行政院國家科學委員會專題研究計畫　成果報告

OFDM 系統之錯誤率與通道容量分析於 QAM 傳輸於雷利衰退通道

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中英文摘要與關鍵詞

摘要

在正交多工系統之下，我們提出所謂事前平均的方法推導錯誤率公式的精確表示。調變的方法為任意長方之 QAM；通道模型為具有頻率選擇性之雷利衰退時變環境。此外，針對此種通訊傳輸，我們亦深入探討其通道容量。

關鍵詞: 錯誤率、正交多工、雷利衰退、通道容量

Abstract

By using what we call the pre-averaging method, an exact closed form expression for the symbol error probability (SEP) is derived for arbitrary rectangular M-QAM signaling in OFDM systems over frequency-selective Rayleigh fading channels. In addition, the channel capacity for the QAM OFDM transmission over Rayleigh fading environment is obtained.

keywords: Symbol error probability, OFDM (Orthogonal frequency division multiplexing), Rayleigh fading, Channel capacity

報告內容

Symbol Error Probability for Rectangular *M*-QAM OFDM Transmission over Rayleigh Fading Channels

Abstract

By using what we call the pre-averaging method, an exact closed form expression for the symbol error probability (SEP) is derived for arbitrary rectangular M-QAM signaling in OFDM systems over frequency-selective Rayleigh fading channels. In addition, the channel capacity for the QAM OFDM transmission over Rayleigh fading environment is obtained.

1. Introduction

A common signaling scheme used for OFDM systems is QAM signaling [1]. There have been numerous research studies to evaluate the error probability performance for QAM transmission in digital communication systems (AWGN channels, multipath fading channels, diversity combining systems). Most of these QAM error probability evaluations are only for square (not rectangular) *M*-QAM cases [2-4]. By far, the mostly discussed fading model is Rayleigh fading. The major difficulty in finding the SEP for QAM in fading is the evaluation of the integral of the squared Gaussian-*Q* function, which many times leads to results containing either hypergeometric functions or unevaluated integrals [3,4]

In this work, by using a pre-averaging method, we successfully avoid the squared Gaussian-Q function integral to obtain exact closed-form SEP (containing no hypergeometric functions nor unevaluated integrals) for OFDM systems employing arbitrary rectangular *M*-QAM over frequency-selective Rayleigh fading channels. We will also obtain the channel capacity for the QAM OFDM system in Rayleigh fading channels.

Section 2 presents the OFDM system model. Section 3 derives the exact closed-form SEP for arbitrary rectangular M-QAM OFDM transmission over frequency-selective Rayleigh fading channels. Section 4 gives simulation results. Then, Section 5 derives the channel capacity for the QAM OFDM transmission over Rayleigh fading channels. Finally, Section 6 draws the conclusion.

2. The OFDM system model

The equivalent channel frequency response at subcarrier frequency $f_k = k/T$ for an OFDM system in frequency-selective fading channels is

$$H_k = \sum_{n=0}^{\upsilon-1} h_n e^{-j2\pi nk/N}\ ,\ \ k = 0,1,2,...,N-1\,, \qquad (1)$$

where T is the duration of a block of N data symbols, h_n is the channel impulse response that spans υ symbols. Assuming independent Rayleigh fading channels, $\{h_n\}$ are independent complex Gaussian RV's with zero means and variances $\{\sigma_n^2\}$ in each real dimension. It is straightforward to verify that $\{H_k\}$ are i.i.d. Gaussian RV's with zero means and variances $\sigma_c^2 = \sum_{n=0}^{\upsilon-1}\sigma_n^2$ in each real dimension for all k. This simply means that OFDM converts a frequency-selective fading channel into flat fading.

The kth subband output of the FFT at the receiver is

$$r_k = H_k X_k + z_k\,, \qquad (2)$$

where $\{X_k\}$ are the transmitted data symbols and $\{z_k\}$ are i.i.d. complex white Gaussian noise RV's with zero means and variances σ_z^2 in each real dimension. By dividing (2) by H_k, we have the kth subband estimate as

$$\hat{X}_k = \frac{r_k}{H_k} = X_k + \frac{z_k}{H_k} = X_k + e_k\,. \qquad (3)$$

We shall assume that perfect channel state estimate is available.

3. SEP for arbitrary rectangular *M*-QAM OFDM transmission over frequency-selective Rayleigh fading channels

To find SEP for fading channels, the usual approach is to first compute the SEP conditioned on a fixed channel realization $|H_k|$ (SEP in AWGN), then average this conditional SEP over channel realization (we call this the post-averaging method as in contrast to our pre-averaging method to be described below) to obtain the final overall SEP. As mentioned earlier, this post-averaging approach will inevitably involve the complicated integration of the squared Gaussian-Q function. What we will do for QAM in OFDM system over Rayleigh fading can avoid this integration. We first average (pre-average) $p(e_{ck}, e_{sk} \,||\, H_k \,|)$, the joint PDF of e_{ck} and e_{sk} (real and imaginary parts of e_k) conditioned on a given channel realization $|H_k|$ to obtain the joint PDF $p_{cs}(e_{ck}, e_{sk})$, then calculate the average SEP from this joint PDF. Straightforward calculations lead to

$$p_{cs}(e_{ck}, e_{sk}) = \frac{(\sigma_z / \sigma_c)^2}{\pi[(\sigma_z / \sigma_c)^2 + e_{ck}^2 + e_{sk}^2]^2}, \quad -\infty < e_{ck}, e_{sk} < \infty. \qquad (4)$$

Assume rectangular M_k-QAM signaling for the *k*th subband with $M_k = 2^{n_k} = M_{ck} M_{sk}$, where M_{ck}-PAM and M_{sk}-PAM are employed respectively for real and imaginary parts of X_k, viz., X_{ck} and X_{sk}. The symbol X_{ck} takes on values from the set $\{(2m_{ck} - 1 - M_{ck})d, m_{kc} = 1,2,..., M_{ck}\}$ with equal probabilities, while X_{sk} takes on values from the set $\{(2m_{sk} - 1 - M_{sk})d, m_{sk} = 1,2,..., M_{sk}\}$ with equal probabilities. Since all subbands have the same $p_{cs}(e_{ck}, e_{sk})$, we will simply drop the subscript *k* for these error variables.

For the 4 corner symbol points of the M_k-QAM constellation, due to constellation symmetry, each point has the identical correct probability given by

$$P_{c1} = \int_{-d}^{\infty}\int_{-d}^{\infty} p_{cs}(e_c, e_s) de_c de_s$$
$$= \frac{1}{4} + \frac{a}{2\sqrt{1+a^2}} + \frac{a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}, \qquad (5)$$

where $a = (\sigma_c / \sigma_z)d$.

For the $2(M_{ck} + M_{sk} - 4)$ border points, not including the corner points, the correct probability is

$$P_{c2} = \int_{-d}^{d}\int_{-d}^{\infty} p_{cs}(e_c, e_s) de_c de_s$$
$$= \frac{a}{2\sqrt{1+a^2}} + \frac{2a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}. \qquad (6)$$

Then, for the $M_k - 2(M_{ck} + M_{sk}) + 4$ inner points, the correct probability is

$$P_{c3} = \int_{-d}^{d}\int_{-d}^{d} p_{cs}(e_c, e_s) de_c de_s$$
$$= \frac{4a}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}. \qquad (7)$$

The overall average SEP can now be obtained as

$$P_{M_k} = \frac{1}{M_k}[4(1-P_{c1}) + 2(M_{ck} + M_{sk} - 4)(1-P_{c2}) + (M_k - 2M_{ck} - 2M_{sk} + 4)(1-P_{c3})]$$
$$= \frac{1}{M_k}[M_k - 1 - \frac{a}{\sqrt{1+a^2}}(M_{ck} + M_{sk} - 2) - \frac{4a}{\pi\sqrt{1+a^2}}(M_k - M_{ck} - M_{sk} + 1)\tan^{-1}\frac{a}{\sqrt{1+a^2}}] \qquad (8)$$

If square QAM is used, $M_{ck} = M_{sk} = \sqrt{M_k}$, (8) becomes

$$P_{M_k} = \frac{1}{M_k}[(M_k - 1) - \frac{2a(\sqrt{M_k} - 1)}{\sqrt{1+a^2}} - \frac{4a(\sqrt{M_k} - 1)^2}{\pi\sqrt{1+a^2}} \tan^{-1}\frac{a}{\sqrt{1+a^2}}]. \qquad (9)$$

Setting $M_{ck} = M_k = 2$ and $M_{cs} = 1$, (8) reduces to the well known result for M_k-PAM [5].

It can be readily shown that

$$a = \frac{\sigma_c d}{\sigma_\eta} = \sqrt{\frac{3\bar{\gamma}_k}{M_{ck}^2 + M_{sk}^2 - 2}}. \qquad (10)$$

Replacing a by $\bar{\gamma}_k$ using (10), we can get the SEP expressions in terms of SNR as is usually preferred.

The block or frame error probability is simply given by

$$P_B = 1 - \prod_{k=0}^{N-1}(1 - P_{M_k}). \qquad (11)$$

Further, if Gray coding is used for each group (subband), we can approximate the average bit error probability by

$$P_b \cong \frac{1}{N}\sum_{k=0}^{N-1}\frac{P_{M_k}}{n_k}. \quad (12)$$

At this point, we must note that the noise or error term $e_k = z_k / H_k$ in (3) is not Gaussian. We need to show that the minimum distance detector used here is optimum.

Let the complex received data symbol and noise samples out of the correlation demodulator be $s_m = s_{mc} + js_{ms}$, $m = 1,2,...,M$, and $e = e_c + je_s$ respectively subscript k dropped). Then the total received data sample is

$$r_t = r_c + jr_s = (s_{mc} + e_c) + j(s_{ms} + e_s). \quad (13)$$

Assume a priori symbol probabilities { $p_m(s_m)$ } are equal for all m. Then, if the joint density function $p_{cs}(e_c, e_s)$ is monotonically decreasing with $|e| = \sqrt{e_c^2 + e_s^2}$, it is readily shown that a minimum distance detector is equivalent to the optimum maximum a posteriori probability (MAP) detector or the maximum-likelihood (ML) detector. Now, applying the above fact, since $p_{cs}(e_{ck}, e_{sk})$ given by (4) is monotonically decreasing with $e_{ck}^2 + e_{sk}^2$, we conclude that our minimum distance detector is indeed optimum.

4. Simulation results

Figure 1 presents the plots of P_{M_k} vs. $\bar{\gamma}_k$ for various combinations of $M_{ck}M_{sk} = M_k = 256$. It is seen that, for a given M_k , the best choice is to use square QAM, and when rectangular QAM is used, then as the difference between M_{ck} and M_{sk} gets larger, the performance gets worse. This can also be readily proven analytically by taking the derivative of P_{M_k} of (8) with respect to M_{ck} and setting the result to zero, meanwhile fixing M_k and $\bar{\gamma}_k$. Also included in Fig. 1 is a curve obtained by Monte Carlo simulations for the square 256-QAM case. It is seen that this curve is in excellent agreement with the theoretical curve.

5. Channel capacity

The channel capacity for a Rayleigh fading channel has been solved by Lee [6]. For average SNR $\bar{\gamma} > 2$, the capacity can be expressed as

$$C = B\log_2 e \cdot [e^{-1/\bar{\gamma}}(\ln\bar{\gamma} + \frac{1}{\bar{\gamma}} - E)] \text{ bits/sec,} \quad (14)$$

where $E = 0.5772157$ is Euler constant. For OFDM systems, with a very small sub bandwidth $\Delta f = 1/T$, the overall channel capacity can be written as

$$C = \sum_{k=0}^{N-1} C_k = \Delta f \sum_{k=0}^{N-1} \log_2 e \cdot [e^{-1/\bar{\gamma}_k}(\ln\bar{\gamma}_k + \frac{1}{\bar{\gamma}_k} - E)] . \quad (15)$$

As $\Delta f \to 0$, (15) can be written as

$$C = (\log_2 e)\int_0^W e^{-1/\bar{\gamma}(f)}(\ln\bar{\gamma}(f) + \frac{1}{\bar{\gamma}(f)} - E)df . \quad (16)$$

With AWGN, we want to find the optimum $\bar{\gamma}(f)$ to get maximum C subject to the power constraint that

$$\int_0^W \bar{\gamma}(f)df = \text{constant.} \quad (17)$$

The maximization is obtained by maximizing the integral

$$\int_0^W [e^{-1/\bar{\gamma}(f)}(\ln\bar{\gamma}(f) + \frac{1}{\bar{\gamma}(f)} - E) + \lambda\bar{\gamma}(f)]df , \quad (18)$$

where λ is a Lagrange multiplier. By use of the calculus of variations, we differentiate the integrand with respect to $\bar{\gamma}(f)$ and then set the result to zero. We get

$$\lambda\bar{\gamma}^3(f)e^{1/\bar{\gamma}(f)} + \bar{\gamma}^2(f) + \bar{\gamma}(f)[\ln\bar{\gamma}(f) - 1 - E] + 1 = 0. \quad (19)$$

This transcendental equation is hard to solve. Fortunately, we need not solve it as evidently by its look, the solution for $\bar{\gamma}(f)$, if exists, will not be a function of f. It will be a value depending only on λ which is again dictated by the constrained power. We thus conclude that the best choice for $\bar{\gamma}_k$ is constant over all subbands.

Now, let the available average power of the transmitter be P_{av}. Then, each subband has the same average power P_{av}/N . Then,

$$\bar{\gamma}_k = \frac{\sigma_0^2 P_{av} T}{\sigma_n^2 N^2}. \quad (20)$$

Substituting (20) into (19), we find the average channel capacity in terms of the transmitted signal power as

$$C = \frac{W}{\ln 2}[\exp(-\frac{\sigma_n^2 N^2}{\sigma_0^2 P_{av} T})(\ln \frac{\sigma_0^2 P_{av} T}{\sigma_n^2 N^2} + \frac{\sigma_n^2 N^2}{\sigma_0^2 P_{av} T} - E)] . \quad (21)$$

It must be noted here that the received SNR $\bar{\gamma}_k$ is an overall average value. For frequency-selective channels, the received SNR during each symbol interval denoted by γ_k is different for different subbands and will vary from one symbol interval to another. Therefore, if one wants to achieve channel capacity by whatever coding means, one must go through the painful process of optimizing signal power distribution by water pouring principle during every symbol interval.

6. Conclusion

By using what we call the pre-averaging method, we derive an exact closed-form SEP expression for arbitrary rectangular *M*-QAM OFDM transmission over Rayleigh fading channels. Monte Carlo simulations are performed to check with the theoretical results. By using the pre-averaging technique, we successfully evade the need for integrating the squared Gaussian-*Q* function which is unavoidable if using post-averaging method adopted by most researchers. As a result, our SEP expression contains no hypergemetric functions nor unevaluated integrals, hence can be easily computed by the computer. We have also obtained the channel capacity in terms of the transmitted signal power for the QAM OFDM transmission over Rayleigh fading channels.

7. References

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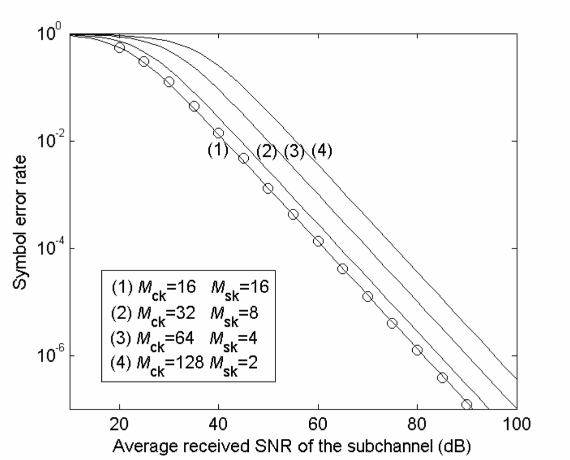

Fig. 1 Rectangular -QAM performance for the OFDM system over Rayleigh fading channels for $M_k = 256$. Monte Carlo simulation curve (marked with O) for square 256-QAM is also incorporated.

參考文獻

本計畫執行期間共發表 4 篇論文[1]-[4]於知名國際性學術研討會，其中[2]即為本計畫主題 OFDM 系統，其內容即如”報告內容”所示；[3]為針對 QAM 系統於無線線通道的分析；[4]為延伸至使用最大比例技術之多接收天線系統；[1]則為雙傳輸-多接收天線系統的情況之下。特別一提，我們將以上研究成果再加以整體、延伸，一篇投稿於 *IEE Proceedings-Commun* 的論文[5]已被接受。

[1] Rainfield Y. Yen, Hong-Yu Liu, and Tu En Lee, “Error Probability for Two-Branch Transmit Diversity Using Arbitrary Rectangular M-QAM over Rayleigh Fading,” *Workshop on Consumer Electronics and Signal Processing*, WCEsp2004, Hsinchu, Taiwan, Nov. 2004.

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計畫成果自評

本計畫執行的成果為高度卓越，一如”參考文獻”中之說明，共發表 4 篇論文於國際性學術研討會，及一篇期刊論文已被接受於 *IEE Proceedings- Communications*，也就是不僅達到原本計畫中預期的目標：分析正交分頻多工的精確錯誤率與通道容量。我們並成功的將我們發展的分析技術運用於其他無線通訊系統中，尤其是將研究焦點投射於多天線系統，包含時-空訊號處理的技術。研究成果顯然超越預期，並受國際學術期刊機構 IEE 的肯定與認可。

於泰國與加拿大發表的論文，分別得到國科會與淡江大學的補助而順利成行。與會中，直接與各國的專家學者相互討論與請詣，讓我們的計畫執行獲得相當大的助力，具體展現於報告中學理上的新發現，此外，我們在發展與驗證理論的過程，亦開發出整套模擬環境的系統雛型，未來有意完成方便與具親和力的介面，使研究成果能普及且益於產學界之研發與教學。

可供推廣之研發成果資料表

□ 可申請專利　■ 可技術移轉　日期：94 年 9 月 23 日

國科會補助計畫	計畫名稱：OFDM 系統之錯誤率與通道容量分析於 QAM 傳輸於雷利衰退通道 計畫主持人：嚴雨田 計畫編號：NSC 93－2213－E－032－017－ 學門領域：電信學門
技術/創作名稱	無線多天線傳輸系統之錯誤率精確評估
發明人/創作人	嚴雨田
技術說明	中文：我們成功發展 OFDM 系統在雷利無線通道下之精確錯誤率公式，且調變採用允許任意長方架構之 QAM。並且更進一步延伸，對於目前熱門的多天線系統亦適用，舉凡採用接收端天線叢集技術的最大比例組合，抑或多輸入與多輸出通道下之系統，我們都已將核心公式與程式建構完成。
	英文：We have successfully developed the exact closed-form symbol error probability for OFDM using arbitrarily rectangular QAM over Rayleigh fading channels. In fact, we also have successfully extended our method to the systems with maximal-ratio combining as well as basic space-time system. The core programs for computer simulation purpose have been finished.
可利用之產業 及 可開發之產品	無線通訊
技術特點	精確錯誤率分析，不需透過數值分析法
推廣及運用的價值	使無線通訊系統好壞之評估數據更為精確，降低傳統因估算誤差所帶來設計上不確定因素的風險

※　1.每項研發成果請填寫一式二份，一份隨成果報告送繳本會，一份送　貴單位研發成果推廣單位（如技術移轉中心）。

※　2.本項研發成果若尚未申請專利，請勿揭露可申請專利之主要內容。

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