互質因子，線性矩陣不等式及低階強健控制器之設計

Coprime Factors, Linear Matrix Inequalities, and Low-Order Controller Design

計畫編號：NSC 88-2213-E-032-027
執行期限：87 年 8 月 1 日至 88 年 7 月 31 日
計畫主持人：蕭照焜 淡江大學航空太空工程學系

摘要

本研究發展一個單輸入單輸出線性離散系統之低階強健控制器的設計方法。利用互質因子及 outer 函數的性質，將低階強健控制器的設計轉化為 zero-placement 的設計，此問題為 convex，並利用線性矩陣不等式來求解。本設計方法可用來解決諸如 pole-placement，模式匹配，及 H_∞ 最佳化等之強健控制問題。

關鍵詞：互質分解，outer 函數，線性矩陣不等式，低階強健控制器設計，強健控制

Abstract

This paper develops a low-order controller design method for linear discrete time-invariant single-input, single-output systems. Using the coprime factors and the properties of discrete outer functions, the low-order controller design becomes a zero-placement problem which is convex. The solutions are obtained using the linear matrix inequality techniques. The proposed design method is used to solve for several optimal control problems, such as pole-placement design, model-matching design, and H_∞ optimization problems.

Keywords - coprime factorization, discrete outer functions, linear matrix inequalities, low-order controller design, robust control design

1. Introduction

One of the most important design issues for many practical control problems is the design of low-order stabilizing controller. The design of low-order controllers to optimize certain H_2, H_∞ and pole-placement involves a biaffine matrix inequality (BMI) which is a non-convex programming problem and cannot be solved in polynomial time. As a result, several researchers [1],[4],[5],[3],[6] have shown that low-order controllers can be obtained by solving iteratively linear matrix inequality (LMI) subproblems, which are convex and can readily be solved using existing convex optimization software such as [2]. These approach include the alternating projection method [4], the rank condition minimization method [5], and the successive substitution method in [3],[6].

In [7], noniterative schemes for designing low-order controllers, for continuous single-input single-output (SISO) systems, to optimize certain performance indices were developed. The key step of the method proposed in [7] is the use of coprime factors such that based on the strictly positive real functions, feedback stabilization using low-order controllers becomes a zero-placement problem which is convex. The design does not involve any iterations so that no convergence consideration is needed.

In this paper, we will focus on the development of noniterative schemes for designing low-order controllers for discrete-time systems. We will only address the discrete linear time-invariant SISO systems in this paper. We establish a sufficient condition which guarantees a discrete transfer function being outer. That is, the zeros are all inside the unit circle of the z-plane. Using the coprime factors and the properties of discrete outer functions, we formulate the low-order controller design problem as a zero-placement problem which is convex. The solutions are obtained using the LMI techniques. The proposed design method is used to solve for several optimal control problems, such as pole-placement design, model matching design, and H_∞ optimization problems.

The paper is organized as follows. In Section 2 we formulate the design of low-order stabilizing controller as an LMI feasibility problem using the coprime factors and the properties of discrete outer function. Section 3 addresses the formulation of pole-placement design. Section 4 discusses the design of model-matching problems. The design of H_∞ suboptimal control problems is presented in Section 5.

2. Design of Low-Order Stabilizing Controllers

Consider a discrete linear time-invariant SISO system $G(z)$ with the minimum state-space realization

$$x(k+1) = Ax(k) + Bu(k) \quad (1a)$$

$$y(k) = Cx(k) \quad (1b)$$

where x is the n-dimensional state variable vector, u is the control input variable, and y is the measured output variable. We formulate a low-order controller design problem as one of finding an n_c-dimensional controller

$$u = -K(z)y \quad (2)$$

with $n_c \leq n\text{-}1$, to place the $n + n_c$ poles of the closed-loop system inside the unit circle of the z-plane, such that the closed-loop system is stable. We perform a coprime factorization of $G(z)$ to obtain

$$G(z) = \frac{N_g(z)}{D_g(z)} \quad (3)$$

where

$$D_g(z) = \frac{d_{gn}(z)}{d_f(z)},\ N_g(z) = \frac{n_{gn}(z)}{d_f(z)} \quad (4)$$

are stable, that is, the roots of $d_f(z)=0$ are within the unit circle of the z-plane. The coefficient of z^n in $d_f(z)$ is set to 1. In particular, the roots of $d_f(z)=0$ are selected to be the desired closed-loop poles of the system as of all the states are available for feedback control. Similarly, we perform a coprime factorization of the controller $K(z)$ to obtain

$$K(z) = \frac{N_c(z)}{D_c(z)} \quad (5)$$

where

$$N_c(z) = \frac{n_{cn}(z)}{d_c(z)},\ D_c(z) = \frac{d_{cn}(z)}{d_c(z)} \quad (6)$$

are stable, that is the roots of $d_c(z)=0$ are within the unit circle of the z-plane. We will also set the coefficient of z^{n_c} in $d_c(z)$ to be 1. In particular, the roots of $d_c(z)=0$ are selected to be the desired poles of the closed-loop system due to the controller. Note that the controller model (2) will not be known a–priori. However, we can still specify $d_c(z)$ based on control bandwidth specifications.

Let the numerators of the controller coprime factors (6) have the polynomial form

$$n_{cn}(z) = a_{n_c} z^{n_c} + a_{n_c-1} z^{n_c-1} + \ldots + a_0 \quad (7)$$

$$d_{cn}(z) = b_{n_c} z^{n_c} + b_{n_c-1} z^{n_c-1} + \cdots + b_0 \quad (8)$$

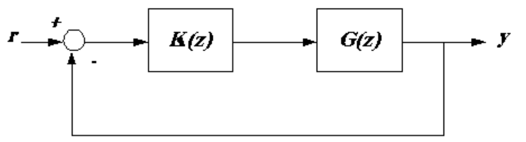

Figure 1. Closed-loop regulation system

We define the vectors a and b to contain the polynomial coefficients

$$a = [a_{nc}, a_{nc-1}, \cdots, a_0], \qquad b = [b_{nc}, b_{nc-1}, \cdots, b_0]$$

(9)

where $b_{nc} = 1$.

Consider the closed-loop regulation system in Figure 1. The closed-loop transfer function $T(z)$ from the command r to the output y is

$$T(z) = \frac{N_g(z)N_c(z)}{D_g(z)D_c(z) + N_g(z)N_c(z)} \quad (10)$$

We denote the denominator of $T(z)$ as

$$Q(z) = D_g(z)D_c(z) + N_g(z)N_c(z) \quad (11)$$
$$= \frac{d_{gn}(z)d_{cn}(z) + n_{gn}(z)n_{cn}(z)}{d_f(z)d_c(z)}$$

The following result shows the dependence of a state-space realization of $Q(z)$ on the coefficients a and b.

Lemma 1. *The function $Q(z)$ (11) has a state-space realization*

$$x_q(k+1) = A_q x_q(k+1) + B_q u_q(k) \quad (12a)$$

$$y_q(k) = C_q(a,b)x_q + u_q \quad (12b)$$

where x_q is an $(n+n_c)$ dimensional state vector, u_q and y_q are input and output variables, respectively, and A_q is Hurwitz. Furthermore, the state matrices have the following properties:

(i) the pair (A_q, B_q) is controllable and is independent of a and b.

(ii) $C_q(a,b)=aS+bR$ is a linear function of a and b, where

$$R = \begin{bmatrix} C_{qd}A_q^{n_c} \\ C_{qd}A_q^{n_c-1} \\ \vdots \\ C_{qd}A_q \\ C_{qd} \end{bmatrix}, \qquad S = \begin{bmatrix} C_{qn}A_q^{n_c} \\ C_{qn}A_q^{n_c-1} \\ \vdots \\ C_{qn}A_q \\ C_{qn} \end{bmatrix}$$

The following result is crucial to the development of our design method.

Lemma 2. *If there exists a symmetric positive definite matrix P, such that the following matrix inequalites are satisfied*

$$\begin{bmatrix} A_q^T P A_q - P & A_q^T P B_q - C_q^T(a,b) \\ B_q^T P A_q - C_q(a,b) & -1 \end{bmatrix} < 0 \quad (13)$$

$$1 - B_q^T P B_q \geq 0 \qquad (14)$$

then all zeros $Q(z)$ (11) are inside the unit circle.

Theorem 1. *If there exist coefficients a and b and a symmetric positive definite matrix P such that the LMIs (13) and (14) are satisfied, then the controller (2) is a low-order stabilizing controller for system (1).*

Theorem 1 gives a practical method for finding a low-order stabilizing controller. The LMIs (13) and (14) together with a set of linear constraints on $C_q(a,b)$ can be solved as a feasibility problem using a convex programming toolbox such as [2]. For each desired order of the controller, the feasibility problem can be solved to see whether such a low-order control exists. Thus this algorithm for computing the coefficients a and b to make $Q(z)$ outer function is convex, versus a non-convex problem without using coprime factors.

Theorem 1 can be used as a building block for solving more complex design problems. We will develop the solutions to a few of these problems in the following sections.

3. Pole-Placement Design

With a low-order controller, we no longer have the freedom to arbitrarily place all the closed-loop system poles. The objective then is to find a low-order controller such that the closed-loop system poles. The objective then is to find a low-order controller such that the closed-loop system poles are close to a set of pre-specified poles. Thus the first step is to set the poles of $Q(z)$ (that is, the roots of $d_f(z)d_c(z)=0$) at the desired locations. Then we propose an optimization problem

$$\min_{K(z)} \left\| W(z)(1 - Q(z)) \right\|_\infty \qquad (15)$$

subject to $Q(z)$ being outer function, where $W(z)$ is a stable weighting function, allowing the emphasis of the placement of the dominant closed-loop poles. We can regard (15) as a regional pole-placement problem, in which the pole-placement regions are determined by the poles of $Q(z)$. We remark that if we set $n_c=n-1$, then (15) becomes the reduced-order observer design in which all the desired poles can be exactly placed.

Let the minimal realization of the state-space model of $W(z)(1-Q(z))$ be given by

$$W(z)(1 - Q(z)) \leftrightarrow \left[\begin{array}{c|c} A_{wq} & B_{wq} \\ \hline C_{wq}(a,b) & 0 \end{array}\right] \qquad (16)$$

where $C_{wq}(a,b)$ is a linear function of a and b. Using the Bounded- Real Lemma [5], the optimization problem (16) can be expressed as

$$\min_{K(z)} \gamma \qquad (17)$$

subject to the LMIs

$$\begin{bmatrix} A_{wq}^T X A_{wq} - X & A_{wq}^T X B_{wq} & C_{wq}^T(a,b) \\ B_{wq}^T X A_{wq} & B_{wq}^T X B_{wq} - \gamma & 0 \\ C_{wq}(a,b) & 0 & -\gamma \end{bmatrix} < 0 \qquad (18)$$

$$X > 0 \qquad (19)$$

and the LMIs (13) and (14). This problem is convex and can be solved using an LMI algorithm in the unknowns P, X, a, b, and γ.

4. Model-Matching Design

In addition to pole-placement design, we can use the low-order controller design idea in Theorem 1 as a building block to obtain sub-optimal designs of many optimal control problems. One of the optimal control problems of interest is the model-matching problem, that is, the design of $u=-K(z)y$ for system (1) so that the closed-loop transfer function $T(z)$ (10) matches as closely possible, in the frequency domain, to a desired stable transfer function $T_d(z)$, which is usually a low-order transfer function incorporating the desire features of the control specifications. Using the coprime factor for the plant $G(z)$ and the controller $K(z)$, We can define this problem as

$$\min_{K(z)} \left\| W(z)(Td(z) - T(z)) \right\|_\infty = \min_{K(z)} \left\| W(z)(T(z) - \frac{N_g(z)N_c(z)}{D_g(z)D_c(z) + N_g(z)N_c(z)}) \right\|_\infty \qquad (20)$$

where $W(z)$ is a stable weighting function.

The optimization problem (20) cannot be directly set up as a convex optimization problem, because the controller coefficients a and b are contained in the denominator $Q(z)$. One way to circumvent this difficulty is to formulate a suboptimal control problem by removing the denominator $Q(z)$ and using it for a pole placement design. One such possibility is to pose the suboptimal control problem of

$$\min_{K(z)} \left\| \begin{bmatrix} W(z)(T_d(z) - N_g(z)N_c(z)) \\ W(z)(1 - (D_g(z)D_c(z) + N_g(z)N_c(z))) \end{bmatrix} \right\|_\infty \qquad (21)$$

If the design (21) yields $D_c(z)$ and $N_c(z)$ such that $Q(z)=1$, then the controller also satisfies (20). Besides (21), there are many other alternatives to develop a suboptimal control problem. We will only deal with (21) in this paper.

In the model-matching design problem, the poles of $T_d(z)$ are the desired dominant poles of the closed-loop system. Thus in performing the coprime factorization of the plant $G(z)$, $d_f(z)$ should contain these desired poles. Construct the minimum state-space realization

$$\begin{bmatrix} W(z)(T_d(z)-N_g(z)N_c(z)) \\ W(z)(1-(D_g(z)D_c(z)+N_g(z)N_c(z))) \end{bmatrix} \leftrightarrow \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m(a,b) & 0 \end{array}\right] \quad (22)$$

where $C_m(a,b)$ is a linear function of a and b.

The suboptimal control problem (21) can be readily solved as an LMI problem of

$$\min_{K(z)} \gamma \quad (23)$$

subject to the LMIs

$$\begin{bmatrix} A_m^T X A_m - X & A_m^T X B_m & C_m^T(a,b) \\ B_m^T X A_m & B_m^T X B_m - \gamma & 0 \\ C_m(a,b) & 0 & -\gamma \end{bmatrix} < 0 \quad (24)$$

$$X > 0 \quad (25)$$

and the LMIs (13) and (14) for $Q(z)$. The LMI variables for this problem are γ, a, b, and the positive matrices X and P.

5. H_∞ Suboptimal Design

Consider the two-input, two-output linear time-invariant system $G(z)$ with the state-space realization

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k) \quad (26a)$$

$$\eta(k) = C_1 x(k) + D_{12} u(k) \quad (26b)$$

$$y(k) = C_2 x(k) + D_{21} w(k) \quad (26c)$$

where x is the n-dimensional state variable vector, w is the disturbance input variable, u is the control input variable, η is the controlled output variable, and y is the measured output variable. Here the triple (A, B_2, C_2) is stabilizable and detectable. For system (26), we propose to develop a design algorithm to obtain a suboptimal low-order controller $u=-K(z)y$ to minimize the H_∞-norm of the closed-loop transfer function from w to η.

To use the result of Theorem 1, we first develop the appropriate coprime factors. We write (26) in transfer function form

$$\begin{bmatrix} \eta \\ y \end{bmatrix} = \begin{bmatrix} \frac{N_{11}(z)}{D_g(z)} & \frac{N_{12}(z)}{D_g(z)} \\ \frac{N_{21}(z)}{D_g(z)} & \frac{N_{22}(z)}{D_g(z)} \end{bmatrix} \begin{bmatrix} \omega \\ u \end{bmatrix} \quad (27)$$

where

$$N_{ij}(z) = \frac{n_{ijn}(z)}{d_f(z)}, \quad i,j = 1,2 \quad (28)$$

and

$$D_g(z) = \frac{d_{gn}(z)}{d_f(z)} \quad (29)$$

are stable transfer functions with $d_{gn}(z)=\det(zI-A)$. In this notation, the closed-loop transfer function of (26) from w to η, denoted as $T_{\eta w}(z)$,using the controller coprime factors (5) is

$$T_{\eta\omega}(z) = \frac{N_{11}(z)D_c(z) + \bar{N}(z)D_c(z)}{D_g(z)D_c(z) + N_{22}N_c(z)} \quad (30)$$

where

$$\bar{N}(z) = \frac{N_{11}(z)N_{22}(z) - N_{12}(z)N_{21}(z)}{D_g(z)} = \frac{\bar{n}_n(z)}{d_f(z)} \quad (31)$$

with $\bar{n}_n(z)$ being a polynomial.

For the same reason as the model-matching problem, we cannot directly minimize $\|T_{\eta w}\|_\infty$ because with the controller parameters in $D_g(z)D_c(z)+N_{22}(z)N_c(z)$, the resulting problem would be non-convex. Thus we formulate a suboptimal H_∞ control problem of

$$\min_{K(z)} \left\| \begin{bmatrix} W(z)(N_{11}(z)D_c(z) + \bar{N}(z)D_c(z)) \\ W(z)(1-(D_g(z)D_c(z)+N_{22}(z)N_c(z))) \end{bmatrix} \right\|_\infty \quad (32)$$

where $W(z)$ is a stable weighting function. Construct the minimum state-space realization

$$\begin{bmatrix} W(z)(N_{11}(z)D_c(z) + \bar{N}(z)D_c(z)) \\ W(z)(1-(D_g(z)D_c(z)+N_{22}(z)N_c(z))) \end{bmatrix} \leftrightarrow \left[\begin{array}{c|c} A_h & B_h \\ \hline C_h(a,b) & 0 \end{array}\right] \quad (33)$$

where $C_h(a,b)$ is a linear function of a and b.

The suboptimal H_∞ optimization problem (32) can be readily solved as an LMI problem of

$$\min_{K(z)} \gamma \quad (34)$$

subject to the LMIs

$$\begin{bmatrix} A_h^T X A_h - X & A_h^T X B_h & C_h^T(a,b) \\ B_h^T X A_h & B_h^T X B_h - \gamma & 0 \\ C_n(a,b) & 0 & -\gamma \end{bmatrix} < 0 \quad (35)$$

$$X > 0 \quad (36)$$

and the LMIs (13) and (14) for $D_g(z)D_c(z)+N_g(z)N_c(z)$. The LMI variables for this problem are γ, *a*, *b*, and the positive matrices *X* and *P*.

6. Conclusion

In this paper we have presented a new low-order controller design method for discrete single-input single-output systems. This method requires only the solution of a convex optimization problem. We established a sufficient condition which guarantees a transfer function being outer. Using the coprime factors and the properties of discrete outer functions, we show that the design of low-order controller can be formulated as a zero-placement problem. And, the solution can be obtained using LMI algorithms.

References

[1] T. Iwasaki and R. E. Skelton, "All Controllers for the General H_∞ Control Problem: LMI Existence Conditions and State Space Formulas," *Automatica*, Vol. 30, pp. 1307-1317, 1994.

[2] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, LMI *Control Toolbox*, The MathWorks, Inc., 1995.

[3] L. El Ghaoui and V. Balkrishnan, "Synthesisof Fixed-Structure Controllers via Numerical Optimization," *Proc. 1994* CDC, pp. 2678-2683.

[4] K. M. Grigoriadis and R. E. Skelton, "Fixed-Order Controller Problems Using Alternating Projection Methods," *Proc. 1994* CDC, pp. 2003-2008.

[5] P. Gahinet and P. Apkarian, "A Linear Matrix Inequality Approach to H_∞ Control," *Int. J. Robust and Nonlinear Control*, vol. 4, pp. 421-448, 1994.

[6] J. K. Shiau and J. H. Chow,"Robust Decentralized State Feedback Controller Design Using an Iterative Linear Matrix Inequality Algorithm," Preprints of 13th IFAC World Congress, Vol. H, pp. 203-208, 1996.

[7] S. Wang and J. H. Chow "Coprime Factors, Strictly Positive Real Functions, LMI and Low-Order Controller Design For SISO Systems", *Proceedings of the 37th IEEE Conference on Decision & Control*, Tampa, Florida,