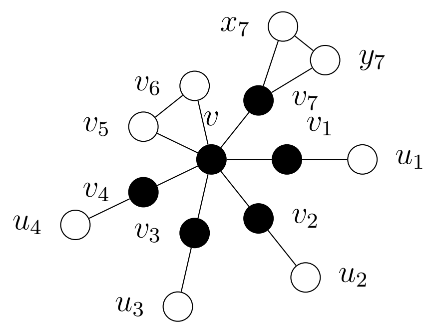

Figure 3.23: A 7-vertex adjacent to four 2-vertices a pendant light 3-vertex and a 3-face.

Obviously, $|L(v) \setminus \{\pi(v_5), \pi(v_6)\}| \geq 3$ and $|S| = 6$. This fact implies that there exists a colour c belonging to $L(v) \setminus \{\pi(v_5), \pi(v_6)\}$ appearing at most twice on the set S. If c appears at most once on the set S, then let $\pi(v) = c$, colour v_i different from $c, \pi(v_5), \pi(v_6)$ (and, without loss of generality, $\pi(y_7)$ if $\pi(x_7) = c$), and colour the remaining vertices the set $\{v_1, v_2, v_3, v_4, v_7\}$ which have not yet been coloured. If c appears exactly twice in S, say $\pi(v_i) = \pi(v_j) = c$ for $(i \neq j) \in \{1, 2, 3, 4, 7\}$, let $\pi(v) = c$, let $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(v_5), \pi(v_6)$ (and, without loss of generality, $\pi(y_7)$ if $\pi(x_7) = c$), let $\pi(v_j) = c_j \in L(v_j) \setminus \{c, c_i, \pi(v_5), \pi(v_6)$, and colour the remaining vertices the set $\{v_1, v_2, v_3, v_4, v_7\}$ which have not yet been coloured.

The resulting colouring is an acyclic list 5-colouring of G. This contradicts the choice of G. □

Proof. ***(C7.5)*** Suppose to the contrary that $[vv_1v_2]$ is a $(7,3,3)$-face such that $d(v_1) = d(v_2) = 3$, v_3, v_4, v_5 are 2-vertices and v_6, v_7 are pendant light 3-vertices of v. By the minimality of $G, G' = G - \{v, v_1, \ldots, v_7\}$ admits an acyclic list 5-colouring π. Let $N(v_1) = (u_1, v, v_2)$ and $N(v_2) = (u_2, v, v_1)$. Let u_3, u_4, u_5 be, respectively, the neighbours of v_3, v_4, v_5. Let $S = \{u_1, u_2, u_3, u_4, u_5, x_6, y_6, x_7, y_7\}$ where x_6, y_6, x_7, y_7 are the neighbours of v_6 and v_7 different from v as in Figure 3.24.

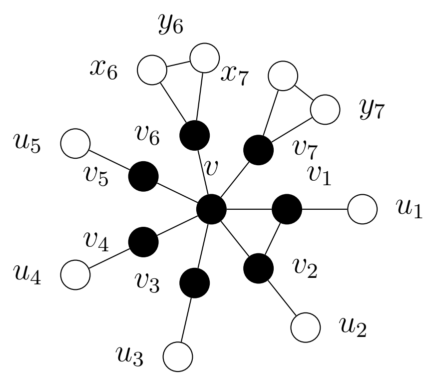

Figure 3.24: A 7-vertex adjacent to three 2-vertices, two pendant light 3-vertex and a $(7,3,3)$-face.

Since $|S| = 9$ and $|L(v)| = 5$, then there exists a colour $c \in L(v)$ which appears at most once on the set S. The colouring of G' can be extended in the following way: colour v with c, v_1 with a colour $c_1 \in L(v_1)$ different from $c, \pi(u_1)$ and $\pi(u_2)$, v_2 with a colour $c_2 \in L(v_2)$ different from $c, c_1, \pi(u_1)$

and $\pi(u_2)$, vi with a colour different from c and $\pi(u_i)$ for $i = 3, 4, 5$, and vj with a colour different from the colours assigned to c, x_j and y_j for $j = 6, 7$. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

3.8 Restrictions on a 8-vertex

Proof. ***(C8.1)*** Suppose that G contains an 8-vertex v adjacent to seven 2-vertices

$v_1, v_2, v_3, v_4, v_5, v_6, v_7$ and one other vertex v_8 as in Figure 3.25.

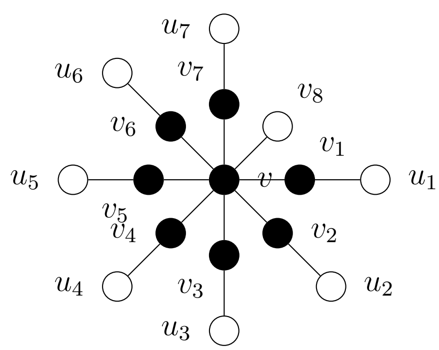

Figure 3.25: An 8-vertex adjacent to seven 2-vertices

Let $G' = G - \{v, v_i\}$ for $1 \leq i \leq 7$. Then G' admits an acyclic list 5-colouring π by the minimality of G. There exists a colour $c \in L(v) \setminus \{\pi(v_8)\}$ which appears at most once in u_i for $1 \leq i \leq 7$, so let $\pi(v) = c$. Assuming $\pi(u_1) = c$, let $\pi(v_i) = c_i \in L(v_i) \setminus \{c, \pi(u_i)\}$ for $2 \leq i \leq 7$ and let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(v_8)\}$. The resulting colouring is an acyclic $L-$colouring of

G. This contradicts the choice of G. □

Proof. ***(C8.2)*** Assume to the contrary that $v_1, v_2, v_3, v_4, v_5, v_6$ are 2-vertices with neighbours $u_1, u_2, u_3, u_4, u_5, u_6$ different from v and let $[vv_7v_8]$ be a 3-face as in Figure 3.26. Note that by (C9.1), no 2-vertex is contained in a 3-face.

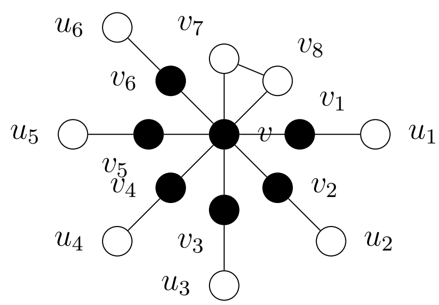

Figure 3.26: An 8-vertex adjacent to six 2-vertices and incident to a 3-face

Let π be an acyclic list 5-colouring of $G - \{v, v_1, \ldots, v_6\}$. Obviously, $\pi(v_7) \neq \pi(v_8)$. Let $S = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. Then there exists a colour $c \in L(v) \setminus \{\pi(v_7), \pi(v_8)\}$ appearing at most twice on the set S, say $\pi(u_1) = \pi(u_2) = c$. Then colour v with c, colour v_1 with a colour c_1 different from $\{c, \pi(v_7), \pi(v_8)\}$ and colour v_2 with a colour c_2 different from $\{c, c_1, \pi(v_7), \pi(v_8)\}$. Finally colour v_i with a colour different from $\{c, \pi(u_i)\}$ for $i = 3, 4, 5, 6$. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

3.9 Restrictions on a 3-face

Proof. ***(C9.1)*** Let $f = [xyz]$ be a 3-face and assume for contradiction that $d(x) = 2$ as in Figure 3.27. By the minimality of G, the graph $G - x$ admits

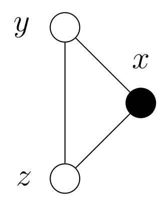

Figure 3.27: A 3-face $[xyz]$ with $d(x) = 2$.

an acyclic L-colouring π. Since $\pi(y) \neq \pi(z)$ in $G - x$, extend this acyclic list 5-colouring letting $\pi(x) = c \in L(x) \setminus \pi(y), \pi(z)$. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

Proof. ***(C9.2)*** Suppose that G contains a 3-face $[xyz]$ with $d(x) = d(y) = 3$ and $d(z) = 5$ as in Figure 3.28. Notice that be Lemma **??**, $d(z) \geq 5$

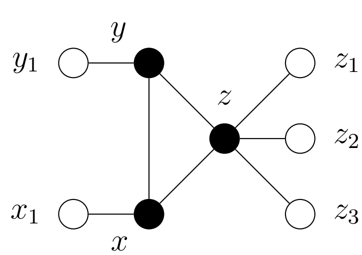

Figure 3.28: A 3-face $[xyz]$ with $d(x) = d(y) = 3$ and $d(z) = 5$.

Let $x_1, y_1,$ and $z_1, z_2, z_3,$ be, respectively, the neighbours of $x, y,$ and z that are not on the face $[xyz]$. Let π be an acyclic list 5-colouring of

$G - \{x, y\}$ and consider the extension of this colouring in the following four cases. If the colours of x_1, y_1, z are pairwise distinct, colour x with a colour $c_x \in L(x) \setminus \{\pi(x_1), \pi(z)\}$ and y with a colour $c_y \in L(y) \setminus \{c_x, \pi(y_1), \pi(z)\}$. By symmetry, consider if $\pi(x_1) = \pi(z)$ and $\pi(y_1) \neq \pi(z)$, colour x with $c_x \in L(x)\setminus\{\pi(z), \pi(z_1), \pi(z_2), \pi(z_3)\}$ and y with a colour $c_y \in L(y)\setminus\{c, \pi(y_1), \pi(z)\}$. If $\pi(x_1) = \pi(y_1)$ and $\pi(z) \neq \pi(x_1)$, colour x with $c_x \in L(x) \setminus \{\pi(x_1), \pi(z)\}$ and y with a colour $c_y \in L(y) \setminus \{c_x, \pi(y_1), \pi(z)\}$. Finally suppose that $\pi(x_1) = \pi(y_1) = \pi(z)$. If z_1, z_2, z_3 have pairwise distinct colours, recolour z with a colour different from the colours the colours assigned to z, z_1, z_2, z_3 and then reduce to the previous case. If at least two of z_1, z_2, z_3 have the same colour, say $\pi(z_1) = \pi(z_2)$, colour x with a colour different from the colours assigned to z, z_2, z_3 , and y with a colour different from the colours assigned to $x, z, z_2, , z_3$. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

Proof. ***(C9.3)*** Suppose that G contains a 3-face $[xyz]$ with $d(x) = 3$ and $d(y) = d(z) = 4$ as in Figure 3.29. Let x_1, y_1, y_2, and z_1, z_2 be, respectively, the neighbours of x, y, and z that are not on the face $[xyz]$. Let π be an acyclic list 5-colouring of $G - x$.

If x_1, y, z have pairwise distinct colours, colour x properly. Otherwise, suppose that $\pi(y) = \pi(x_1) \neq \pi(z)$. In this case, colour x with a colour different from the colours assigned to y, z, y_1, y_2. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G. □

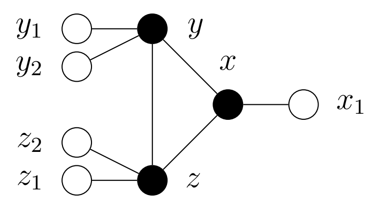

Figure 3.29: A 3-face $[xyz]$ with $d(x) = 3$ and $d(y) = d(z) = 4$.

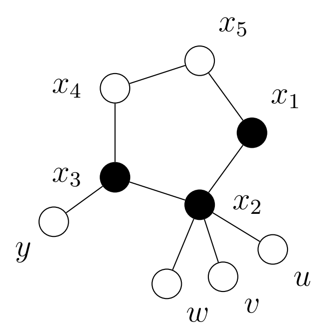

Figure 3.30: A 5-face $[x_1, \ldots, x_5]$ with $d(x_1) = 2, d(x_2) = 5$ and $d(x_3) = 3$.

3.10 Restrictions on a 5-face

Suppose to the contrary that G contains a 5-face f with boundary vertices $x_1, \ldots, x_5$ such that $d(x_1) = 2, d(x_2) = 5$ and $d(x_3) = 3$. Let u, v, w be the neighbours of x_2 different from x_1 and x_3. Let y be the neighbour of x_3 different from x_2 and x_4 as in Figure 3.30.

By the minimality of G, the graph $G - x_1$ admits an acyclic L-colouring π. To extend the colouring of $G - x_1$ to G, consider the following. If $\pi(x_2) \neq \pi(x_5)$, let x_1 be coloured with $c \in L(x_1) \setminus \{\pi(x_2), \pi(x_5)\}$. If $\pi(x_2) = \pi(x_5)$, then let x_1 be coloured with $c \in L(x_1) \setminus \{\pi(x_2), \pi(x_3), \pi(u), \pi(v), \pi(w)\}$. If there is no such colour c and if x_1 cannot be acyclically coloured with $\pi(x_3)$, $L(x_1) = \{\pi(x_2), \pi(x_3), \pi(u), \pi(v), \pi(w)\}$ and $\pi(y) = \pi(x_2)$. If $L(x_2) \neq L(x_1)$, recolour x_2 with a colour in $L(x_2) \setminus L(x_1)$ and give a proper colouring to x_1. If $L(x_2) = L(x_1)$ consider that $\pi(x_4) \neq \pi(x_2)$ and $\pi(x_3)$. Recolour x_2 with $\pi(x_3)$, let x_3 be recoloured with a colour different than those assigned to y, x_4, x_2, and give a proper colouring to x_1. The resulting colouring is an acyclic $L-$colouring of G. This contradicts the choice of G.

Chapter 4

Proof of Theorem

4.1 New Reducible Configurations

The list of reducible figures provided in previous work is not sufficient to ensure that a proof using the discharging method would lead to a contradiction. A closer look at the possible structure of a minimum counterexample G need exploration. Many shortcomings during the discharging portion of this proof came about when

- a 6-vertex had many neighbours of degree 2;
- a vertex had degree 9, 10 and 11.

The following proofs reveal more about the structure G in the same manor as in the previous chapter.

4.1.1 Weak 6-vertex

Definition 1. *We define a weak 6-vertex as a vertex of degree 6 with the following properties:*

1. *v is adjacent to exactly four 2-vertices*
2. *v is incident to either*
   - *three 4-faces or*
   - *two 4-faces containing all four 2-vertices in $N(v)$, and one 5-face.*

Since, by assumption, a 4-face is not adjacent to a 4- or 5-face, and by condition (C6.3), v is not incident to a 3-face, then the number 4- and 5-faces incident to v is less than or equal to 3. Figures 4.1 and 4.2 show the three possible configurations of a weak 6-vertex.

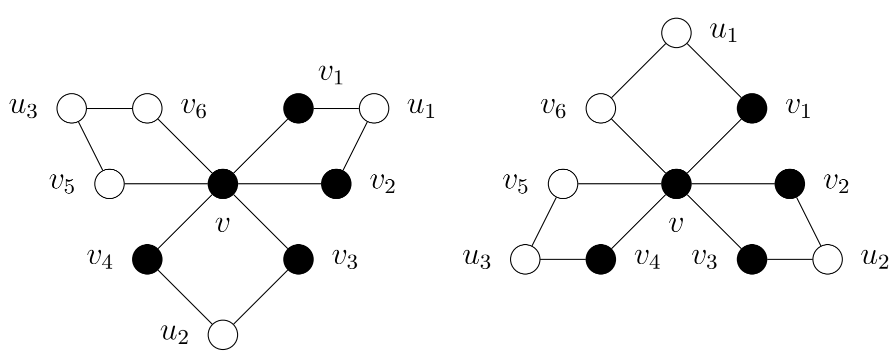

Figure 4.1: A weak 6-vertex incident to three 4-faces.

As a consequence of this definition, a weak 4- or 5-face is a 4- or 5-face which is incident to a weak 6-vertex. Let $w_6(f)$ denotes the number of weak

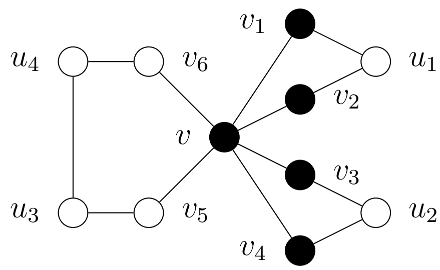

Figure 4.2: A weak 6-vertex incident to a 5-face.

6-vertices incident to f.

The following lemma will prove the non-adjacency of two weak 6-vertices.

Lemma 2. *Every weak 4-face [wxyz]; where w is a weak 6-vertex, satisfies one of the following:*

(A1.1) x is not a weak 6-vertex.

(A1.2) If $d(x) = 2$ and $d(z) \neq 2$, then y is not a weak 6-vertex.

Proof(A1.1) Assume otherwise and let x be a weak 6-vertex as in Figure 4.3. Let $w_1 \dots w_4$ be the neighbours of w different from x and z. Let $d(w_1) = d(w_2) = d(w_3) = 2$. Since w is weak, then either $d(w_4) = 2$ or $d(z) = 2$. With a similar labelling, let $x_1 \dots x_4$ be the neighbours of x different form w and y, and let $d(x_1) = d(x_2) = d(x_3) = 2$. Again, since x is weak, then either $d(x_4) = 2$ or $d(y) = 2$. By (C2), it is not the case that $d(y) = d(z) = 2$. Let $G' = G - \{w, w_1, w_2, w_3, x, x_1, x_2, x_3\}$. Then G' admits an L-colouring π by the minimality of G. Let S denote the set of colours which π has assigned to $\{y, z, w_4, u_1, u_2\}$. If there exists a colour $c \in L(w) \setminus S$ then colour w with c. Otherwise, consider the following two cases.

Assume first that $d(w_4) = 2$. Colour w with $\pi(w_4)$, recolour w_4 different from w and u_2, and colour each of w_1, w_2, w_3, differently than its two neighbours. Now, if $d(z) = 2$, then let $\pi(w) = \pi(z)$, let z be recoloured with a colour different from $\pi(w)$ and $\pi(y)$ and colour each of w_1, w_2, w_3 differently than its two neighbours. All that remains is to colour x, x_1, x_2, x_3.

Let T denote the set of colours which have been assigned to $\{x_4, y, v_1, v_2, \pi(w)\}$. If there exists a colour $c' \in L(x) \setminus T$, then colour x with c' and colour each of x_1, x_2, x_3 differently from their neighbours. Otherwise, consider the following two cases. First assuming that $d(x_4) = 2$. Colour x with $\pi(x_4)$ and give a proper colouring to x_1, x_2, x_3.

Assuming now that $d(y) = 2$, let $\pi(x) = \pi(y)$, let y be recoloured with a colour different from $\pi(x)$ and $\pi(z)$ and colour each of x_1, x_2, x_3 differently than its two neighbours. Since the colouring of G' has been extended to G, then this contradicts our choice of G.

($A1.2$) Let $d(x) = 2$, $d(z) > 2$ and assume that both w and y are weak 6-vertices, as in Figure 4.4. Let $w_1, \ldots, w_4$ be the neighbours of w different from x and z. Let $d(w_1) = d(w_2) = d(w_3) = 2$. With a similar labelling,

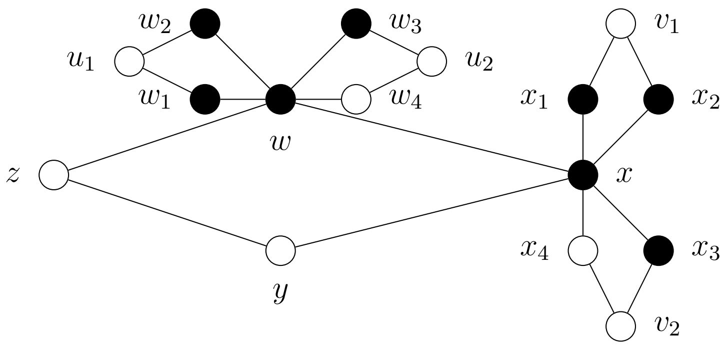

Figure 4.3: A weak 4-face incident with adjacent weak 6-vertices.

let $y_1, \ldots, y_4$ be the neighbours of y different form x and z, and let $d(y_1) = d(y_2) = d(y_3) = 2$.

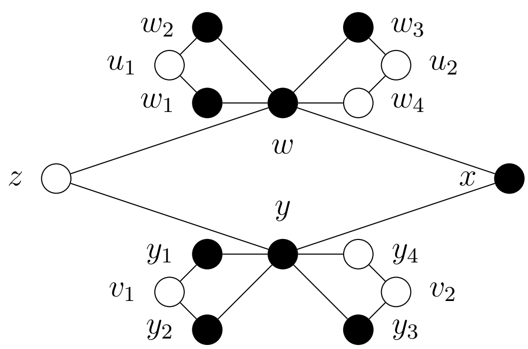

Figure 4.4: A weak 4-face incident with non adjacent weak 6-vertices.

Let $G' = G - \{w, w_1, w_2, w_3, x, y, y_1, y_2, y_3\}$. Then G' admits an L-colouring π by the minimality of G. Let S denote the set of colours which π has assigned to $\{z, w_4, u_1, u_2\}$. Colour w with $c \in L(w) \setminus S$. Let T denote the set of colours which π has assigned to $\{z, y_4, v_1, v_2\}$.

Colour y with $c' \in L(y) \setminus T$. Let $\pi(x) = a \in L(x) \setminus \{\pi(z), c, c'\}$. Now let $w_1, w_2, w_3, y_1, y_2, y_3$ each be coloured with a colour different from its neighbours. Since the colouring of G' has been extended to G, then this contradicts our choice of G.

□

Lemma 3. *Let f be a weak 5-face. Then f is not incident to two adjacent weak 6-vertices.*

Proof. Assume the contrary, that a 5-face $[vwxyz]$ has two adjacent weak 6-vertices v and w as in Figure 4.5. Let $v_1, \ldots, v_4$ be the neighbours of v different from w and z. Let $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$. With a similar labelling, let $w_1, \ldots, w_4$ be the neighbours of w different form v and x, and let $d(w_1) = d(w_2) = d(w_3) = d(w_4) = 2$.

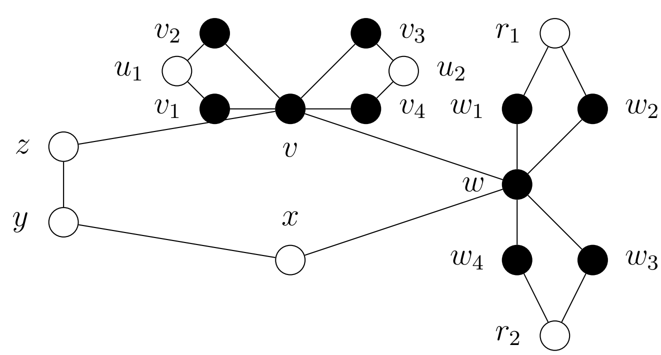

Figure 4.5: A weak 5-face incident with adjacent weak 6-vertices.

Let $G' = G - \{v, v_1, v_2, v_3, v_4, w, w_1, w_2, w_3, w_4\}$. Then G' admits an acyclic L-colouring π by the minimality of G. Let S denote the set of colours which

π has assigned to $\{z, x, u_1, u_2\}$. Colour v with $c \in L(v) \setminus S$. Let T denote the set of colours which π has assigned to $\{x, r_1, r_2\}$ and c. Colour w with $c' \in L(w) \setminus T$. There is a proper colouring of $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$ by colouring each differently from its two neighbours. Since the colouring of G' has been extended to G, then this contradicts our choice of G. □

4.1.2 Restrictions On A 6-vertex

The following lemmas are new and explore some necessary conditions in which a 6-vertex may appear in G.

Lemma 4. *Let v be a vertex of degree 6.*

(B1.1) If $n_2(v) = 3$ and incident to a $(3, 5^+, 6)$-face, and two 4-faces, then $n_3(v) = 1$.

(B1.2) If $n_2(v) = 3$ and incident to a $(3, 5^+, 6)$-face, and two 4-faces, then $w_6(v) = 0$.

(B2) If $n_2(v) = 2$ and v is incident to a $(6, 3, 4^-)$-face, then $t(v) = 1$.

Proof. $(B1.1)$ Suppose that v is a 6-vertex v with neighbours $v_1, \ldots, v_6$ where $d(v_1) = d(v_2) = d(v_3) = 2$ and $d(v_5) = 3$. Suppose to the contrary that $d(v_4) = 3$. Let $[vv_1wv_2]$ and $[vv_3xv_4]$ be two 4-faces and, by $(C9.1)$, let $[v_5v_6v]$ be a $(3, 5^+, 6)$-face as in Figure 4.6. Let y be the neighbour of v_4 different from v and x. Let z be the neighbour of v_5 different from v and v_6.

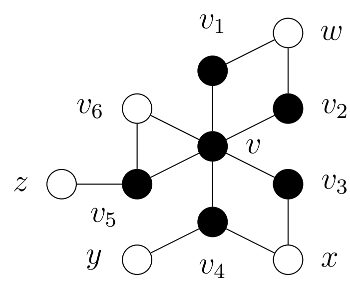

Figure 4.6: A degree 6-vertex v with $n_2(v) = 3$, $n_3(v) = 2$ and incident to a $(3, 5^+, 6)$-face and two 4-faces.

Consider the graph $G' = G - \{v_1, v_2, v_3\}$. By the minimality of G, G' admits an acyclic L-colouring π. Let $\alpha = |\{\pi(v_4), \pi(v_5), \pi(v_6)\}|$ and consider the following two cases:

$\alpha = 3$ If there exists a colour $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$ which does not appear in $\pi(w)$ and $\pi(x)$, then recolour v with c and properly colour v_1, v_2 and v_3 with an admissible colour different from their two neighbours (note that if $\pi(v)$ does not appear in $\pi(w)$ and $\pi(x)$, then no recolouring of v occurs). Otherwise each colour in $L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$ appears exactly once in $\pi(w)$ and $\pi(x)$. Recolour v with $\pi(x)$ (recolouring will not occur if $\pi(v) = \pi(x)$), properly colour v_1 and v_2 with an admissible colour different from $\pi(w)$ and $\pi(v)$ and colour v_3 with an admissible colour different from $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$.

$\alpha = 2$ If $\pi(v)$ appears at most once in $\pi(w)$ and $\pi(x)$, then if $\pi(x)$=$\pi(v)$

colour v_1, v_2 different from $\pi(v)$, $\pi(w)$ and colour v_3 different from $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$. Otherwise, $\pi(w){=}\pi(v)$. Colour v_3 different from $\pi(v), \pi(x)$, colour v_1 different from $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6)$ and v_2 different from $\pi(v), \pi(v_4), \pi(v_5), \pi(v_6), \pi(v_1)$.

Since $\pi(v_4)$ is coloured the same as one of $\pi(v_5)$ or $\pi(v_6)$, then if $\pi(v) = \pi(w) = \pi(x)$ recolour v with a colour $c \in L(v) \backslash \{\pi(v), \pi(v_4), \pi(v_5), \pi(v_6), \pi(y)\}$ and give a proper colouring to v_1, v_2 and v_3 with colours different from their two neighbours.

Since the colouring of G' has been extended to G, then this contradicts our choice of G.

$(B1.2)$ Suppose that v is a 6-vertex v with neighbours $v_1, \ldots, v_6$ where $d(v_1) = d(v_2) = d(v_3) = 2$ and $d(v_5) = 3$. Suppose to the contrary that v_4 is a weak 6-vertex. Let $[vv_1wv_2]$ and $[vv_3xv_4]$ be two 4-faces. Since a weak 6-vertex is not incident to a 3-face and by $(C9.1)$, let $[v_5v_6v]$ be a $(3, 5^+, 6)$-face as in Figure 4.7. Let $u_1, \ldots, u_4$ be the neighbours of v_4 different from v and x such that all are 2-vertices. Let z be the neighbour of v_5 different from v and v_6.

Consider the graph $G' = G - \{v, v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$. By the minimality of G, G' admits an acyclic L-colouring π. Let $\pi(v) = c \in L(v) \setminus \{\pi(v_5), \pi(v_6), \pi(w), \pi(x)\}$, let $v_4 = c_4 \in L(v_4) \setminus \{c, \pi(x), \pi(y_1), \pi(y_2)\}$. Since both $\pi(w)$ and $\pi(x) \neq \pi(v)$ and both $\pi(y_1)$ and $\pi(y_2) \neq \pi(v_4)$, colour each of $v_1, \ldots, v_3, u_1, \ldots, u_4$ with an admissible colour which is

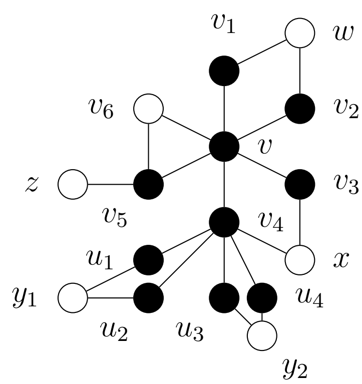

Figure 4.7: A degree 6-vertex v with $n_2(v) = 3$, $w_6(v) = 1$ and incident to a $(3, 5^+, 6)$-face and two 4-faces.

different from that of its two neighbours. Since the colouring of G' has been extended to G, then this contradicts our choice of G.

$(B2)$ Suppose to the contrary that v is adjacent to two vertices with degree 2, say v_1 and v_2, a $(6, 3, 4)$-face $[vv_3v_4]$ and, by $(C9.1)$, a second 3-face $[vv_5v_6]$. Let u_i be the neighbour of v_i different from v for $i = \{1, 2\}$ and u_3 be the neighbour of v_3 different from v and v_4. Let x (and y) be the neighbours of v_4 different from v and v_3 as shown in Figure 4.8.

Consider the graph $G' = G - \{v, v_1, v_2, v_3\}$. By the minimality of G, G' admits an acyclic L-colouring π. Let $\alpha = |\{\pi(v_4), \pi(v_5), \pi(v_6)\}|$ and consider the following two cases:

- If $\alpha = 3$ then there exits a colour $c \in L(v) \setminus \{\pi(v_4), \pi(v_5), \pi(v_6)\}$ which appears at most once in the colouring of u_1, u_2 and u_3.

  ◇ If $\pi(u_1) = c$, then let $\pi(v) = c$, let $\pi(v_1)$ be coloured with $c_1 \in$

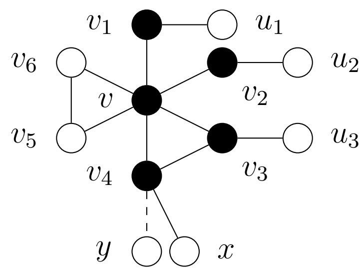

Figure 4.8: A degree 6-vertex v with $n_2(v) = 2$ and $t(v) = 2$.

$L(v_1) \setminus \{c, \pi(v_4), \pi(v_5), \pi(v_6)\}$, let $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, \pi(u_2)$ and let $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_4), \pi(u_3)$.

- ⋄ If $\pi(u_2) = c$, then with a similar argument to $\pi(u_1) = c$, let $\pi(v) = c$, let $\pi(v_2)$ be coloured with $c_2 \in L(v_2) \setminus \{c, \pi(v_4), \pi(v_5), \pi(v_6)\}$, let $\pi(v_1) = c_1 \in L(v_1) \setminus \{c, \pi(u_1)$ and let $\pi(v_3) = c_3 \in L(v_3) \setminus \{c, \pi(v_4), \pi(u_3)$.
- ⋄ Otherwise $\pi(u_3) = c$. Let $\pi(v) = c$, let $\pi(v_i)$ be a colour different from c and $\pi(u_i)$ for $i = \{1, 2\}$, and let $\pi(v_3)$ be a colour different from $c, \pi(v_4), \pi(v_5), \pi(v_6)$.

- If $\alpha = 2$, then consider the following.
  - ⋄ If $\pi(x) \neq \pi(y)$ then recolour v_4 with a colour different from $\pi(x)$, $\pi(y), \pi(v_5), \pi(v_6)$, then colour v, v_1, v_2 and v_3 as above, since $\pi(v_4)$, $\pi(v_5)$ and $\pi(v_6)$ are now pairwise distinct.
  - ⋄ Otherwise $\pi(x) = \pi(y)$. Since $|\{\pi(v_4), \pi(v_5), \pi(v_6), \pi(x), \pi(y)\}| \leq 3$, then there exits a colour $c \in L(v)\setminus\{\pi(v_4), \pi(v_5), \pi(v_6), \pi(x), \pi(y)\}$

which appears at most once in the colouring of u_1, u_2 and u_3. With a similar argument to the case when $\alpha = 3$, let $\pi(v) = c$, colour two of v_1, v_2 and v_3 with a colour different from it neighbours if v is the only neighbour coloured with c, and colour one of v_1, v_2 and v_3 with a colour different from $c, \pi(v_4), \pi(v_5), \pi(v_6)$ if two of its neighbours are coloured with c. This contradicts our chose of G, as we can obtain an acyclic colouring of G.

□

4.1.3 Restrictions On A 9-, 10-, 11-vertex

Lemma 5. *Let v be a 9-, 10- or 11-vertex. Then $n_2(v) \leq d(v) - 2$.*

Proof. Let v be a 9- 10- or 11-vertex and let $N(v) = \{v_1, \dots v_k\}$ for $9 \leq k \leq 11$ be the neighbours of v. Assume for contradiction that $n_2(v) = d(v) - 1$, with $d(v_2) = \dots = d(v_k) = 2$. Let $u_1, \dots, u_k$ be the neighbour of $v_1, \dots v_k$ different than v, as in Figure 4.9.

Let $G' = G - \{v, v_2, \dots, v_k\}$. Then G' admits an acyclic L-colouring π by the minimality of G. There exists a colour $c \in L(v) \setminus \{\pi(v_1)\}$ which appears at most twice in $u_2, \dots, u_k$. Let $\pi(v) = c$ and assume without loss of generality that $\pi(u_2) = \pi(u_3) = c$. Let $\pi(v) = c$, let $\pi(v_2) = c_2 \in L(v_2) \setminus \{c, \pi(v_1)\}, \pi(v_3) = c_3 \in L(v_3) \setminus \{c, c_2, \pi(v_1)\}$ and colour each of $v_4, \dots v_k$ with a colour different $c, \pi(u_i)$ for $i = 4, \dots, k$. Since the colouring of G' has been extended to G, then this contradicts our choice of G.

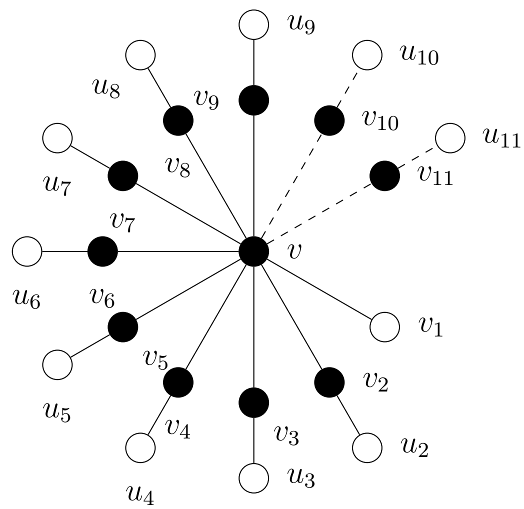

Figure 4.9: A 9-, 10-, 11-vertex with $n_2(v) = d(v) - 1$.

□

4.2 Initial Charging

In order to complete the proof, we suppose that G is a counterexample to Theorem 1 with the least number of vertices. Let L be a list assignment such that $|L(v)| \geq 5$ for all $v \in V(G)$. Thus, G satisfies Lemma 1 to 5. Since G is a planar graph, then; using Euler's formula, G has the characteristic that $|V(G)| - |E(G)| + |F(G)| = 2$. With some arithmetic manipulation, the relation yields $(4|E| - 6|V|) + (2|E| - 6|F|) = -12$. When summing the degree of the vertices and the degrees of the faces of G, we can derive the following identities: $\sum d(v) = 2|E(G)|$ and $\sum d(f) = 2|E(G)|$, since each edge is counted twice in each summation.

$$\begin{aligned}
&\sum_{v\in V(G)}(2d(v)-6)+\sum_{f\in F}(d(f)-6) \\
&=4|E|-6|V|+2|E|-6|F| \\
&=6(|E|-|V|-|F|)=-12.
\end{aligned}$$

If G is assigned charges of $2d(v)-6$ to each vertex $v \in V(G)$ and is assigned charges of $d(f)-6$ to each face $F \in V(G)$, where $F(G)$ is the set of faces G, then the total charge assigned to G is -12. We define a charge function ch by $ch(v) = 2d(v)-6$ for all $v \in V(G)$, and $ch(f) = d(f)-6$ if $f \in F(G)$. It follows from the above identity that the total sum of charges $ch(G)-12$. We design appropriate discharging rules and redistribute charges accordingly. Once the discharging is finished, a new charge function ch^* is produced. However, the total sum of charges is kept fixed when the discharging is in process. Nevertheless, after the discharging is complete, the new charge function $ch^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. This leads to the following obvious contradiction,

$$0 \leq \sum_{x\in V(G)\cup F(G)} ch^*(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12.$$

For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the amount of charges transferred from x to y. Suppose that $f = [v_1v_2v_3]$ is a $3-$face with $d(v_1) \leq$

$d(v_2) \leq d(v_3)$. We use $(d(v_1), d(v_2), d(v_3)) \to (c_1, c_2, c_3)$ to denote that the vertex v_i gives f the amount of charge c_i for $i = 1, 2, 3$. For example, $R2(d)$ in section 4.3 states as follows:

$$(4^+, 4^+, 4^+) \to (1, 1, 1).$$

This discharging rule states that each 4^+-vertex gives 1 charge to each incident 3-face. In other words, each 3-face having all incident vertices with degrees at least 4 receives 1 charge from each of these vertices. Our discharging rules are as follows:

4.3 Discharging Rules

R1 Every 5^+-vertex v gives

- (a) 1 charge to each adjacent 2-vertex
- (b) $\frac{1}{2}$ charge to each adjacent pendant light 3-vertex.

R2 Let $f = [v_1v_2v_3]$ be a 3-face with $d(v_1) \leq d(v_2) \leq d(v_3)$. Then

- (a) $(3, 3, 6^+) \to (\frac{1}{2}, \frac{1}{2}, 2)$;
- (b) $(3, 4, 5^+) \to (\frac{1}{2}, 1, \frac{3}{2})$;
- (c) $(3, 5^+, 5^+) \to (\frac{1}{2}, \frac{5}{4}, \frac{5}{4})$;
- (d) $(4^+, 4^+, 4^+) \to (1, 1, 1)$.

R3 Let $f = [v_1v_2v_3v_4]$ be a weak 4-face with w being a weak 6-vertex and $d(x) = 2$. By Definition 1, f has two possible configurations. Then

(a) $(6, 2, 6^+, 2) \to (1, 0, 1, 0)$;

(b) $(6, 2, 5^+, 4^+) \to (\frac{1}{2}, 0, \frac{3}{4}, \frac{3}{4})$;

R4 Let f be a 4^+-face in G with boundary vertices $v_1, \ldots, v_k$, for $k = d(f)$. Let $\Gamma(f)$ be the set of vertices $\{v_1, \ldots, v_k\} \setminus \{v_i\}$ such that v_i is a 2-, 3- or weak 6-vertex in the boundary of f. Let $\gamma(f) = |\Gamma(f)|$. Then each vertex $v \in \Gamma(f)$ gives

(a) $\frac{2}{\gamma(f)}$ to each 4-face which is not weak;

(b) $\frac{1}{\gamma(f)}$ to each 5-face.

4.4 Proof of Theorem 1

For each $x \in V(G) \cup F(G)$ we will show that $ch^*(x) \geq 0$, which produces a contradiction.

Proof. **4.4.1 Faces**

$d(f) = 3$ An initial charge of $ch(f) = -3$ is assigned to all 3-faces $f = [xyz] \in G$. By $(C9.1)$, a 2-vertex is not incident to a 3-face. By $(C9.2)$, G does not contain a $(3, 3, 5^-)$-face. So all 3-faces with $n_3(f) = 2$ are $(3, 3, 6^+)$-faces. By R2(a), $ch^*(f) = ch(f) + \tau((x, y, z) \to (f)) = -3 + \frac{1}{2} + \frac{1}{2} + 2 = 0$.

By $(C9.2)$, G does not contain a $(3,4,4)$-face. If $d(x) = 3$ and $d(y) = 4$, then z is a 5^+-vertex. By R2(b), $ch^*(f) = ch(f) + \tau((x,y,z) \to (f)) = -3 + \frac{1}{2} + 1 + \frac{3}{2} = 0$.

If f is a $(3,5^+,5^+)$-face, then by R2(c), $ch^*(f) = ch(f) + \tau((x,y,z) \to (f)) = -3 + \frac{1}{2} + \frac{5}{4} + \frac{5}{4} = 0$.

Otherwise, f is a $(4^+,4^+,4^+)$-face. By R2(d), $ch^*(f) = ch(f) + \tau((x,y,z) \to (f)) = -3 + 1 + 1 + 1 = 0$.

$d(f) = 4$

Observation 1. *Let $f = [wxyz]$ be a 4-face. If f is not weak then by $(C2)$ and $(C3.1)$, $n_2(f) + n_3(f) \leq 2$ and hence $\Gamma(f) \geq 2$. Otherwise, if f is weak, then consider the following two cases. If $n_2(f) = 2$, then $\Gamma(f) \leq 1$. If $n_2(f) = 1$, then by $(C2)$ and Lemma 1, $\Gamma(f) = 2$.*

An initial charge of $ch(f) = -2$ is assigned to all 4-faces $f = [wxyz] \in G$. Consider first if f is weak. If $\gamma(f) = 0$ or 1, then by R3(a), $ch^*(f) = ch(f) + \tau((w,x,y,z) \to (f)) = -2 + 1 + 0 + 1 + 0 = 0$. Otherwise, by Observation 1, $\gamma(f) = 2$. By R3(b) $ch^*(f) = ch(f) + \tau((w,x,y,z) \to (f)) = -2 + \frac{1}{2} + 0 + \frac{3}{4} + \frac{3}{4} = 0$. If, f is not weak, then by R4(a), $ch^*(f) = ch(f) + \tau(\Gamma(f) \to (f)) = -2 + \frac{2}{\gamma(f)} * \gamma(f) = 0$.

$d(f) = 5$

Observation 2. *Let $f = [vwxyz]$ be a 5-face. If f is not weak, then by $(C2)$, $n_2(f) \leq 2$. If $n_2(f) = 2$, then $\Gamma(f) = 3$. If $n_2(f) \leq 1$,* *then by* $(C3.1)$*,* $\Gamma(f) \geq 2$*. If* w *is a weak 6-vertex then by* $(C6.2)$ *and Definition 1, then vertices* v *and* x *are not 2-, 3-, or weak 6-vertices. Hence,* $\gamma(f) \neq 0$*.*

An initial charge $ch(f) = -1$ is assigned to all 5-faces $f \in G$. Since by Observation 2 and by R4(b), $ch^*(f) = ch(f) + \tau(\Gamma(f) \to (f)) = -1 + \frac{1}{\gamma(f)}\gamma(f) = 0$.

$d(f) \geq 6$ An initial charge $ch(f) \geq 0$ is assigned to all 6^+-faces $f \in G$. Then $ch^*(f) = ch(f) \geq 0$, as the discharging rules do not transfer charges from faces to either vertices or faces.

4.4.2 Vertices

$d(v) = 2$ An initial charge $ch(v) = -2$ is assigned to all 2-vertices $v \in G$. By $(C2)$, if u and w are the two neighbours of v with $d(u) \leq d(w)$, then $5 \leq d(u) \leq d(w)$. By R1(a), both u and w give 1 charge to v. Hence, $ch^*(v) = ch(v) + \tau((u,w) \to v) = -2 + 1 + 1 = 0$.

$d(v) = 3$ An initial charge $ch(v) = 0$ is assigned to all 3-vertices $v \in G$. If v is a pendant light 3-vertex of a vertex u, and is incident to a 3-face f, then by R1(b) and R2(a),(b),(c), $ch^*(v) = ch(v) + \tau(u \to v) - \tau(v \to f) = 0 + \frac{1}{2} - \frac{1}{2} = 0$. Otherwise, v is not incident to a 3-face, hence $ch^*(v) = ch(v) = 0$.

Observation 3. *By Observation 1, R3 and R4(a), every* 4^+*-vertex* $v \in G$ *gives at most 1 charge to each incident 4-face. Since, by assumption,* *a 4-face is not adjacent to a 3-, 4- or 5-face, then every 4^+-vertex $v \in G$, on average, gives at most $\frac{1}{2}$ charge to each of two adjacent faces f_1 and f_2 is one of these faces is a 4-face. By Observation 2 and R4(b), since $\gamma \geq 2$ for all 5-faces in G, then every $v \in G$ gives at most $\frac{1}{2}$ charges to each incident 5-face. Similarly, since a 3-face is not adjacent to a 4- or 5-face, then a pendant light 3-vertex v of u is incident to one 3-face and two 6^+-faces. And, like the 5-face, a pendant light 3-vertex requires only $\frac{1}{2}$ charges from its neighbour u. Hence, a vertex having $\frac{d(v)}{2}$ charges available to distribute to its incident faces has sufficient charge if v is not incident a 3-face.*

If, however, v is incident to 3-faces, then for any 6^+-vertex, v distributes up to 2 charges to a 3-face by R3. Since, by the assumption that a 3-face is not adjacent to a 3-, 4- or 5-face, then every $v \in G$, on average, at most 1 to each of two adjacent faces f_1 and f_2 is one of these faces is a 3-face. Hence, a vertex v which has at least 1 charge available to distribute to each of $d(v)$ incident faces has sufficient charges for any configuration which contains $\lfloor \frac{d(v)}{2} \rfloor$ 3-faces.

Lastly, since a 2-vertex is not incident to a 3-face by $(C9.1)$, then we consider a vertex v which is incident a 4-face containing two 2-vertices. By R1(a), R3(a) and R4(a), every 6^+-vertex v distributes 1 charge to each 2-vertex and 4-face. Since, by assumption a 4-face is not adjacent to a 3-, 4- or 5-face, then a vertex with $\left\lfloor \frac{3d(v)}{2} \right\rfloor$ has sufficient charge for every possible configuration.

In the following lemmas we define $N_2(v)$ as the set of vertices $v_i \in N(v)$ such that $d(v_i) = 2$. Similarly, $N_3(v)$ as the set of vertices $v_i \in N(v)$ such that v_i is a pendant light 3-vertex of v. Lastly, we define $\{f_1, \ldots, f_k\}$ is the set of faces incident with one common vertex v, where $k = d(v)$.

$d(v) = 4$ An initial charge $ch(v) = 2$ is assigned to all 4-vertices $v \in G$. By $(C2)$ and $(C4)$, v in not adjacent to a 2-vertex or a pendant light 3-vertex. By $(C9.3)$, v is not incident a $(3, 4, 4)$-face. By Observation 3, R2(b)(d) and R4, $ch^*(v) \geq ch(v) - \tau(v \to (f_1, \ldots, f_k)) = 2 - 4(\frac{1}{2}) = 0$.

$d(v) = 5$ An initial charge $ch(v) = 4$ is assigned to all 5-vertices $v \in G$. By $(C5.1)$, v is adjacent to at most one 2-vertex. Consider the following two cases:

$n_2(v) = 1$ By $(C5.2)$, if v is adjacent to a 2-vertex u, then v is not adjacent to a pendant light 3-vertex. Since, by $(C5.3)$, v distributes one charge to each incident 3-face as per R3(d). Hence, as in Observation 3, $ch^*(v) \geq ch(v) - \tau(v \to u) - \tau(v \to (f_1, \ldots, f_k)) = 4 - 1(1) - 5(\frac{1}{2}) = \frac{1}{2}$ by R1(a), R4 and by Observation 2.

$n_2(v) = 0$ If v is not incident to a 3 or 4-face, then v is incident to at most five 5-faces which require charge, meaning $ch^*(v) \geq ch(v) - \tau(v \to (f_1, \ldots, f_k)) = ch(v) - 5(\frac{1}{2}) = \frac{3}{2}$ by R4(b) and Observation 2. If v is incident to one 4-face, then v is incident to at most three 5-faces or three pendant light 3-vertices by $(C5.4)$. By R1(b) and R4(a)(b), $ch^*(v) \geq ch(v) - \tau(v \to (f_1, \ldots, f_k)) = ch(v) - 1(1) - 3(\frac{1}{2}) = \frac{3}{2}$.

If v is incident to two 4-faces, then v is not incident to a 5-face, but adjacent to at most one pendant light 3-vertex. By R1(b) and R4(a), $ch^*(v) \geq ch(v) - \tau(v \to (f_1, \ldots, f_k)) - \tau(v \to N_3(v)) = ch(v) - 2(1) - 1(\frac{1}{2}) = \frac{3}{2}$. If v is incident to one 3-face f_1, then v is incident to at most three 5-faces or at most two pendant light 3-vertices by (C5.5) and (C5.6). Since by (C9.2) $f_1 \neq (3,3,5)$-face, then by R1(b), R2(b)(c)(d) and R4(b), $ch^*(v) \geq ch(v) - \tau(v \to N_3(v)) - \tau(v \to (f_1, \ldots, f_k)) = ch(v) - 1(\frac{3}{2}) - 3(\frac{1}{2}) = 1$. If v is incident to two 3-faces, then v is not incident to a 5-face, but adjacent to at most one pendant light 3-vertex. Hence, $ch^*(v) \geq ch(v) - \tau(v \to (f_1, \ldots, f_k)) - \tau(v \to N_3(v)) = ch(v) - 2(\frac{3}{2}) - 1(\frac{1}{2}) = \frac{1}{2}$.

$d(v) = 6$ An initial charge $ch(v) = 6$ is assigned to all 6-vertices $v \in G$. By (C6.1), v is adjacent to at most four 2-vertex. Consider the following cases:

$n_2(v) = 4$ If v is weak, then by Definition 1, it is incident to either three 4-faces or to two 4-faces and one 5-face. By R1(a) and R3(a), either $ch^*(v) = ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 4 - 1 - 1 = 0$ if every vertex in $N_2(v)$ are contained in only two 4-faces. Otherwise by R1(a) and R3(b), $ch^*(v) = ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 4 - 1 - \frac{1}{2} - \frac{1}{2} = 0$. If v is not weak, then by R1, it distributes one to each incident 2-

vertex, leaving two charges to distribute amongst its incident faces. As a consequence of $(C6.2)$ and by $(C6.3)$, $p_3(v) = 0$ and v is not incident to a 3-face. By Definition 1, v is incident to at most two 4-faces (else, it would be weak), so consider the following cases: If v is incident to two 4-faces, then $ch^*(v) = ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 4 - 1 - 1 = 0$ if v is incident to four 6^+-faces. Otherwise, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 4 - 1(1) - 1(\frac{2}{3}) - 1(\frac{1}{3}) = 0$ If v is incident to a 5-face. If v is incident one 4-face then by assumption, v is incident to at most three 5-faces. By $(C2)$, $(C3.1)$ and Definition 1, each of these 5-faces has $\Gamma \geq 3$. By R1(a) and R4(b), $ch^*(v) \geq ch(v) - 4(1) - 1(1) - 3(\frac{1}{3}) = 0$.

$n_2(v) = 3$ Then by $(C6.8)$ and $(C9.1)$, v is incident to at most one 3-face which is not a $(3, 4^-, 6)$-face. Consider the following two cases:

If v is incident to a $(3, 5^+, 6)$-face, then by $(C6.4)$, $p_3(v) = 0$. If v is not incident to a 4-face, then v is incident to at most three 5-faces. Hence $ch^* \geq ch(v) - 3(1) - 1(\frac{5}{4}) - 3(\frac{1}{2}) = \frac{1}{4}$ by R2(a) and R4(b). Otherwise, v is incident to at most two 4-faces. By Lemma 4, R2 and R4, $ch^* \geq ch(v) - 3(1) - 1(\frac{5}{4}) - 1(1) - 1(\frac{2}{3}) = \frac{1}{12}$.

$n_2(v) = 2$ By R1(a), v has four charges to distribute to its adjacent pendant light 3-vertices and its incident faces. By $(C6.5)$ and $(C6.6)$ and by lemma 4, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 2(1) - 2(\frac{5}{4}) - 2(\frac{1}{2}) = \frac{1}{2}$ since by assumption, a 3-face is not adjacent to a 3-, 4- or 5-face.

$n_2(v) = 1$ Then by R1(a), R2, R4, and Observation 3 then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 1(1) - 6(\frac{1}{2}) = 2$ if v is not incident to a 3-face and $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to \{f_1, \ldots, f_k\}) = 6 - 1(1) - 4(1) - 2(\frac{1}{2}) = 0$.

$n_2(v) = 0$ If $n_2(v) = 0$, then v has at least one charge to give to each incident face. By the assumption that G contains no adjacent small cycles and by R2 and R4, this is sufficient charge. Hence $ch^*(v) \geq 0$ for every 6-vertex in G.

$d(v) = 7$ An initial charge of $ch(v) = 8$ is assigned to each 7-vertex in G. By $(C7.1)$, since $n_2(v) \leq 5$, consider the following cases:

$n_2(v) = 5$ By $(C7.3)$, $n_3(v) = 0$ and $t(v) = 0$. If v is not incident to a 4-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 5(1) - 7(\frac{1}{3}) = \frac{1}{3}$, as per Observation 2. Otherwise, v is incident to at most three 4-faces. Hence, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 5(1) - 3(1) = 0$ since by assumption no 4-face is adjacent to a 3-, 4- or 5-face.

$n_2(v) = 4$ By $(C7.2)$ and $(C7.4)$, consider the following three cases. Since v is incident to at most one 3-face, then if v is incident to a $(3, 3, 7)$-face, it is not adjacent to a pendant light 3-vertex. Hence, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 4(1) - 1(2) - 4(\frac{1}{2}) = 0$ by Observation 3. If $t(v) = 0$, then by R1(a)(b) and

R4(a)(b), $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 7(\frac{1}{2}) = \frac{1}{2}$ if v is adjacent to pendant light 3-vertices, and incident to 5^--faces.

$n_2(v) = 3$ By $(C9.1)$, v is incident to at most two 3-faces. If $t(v) = 2$ then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 3(1) - 2(2) - 2(\frac{1}{2}) = 0$ since by assumption, a 3-face is not incident a 3-, 4- or 5-face. if $t(v) = 1$, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 3(1) - 1(2) - 6(\frac{1}{2}) = 0$. Hence, by Observation 3, v has sufficient charge to distribute to at most one adjacent pendant light 3-vertex (as per $C7.5$), and any configuration of 5^--faces.

$n_2(v) = 2$ By R1(a)(b) and R4, if v is not incident a 3-face, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 2(1) - 7(\frac{1}{2}) = \frac{5}{2}$ by Observation 3. Otherwise, by $(C9.1)$, $t(v) \leq 2$. If $t(v) = 2$, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 2(1) - 2(2) - 2(\frac{1}{2}) = 1$, by R1(a), R2(c) and R4, and by the assumption that a 3-face is not adjacent to a 3-, 4- or 5-face. If $t(v) = 1$, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 2(1) - 1(2) - 4(\frac{1}{2}) = 2$.

$n_2(v) \leq 1$ If $n_2(v) \leq 1$, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 8 - 1(1) - 7(1) = 0$. Since v has at least one charge to give to each incident face, then by Observation 3, v has sufficient charge.

$d(v) = 8$ An initial charge of $ch(v) = 10$ is assigned to each 8-vertex in G. By $(C8.1)$, $n_2(v) \leq 6$. If $n_2(v) = 6$, then by Observation 3, R1(a), R2(c) and R4(a)(b), $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 10 - 6(1) - 8(\frac{1}{2}) = 0$, since v is not incident a 3-face by $(C8.2)$. If $n_2(v) = 5$, then $t(v) \leq 1$. If v is incident to a 3-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 10 - 5(1) - 1(2) - 5(\frac{1}{2}) = \frac{1}{2}$ since, by assumption, a 3-face is not adjacent to a 3-, 4- or 5-face. Otherwise, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 10 - 5(1) - 8(\frac{1}{2}) = 1$ if v is not incident to a 3-face. If $n_2(v) = 4$, then $t(v) \leq 2$. If v is incident to two 3-faces then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 10 - 4(1) - 2(2) - 4(\frac{1}{2}) = 0$ since, by assumption, a 3-face is not adjacent to a 3-, 4- or 5-face and not every incident face of v requires charge distribution. If v is incident to at most one 3-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 10 - 4(1) - 1(2) - 7(\frac{1}{2}) = \frac{1}{2}$.

$d(v) = 9$ An initial charge of $ch(v) = 12$ is assigned to each 9-vertex in G. By Lemma 5, $n_2(v) \leq 7$. By Observation 3, R1(a), R2(c) and R4, if $n_2(v) = 7$, then v is incident to at most one 3-face by $(C9.1)$. If v is incident to a 3-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 12 - 7(1) - 1(2) - 6(\frac{1}{2}) = 0$ since a 3-face is not adjacent to a 4- or 5-face. Otherwise, if v is not incident a 3-face, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 12 - 7(1) - 9(\frac{1}{2}) = \frac{1}{2}$. If $n_2(v) = 6$, then v is still incident to at most one 3-face, so consider that

$n_2(v) = 5$. Then v is incident to at most two 3-faces, hence $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 12 - 5(1) - 2(2) - 4(\frac{1}{2}) = 1$ since at least three faces incident to v are 6^+-faces. If $n_2(v) = 4$ then again v is incident to at most two 3-faces. By R2, $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 12 - 4(1) - 2(2) - 7(\frac{1}{2}) = \frac{1}{2}$. Finally, if $n_2(v) \leq 3$ then by Observation 3, R1(a), R2 and R4, $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 12 - 3(1) - 9(1) = 12 - 12 = 0$.

$d(v) = 10$ An initial charge of $ch(v) = 14$ is assigned to each 10-vertex in G. By Lemma 5, $n_2(v) \leq 8$. By Observation 3, R1(a), R2(c) and R4, if $n_2(v) = 8$, then v is incident to at most one 3-face by $(C9.1)$. If v is incident to a 3-face then $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 14 - 8(1) - 1(2) - 7(\frac{1}{2}) = \frac{1}{2}$ since a 3-face is not adjacent to a 4- or 5-face. Otherwise, if v is not incident a 3-face, then $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 14 - 8(1) - 10(\frac{1}{2}) = 1$. If $n_2(v) = 7$, then v is still incident to at most one 3-face, so consider that $n_2(v) = 6$. Then v is incident to at most two 3-faces, hence $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 14 - 6(1) - 2(2) - 8(\frac{1}{2}) = 0$. If $n_2(v) = 5$ then again v is incident to at most two 3-faces. By R2, $ch^*(v) \geq ch(v) - \tau(v \rightarrow N_2(v)) - \tau(v \rightarrow (f_1, \ldots, f_k)) = 14 - 5(1) - 2(2) - 8(\frac{1}{2}) = 1$. Finally, if $n_2(v) \leq 4$ then by Observation 3, R1(a), R2 and R4, $ch^*(v) \geq ch(v) - 4(1) - 10(1) = 14 - 14 = 0$.

$d(v) = 11$ An initial charge of $ch(v) = 16$ is assigned to each 11-vertex in G. By

Lemma 5, $n_2(v) \leq 9$. By Observation 3, R1(a), R2(c) and R4, if v is incident to a 3-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 16 - 9(1) - 1(2) - 10(\frac{1}{2}) = 0$. If v is not incident to a 3-face then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 16 - 9(1) - 11(\frac{1}{2}) = \frac{3}{2}$. If $n_2(v) = 8$, then $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 16 - 8(1) - 1(2) - 10(\frac{1}{2}) = \frac{3}{2}$ if v is incident to one $(3,3,11)$-face. If $n_2(v) = 7$, then v is incident to at most two 3-faces. Hence, if all other incident faces are 4- or 5-faces, then by Observation 3, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 16-7(1)-2(2)-9\frac{1}{2} = \frac{1}{2}$. If $n_2(v) = 6$, then again v is incident to at most two 3-faces. Hence, if all other incident faces are 4- or 5-faces, then by Observation 3, $ch^*(v) \geq ch(v) - \tau(v \to N_2(v)) - \tau(v \to (f_1, \ldots, f_k)) = 16 - 6(1) - 2(2) - 9\frac{1}{2} = \frac{3}{2}$. If $n_2(v) \leq 5$, then v has at least one charge to give to each incident face. By Observation 3, v has sufficient charge to distribute to its incident faces.

$d(v) \geq 12$ An initial charge of $ch(v) \geq 18$ is assigned to each 12^+-vertex in G. By Observation 3, $ch^*(v) \geq ch(v) - \left\lfloor \frac{3d(v)}{2} \right\rfloor \geq 18 - \left\lfloor \frac{3(12)}{2} \right\rfloor = 0$.

□

It was discusses earlier that

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F} (d(f) - 6) = -12.$$

is the total charge on the graph G because of the fact that G is planar. We have shown through careful redistribution of charges demonstrated in Lemmas 4.4.1 to 4.4.2 yields the desired contradiction that

$$0 \leq \sum_{x \in V(G) \cup F(G)} ch^*(x) = \sum_{x \in V(G) \cup F(G)} ch(x) = -12.$$

This last equation clearly shows that the assumption of the existence of a minimum counterexample G was incorrect. Without the existence of such a graph, we have succeeded in proving that every planar graph which does not contain a 3- or 4-cycle adjacent to a 3-, 4- and 5-cycle is acyclically 5-choosable.

Chapter 5

Conclusion

An improvement on this work will need further knowledge of the structure of G, especially the understanding of if and when triangles and 4-cycles can intersect, as most previous work does not allow intersections of such configurations. It was recently asked by Borodin and Ivanova [Bor13]

Is it true that every planar graph satisfies $\chi(G) = \chi^l(G)$?

One might then ask about the relationship between $\chi^l(G)$ and $\chi^l_a(G)$.

What, if any, is the connection between the k-choosability of planar graphs and general graphs?

Some progress has also been made in finding a bound for the k-choosability of general graphs for $3 \leq k \leq 5$. In this case, the maximum average degree, $Mad(G)$ of a graph G is considered [MOR06]. Some recent results have found bounds of

- Every graph G with $Mad(G) < \frac{8}{3}$ is acyclically 3-choosable

- Every graph G with $Mad(G) < \frac{19}{6}$ is acyclically 4-choosable
- Every graph G with $Mad(G) < \frac{24}{7}$ is acyclically 5-choosable

where

$$Mad(G) = \{max\frac{2|E(H)|}{|V(H)|}, \forall H \subseteqq G\}.$$

These answers is still far off but strides are still being made. I myself have found many results on the restrictions of intersecting triangles and the restrictions on the degrees of the vertices of a 4-cycle in an attempt to allow a 3- and 4-cycle to exist adjacent a 5-cycle. There has been some difficulty in disproving the existence of some suns, i.e., a polygon with n sides adjacent to n triangles but early discharging attempts look promising.

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Appendices

Appendix A

List of Symbols

Symbol	Description
v	The end point of an edge.
$k - vertex$	A vertex with exactly k neighbours.
$k^+ - vertex$	A vertex with at least k neighbours.
$k^- - vertex$	A vertex with at most k neighbours.
$N(v)$	The set of neighbours adjacent to v.
$n_i(v)$	The number neighbours of v having degree i.
$t(v)$	The number of 3-faces incident to a vertex v.
f	A polygon enclosed by its outer edges.

$k - face$	A face with exactly k edges in its boundary.
$k^+ - face$	A face with at least k edges in its boundary.
$k^- - face$	A face with at most k edges in its boundary.
$n_j(f)$	The number of incident vertices with degree j.
(a_1, a_2, a_3)-face	A 3-face with vertices having degrees a_1, a_2 and a_3 respectively.
$p_3(u)$	The number of pendant light 3-vertices of a vertex u.
$(\alpha,\ \beta)$-path	A path coloured with two alternating colours α and β.
$w_6(f)$	The number of weak 6-vertices incident a 4- or 5-face.
$N_2(v)$	The set of 2-vertices in the neighbourhood of v.
$N_3(v)$	The set of pendant light 3-vertices in the neighbourhood of v.
$\tau(x \to y)$	The amount of charge redistributed from $x \in V(G) \cup F(G)$ to $y \in V(G) \cup F(G)$.

List of Figures