行政院國家科學委員會專題研究計畫　成果報告

線性及非線性貝龍--佛羅貝紐斯理論(3/3)

計畫類別：個別型計畫
計畫編號：NSC94-2115-M-032-001-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：淡江大學數學系

計畫主持人：譚必信

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報告類型：完整報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中　華　民　國 95 年 10 月 30 日

目　　録

摘 要

這三年期專題研究計畫是延伸及擴展本人 (及共同作者) 以前在有限錐正算子幾何譜理論、最終非負方陣、錐體正變換的有向圖等方面的工作。貝龍—佛羅貝紐斯理論的多樣方向, 線性的或非線性的, 都被納入研究。所考慮的主要是下列四個項目: (I) 一對非線性影射的哥拉斯—威蘭集合; (II) 線性錐體正變換的有向圖及本原性; (III) (最終) 非負矩陣的貝龍—佛羅貝紐斯理論; 及 (IV) 線性錐體正變換的佛羅貝紐斯理論。特別是在第二及第四項我們獲得重大的進展。

首先, 我們統合出一套方法, 用以重証非負矩陣組合譜理論的四個重要結果, 並把它們推廣至有限面錐體 (或一般的常態錐體) 場合。爲達成此項目標, 我們引入了錐體正變換譜錐體的概念, 並且結合 Hartwig、Neumann 和 Rose 等人的代數兼分析方法及我們以前在錐體正變換幾何譜理論所發展出的一套方法。

接著, 我們研究線性錐體正變換的局部貝龍—佛羅貝紐斯理論。對任一 n 階實矩陣 A, 任一 $\mathbb{R}^n$ 中的向量 x 及任一非負整數 k, 我們以 $w_k(A,x)$ 表示由 A^kx, $A^{k+1}x, \ldots$ 等向量產生的錐體, 並稱之爲 A–循環錐體。我們獲得錐體 $w_k(A,x)$ 或其閉包能滿足下列一個或多個性質的等價條件: (i) 非零, (ii) 尖的, 或 (iii) 非線性子空間。答案則跟把 x 寫成 A 的廣義特徵向量之和的寫法有關。當 A 滿足 Perron-Schaefer 條件時, 我們可以取有限個 A–循環錐體的閉包之和以建構 A–不變常態錐體。利用定義在 $\mathcal{M}_n(\mathbb{R})$ 的線性算子 $L_A(X) = AX$, 我們不難獲得 Hans Schneider 在 80 年代初所得到的內在貝龍—佛羅貝紐斯理論。以

$\{\exp(tA)x : t \geq 0\}$ 代替 $\{A^i x : x \geq 0\}$, 我們也可以發展出一套互正矩陣的局部貝龍–佛羅貝紐斯理論。

對線性錐體正變換的有向圖我們也獲得甚多良好的結果。若 K 爲 $\mathbb{R}^n$ 上的常態錐體, 我們以 $\pi(K)$ 表示: 滿足 $AK \subseteq K$ 的 n 階實方陣 A 所構成的集合, 此集合爲 $\mathcal{M}_n(\mathbb{R})$ 上的常態錐體。若 $A \in \pi(K)$, 以 $(\mathcal{E}, \mathcal{P}(A,K))$ 表示有向圖, 其頂點集合 $\mathcal{E}$ 是由 K 的極端射線所構成, (E_1, E_2) 爲該圖的邊, 若 $E_2 \subseteq \Phi(AE_1)$, 其中 $\Phi(S)$ 表示 K 由子集 S 產生的 (錐) 面。對有向圖 $(\mathcal{E}, \mathcal{P}(A,K))$ 及與 A 有關的兩個幾何物體, 即由 A 在 $\pi(K)$ 所產生的面及暴露面, 我們獲得一些結果。我們發現若 K 爲常態錐體, 則其對偶算子爲 1–1 對應的充要條件爲: 對任意 $A, B \in \pi(K)$, $(\mathcal{E}(K), \mathcal{P}(A,K)) = (\mathcal{E}(K), \mathcal{P}(B,K))$ 若且唯若 $(\mathcal{E}(K^*), \mathcal{P}(A^T, K^*)) = (\mathcal{E}(K^*), \mathcal{P}(B^T, K^*))$。另外, 我們也証得若 K 爲組合等價已知, 則有向圖 $(\mathcal{E}, \mathcal{P}(A,K))$ 可完全決定 A 的 K–不可約性或 K–本原性。

在研究 K–本原指數的上界時, 有向圖 $(\mathcal{E}, \mathcal{P}(A,K))$ 是很有用的工具。對有限面錐體 K, 以 $\gamma(K)$ 表示 $\max\{\gamma(A) | A$ 爲 K–本原$\}$。其中 $\gamma(A)$ 爲 K–本原矩陣 A 的本原指數。我們可証明若 K 爲非單純有限面錐體, 且恰有 m 個極端射線, 則當 m 爲奇數時 $\gamma(K)$ 的最大值爲 $m^2 - 3m + 3$; 若 m 爲偶數, 則最大值爲 $m^2 - 3m + 2$。還有, 這些最大值是當 K 爲最小錐體, 且其極端向量滿足的線性關係兩邊的項數不超過 1 時達到。當 K 爲三維且恰擁有 m 個極端射線時, $\gamma(K)$ 的最大值則爲 $2m - 1$。若 K 爲 n 維, 且恰有 m 個極端射線時, $\gamma(K)$ 的最大值爲何? 這個問題尚未解決, 但對此我們已提出一個合理的猜想。

關鍵詞: 錐體正變換、有向圖、K–本原、K–不可約、K–本原指數、譜錐體、非負矩陣、梯階特徵數、高度特徵數、優先基底定理、優勢關係、主要分量、局部貝龍—佛羅貝紐斯理論、循環錐體、局部貝龍—謝弗條件、互正矩陣、譜橫坐標、有限面錐體、最小錐體。

Abstract

This three-year project is a continuation and extension of the previous work done by this investigator (and his collaborators) on the Geometric Spectral Theory of Positive Linear Operators (in finite dimensions), eventually nonnegative matrices, digraphs of cone-preserving maps, etc. Various aspects of the Perron-Frobenius theory of maps, linear or non-linear, are treated. We focus on the following four topics: (I) Collatz-Wielandt sets associated with a pair of nonlinear maps; (II) Digraph and primitivity for a linear cone-preserving map; (III) Perron-Frobenius theory of (eventually) nonnegative matrices; and (IV) Perron-Frobenius theory of linear cone-preserving maps. In particular, substantial contribution has been made on topics (II) and (IV).

A unified treatment is offered to reprove known results on the following four highlights of the combinatorial spectral theory of nonnegative matrices, or to extend (or partly extend) the results to the setting of a linear map preserving a polyhedral proper (or proper) cone: the preferred-basis theorem, equivalent conditions for equality of the (graph-theoretic) level characteristic and the (spectral) height characteristic, the strong majorization relation between the two characteristics, and the relation between the combinatorial properties of a nonnegative matrix and the positivity of the individual entries in its principal components. This is achieved by employing the new concept of spectral cone of a cone-preserving map and combining the cone-theoretic methods developed in our previous papers on the geometric spectral theory of cone-preserving maps with the algebraic-analytic method introduced by Hartwig, Neumann and Rose and further exploited by Neumann and Schneider for nonnegative matrices.

A local Perron-Frobenius theory for a linear cone-preserving map is pursued. For an $n \times n$ real matrix A, a vector $x \in \mathbb{R}^n$, and any nonnegative integer k, denote by $w_k(A, x)$ the convex cone generated by $A^k x, A^{k+1}x, \ldots$ in $\mathbb{R}^n$ and refer to it as an A-cyclic cone. Equivalent conditions are obtained for $w_k(A, x)$ or its closure to satisfy one or a combination of the following properties: (i) nonzero, (ii) pointed, or (iii) not a linear subspace. The answers are given in terms of the representation of x as a sum of generalized eigenvectors of A; namely, whether or not in the representation there is a generalized eigenvector that corresponds to $\rho_x(A)$ (the local spectral radius of A at x), a positive eigenvalue, a nonnegative eigenvalue, or the zero eigenvalue and with a corresponding generalized eigenvector of order not less than k. In particular, it is shown that $\operatorname{cl} w_k(A, x)$ is pointed if and only if A satisfies the local Perron-Schaefer condition at x. In case A satisfies the Perron-Schaefer condition, A-invariant proper cones are

constructed by taking the sum of the closures of finitely many A-cyclic cones. The intrinsic Perron-Frobenius theorems due to Hans Schneider are recovered by looking at the linear map L_A on the space of $n \times n$ real matrices given by $\mathrm{L}_A(X) = AX$. Similar results for the convex cone generated by $\{\exp(tA)x : t \geq 0\}$ (instead of $\{A^i x : i \geq 0\}$) are proved and local versions of Perron-Frobenius type theorems for cross-positive matrices (on K) are obtained; the concept of spectral pair (of a vector or of a face) for a cone-preserving map now becomes spectral abscissa for a cross-positive matrix. Some partial results are also obtained for the question of characterizing real square matrices A for which there is a proper cone K such that A is an automorphism on K (i.e., $AK = K$).

Various aspects of a general nature on the digraph of a cone-preserving map are investigated. For a proper cone K in $\mathbb{R}^n$, let $\pi(K)$ denote the proper cone in $\mathcal{M}_n(\mathbb{R})$ which consists of all real matrices A that satisfy $AK \subseteq K$. For $A \in \pi(K)$, let $(\mathcal{E}, \mathcal{P}(A, K))$ denote the digraph with vertex set $\mathcal{E}$, consisting of the extreme rays of K, such that (E_1, E_2) is an arc if and only if $E_2 \subseteq \Phi(AE_1)$, where $\Phi(S)$ denotes the face of K generated by S. The digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and two geometric objects associated with A, namely, the face and the exposed face of $\pi(K)$ generated by A are related. In particular, it is shown that for any $A, B \in \pi(K)$, the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(B, K))$ are equal if and only if the collection of simple faces of $\pi(K)$ containing A and the corresponding collection for B are the same; when A, B generate the same exposed face, they are both K-primitive or both not K-primitive, and if they both are, then they have the same exponent. (An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$.) It is proved that for a proper cone K, in order that we have for any $A, B \in \pi(K)$, the digraphs of A, B are equal if and only if the digraphs of A^T, B^T (as cone-preserving maps for the dual cone of K) are equal, it is necessary and sufficient that the duality operator of K (which is a natural map from the face lattice of K to the face lattice of its dual cone) be bijective (which is the case if K is polyhedral). It is also shown that the K-irreducibility and the K-primitivity of A are each completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ once K is known up to combinatorial equivalence.

The usefulness of the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is demonstrated in the study of the specific problem of determining bounds for the exponents of K-primitive matrices. For a polyhedral proper cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is denoted by $\gamma(K)$. It is proved that for the class of non-simplicial polyhedral cones K with m extreme rays (and of arbitrary dimension within the natural boundaries), the maximum value of $\gamma(K)$ is $m^2 - 3m + 3$ if m is odd and is $m^2 - 3m + 2$ if m is even; furthermore, the bounds are attained at minimal cones with the property that in the essentially unique linear relation on their extreme vectors the number of vectors on the two sides differ by at most 1, and also if A is an optimal K-primitive matrix then the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by one of the two primitive digraphs on m vertices for which the length of the shortest circuit is equal to $m-1$. It is also proved that the maximum value of $\gamma(K)$ as K runs through all 3-dimensional polyhedral cones with m extreme rays is $2m-1$. To answer the question of what the maximum value of $\gamma(K)$ is, as K runs through all n-dimensional polyhedral cones with m extreme rays, a conjecture is posed and confirmed for some specific cases.

Keywords: Cone-preserving map, digraph, K-primitive, K-irreducible, K-exponent, spectral cone, nonnegative matrix, level characteristic, height characteristic, preferred-basis theorem, majorization relation, principal components, local Perron-Frobenius theory, cyclic cone, local Perron-Schaefer condition, cross-positive matrices, spectral abscissa, polyhedral cone, minimal cone.

1. Motivation and Aims

This three-year project is a continuation and extension of the previous work done by this author (and his co-authors) on the Geometric Spectral Theory of Positive Linear Operators (in finite dimensions) ([TW], [T1], [TS1], [TS2], [TS3]; for surveys, see [T3] and [TS4]), eventually nonnegative matrices ([ZT]), digraphs of cone-preserving maps ([BT] and [TB]), etc. Various aspects of the Perron-Frobenius theory of maps, linear or non-linear, are treated. In particular, we focus on the following four topics:

(I) Collatz-Wielandt sets associated with a pair of nonlinear maps;

(II) Digraph and primitivity for a liner cone-preserving map;

(III) Perron-Frobenius theory of (eventually) nonnegative matrices;

(IV) Perron-Frobenius theory of linear cone-preserving maps.

To be more specific, the following four problems (or research directions) are being considered:

Problem 1. Let K_1, K_2 be proper cones in possibly different finite dimensional real vector spaces. Let $F, G : K_1 \to \operatorname{span} K_2$ be homogeneous, nonlinear maps. We extend the definitions of the four Collatz-Wielandt sets associated with a single linear map A that preserves a proper cone K to the pair (F, G) in the natural way; for instance, define

$$\Omega(F, G) = \{\omega \geq 0 : \exists x \in K_1 \setminus \{0\}, Gx - \omega Fx \in K_2\},$$

etc. Find appropriate conditions on F, G (such as (K_1, K_2)-monotonicity/nonnegativity, K_2-convexity/concavity, K_2-(upper or lower) semicontinuity, etc.] so that the known results on the Collatz-Wielandt sets associated with a single linear cone-preserving map still hold (fully or partially). Unify and/or extend as far as possible the existing Perron-Frobenius theory of nonlinear maps in the context of Collatz-Wielandt sets.

Problem 2. Let K be a proper cone and let $\pi(K)$ be the cone of all matrices A such that $AK \subseteq K$. Let $G(A)$ denote the digraph with vertex set equal to the set of all extreme rays of K and such that for any extreme rays F_1, F_2 of K, (F_1, F_2) is an arc if and only if $\Phi(AF_1) \supseteq F_2$, where $\Phi(S)$ denotes the face of K generated by S. We call $A \in \pi(K)$ K-primitive if there exists a positive integer m such that $A^m(K\setminus\{0\}) \subseteq \operatorname{int} K$; in this case, the smallest such m will be referred to as the K-exponent of A. For $A \in \pi(K)$, study the K-primitivity and K-exponent of A in terms of its associated digraph $G(A)$.

Problem 3. Improve/complete the recent work of Zaslavasky, McDonald and Naqvi on the characterization of the Jordan structure of seminonnegative (eventually) nonnegative matrices and the Jordan structure of the peripheral spectrum of a nonnegative matrix.

Problem 4. Continue some of my not completed work on the spectral theory of linear cone-preserving maps and study the open problems given in my review paper [T3].

2. Results and Discussions

In my first year of carrying out the research project, I worked on Problem 4 and completed paper [T4]. A list of the results obtained can be found in my first intermediate report. I don't want to repeat here. I'll attach the full paper [T4] in the Appendix.

In my second year I continued my work on Problem 4 and also worked on Problem 2, resulting in respectively the papers [T5] and [T6], but they have not been completed yet. I also did some partial work on Problem 3. A list of the results can be found in my second intermediate report. Again I don't want to repeat, but I'll attach the drafts of these papers in the Appendix. I want to add that in the meantime, I have also done some work concerning the omega limit sets for a cone-preserving linear map. Essentially, I reproved the results of Lins and Nussbaum [LN] in an easier way (using my cone-theoretic methods) and obtained more results. I gave a talk entitled "The omega limit set of a point under a cone-preserving map" in a local conference and also at the recent 13th ILAS Conference held at Amsterdam. A draft of my talk will be given in the Appendix. I intend to include this work as one section of [T5]. Let me just give the main result I have obtained.

If f is a self-map on a set D and x is in D, by the omega limit set of x under f we mean the set which consists of the limits of convergent subsequences of the sequence $(f^k x)_{k=1}^{\infty}$.

Theorem. Let K be a proper cone in $\mathbb{R}^n$. Let $A \in \pi(K)$ be non-nilpotent. Define T on $(S^{n-1} \cap K) \setminus \mathcal{N}(A)$, where S^{n-1} denotes the Euclidean sphere of $\mathbb{R}^n$, by $Tx = \frac{Ax}{\|Ax\|}$. Let Λ denote the set consisting of peripheral eigenvalues of A with index equal to that of the spectral radius $\rho(A)$, and let $M = \mathbb{R}^n \cap \oplus_{\lambda \in \Lambda} [(\lambda I_n - A)^{\nu-1} \mathcal{N}((\lambda I_n - A)^{\nu})]$, where we use ν to denote the index of the spectral radius. Let $x \in S^{n-1} \cap K$ be such that the A-invariant face of K generated by x is K. Then

(i) $\Phi(\omega(x;T)) = \Phi(M \cap K)$, and

$$\omega(x;T) \subseteq \begin{cases} \mathrm{ri}(M \cap K) & \text{if } \mathrm{orbit}_T(x) \cap \mathrm{int}\, K \neq \emptyset \\ \mathrm{rbd}(M \cap K) & \text{otherwise} \end{cases},$$

where $\mathrm{orbit}_T(x)$ denotes the orbit of x under T, i.e., $\{T^j x : j = 0, 1, ...\}$.

(ii) If A has no eigenvector in $\mathrm{int}\, K$, then $\Phi(M \cap K) \subseteq \partial K$.

(iii) If $\frac{1}{\rho(A)}\Lambda$ consists of roots of unity (in particular, if K is polyhedral), then $\omega(x;T)$ is a finite set.

In the last months of the second year and in the third year, I, together with Raphael Loewy (of Israel Institute of Technology), continued the work on Problem 2. We got some exciting results, which are intended for the paper [LT]. Below I give a brief description of what has been done. A draft of [LT] will be given in the Appendix.

We consider the following problem posed by Stephen Kirkland in 1999 at the 8th International Linear Algebra Conference:

> If K is a polyhedral (proper) cone in $\mathbb{R}^n$ with m extreme rays, what is the maximum value of the exponents of K-primitive matrices ?

Here by a K-primitive matrix we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. In view of Wielandt's sharp bound for exponents of (nonnegative) primitive matrices of a given order, Kirkland conjectured that $m^2 - 2m + 2$ is an upper bound for the maximum value.

For $A \in \pi(K)$, we work with the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ defined in the following way: Its vertex set is $\mathcal{E}$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. (Here $\Phi(y)$ denotes the face of K generated by y.) When K is the nonnegative orthant $\mathbb{R}^n_+$, $(\mathcal{E}, \mathcal{P}(A, K))$ becomes the usual digraph associated with A^T, the transpose of A.

When the polyhedral cone K is a nonnegative orthant (or a simplicial cone), the study of K-primitive matrices is reduced to the classical nonnegative matrix case. The general polyhedral cone case differs from the nonnegative matrix case in at least the following two respects. First, in the nonnegative matrix case the (distinct) extreme vectors of the underlying cone are linearly independent, whereas in the general polyhedral cone case the extreme vectors of of the underlying cone satisfy certain nonzero linear relations. Second, in the nonnegative matrix case, given any (finite) digraph, it is always possible to find a nonnegative matrix with the given digraph as its usual associated digraph. On the other hand, in the general polyhedral cone case, we are confronted with the realization problem. Usually it is not easy to tell whether there is a polyhedral cone K for which there is a K-nonnegative matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by a prescribed digraph. As expected, and also illustrated by our work, the study of the polyhedral cone case is much more difficult than the classical nonnegative matrix case.

To begin with, we obtain an upper bound for the local exponents, and hence also an upper bound for the exponent, of a K-primitive matrix A in terms of the lengths of circuits in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and the degree of the minimal polynomial of A. The result suggests that for a K-primitive matrix A, the larger is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A, K))$ the larger is the value of $\gamma(A)$. And we note that the length of the shortest circuit is at most $m - 1$. Then we identify those digraphs on m vertices, with the length of the shortest circuit equal to $m - 1$, that may be realized as $(\mathcal{E}, \mathcal{P}(A, K))$ for some K-primitive matrix A, where K is a polyhedral cone with m extreme rays. We find that, up to graph isomorphism, there are two of them, represented by Figure 1 or Figure 2, as given below:

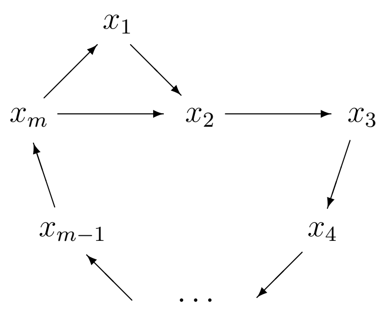

Figure 1.

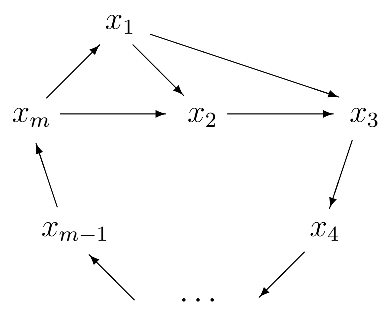

Figure 2.

(For simplicity, we label the vertex $\Phi(x_i)$ simply by x_i.).

We also obtain a geometric property on polyhedral cones K with m extreme rays, for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2 — they are either indecomposable or are the direct sum of a ray and an indecomposable minimal cone of a special kind, and when it is the latter, the digraph must be given by Figure 2. Here by a minimal cone we mean a polyhedral cone whose number of extreme rays equals the dimension of the cone plus 1; equivalently, it is a polyhedral cone whose extreme vectors satisfy, up to multiples, a unique linear relation.

Next, we prove that for a polyhedral cone K with m extreme rays, if A is a K-primitive matrix, then

$$\gamma(A) \leq (m_A - 1)(m - 1) + 1,$$

where m_A is the degree of the minimal polynomial of A; moreover, the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. As a consequence, if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A) : A \text{ is } K\text{-primitive}\}$, does not exceed $(n-1)(m-1)+1$. As a by-product, we answer in the affirmative the conjecture posed by Kirkland mentioned at the beginning.

Then we prove that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even. We also determine (up to linear isomorphism) the minimal cones K (and also the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attains the maximum value. It is found that in the (essentially) unique linear relation on the extreme vectors of an optimal minimal cone the number of vectors on the two sides of the relation differ by at most 1, and furthermore if K is an optimal minimal cone and A is a K-primitive matrix such that $\gamma(A)$ attains the maximum value then necessarily the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2.

We also prove that the maximum value of $\gamma(K)$ as K runs through all 3-dimensional polyhedral cones with m extreme rays is $2m-1$. In fact, we show that if K is a 3-dimensional polyhedral cone with m extreme rays and A is a K-primitive matrix, then $\gamma(A) = 2m-1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. For every positive integer $m \geq 5$ we also demonstrate the existence of a 3-dimensional polyhedral cone K with m extreme rays for which there does not exist a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since every two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, this means that the exponents of combinatorially equivalent cones may be different.

For every pair of positive integers m, n with $3 \leq n \leq m$ we are able to construct an n-dimensional polyhedral cone K with m extreme rays for which there is a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.

Kirkland's problem has not been resolved yet, but we have made some very nice contribution. We have found some evidence that support the following:

Conjecture. The maximum value of $\gamma(K)$ as K runs through all n-dimensional polyhedral cones K with m extreme rays equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and equals $(n-1)(m-1)$ when m is odd and n is even.

3. Self-evaluation of Performance

Seeds have been sown, but it will take time to ripen. For the time being, visible outcomes include the four long (or very long) papers [T4],[T5],[T6],[LT] given in the reference list. Paper [T4] has already appeared, whilst the other three papers remain to be completed.

Paper [T4] provides a unified cone-theoretic (geometric) treatment for four highlights of the combinatorial spectral theory of nonnegative matrices, which were previously obtained mainly by matrix-theoretic and graph-theoretic methods. It covers the work of more than 10 people in about two dozen papers. Not only does the paper re-prove known results, it also yields new results — extensions to the setting of cone-preserving maps, especially for polyhedral cones. It demonstrates the powerfulness of the cone-theoretic methods and gives hopes for further work in the cone-preserving map setting. Paper [1] is the sixth of a sequence of papers on the Geometric Spectral Theory of Positive Operators (in finite dimensions); the previous papers in the sequence are [TW], [T1], [TS1], [TS2], [TS3]. (For a survey, see [T3]. A more updated survey is also given in [TS4].) The next paper in the sequence is going to be [T6]. The work is about 70% done, and at present I have already got enough material to write several papers (for journals in the SCI list). But to keep the high standard of the papers in the sequence, I would rather make a thorough study and publish the work as a whole. (There is no point to break a large jewel into smaller pieces !) Even when the paper [T6] is completed, my Problem 4 is far from being completed — then it is only about 20% done, as I have hardly touched upon the open problems I posed in my review paper [T3].

Papers [T6] and [LT] are respectively about digraphs for a cone-preserving map and the study of exponents of K-primitive matrices (using one of the digraphs as a tool). The work in [T6] is about 90% done (it is only a matter of wrapping it up), and that in [LT] is about 75% done. We have hardly tried the open questions I posed for Problem 2, and in working on the paper [LT] we have found new questions. So I would say Problem 2 is only about 30% done.

I want to add that in [LT] we deal with the polyhedral cone case, not with proper cone in general. We have found that the situation for the polyhedral cone case is much more involved than the simplicial cone case, or equivalently, the classical nonnegative matrix case, and we have been able to deal with only the minimal cones (which, besides the simplicial cones, are the next simplest cones) and only for certain problems, and already the solution is very interesting and nontrivial. Given that so much work can be (and has been) done on the primitive nonnegative matrices (see, for instance, [BR], [HN], [S] and the references therein), I expect that considerably more work can be done on primitive matrices over polyhedral cones or proper cones in general. What we have achieved is only the tip of the iceberg. It is a new area !

I have done some partial work on Problem 3, but not substantial enough for publication; I didn't try much. I have not worked on Problem 1; I just couldn't find the time for it.

One of the main problems I have been facing is the lack of time. The teaching load and other academic duties (refereeing and editorial jobs) are too much for me. I have two Ph.D. students, and they have been taking some of my graduate course, in which I teach them nonnegative matrix theory and cone-preserving maps. But my open problems might be too hard for them — of over a dozen open problems I ever posed in journals, only two have been solved by other people, and it takes time for them to mature. So I can't count on them, at least for a couple of years.

To conclude I would say in these three years I have accomplished less than 20% of what I intended to do. But already I have produced some very good work. I would say my project is like a gold mine, and each of the four problems alone is worth a three-year project. How well have I done ? I leave it to you to judge.

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THE PERRON GENERALIZED EIGENSPACE AND THE SPECTRAL CONE OF A CONE-PRESERVING MAP

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August 23, 2004

Abstract. A unified treatment is offered to reprove known results on the following four highlights of the combinatorial spectral theory of nonnegative matrices, or to extend (or partly extend) the results to the setting of a linear map preserving a polyhedral proper (or proper) cone: the preferred-basis theorem, equivalent conditions for equality of the (graph-theoretic) level characteristic and the (spectral) height characteristic, the majorization relation between the two characteristics, and the relation between the combinatorial properties of a nonnegative matrix and the positivity of the individual entries in its principal components. This is achieved by employing the new concept of spectral cone of a cone-preserving map and combining the cone-theoretic methods developed in our previous papers on the geometric spectral theory of cone-preserving maps with the algebraic-analytic method introduced by Hartwig, Neumann and Rose and further exploited by Neumann and Schneider for nonnegative matrices.

*Research supported by the National Science Council of the Republic of China

AMS classification: 15A48.

Key words: Cone-preserving map, nonnegative matrix, level characteristics, height characteristic, semi-distinguished invariant face, Perron generalized eigenspace, principal component, spectral cone, preferred-basis theorem, majorization relation.

1. Introduction

This is the sixth of a sequence of papers (namely, [T-W], [Tam2], [T-S1], [T-S2], [T-S3] and the current paper) studying the classical Perron-Frobenius theory of (entrywise, square) nonnegative matrices and its generalizations to cone-preserving maps in the finite-dimensional setting from a cone-theoretic viewpoint. In this paper we provide a unified approach to treat several topics of interest in the combinatorial spectral theory of nonnegative matrices. We extend two theorems for nonnegative matrices to the setting of cone-preserving maps, one of which is about equivalent conditions for equality of the level characteristic and the height characteristic, and the other is about a majorization relation between the two characteristics. We also provide cone-theoretic proofs for the preferred-basis theorem and for a result about the nonnegativity structure of the principal components of a nonnegative matrix.

One early focus of interest in the study of nonnegative matrices is the questions of existence of a nonnegative basis for the Perron generalized eigenspace of the matrix and the properties of such a basis. The first approach to the questions is matrix combinatorial and uses the Frobenius normal form. In [H-N-R] Hartwig, Neumann and Rose give an algebraic-analytic proof, which utilizes the resolvent expansion but does not involve the Frobenius normal form, for the existence of a nonnegative basis. Their approach is further developed by Neumann and Schneider [N-S1, 2, 3], and the connection between the two approaches is also examined in detail.

In this paper we employ the new concept of spectral cone of a cone-preserving map and combine the cone-theoretic methods developed in our sequence of papers with the algebraic-analytic method of Hartwig, Neumann and Rose.

We now describe the contents of this paper in some detail. Unless specified otherwise, hereafter in this section we always use K to denote a proper (i.e., closed full pointed convex) cone in $\mathbb{R}^n$, and use A to denote an $n \times n$ real matrix that preserves K, i.e., $AK \subseteq K$. We also use $\pi(K)$ to denote the set of all matrices that preserve K.

Section 2 contains most of the definitions which we use in the paper.

In Section 3, we first clarify the logical relations between several natural conditions (involving the resolvent $(\lambda I - A)^{-1}$, the principal components $Z_A^{(k)}$ or transform principal components $J_A^{(k)}(\varepsilon)$), each of which is sufficient for the Perron generalized eigenspace of A to possess a K-semipositive basis (i.e., one made up of vectors from K). Next, we derive some new properties of A in case K is polyhedral (finitely generated). This is achieved by introducing a cone-duality argument into the method of Hartwig, Neumann and Rose — since $\pi(K)$ is itself a proper cone, in order to prove $AK \subseteq K$, it suffices to show that $\langle A, B \rangle \geq 0$ for all $B \in \pi(K)^*$. When K is polyhedral, we obtain, in particular, the following property of A, which is crucial for the later developments in this paper: For $k = 0, 1, \ldots, \nu_\rho - 1$, where ν_ρ denotes the index of the spectral radius of A, and for sufficiently small $\varepsilon > 0$, we have $J_A^{(k)}(\epsilon) \in \pi(K)$. From this, we recover the known result that the Perron generalized eigenspace of A always has a K-semipositive basis. From another property of A (again when K is polyhedral), we deduce a result of Meyer and Stadelmaier [M-S] on the characterization of a singular M-matrix in terms of the Drazin pseudo-inverse of the matrix. A by-product of our approach is that if A is cross-positive on a proper cone K, then the $(\nu_\tau - 1)$th component of A corresponding to its spectral abscissa τ, where ν_τ denotes the index of τ, belongs to $\pi(K)$. This extends the corresponding result due to Hans Schneider [Sch1] for the case when $A \in \pi(K)$.

In Section 4, we introduce and study the spectral cone of A, i.e., the set $C(A, K) := \{x \in K : (A - \rho(A)I)^j x \in K \text{ for all positive integers } j\}$. Conceivably, this set can be useful in the study of K-semipositive Jordan chains for A, and in the study of preferred-basis of a nonnegative matrix (as each vector in such a basis clearly belongs to the spectral cone). We show that when K is polyhedral, for sufficiently small $\epsilon > 0$, we have the inclusion relations $J_A^{(0)}(\epsilon)K \subseteq C(A, K) \subseteq E(A) \cap K$, which implies that $C(A, K)$ is a polyhedral full subcone of $E(A) \cap K$, where $E(A)$ denotes $\mathcal{N}((A - \rho(A)I)^n)$, the Perron generalized eigenspace of A.

In Section 5, we consider the problem of extending the equivalent conditions for equality of the height and the level characteristics of a nonnegative matrix (corresponding to its spectral radius) to the setting of a cone-preserving map. When K is polyhedral, we define $\lambda(A)$, the level characteristic of A, to be the ν_ρ-tupe $(\lambda_1, \ldots, \lambda_{\nu_\rho})$ given by:

$$\lambda_k = \dim \operatorname{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K] - \dim \operatorname{span}[\mathcal{N}((A - \rho(A)I)^{k-1}) \cap K].$$

It is known that in the nonnegative matrix case the above definition is equivalent to the usual definition for level characteristic (see Remark 2.2 and the paragraph following it). Similarly, we also define the level of a vector in $E(A)$, and then extend the definitions of level basis, height-level basis, peak vector, etc. from a nonnegative matrix to a cone-preserving map on a polyhedral proper cone; indeed, these definitions also hold in the setting of a proper cone, provided that we take suitable precautions. For a general proper cone K, we give in the main theorem of this section (Theorem 5.9) five conditions each equivalent to the condition that the level characteristic and the height characteristic of A are the same. In the same theorem we also give two conditions, both expressed in terms of the spectral cone of A, which are each equivalent to the condition that $E(A)$ has a K-semipositive Jordan basis. We show that conditions in the second group of equivalent conditions imply those in the first group, and also that when K is polyhedral, the conditions in the two groups are all equivalent. For a proper cone K, in terms of the spectral cone of A, we determine the maximum number of K-semipositive Jordan chains for A of length k whose union is a linearly independent subset of $E(A)$ for $k = 1, \ldots, \nu_\rho$. We thus extend and clarify the previous work of other people in this direction for nonnegative matrices ([H-S2], [Her1], [N-S3]).

In Section 6, we take up the task of offering a cone-theoretic proof for the Hershkowitz-Richman-Rothblum-Schneider preferred-basis theorem. The existence of such a proof was anticipated by this author in [Tam2] when he offered a cone-theoretic proof for the nonnegative-basis theorem. Strictly speaking, our proof is not entirely cone-theoretic; we settle the hardest part by a cone-theoretic argument, but for the remaining parts, our argument is matrix-theoretic and depends on the combinatorial structure of a nonnegative matrix. We observe that Rothblum [Rot] establishes part(2) of his main theorem (which is now known as the Rothblum index theorem) by proving a stronger result, and also that once the result is established, it is not difficult to obtain the preferred-basis theorem. (For more detailed descriptions, see our Section 6.) Our contribution is to provide an extension of the said result in the setting of a linear map preserving a polyhedral proper cone (see Theorem 6.5). It is interesting to note that our latter result is expressed in terms of the concepts of a semi-distinguished invariant face and a spectral cone, and its proof relies on a number of facts, one of which is a lemma crucial for our proof of an extension of the Rothblum index theorem to a linear map preserving a polyhedral proper cone (see [T-S2, Theorem 5.1 and Lemma 6.2]).

In Section 7, we show that the known majorization relation between the level characteristic and the height characteristic of a nonnegative matrix can be extended to the setting of a linear map that preserves a polyhedral proper cone. Our proof relies on the following property of such a map, which is not shared by a general cone-preserving map: For $k = 1, \ldots, \nu_\rho$,

$$(A - \rho(A)I)\,\mathrm{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K] \subseteq \mathrm{span}[\mathcal{N}((A - \rho(A)I)^{k-1}) \cap K].$$

Thus, we also find an alternative short proof for the original majorization result for a nonnegative matrix.

In Section 8, based on the preferred-basis theorem (or rather, a weak version of it) and on the nonnegativity of the 0th transform principal component, we offer a conceptual proof for a fundamental result about the nonnegativity structure of the principal components of a nonnegative matrix given by Neumann and Schneider [N-S1]. Again we are able to formulate a critical part of our argument in the setting of a linear map preserving a polyhedral proper cone (see Theorem 8.3).

In Section 9, among our final remarks, we conjecture a cone version of the nonnegative-basis theorem.

Earlier versions of this work (which consist of most parts of Sections 3 and 5 of the present paper) were reported by this author in two talks, which both bore the title “On semipositive bases for a cone-preserving map”, one at the Second International Summer School on Linear Algebra and Applications held at Guanzhou, China on August 29-September 2, 1994 and the other one at the ILAS Conference held at Haifa, Israel on June 25–29, 2001. In subsection 6.2 of the expository-research paper [Tam4], there was also an announcement of results about this work, namely, Corollary 3.4 and part of Theorem 5.9 of this paper. As considerably more new material was found subsequently, the title was altered to reflect the change in contents.

2. Preliminaries

As for previous papers in this sequence we work in the setting of a finite-dimensional vector space. A familiarity with convex cones, cone-preserving maps, and graph-theoretic properties of nonnegative matrices is assumed. For convenience, we collect in this section the definitions and notations that are used throughout the paper. A few more definitions and notations will be introduced in later sections.

We use the terms "matrix" and "linear mapping" interchangeably. The *nullspace* and the *range space* of a matrix A are denoted respectively by $\mathcal{N}(A)$ and $\mathcal{R}(A)$. The spectral radius of A is denoted by $\rho(A)$. By the *index* of A, denoted by $\nu(A)$, we mean the smallest nonnegative integer k such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$. For any eigenvalue λ of A, we denote by $\nu_\lambda(A)$ (or ν_λ) the *index of* λ *as an eigenvalue* of A, i.e., $\nu(A - \lambda I)$. We use $E_\lambda^{(0)}$ (or $E_\lambda^{(0)}(A)$ when there is the need to indicate the dependence on A) to denote the projection of $\mathbb{C}^n$ onto the generalized eigenspace $\mathcal{N}((\lambda I - A)^n)$ along the direct sum of other generalized eigenspaces of A, and define the components of A corresponding to λ by

$$E_\lambda^{(k)} = (A - \lambda I)^k E_\lambda^{(0)} \text{ for } k = 0, 1, \ldots .$$

For background material on the component matrices see Lancaster and Tismenetsky [L-T, Chapter 9] and, in the case of nonnegative matrices, see Neumann and Schneider [N-S1].

If K is a proper (i.e., closed pointed full convex) cone in $\mathbb{R}^n$, we denote by $\pi(K)$ the set of all $n \times n$ real matrices A such that $AK \subseteq K$. Matrices in $\pi(K)$ are referred to as cone-preserving maps on K. Hereafter, we always use K to denote a proper cone in $\mathbb{R}^n$ for some positive integer n.

Let $A \in \pi(K)$. We call the generalized eigenspace $\mathcal{N}((A - \rho(A)I)^n)$ the *Perron generalized eigenspace* of A and denote it simply by $E(A)$. (In case A is a nonnegative matrix, the space $E(A)$ is also referred to as the *Perron eigenspace* or *algebraic eigenspace* of A by other authors.) We denote the principal eigenprojection (also known as the Perron eigenprojection) and the kth principal component of A, i.e., $E_{\rho(A)}^{(0)}$ and $E_{\rho(A)}^{(k)}$, simply by $Z_A^{(0)}$ and $Z_A^{(k)}$ respectively. For convenience, we often denote $\nu_{\rho(A)}(A)$ by ν_ρ. By considering the Jordan canonical form of A, one readily shows that for all ε in a sufficiently small punctured disk centered at the origin of the complex plane, the following expansions are valid:

$$((\rho(A) + \varepsilon)I - A)^{-1} = Z_A^{(0)}/\varepsilon + Z_A^{(1)}/\varepsilon^2 + \cdots + Z_A^{(\nu_\rho - 1)}/\varepsilon^{\nu_\rho} + R(\varepsilon), \tag{2.1}$$

and

$$(\rho(A)I - A + \varepsilon Z_A^{(0)})^{-1} = Z_A^{(0)}/\varepsilon + Z_A^{(1)}/\varepsilon^2 + \cdots + Z_A^{(\nu_\rho - 1)}/\varepsilon^{\nu_\rho} + R(0), \tag{2.2}$$

where $R(\varepsilon)$ is an analytic operator in the corresponding nonpunctured disk satisfying $Z_A^{(0)}R(\varepsilon) = R(\varepsilon)Z_A^{(0)} = 0$. The algebraic-analytic approach of Hartwig, Neumann and Rose [H-N-R] depends much on the expansion (2.1).

For $k = 0, \ldots, \nu_\rho - 1$, following Neumann and Schneider [N-S3], we define the *kth transform principal component of A* by:

$$J_A^{(k)}(\varepsilon) = Z_A^{(k)} + Z_A^{(k+1)}/\varepsilon + \cdots + Z_A^{(\nu_\rho-1)}/\varepsilon^{\nu_\rho-k-1} \quad \text{for all} \quad \varepsilon \in \mathbb{C}\backslash\{0\}.$$

Then straightforward calculations yield

$$J_A^{(k)}(\varepsilon) = \varepsilon Z_A^{(k)}((\rho(A)+\varepsilon)I - A)^{-1}, \tag{2.3}$$

and

$$J_A^{(k)}(\varepsilon) = (A - \rho(A)I)^k J_A^{(0)}(\varepsilon) \tag{2.4}$$

for $k = 0, \ldots, \nu_\rho - 1$ (and with appropriate restrictions on ε).

Indeed, for any eigenvalue λ of A, the following expansion, which is similar to (2.1), (and also the one similar to (2.2)) is valid:

$$((\lambda+\varepsilon)I - A)^{-1} = E_\lambda^{(0)}/\varepsilon + E_\lambda^{(1)}/\varepsilon^2 + \cdots + E_\lambda^{(\nu_\lambda-1)}/\varepsilon^{\nu_\lambda} + R(\varepsilon). \tag{2.5}$$

[When $\lambda \neq \rho(A)$, the operator $R(\varepsilon)$ that appears in (2.5) is different from the one that appears in (2.1) but they have similar properties.]

If we define the kth transform component of A corresponding to λ by

$$F_\lambda^{(k)}(\varepsilon) = E_\lambda^{(k)} + E_\lambda^{(k+1)}/\varepsilon + \cdots + E_\lambda^{(\nu_\lambda-1)}/\varepsilon^{\nu_\lambda-k-1},$$

then we also have the relations corresponding to (2.3) and (2.4).

We would like to point out that if A is a nonnegative matrix, then the left side of (2.1) is in fact the resolvent of the minus (singular) M-matrix associated with A (i.e., the matrix $A - \rho(A)I$). In [N-S1,2,3] Neumann and Schneider formulate their results in terms of minus M-matrices.

We denote by $\geq^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geq^K y$ if and only if $x - y \in K$. Sometimes we also write $x \gg^K 0$ (respectively, $x >^K 0$) for $x \in \operatorname{int} K$ (respectively, $x \geq^K 0$ and $x \neq 0$) and call the vector x *K-strictly positive* (respectively, *K-semipositive*). We reserve the notation $\geq$, $>$, $\gg$ and the terms *semipositive*, *strictly positive*, for the componentwise ordering, that is, for the special case when K is equal to the nonnegative orthant $\mathbb{R}^n_+$. A subset of $\mathbb{R}^n$ is said to be *K-semipositive* if it is composed of K-semipositive vectors. It is clear that a subspace W of $\mathbb{R}^n$ contains a K-semipositive basis if and only if $W = \operatorname{span}(W \cap K)$, or equivalently, $W \cap K$ is a full cone in W.

A closed, pointed cone is called *polyhedral* if it has finitely many extreme rays. For any nonempty subset S of $\mathbb{R}^n$ we use S^* to denote the dual of S in $\mathbb{R}^n$, i.e., $S^* = \{z \in \mathbb{R}^n : \langle z, x\rangle \geq 0 \text{ for all } x \in S\}$, where $\langle\ ,\ \rangle$ is the usual inner product of $\mathbb{R}^n$ given by $\langle x, z\rangle = z^T x$. The set S^* is always a closed convex cone. When K is a cone, K^* is called the *dual cone* of K. It is well-known (see [S-V], [Tam 3]) that if K is a proper (respectively, polyhedral proper) cone in $\mathbb{R}^n$, then K^* is a proper (respectively, polyhedral proper) cone in $\mathbb{R}^n$ and $\pi(K)$ is a proper (respectively, polyhedral proper) cone in $\mathcal{M}_n(\mathbb{R})$, the space of $n \times n$ real matrices; also, we have $(K^*)^* = K$.

By a *Jordan chain for* A ($\in \pi(K)$) of length k (corresponding to $\rho(A)$) we mean a sequence of k nonzero vectors

$$x, (A - \rho(A)I)x, \ldots, (A - \rho(A)I)^{k-1}x$$

such that $(A - \rho(A)I)^k x = 0$; if all vectors in the sequence lie in K, we refer to the sequence as a *K-semipositive Jordan chain.* A basis for $E(A)$ is called a (K-semipositive) *Jordan basis for* A if it is composed of (K-semipositive) Jordan chains for A.

Suppose the Jordan blocks of A associated with $\rho(A)$ have sizes $\sigma_1, \ldots, \sigma_s$, where $\sigma_1 \geq \cdots \geq \sigma_s > 0$. The sequence $\sigma = (\sigma_1, \ldots, \sigma_s)$ is called the *Segré characteristic* of A associated with $\rho(A)$. The *Jordan diagram* of A for $\rho(A)$, denoted by $J(A)$, is the diagram formed by s columns of stars such that the jth column (from the left) has σ_j stars. The sequence $\eta = (\eta_1, \ldots, \eta_p)$ (or written as $\eta(A) = (\eta_1(A), \ldots, \eta_p(A))$ when there is the need to indicate the dependence on A) of row lengths of $J(A)$ (read upwards) is called the *height characteristic* of A (associated with $\rho(A)$). (In other words, η is the conjugate sequence of σ.) Clearly, $\eta_1 \geq \cdots \geq \eta_p > 0$. As is well-known, $p = \nu_{\rho(A)}(A)$ and $\eta_1 + \cdots + \eta_k = \dim \mathcal{N}((A - \rho(A)I)^k)$ for $k = 1, \ldots, p$. Hence,

$$\eta_k = \dim \mathcal{N}((A - \rho(A)I)^k) - \dim \mathcal{N}((A - \rho(A)I)^{k-1}).$$

Also, it is readily seen that $\eta_k = \dim(A - \rho(A)I)^{k-1}\mathcal{N}((A - \rho(A)I)^k)$.

Let $A \in \pi(K)$. By the *height* of a vector $x \in E(A)$, denoted by $ht(x)$, we mean the smallest nonnegative integer k such that $(A - \rho(A)I)^k x = 0$. (As usual, we adopt the convention $(\rho(A)I - A)^0 = I$.) A basis $\mathcal{B}$ for $E(A)$ is called a *height basis for* A if the number of vectors in $\mathcal{B}$ of height k equals η_k for $k = 1, \ldots, \nu_{\rho(A)}(A)$.

Let A be an $n \times n$ real matrix. It is known that a necessary and sufficient condition for the existence of a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ is that, for each eigenvalue λ with modulus $\rho(A)$, we have $\nu_\lambda(A) \leq \nu_{\rho(A)}(A)$. (Then clearly $\rho(A)$ is an eigenvalue of A.) The condition is now usually referred to as the *Perron-Schaefer condition* (see [Sch1, the paragraph following Theorem 1.1]).

If $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$, we can write $x = x_1 + \cdots + x_m$, where $x_1, \ldots, x_m$ are generalized eigenvectors of A corresponding respectively to the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$.

Then we define $\rho_x(A)$, *local spectral radius of A at x*, to be the quantity $\max_{1\le i\le m}|\lambda_i|$. We also set $\mathrm{ord}_A(x) = \max\{\mathrm{ord}_A(x_i) : |\lambda_i| = \rho_x(A)\}$, where $\mathrm{ord}_A(x_i)$ denotes the order of the generalized eigenvector x_i, and refer to $\mathrm{ord}_A(x)$ as the *order of x relative to A*. [We reserve the term "height" for the order of a generalized eigenvector that corresponds to the spectral radius.] The ordered pair $(\rho_x(A), \mathrm{ord}_A(x))$, denoted by $\mathrm{sp}_A(x)$, is called the *spectral pair of x relative to A*. It was first introduced in [T-S2] and has proved to be a useful concept. In particular, in [T-S2, Lemma 4.3] the following was observed:

Let $A \in \pi(K)$. For any $x \in \mathrm{int}\,K$, we have $\mathrm{sp}_A(x) = (\rho(A), \nu_{\rho(A)}(A))$.

It was also proved in [T-S2, Theorem 4.7] that A ($\in \pi(K)$) always satisfies the *local Perron-Schaefer condition*:

For any $0 \neq x \in K$, there is a generalized eigenvector y of A corresponding to $\rho_x(A)$ that appears as a term in the representation of x as a sum of generalized eigenvectors of A. Furthermore, we have $\mathrm{ord}_A(x) = \mathrm{ord}_A(y)$.

We now give a sketch of the combinatorial spectral theory of nonnegative matrices, which has been a source of motivation for our work. For excellent surveys on the subject, see [Sch2], [Her2] and [Her3].

Let P be an $n \times n$ nonnegative matrix. We use $G(P)$ to denote the usual digraph of P, i.e., its vertex set is $\langle n\rangle := \{1, \dots, n\}$ and (i, j) is an arc if and only if $p_{ij} \neq 0$. By a *class* of P we mean the vertex set of a strongly connected component of $G(P)$. The term "class" was introduced by Rothblum [Rot]. If α, β are classes of P, we say α *has access to* β (or β *has access from* α) if either $\alpha = \beta$ or there is a path in $G(P)$ from some (and hence from all) vertex in α to some (and hence all) vertex in β; in which case we write $\alpha >= \beta$. We write $\alpha >- \beta$ if $\alpha >= \beta$ and $\alpha \neq \beta$. Sometimes we also write $i >= \alpha$, where $i \in \langle n\rangle$ and α is a class, with the obvious meaning. A class α is *initial* if it has no access from any other class. Similarly, we can define the concept of a *final class*. If $\mathcal{K}$ is a nonempty collection of classes of P, then a class $\alpha \in \mathcal{K}$ is said to be *final in* $\mathcal{K}$ if it has no access to any other classes of $\mathcal{K}$. (Actually, the preceding definitions are valid for a square complex matrix P; the nonnegativity of P is not required.) We call a class α *basic* if $\rho(P_{\alpha\alpha}) = \rho(P)$, where we use $P_{\alpha\beta}$ to denote the submatrix of P with rows indexed by α and columns indexed by β. A class α of P is *distinguished* (respectively, *semi-distinguished*) if $\rho(P_{\alpha\alpha}) > \rho(P_{\beta\beta})$ (respectively, $\rho(P_{\alpha\alpha}) \geq \rho(P_{\beta\beta})$) for any class $\beta >- \alpha$. A sequence of classes $\alpha_1, \dots, \alpha_k$ is said to form a *chain* from α_1 to α_k if $\alpha_i >- \alpha_{i+1}$ for $i = 1, \dots, k-1$. The *length of a chain* is the number of basic classes it contains. The *level* of a class β, denoted by $\mathrm{lev}(\beta)$, is the length of the longest chain of classes that terminates in β. Following [N-S1], we say the *singular distance* from a class α to a class β is m and we write $d(\alpha, \beta) = m$ if m is the length of the longest chain from α to β. If α has no access to β, we set $d(\alpha, \beta) = -1$. (Rothblum [Rot] uses the term "height" instead of "level" for classes. He also says α has access to β in m steps to mean $d(\alpha, \beta) = m$.) If $j, k \in \langle n\rangle$, by the singular distance from j to k, denoted by $d_{j,k}$, we mean the quantity $d(\alpha, \beta)$, where α, β are the classes that contain j and k respectively.

For $x = (\xi_1, \ldots, \xi_n)^T$, by the *support of* x, we mean the set $\mathrm{supp}(x) := \{i \in \langle n \rangle : \xi_i \neq 0\}$. By the *level* of x (with respect to P), denoted by $\mathrm{lev}(x)$, we mean $\max_\alpha \mathrm{lev}(\alpha)$, where the maximum is taken over all classes α such that $x_\alpha \neq 0$ (or, equivalently, $\alpha \cap \mathrm{supp}(x) \neq \emptyset$). (By convention, the level of the zero vector is 0.) A vector $x \in E(P)$ is called a *peak vector* if $\mathrm{lev}(x) = \mathrm{ht}(x)$.

Theorem 2.1. *Let P be a nonnegative matrix with m basic classes $\alpha_1, \ldots, \alpha_m$.*

(i) *The Perron generalized eigenspace $E(P)$ of P contains a semipositive subset $\{x^{(\alpha_1)}$, ..., $x^{(\alpha_m)}\}$ with the property that $x_j^{(\alpha_i)} > 0$ if and only if $j >\!\!= \alpha_i$, where $x_j^{(\alpha_i)}$ denotes the jth component of the vector $x^{(\alpha_i)}$. Furthermore, any such semipositive subset forms a basis for $E(P)$.*

(ii) *There exists a basis for $E(P)$ which, in addition to the property described in* (i), *satisfies the following*: *For $i = 1, \ldots, m$, we have*

$$(P - \rho(P)I)x^{(\alpha_i)} = \sum_{k=1}^{m} c_{ik} x^{(\alpha_k)},$$

where c_{ik} is positive if $\alpha_k >\!\!- \alpha_i$ and equals 0 otherwise.

Part(i) of Theorem 2.1, now usually referred to as the nonnegative-basis theorem, was due to Rothblum and first appeared in [Rot, Theorem 3.1(1)]. For convenience, we will call a vector $x^{(\alpha_i)} \in E(P)$ a *strongly combinatorial vector associated with* α_i or simply a *strongly* α_i-*combinatorial vector* if it has the properties as described in Theorem 2.1(i). A (semipositive) basis for $E(P)$ which is composed of strongly combinatorial vectors, one vector for each basic class, is referred to as a (nonnegatively) *strongly combinatorial basis* by Neumann and Schneider [N-S3]. Part(ii) of the theorem, which is a strengthening of part (i), is now called the *preferred-basis theorem* and first appeared in Richman and Schneider [R-S] (in the setting of a singular M-matrix). A basis for $E(P)$ whose vectors satisfy the conditions given in (ii) (and (i)) is now called a *preferred-basis.* In [H-S1], the preferred-basis theorem is re-proved and extended to the class of Z-matrices.

The above-mentioned [Rot, Theorem 3.1] has three parts. Part(2) of the result is now referred to as the Rothblum index theorem, which says that the length of the longest chain in the reduced graph $R(P)$ (see [H-S2, Definition 2.15]) is equal to $\nu_{\rho(P)}(P)$, the index of $\rho(P)$ as an eigenvalue of P.

By the Rothblum index theorem, the maximum level of the classes of a nonnegative matrix P is equal to $\nu_{\rho(P)}(P)$. For each $i = 1, \ldots, \nu_{\rho(P)}(P)$, let $\lambda_i(P)$ (or simply λ_i if there is no danger of confusion) denote the number of basic classes of P of level i. We call the ν-tuple $(\lambda_1, \ldots, \lambda_\nu)$, where $\nu = \nu_{\rho(P)}(P)$, the *level characteristic of* P and denote it by $\lambda(P)$. We also call the ν-tuple $\xi(P) = (\xi_1(P), \ldots, \xi_\nu(P))$, where $\xi_k(P) = \dim(P - \rho(P)I_n)^{k-1}\,\mathrm{span}(\mathbb{R}^n_+ \cap \mathcal{N}((P - \rho(P)I)^k)$, the *peak characteristic* of P.

A basis for $E(P)$ is called a *level basis for P* if the number of basis elements of level j equals λ_j for all j, $j = 1, \ldots, \nu_{\rho(P)}(P)$. A basis for $E(P)$ which is both a height basis and a level basis is called a *height-level basis.* A basis for $E(P)$ that consists of peak vectors is called a *peak basis* for $E(P)$.

Remark 2.2. Let P be an $n \times n$ nonnegative matrix. For any integer $k = 1, \ldots, \nu_\rho$, the subspaces $\operatorname{span}[\mathcal{N}((P - \rho(P)I)^k) \cap \mathbb{R}^n_+]$ and $\{x \in E(P) : \operatorname{lev}(x) \leq k\}$ are the same and are also equal to $\operatorname{span}\{x^{(\alpha_j)}\colon\ 1 \leq j \leq m,\ \operatorname{lev}(\alpha_j) \leq k\}$, where $\alpha_1, \ldots, \alpha_m$ are the basic classes of P and each $x^{(\alpha_j)}$ is a strongly α_j-combinatorial vector. Consequently, for each $k = 1, \ldots, \nu_{\rho(P)}(P)$, we have $\dim \operatorname{span}[\mathcal{N}((P - \rho(P)I)^k) \cap \mathbb{R}^n_+] = \lambda_1 + \cdots + \lambda_k$.

The above result is known (see [N-S3, (3.3)]); it follows from the fact that every nonnegative vector in $E(P)$ is a peak vector, which in turn is a consequence of the preferred-basis theorem. At the end of Section 7, we will show how to obtain this remark by our cone-theoretic approach.

In view of Remark 2.2, we have

Remark 2.3. Let P be an $n \times n$ nonnegative matrix. For any $0 \neq x \in E(P)$, $\operatorname{lev}(x)$ equals the smallest positive integer k such that $x \in \operatorname{span}[\mathcal{N}((P - \rho(P)I)^k) \cap \mathbb{R}^n_+]$.

3. Nonnegative components and K-semipositive bases

In [Sch1, Theorem 5.2], based on a method due to Birkhoff [Bir] and using the concept of an annihilating polynomial, Schneider proves that if $A \in \mathcal{M}_n(\mathbb{C})$ satisfies the Perron-Schaefer condition, then the principal component $Z_A^{(\nu_\rho - 1)}$ of A belongs to the convex cone $\operatorname{cl}\omega_k(A)$ for all nonnegative integers k, where $\omega_k(A)$ is defined to be the positive hull (i.e., the set of all possible nonnegative linear combinations) of $\{A^i : i = k, k+1, \ldots\}$. It is also known that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ satisfies the Perron-Schaefer condition if and only if there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ (see [Sch1, Theorems 1.1 and 1.2]). So we have the following useful known result:

Theorem 3.1. *If $A \in \pi(K)$, then $Z_A^{(\nu-1)} \in \pi(K)$, where $\nu = \nu_{\rho(A)}(A)$.*

We would like to take this opportunity to offer the following short direct proof of the above mentioned result of Schneider, one which does not rely on Birkhoff's method:

Let $J_k(\lambda)$ denote the $k \times k$ (upper triangular) elementary Jordan matrix corresponding to λ. Note that when m is a large positive integer, the dominating term of $J_q(r+1)^m J_q(r)^k$, where r is a positive real number and k is a fixed nonnegative integer, occurs at its $(1,q)$-entry and is of the same order of magnitude as $[(q-1)!]^{-1}m^{q-1}(r+1)^{m-q+1}r^k$. Hence, $\lim_{m\to\infty}[(q-1)!]m^{-(q-1)}r^{-k}(r+1)^{-(m-q+1)}J_p(\lambda+1)^m J_p(\lambda)^k$ equals the $q\times q$ matrix with 1 at its $(1,q)$-entry and 0 elsewhere if $p=q$ and $\lambda=r$, and equals the zero matrix if $|\lambda|<r$, or $p<q$ and $|\lambda|=r$, or $p=q$, $|\lambda|=r$ and $\lambda\neq r$.

Denote $\rho(A)$ by ρ. By considering the Jordan form of A, we have

$$Z_A^{(\nu-1)} = \lim_{m\to\infty}[(\nu-1)!]m^{-(\nu-1)}\rho^{-k}(\rho+1)^{-(m-\nu+1)}(A+I)^m A^k,$$

and hence $Z_A^{(\nu-1)} \in \operatorname{cl}\omega_k(A)$.

If our aim is to establish Theorem 3.1 only, then we have an even shorter proof: By (2.1), we have, $\lim_{\varepsilon\downarrow 0}\varepsilon^\nu((\rho(A)+\varepsilon)I-A)^{-1} = Z_A^{(\nu-1)}$. But $((\rho(A)+\varepsilon)I-A)^{-1}\in\pi(K)$ for all $\varepsilon>0$, so $Z_A^{(\nu-1)}\in\pi(K)$.

A variant of the argument is to use cone-duality:

Assume to the contrary that $Z_A^{(\nu-1)}\notin\pi(K)$. Then we can find some $C\in\pi(K)^*$ such that $\langle Z_A^{(\nu-1)},C\rangle<0$. (Here we use the usual inner product of $\mathcal{M}_n(\mathbb{R})$ given by $\langle A,B\rangle=\operatorname{tr}\langle B^TA\rangle$.) In view of (2.1) (and the continuity of $R(\varepsilon)$ at $\varepsilon=0$), it is clear that for sufficiently small $\varepsilon>0$, we have $\langle((\rho(A)+\varepsilon)I-A)^{-1},C\rangle<0$, so $((\rho(A)+\varepsilon)I-A)^{-1}\notin\pi(K)$, which is a contradiction.

Later we will incorporate the above duality argument into the algebraic-analytic method of Hartwig-Neumann-Rose.

We would also like to point out that Theorem 3.1 can be extended to a wider class of matrices, namely $\Sigma(K)$, the class of cross-positive matrices on K. Recall that $A\in\mathcal{M}_n(\mathbb{R})$ is said to be *cross-positive* on K if for all $y\in K$, $z\in K^*$ such that $\langle z,y\rangle=0$ we have $\langle z,Ay\rangle\geq 0$. In ([S-V], Theorem 6]) it is shown that if $A\in\Sigma(K)$, then $\tau(A):=\max\{\operatorname{Re}\lambda : \lambda\in\sigma(A)\}$, the spectral abscissa of A, must be an eigenvalue of A. It is also known that for any $A\in\mathcal{M}_n(\mathbb{R})$, $A\in\Sigma(K)$ if and only if $(\lambda I-A)^{-1}\in\pi(K)$ for all $\lambda>\tau(A)$ (see [Els, Satz 1] or [B-N-S, p.74, Theorem 3.11]). So, with slight modifications (and by applying expansion (2.5) with $\lambda=\xi(A)$), our above proofs for Theorem 3.1 also lead to the following new result:

Theorem 3.2. *If $A\in\Sigma(K)$, then $E_\tau^{(\nu_\tau-1)}(A)\in\pi(K)$, where τ denotes the spectral abscissa of A and $\nu_\tau=\nu_\tau(A)$.*

Let $\pi_1(K)$ denote the set $\{A : A+\alpha I\in\pi(K)$ for some real $\alpha\}$. It is known that we always have $\Sigma(K)=\operatorname{cl}\pi_1(K)$, and that for $n\geq 3$ the equality $\Sigma(K)=\pi_1(K)$ fails for "almost all" K in the sense of Baire category, and also that $\rho(A)=\tau(A)$ for any $A\in\pi(K)$ (see [G-K-T]). So Theorem 3.2 is truly an improvement of Theorem 3.1.

Note, however, that Schneider's result mentioned at the beginning of this section cannot be extended to the class $\Sigma(K)$. In other words, when $A\in\Sigma(K)$, we need not have

$E_\tau^{(\nu_\tau - 1)} \in \operatorname{cl}\omega_0(A)$. For instance, let K be the 3-dimensional ice-cream $\{x = (\xi_1, \xi_2, \xi_3)^T : \xi_3 \geq \sqrt{(\xi_1^2 + \xi_2^2)}\}$ and take

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As can be readily checked, $A \in \Sigma(K)$. (We are borrowing from [B-N-S, p.62, Exercise 1.6(i)].) By direct calculation, $\sigma(A) = \{0, i, -i\}$. So we have $\tau = 0$, $\nu_\tau(A) = 1$ and also $E_\tau^{(\nu_\tau - 1)} = E_\tau^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is clear that $E_\tau^{(0)} \in \pi(K)$ but $E_\tau^{(0)} \notin \operatorname{cl}\omega_0(A)$.

Theorem 3.3. *Let K be a proper cone in $\mathbb{R}^n$, and let $A \in \pi(K)$. For any integer $k = 0, \ldots, \nu_{\rho(A)}(A) - 1$, consider the following conditions:*

(a) $Z_A^{(k)} \in \pi(K)$.

(b) $J_A^{(k)}(\varepsilon) \in \pi(K)$ *for all positive* ε (*or, for all sufficiently large* ε).

(c) $J_A^{(k)}(\varepsilon) \in \pi(K)$ *for all sufficiently small positive* ε.

(d) $J_A^{(k)}(\varepsilon) \in \pi(K)$ *for at least one positive* ε.

(e) $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$ *contains a K-semipositive basis.*

Then conditions (a) *and* (b) *are equivalent, conditions* (c) *and* (d) *are also equivalent, and the following implications hold:* (a) $\Longrightarrow$ (c) $\Longrightarrow$ (e).

Proof. For $\varepsilon > 0$, we always have $((\rho(A) + \varepsilon)I - A)^{-1} \in \pi(K)$. So, by (2.3), we have $J_A^{(k)}(\varepsilon) \in \pi(K)$ for all $\varepsilon > 0$, whenever $Z_A^{(k)} \in \pi(K)$. This proves (a) $\Longrightarrow$ (b).

Since $\lim_{\varepsilon \to \infty} J_A^{(k)}(\varepsilon) = Z_A^{(k)}$ and $\pi(K)$ is a closed cone, the implication (b) $\Longrightarrow$ (a) clearly also holds.

The implications (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d) are obvious.

Now, suppose that (d) holds, i.e., we have $J_A^{(k)}(\varepsilon_0) \in \pi(K)$ for some $\varepsilon_0 > 0$. Consider any ε, $0 < \varepsilon < \varepsilon_0$. Making use of (2.3) and the identity $C^{-1} - D^{-1} = C^{-1}(D - C)D^{-1}$, we have

$$\begin{aligned}
& \varepsilon^{-1} J_A^{(k)}(\varepsilon) - \varepsilon_0^{-1} J_A^{(k)}(\varepsilon_0) \\
= \; & Z_A^{(k)}[((\rho(A) + \varepsilon)I - A)^{-1} - ((\rho(A) + \varepsilon_0)I - A)^{-1}] \\
= \; & (\varepsilon_0 - \varepsilon)((\rho(A) + \varepsilon)I - A)^{-1}((\rho(A) + \varepsilon_0)I - A)^{-1} Z_A^{(k)} \\
= \; & (\varepsilon_0 - \varepsilon)\varepsilon_0^{-1}((\rho(A) + \varepsilon)I - A)^{-1} J_A^{(k)}(\varepsilon_0).
\end{aligned}$$

But $J_A^{(k)}(\varepsilon_0) \in \pi(K)$, so we have $J_A^{(k)}(\varepsilon) \in \pi(K)$. This establishes (d) $\Longrightarrow$ (c). Note that the range space of $J_A^{(k)}(\varepsilon_0)$ is the same as that of $Z_A^{(k)}$, namely, $(A - \rho(A)I)^k E(A)$ or $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$. Since K is a full cone in $\mathbb{R}^n$, $J_A^{(k)}(\varepsilon_0)K$ is a full cone in $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$. But $J_A^{(k)}(\varepsilon_0)K \subseteq K$, hence $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$ contains a K-semipositive basis. This establishes (d) $\Longrightarrow$ (e). ■

From the proof, it is clear that condition (b) of Theorem 3.3 can also be replaced by the following: $J_A^{(k)}(\varepsilon) \in \pi(K)$ for all ε in an unbounded subset of the interval $(0, \infty)$.

We would like to compare condition (a) of Theorem 3.3 for different k's. By the definition of the principal components of A and the fact that A and $Z_A^{(0)}$ commute, we have $Z_A^{(k)} = (Z_A^{(1)})^k$ for all positive integers k. So if $Z_A^{(1)}$ belongs to $\pi(K)$, then so does $Z_A^{(k)}$ for $k = 2, \ldots, \nu_\rho$. In general, the conditions "$Z_A^{(0)} \in \pi(K)$" and "$Z_A^{(1)} \in \pi(K)$" are logically independent. Note that the condition "$Z_A^{(1)} \in \pi(K)$" amounts to "$(A - \rho(A)I)(Z_A^{(0)}K) \subseteq E(A) \cap K$", which is rather stringent, because the set $Z_A^{(0)}K$ includes $E(A) \cap K$ and now we require that $A - \rho(A)I$ maps the larger set into the smaller set. But, in the light of the properties of the spectral cone introduced in the next section, we can conceive how this stringent condition may be met.

The equivalence of conditions (a) and (b) of Theorem 3.3 for the nonnegative matrix case (i.e., when $K = \mathbb{R}^n_+$) is known and can be found in [N-S1, Section 6], where they formulate the result in terms of a minus M-matrix.

By (2.3), we have $J_A^{(0)}(\varepsilon) = \varepsilon((\rho(A) + \varepsilon)I - A)^{-1} Z_A^{(0)}$. So, when $k = 0$, Theorem 3.3 reduces to the following result, which was announced in [Tam4, Subsection 6.2]:

Corollary 3.4. *Let K be a proper cone in $\mathbb{R}^n$, and let $A \in \pi(K)$. Consider the following conditions*:

(a) $Z_A^{(0)} \in \pi(K)$.

(b) $(\lambda I - A)^{-1} Z_A^{(0)} \in \pi(K)$ *for all* $\lambda > \rho(A)$ (*or, for all sufficiently large* λ).

(c) $(\lambda I - A)^{-1} Z_A^{(0)} \in \pi(K)$ *for all* $\lambda > \rho(A)$, *sufficiently close to* $\rho(A)$.

(d) $(\lambda I - A)^{-1} Z_A^{(0)} \in \pi(K)$ *for at least one* $\lambda > \rho(A)$.

(e) $E(A)$ *contains a* K*-semipositive basis.*

Then conditions (a) *and* (b) *are equivalent, conditions* (c) *and* (d) *are also equivalent, and the following implications hold*: (a) $\Longrightarrow$ (c) $\Longrightarrow$ (e).

Remark 3.5. In Theorem 3.3, if we replace the hypothesis "$A \in \pi(K)$" by "$A \in \Sigma(K)$", then the result still holds, provided that in the conclusion we replace $\rho(A)$, $Z_A^{(k)}$, $J_A^{(k)}(\varepsilon)$ and $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$ respectively by $\tau(A)$, $E_{\tau(A)}^{(k)}$, $F_{\tau(A)}^{(k)}$ and $\mathcal{R}((A - \tau(A)I)^k) \cap \mathcal{N}((A - \tau(A)I)^n)$ but keeping $\pi(K)$ unchanged.

A similar remark also holds for Corollary 3.4.

Next, we give two special properties of a cone-preserving map on a polyhedral proper cone.

Theorem 3.6. *Let K be a polyhedral proper cone in $\mathbb{R}^n$, and let $A \in \pi(K)$. Then*:

(i) *For $k = 0, \ldots, \nu_{\rho(A)}(A) - 1$, $J_A^{(k)}(\varepsilon) \in \pi(K)$ for all sufficiently small positive ε.*

(ii) *$\varepsilon^{-1} J_A^{(0)}(\varepsilon) + R(0) \in \pi(K)$ for all sufficiently small positive ε, where $R(\varepsilon)$ denotes the analytic operator that appears in* (2.1).

Proof. First, we contend that for any nonzero matrix C in $\pi(K)^*$, we have $\langle J_A^{(k)}(\varepsilon), C\rangle \geq 0$ for all sufficiently small positive ε. By definition, $J_A^{(k)}(\varepsilon) = Z_A^{(k)} + Z_A^{(k+1)}/\varepsilon + \cdots + Z_A^{(\nu-1)}/\varepsilon^{\nu-k-1}$. If $\langle Z_A^{(i)}, C\rangle = 0$ for $i = k, \ldots, \nu - 1$, clearly there is no problem. So suppose that there exists $i \geq k$ such that $\langle Z_A^{(i)}, C\rangle \neq 0$ and let j be the largest such i. If $\langle Z_A^{(j)}, C\rangle < 0$, then for all sufficiently small positive ε, in view of the expansion (2.1) for $((\rho(A)+\varepsilon)I - A)^{-1}$, we have $\langle(\rho(A)+\varepsilon)I - A)^{-1}, C\rangle < 0$ and hence $((\rho(A)+\varepsilon)I - A)^{-1} \notin \pi(K)$, which is a contradiction. So $\langle Z_A^{(j)}, C\rangle > 0$ and by the definition of $J_A^{(k)}(\varepsilon)$ again, we have $\langle J_A^{(k)}(\varepsilon), C\rangle > 0$ for all sufficiently small positive ε. This establishes our contention. Now, each matrix in $\pi(K)^*$ can be expressed as a nonnegative linear combination of the extreme matrices of $\pi(K)^*$. But $\pi(K)^*$ has (up to multiples) only finitely many extreme matrices (as K is polyhedral), in view of our proved contention, it follows that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have $\langle J_A^{(k)}(\varepsilon), C\rangle \geq 0$ for all $C \in \pi(K)^*$. In other words, $J_A^{(k)}(\varepsilon) \in \pi(K)$ for all sufficiently small positive ε. This proves part (i).

To prove part (ii), it suffices to show that for any $0 \neq C \in \pi(K)^*$, we have $\langle \varepsilon^{-1} J_A^{(0)}(\varepsilon) + R(0), C\rangle \geq 0$ for all sufficiently small positive ε. In view of the result of part (i), clearly there is no problem when $\langle R(0), C\rangle \geq 0$. When $\langle R(0), C\rangle < 0$, again making use of the expansion (2.1), we can show that then $\langle Z_A^{(i)}, C\rangle \neq 0$ for some i, $0 \leq i \leq \nu - 1$ and also that if j is the largest such i, then necessarily we have $\langle Z_A^{(j)}, C\rangle > 0$. Then by the definition of $J_A^{(0)}(\varepsilon)$, it is clear that the inequality $\langle J_A^{(k)}(\varepsilon) + R(0), C\rangle > 0$ holds for all sufficiently small positive ε. The proof is complete. ■

Again Theorem 3.6 still holds if in the hypothesis we replace "$A \in \pi(K)$" by "$A \in \Sigma(K)$" and in the conclusion we replace $\rho(A)$ and $J_A^{(k)}(\varepsilon)$ respectively by $\tau(A)$ and $F_{\tau(A)}^{(k)}$. The same proof carries over. However, the result is not much better than Theorem 3.6. The point is, when K is polyhedral, $\Sigma(K) = \pi_1(K)$ (see [S-V, Theorem 8] or [B-N-S, Chapter 4, Theorem 3.41]). So there exists $\alpha \in \mathbb{R}$ such that $A + \alpha I \in \pi(K)$. Then $\rho(A + \alpha I) = \tau(A + \alpha I) = \tau(A) + \alpha$ and $\nu_{\tau(A)}(A) = \nu_{\rho(A+\alpha I)}(A + \alpha I)$, and by a little calculation we obtain

$Z^{(k)}_{A+\alpha I} = E^{(k)}_{\tau(A)}(A)$ and $J^{(k)}_{A+\alpha I}(\varepsilon) = F^{(k)}_{\tau(A)}(\varepsilon)$ for $k = 0, \ldots, \nu_{\tau(A)}(A) - 1$. So the result can be derived by applying Theorem 3.6 to $A + \alpha I$.

In terms of the principal components of A, we have the following:

Corollary 3.7. *Let K be a polyhedral proper cone, and let $A \in \pi(K)$. Then:*

(i) *For $k = 0, \ldots, \nu_\rho - 1$, there exist positive numbers $\delta_{kk}, \ldots, \delta_{k,\nu_\rho-1}$ such that*

$$\delta_{kk} Z^{(k)}_A + \delta_{k,k+1} Z^{(k+1)}_A + \cdots + \delta_{k,\nu_\rho-1} Z^{(\nu_\rho-1)}_A \in \pi(K).$$

(ii) *For $k = 0, \ldots, \nu_\rho - 1$, we have $(\lambda I - A)^{-1} Z^{(k)}_A \in \pi(K)$ for all $\lambda > \rho(A)$, sufficiently close to $\rho(A)$.*

(iii) *For any sufficiently small $\varepsilon > 0$, we have $(\rho(A)I - A + \varepsilon Z^{(0)}_A)^{-1} \in \pi(K)$.*

(iv) *For $k = 0, \ldots, \nu_\rho - 1$, $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$ contains a K-semipositive basis.*

Proof. (i) follows from Theorem 3.6(i) and the definition of $J^{(k)}_A(\varepsilon)$.

(ii) follows from Theorem 3.6(i) and the equality relation (2.3), whereas (iv) follows from Theorem 3.6(i) and Theorem 3.3, (c) $\Longrightarrow$ (e).

In view of the expansion (2.2), we have $(\rho(A)I - A + \varepsilon Z^{(0)}_A)^{-1} = \varepsilon^{-1} J^{(0)}_A(\varepsilon) + R(0)$. So (iii) follows from Theorem 3.6(ii). ∎

In [H-N-R] it is shown that, in the nonnegative matrix case, Corollary 3.7(ii) always holds.

By Corollary 3.7(iv) with $k = 0$, we recover the known result that if K is a polyhedral proper cone, then for any $A \in \pi(K)$, $E(A)$ contains a K-semipositive basis ([Tam2, Theorem 7.5(i)]). The proof of the latter result as given in [Tam2] relies on the corresponding result for a nonnegative matrix and uses the minimal generating matrix (for a polyhedral cone) as a tool. The present proof does not assume the corresponding nonnegative matrix result, thus keeping our cone-theoretic treatment self-contained.

Corollary 3.7(i) gives a sort of geometric condition concerning the joint position of the matrices $Z^{(0)}_A, \ldots, Z^{(\nu-1)}_A$ relative to $\pi(K)$. In fact, for $A \in \pi(K)$, where K is a proper cone, the conditions given in Theorem 3.6(i) and Corollary 3.7(i) are equivalent. More generally, we have the following:

Theorem 3.8. *Let K be a proper cone in $\mathbb{R}^n$, and let $x_1, \ldots, x_p$ be nonzero vectors of $\mathbb{R}^n$. Consider the following conditions:*

(a) *For each i, $i = 1, \ldots, p$, K contains a linear combination of $x_i, \ldots, x_p$ with positive coefficients.*

(b) *For each* i, $i = 1, \ldots, p-1$, *we have* $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-i} x_i \in K$ *for all sufficiently small* $\varepsilon > 0$.

(c) $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-1} x_1 \in K$ *for all sufficiently small* $\varepsilon > 0$.

(d) $x_p \in K$ *and* $x_i \in [(\mathrm{span}\{x_{i+1}, \ldots, x_p\})^\perp \cap K^*]^*$ *for* $i = 1, \ldots, p-1$.

Conditions (a) *and* (b) *are always equivalent and we have* (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d). *When* K *is polyhedral, conditions* (a)–(d) *are all equivalent.*

Proof. (a) $\Longrightarrow$ (b): By condition (a), $x_p \in K$ and for each i, $i = 1, \ldots, p-1$, there exists a vector y_i in K which can be written as a positive linear combination of $x_i, \ldots, x_p$. By simple algebraic manipulations, one can show that for sufficiently small $\varepsilon > 0$ and $i = 1, \ldots, p-1$, $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-i} x_i$ can be written as a positive linear combination of $y_i, \ldots, y_{p-1}$, and x_p, and hence it belongs to K.

The implications (b) $\Longrightarrow$ (a) and (b) $\Longrightarrow$ (c) clearly both hold.

(c) $\Longrightarrow$ (d): Since $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-1} x_1 \in K$ for all sufficiently small $\varepsilon > 0$ and $\lim_{\varepsilon \to 0}(x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-1} x_1) = x_p$, clearly $x_p \in K$. Assume that for some i, $1 \leq i \leq p-1$, $x_i \notin [(\mathrm{span}\{x_{i+1}, \ldots, x_p\})^\perp \cap K^*]^*$. Then there exists $0 \neq z \in K^*$ such that $\langle z, x_{i+1} \rangle = \cdots = \langle z, x_p \rangle = 0$ and $\langle z, x_i \rangle < 0$. But then we have

$$\langle z, x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-1} x_1 \rangle = \varepsilon^{p-i} \langle z, x_i + \varepsilon x_{i-1} + \cdots + \varepsilon^{i-1} x_1 \rangle < 0$$

for all sufficiently small $\varepsilon > 0$, which contradicts (c).

Now suppose, in addition, that K is polyhedral.

(d) $\Longrightarrow$ (a): According to (d), we have $x_{p-1} \in [(\mathrm{span}\{x_p\})^\perp \cap K^*]^*$. By definition, $(\mathrm{span}\{x_p\})^\perp \cap K^*$ equals $d_K(\Phi(x_p))$, where $\Phi(x_p)$ is the face of K generated by x_p and $d_K(\Phi(x_p))$ denotes its dual face in K^*, i.e., the set $\{z \in K^*\colon \langle z, y \rangle = 0 \text{ for all } y \in \Phi(x_p)\}$. By [Tam1, Corollary 3.2], $[d_K(\Phi(x_p))]^* = \mathrm{cl}[\mathrm{cone}(x_p, K)]$, where for a convex set C and any point $y \in C$ we use $\mathrm{cone}(y, C)$ to denote the cone of C at y, i.e., the convex cone $\{\alpha(x - y) : x \in C \text{ and } \alpha \geq 0\}$. Since K is polyhedral, $\mathrm{cone}(x_p, K)$ is a closed set. So there exists a positive number α and a vector $x \in K$ such that $x_{p-1} = \alpha(x - x_p)$; hence K contains a positive linear combination of x_p and x_{p-1}, namely, the vector $y_{p-1} := x_{p-1} + \alpha x_p$.

Next, by condition (d) we also have $x_{p-2} \in ((\mathrm{span}\{x_{p-1}, x_p\})^\perp \cap K^*)^*$. Using the fact that for any $z \in K^*$ and any $w_1, w_2 \in K$, we have, $\langle z, w_1 + w_2 \rangle = 0$ if and only if $\langle z, w_1 \rangle = \langle z, w_2 \rangle = 0$, one can readily check that for any $z \in K^*$, z is orthogonal to both x_{p-1} and x_p if and only if z is orthogonal to the vector $y_{p-1} + x_p$ of K. So we have $x_{p-2} \in [d_K(\Phi(y_{p-1} + x_p))]^*$. Repeating the above argument (with x_{p-2} and $y_{p-1} + x_p$ playing respectively the roles of x_{p-1} and x_p), we find that there exists a positive number β such that the vector $y_{p-2} := x_{p-2} + \beta(y_{p-1} + x_p)$ belongs to K and is a positive linear combination of x_{p-2}, x_{p-1} and x_p.

Similarly, the condition $x_{p-3} \in [(\text{span}\{x_{p-2}, x_{p-1}, x_p\})^{\perp} \cap K^*]^*$ can be rewritten as $x_{p-3} \in [d_K(\Phi(u))]^*$, where u is the vector $x_p + y_{p-1} + y_{p-2}$ of K, and then we can deduce that K obtains a positive linear combination of $x_{p-3}, \ldots, x_p$. By repeating the argument, we obtain condition (a). ■

In Theorem 3.8 if we take K to be $\pi(K)$ and $x_1, x_2, \ldots, x_p$ to be respectively $Z_A^{(0)}$, $Z_A^{(1)}, \ldots, Z_A^{(\nu-1)}$ (with $p = \nu := \nu_{\rho(A)}(A)$ for some fixed $A \in \pi(K)$), then condition (c) of Theorem 3.8 becomes:

$$Z_A^{(\nu-1)} + \varepsilon Z_A^{(\nu-2)} + \cdots + \varepsilon^{\nu-1} Z_A^{(0)} \in \pi(K) \text{ for all sufficiently small } \varepsilon > 0,$$

which is a restatement of the condition that $J_A^{(0)}(\varepsilon) \in \pi(K)$ for all sufficiently small $\varepsilon > 0$, i.e., condition (c) of Theorem 3.3 with $k = 0$. According to Theorem 3.6(i), the latter condition is always satisfied if K is polyhedral. But, for a general proper cone K, as we will show in Example 3.10, the condition need not be fulfilled. With the same specialization, condition (d) of Theorem 3.8, which is weaker than the above one, becomes:

$$Z_A^{(i)} \in [(\text{span}\{Z_A^{(i+1)}, \ldots, Z_A^{(\nu-1)}\})^{\perp} \cap \pi(K)^*]^* \text{ for } i = 0, 1, \ldots, \nu - 2.$$

(We have deliberately left out the condition $Z_A^{(\nu-1)} \in \pi(K)$, which is always satisfied.) The interesting thing is, as the argument given in the proof of Theorem 3.6(i) shows, the latter set of condition is always satisfied by any $A \in \pi(K)$ for a general proper cone K [and indeed by any $A \in \Sigma(K)$, provided that we replace $Z_A^{(i)}$ by $E_{\tau(A)}^{(i)}$ and ν by $\nu_{\tau(A)}(A)$ and keeping $\pi(K)^*$ unchanged.]

Now we would like to give further remarks concerning the conditions that appear in Theorem 3.8. From the proof of the theorem one can readily see that if condition (b) is satisfied, then for $i = 1, \ldots, p-1$ and for all sufficiently small $\varepsilon > 0$, $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-i} x_i$ belongs to K and generates the face $\Phi(y_i + \cdots + y_{p-1} + x_p)$ of K. Note, however, that when we have only condition (c), the face generated by $x_p + \varepsilon x_{p-1} + \cdots + \varepsilon^{p-1} x$ does depend on ε. (See the example given in the paragraph following.) In condition (d), the subcondition $x_{p-1} \in [(\text{span}\{x_p\})^{\perp} \cap K^*]^* = [d_K(\Phi(x_p))]^*$ (and $x_p \in K$) amounts to saying that x_{p-1} is subtangential to K at x_p (as $-d_K(\Phi(x_p))$ is equal to the normal cone to K at x_p) (see [B-N-S, p.65, Definition 2.1]). In general, the latter condition is weaker than the condition that $x_p + \varepsilon x_{p-1} \in K$ for some (and hence for all sufficiently small) $\varepsilon > 0$. So the missing implication (d) $\Longrightarrow$ (c) of Theorem 3.8 does not hold. [For a counter-example, consider the 3-dimensional ice-cream cone $K = \{(\xi_1, \xi_2, \xi_3) : \xi_3 \geq \sqrt{(\xi_1^2 + \xi_2^2)}\}$ and take $x_1 = (0, 1, 0)^T$ and $x_2 = (1, 0, 1)^T$ (and with $p = 2$).]

To show that when K is a general proper cone, the implication (c) $\Longrightarrow$ (b) does not hold for the conditions (b), (c) of Theorem 3.8, consider the proper cone K in $\mathbb{R}^3$ given by:

$$K = \{\alpha(\xi_1, \xi_2, 1) : \alpha \geq 0, \sqrt{\xi_1} \geq \xi_2 \geq 0\} \cup \operatorname{pos}\{e_1\},$$

where e_i denotes the ith standard unit vector of $\mathbb{R}^3$ and $\operatorname{pos}(S)$ denotes the positive hull of S. We readily check that $e_3 + \varepsilon e_2 + \varepsilon^2 e_1 \in K$ for all sufficiently small $\varepsilon > 0$. However, $e_3 + \varepsilon e_2 \notin K$ for any $\varepsilon > 0$. Note also that, for any $\varepsilon > 0$, $e_3 + \varepsilon e_2 + \varepsilon^2 e_1$ is an extreme vector of K. So, in this case, it is far from being true that $e_3 + \varepsilon e_2 + \varepsilon^2 e_1$ generates the same face for all sufficiently small $\varepsilon > 0$.

In [M-S, Theorem 1] Meyer and Stadelmaier obtain the following result as an extension of the well-known fact that a Z-matrix is a nonsingular M-matrix if and only if it is inverse-nonnegative:

If A is an M-matrix, then there is a number $c > 0$ such that $A + t(I - AA^D)$ is inverse-nonnegative when $t \in (0, c)$. The converse is also true if A is a Z-matrix.

In the above A^D denotes the Drazin pseudo-inverse of A. The proof given in [M-S] for the first part of the result is rather involved. We are going to give a simple proof for the result and also for the corresponding result for an irreducible M-matrix ([M-S, Theorem 2]). Indeed, we are able to establish these results in the more general setting of a linear map preserving a polyhedral proper cone.

Recall that a matrix $A \in \pi(K)$ is said to be *K-irreducible* if A has no eigenvector in ∂K, the boundary of K. We denote by $\pi^+(K)$ the set $\{A \in \pi(K) : A(K \backslash \{0\}) \subseteq \operatorname{int} K\}$. It is known that $\pi^+(K) = \operatorname{int} \pi(K)$ (see [Bar, Proposition 1]).

Corollary 3.9. *Let K be a polyhedral proper cone in $\mathbb{R}^n$, and let $A \in \pi(K)$. Let $B = A - \lambda I$, where λ is a real number. Denote by Z the projection of $\mathbb{R}^n$ onto $\mathcal{N}(B^n)$ along $\mathcal{R}(B^n)$. Then*

(i) *$\lambda \geq \rho(A)$ if and only if there is a number $c > 0$ such that $(\varepsilon Z - B)^{-1} \in \pi(K)$ for all $\varepsilon \in (0, c)$.*

(ii) *A is K-irreducible and $\lambda \geq \rho(A)$ if and only if there is a number $c > 0$ such that $(\varepsilon Z - B)^{-1} \in \pi^+(K)$ for all $\varepsilon \in (0, c)$.*

Proof. (i): "Only if" part: If $\lambda > \rho(A)$, then B is nonsingular and $Z = 0$, and we have $(\varepsilon Z - B)^{-1} = (-B)^{-1} = (\lambda I - A)^{-1} \in \pi(K)$.

If $\lambda = \rho(A)$, then $Z = Z_A^{(0)}$ and by Corollary 3.7(iii) we have $(\varepsilon Z - B)^{-1} = (\rho(A)I - A + \varepsilon Z_A^{(0)})^{-1} \in \pi(K)$ for all sufficiently small $\varepsilon > 0$.

"If" part: By the Perron-Frobenius theorem, there exists $0 \neq x \in K$ such that $Ax = \rho(A)x$. Suppose $\lambda \neq \rho(A)$. Then $x \in \mathcal{R}(B^n)$ as $x = B((\rho(A) - \lambda)^{-1}x)$, hence we have

$(\varepsilon Z - B)x = -Bx = (\lambda - \rho(A))x$, or $(\varepsilon Z - B)^{-1}x = (\lambda - \rho(A))^{-1}x$. But $(\varepsilon Z - B)^{-1}x \in K$ for all sufficiently small positive ε, so we must have $\lambda > \rho(A)$.

(ii): "Only if" part: If $\lambda > \rho(A)$, then $(\varepsilon Z - B)^{-1} = (\lambda I - A)^{-1} = \sum_{i=0}^{\infty} A^i/\lambda^{i+1} \in \pi^+(K)$, as A is K-irreducible.

Suppose $\lambda = \rho(A)$. Since A is K-irreducible, $\nu_{\rho(A)}(A) = 1$. In view of (2.2), we have $(\varepsilon Z - B)^{-1} = (\rho(A)I - A + \varepsilon Z_A^{(0)})^{-1} = Z_A^{(0)}/\varepsilon + R(0)$. Let y and z denote respectively the Perron vector of A and A^T. Then $y \in \operatorname{int} K$ and $z \in \operatorname{int} K^*$. Normalize y and z so that $z^T y = 1$. Then $Z_A^{(0)}$ equals yz^T and belongs to $\pi^+(K)$. But $\pi^+(K) = \operatorname{int}\pi(K)$, it follows that we have $(\varepsilon Z - B)^{-1} \in \pi^+(K)$ for all sufficiently small $\varepsilon > 0$.

"If" part: By the "if" part of part(i), we have $\lambda \geq \rho(A)$. Assume to the contrary that A is K-reducible. Then there exists a nonzero vector $x \in \partial K$ such that $Ax = \rho(A)x$. Consider any $\varepsilon > 0$. If $\lambda > \rho(A)$, then, as shown in the proof of the "if" part of part(i), we have $(\varepsilon Z - B)^{-1}x = (\lambda - \rho(A))^{-1}x \in \partial K$. On the other hand, if $\lambda = \rho(A)$, then $(\varepsilon Z - B)x = \varepsilon Z_A^{(0)}x = \varepsilon x$ and hence $(\varepsilon Z - B)^{-1}x = \varepsilon^{-1}x \in \partial K$. In any case, we have $(\varepsilon Z - B)^{-1} \notin \pi^+(K)$, which is a contradiction. ■

Recall that if A is an $m \times n$ matrix and $k = \nu_0(A)$, then the Drazin pseudo-inverse A^D of A is given by: $A^D|_{\mathcal{N}(A^k)} = 0$ and $A^D|_{\mathcal{R}(A^k)} = (A|_{\mathcal{R}(A^k)})^{-1}$. Hence, $I - AA^D$ is the projection onto $\mathcal{N}(A^k)$ along $\mathcal{R}(A^k)$. In particular, if A is a singular M-matrix, say, $A = \rho(P)I - P$, where $P \geq 0$, then $I - AA^D$ is equal to $Z_P^{(0)}$ and we have $A + t(I - AA^D) = tZ_P^{(0)} - (P - \rho(P)I)$. Now it is readily seen that Corollary 3.9 is an extension of [M-S, Theorems 1 and 2].

In [N-S1, Theorem 3.6] Neumann and Schneider provide a necessary and sufficient condition for the principal eigenprojection $Z_A^{(0)}$ of a nonnegative matrix A to be nonnegative. So, even in the nonnegative matrix case, condition (a) of Corollary 3.4 is not always fulfilled. On the other hand, by Corollary 3.7(ii), condition (c) of Corollary 3.4 is always satisfied whenever K is polyhedral. Thus, for the conditions (a), (c) of Corollary 3.4, the implication (c) $\Longrightarrow$ (a) does not hold in general.

We end this section with a multi-purpose example, which shows in particular that for the conditions (d), (e) of Corollary 3.4, the implication (e) $\Longrightarrow$ (d) does not hold, indeed, not even if we replace (e) by the stronger condition that $E(A)$ contains a K-semipositive Jordan basis.

Example 3.10. Let α be a real number such that $0 < \alpha < \frac{1}{2}$. Let C_0 be the unbounded closed convex set in $\mathbb{R}^2$ with $\{(0,0)^T, (k, \alpha^{k-1})^T, k = -1, -2, \ldots\}$ as its set of extreme points and with $\operatorname{pos}\{(1,0)^T, (0,1)^T\}$ as its recession cone. (For the definition and properties of the recession cone of a convex set, see [Roc].) Let K be the proper cone in $\mathbb{R}^3$ given by:

$$K = \left\{\lambda \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in C_0, \lambda \geq 0\right\} \cup \operatorname{pos}\{(0,1,0)^T, (0,0,1)^T\}.$$

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$.

It is straightforward to verify that $A \in \pi(K)$. (The matrix A and the cone K have appeared in [T-S1, Example 6.2]). Clearly, $\rho(A) = 1$ and $E(A)$ equals span$\{(1,0,0)^T, (0,1,0)^T\}$. Since K contains both of the vectors $(1,0,0)^T$ and $(0,1,0)^T$, $E(A)$ contains a K-semipositive Jordan basis for A. By a straightforward calculation we obtain

$$Z_A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [(1+\varepsilon)I - A]^{-1} Z_A^{(0)} = \begin{bmatrix} \varepsilon^{-1} & 0 & 0 \\ \varepsilon^{-2} & \varepsilon^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for any $\varepsilon > 0$. Now $(1, k, \alpha^{k-1})^T$ is an extreme vector of K, whenever k is a negative integer. With $\varepsilon > 0$ fixed, the vector

$$[(1+\varepsilon)I - A]^{-1} Z_A^{(0)} (1, k, \alpha^{k-1})^T = (\varepsilon^{-1}, \varepsilon^{-2} + k\varepsilon^{-1}, 0)^T$$

does not belong to K if k is a negative integer, sufficiently large in absolute value. So $\varepsilon^{-1} J_A^{(0)}(\varepsilon) = [(1+\varepsilon)I - A)]^{-1} Z_A^{(0)} \notin \pi(K)$ for any $\varepsilon > 0$.

In the above, we have demonstrated that part(i) of Theorem 3.6 is invalid if K is a nonpolyhedral proper cone. We have also shown that, in general, for the conditions (d), (e) of Corollary 3.4, the implication (e) $\Longrightarrow$ (d) does not hold.

Next, we show that A satisfies the condition of Corollary 3.7(iii). A straightforward calculation yields

$$(I - A + \varepsilon Z_A^{(0)})^{-1} = \begin{bmatrix} \varepsilon^{-1} & 0 & 0 \\ \varepsilon^{-2} & \varepsilon^{-1} & 0 \\ 0 & 0 & (1-\alpha)^{-1} \end{bmatrix} \text{ for } \varepsilon \neq 0.$$

Since $0 < \alpha < 1$, we have $\alpha^{1/\varepsilon} < \varepsilon\alpha/(1-\alpha)$ for all sufficiently small $\varepsilon > 0$. We contend that for any such ε, we have $(I - A + \varepsilon Z_A^{(0)})^{-1} \in \pi(K)$.

Clearly, the matrix $(I - A + \varepsilon Z_A^{(0)})^{-1}$ sends the extreme vectors $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ each into $\mathbb{R}^3_+$, and hence into K. For any negative integer k, we have

$$(I - A + \varepsilon Z_A^{(0)})^{-1} (1, k, \alpha^{k-1})^T = \varepsilon^{-1}(1, \varepsilon^{-1} + k, \varepsilon\alpha^{k-1}(1-\alpha)^{-1})^T.$$

If $\varepsilon^{-1} + k \geq 0$, then $(I - A + \varepsilon Z_A^{(0)})^{-1}(1, k, \alpha^{k-1})^T \in \mathbb{R}^3_+ \subseteq K$. So suppose that $\varepsilon^{-1} + k < 0$. Then

$$\frac{\varepsilon}{1-\alpha} > \alpha^{\varepsilon^{-1}-1} \geq \alpha^{[\varepsilon^{-1}]},$$

where $[\varepsilon^{-1}]$ denotes the largest integer less than or equal to ε^{-1}; hence

$$\frac{\varepsilon\alpha^{k-1}}{1-\alpha} \geq \alpha^{[\varepsilon^{-1}]+k-1}.$$

Thus, the coordinates of the point $(\varepsilon^{-1}+k, \varepsilon\alpha^{k-1}(1-\alpha)^{-1})^T$ are each greater than or equal to the corresponding coordinate of the extreme point $([\varepsilon^{-1}]+k,\ \alpha^{[\varepsilon^{-1}]+k-1})^T$ of the convex set C_0 (which was used to define K). But $(0,1)^T$ belongs to the recession cone of C_0, hence $(\varepsilon^{-1}+k, \varepsilon\alpha^{k-1}(1-\alpha)^{-1})^T \in C_0$ and so $(I-A+\varepsilon Z_A^{(0)})^{-1}(1,k,\alpha^{k-1})^T \in K$. We just showed that $(I-A+\varepsilon Z_A^{(0)})^{-1} \in \pi(K)$ for all sufficiently small $\varepsilon > 0$, i.e., A satisfies the condition of Corollary 3.7(iii).

The preceding discussion shows that for a general proper cone K the condition given in part (ii) of Theorem 3.6 does not imply the one given in part (i).

For any matrix C, we have $C \in \pi(K)$ if and only if $C^T \in \pi(K^*)$. And since $((I-A+\varepsilon Z_A^{(0)})^{-1})^T = (I-A^T+\varepsilon Z_{A^T}^{(0)})^{-1}$, in this case, A^T (as an element of $\pi(K^*)$) also satisfies the condition of Corollary 3.7(iii) (with A and K replaced by A^T and K^* respectively). Note, however, that

$$E(A^T) = \operatorname{span}\{(1,0,0)^T, (0,0,1)^T\} \neq \operatorname{span}\{(1,0,0)^T\} = \operatorname{span}(K^* \cap E(A^T)).$$

So $E(A^T)$ does not contain a K^*-semipositive basis. This shows that for a general proper cone K, the condition of Corollary 3.7(iii) is not sufficient for the existence of a K-semipositive basis for $E(A)$, and also that when $E(A)$ has a K-semipositive basis, $E(A^T)$ may not have a K^*-semipositive basis.

Whether a cone-preserving map satisfies the conditions of Corollary 3.4 (or Corollary 3.7) depends not only on the map but also on the underlying cone.

For our matrix A, if we take C_1 to be the closed convex set in $\mathbb{R}^2$ with extreme points $(0,0)^T$ and $(-1,\alpha^{-2})^T$, and with recession cone $\operatorname{pos}\{(1,0)^T,\ (0,1)^T\}$, and if we take K_1 to be the proper cone of $\mathbb{R}^3$ given by:

$$K_1 = \left\{\lambda\begin{pmatrix}1\\x\end{pmatrix} : x \in C_1,\ \lambda \geq 0\right\} \cup \operatorname{pos}\{(0,1,0)^T, (0,0,1)^T\},$$

then it is straightforward to verify that $A \in \pi(K_1)$, and also that in this case A satisfies the conditions of Corollary 3.7(ii), (iii) (with K replaced by K_1). We omit the details.

In view of Corollary 3.7, certainly A satisfies all of the conditions of Corollary 3.7 if we take K to be $\mathbb{R}^3_+$.

In view of Theorem 3.6(i), one may ask whether it is true that if K is polyhedral, then for any $A \in \pi(K)$, we have $J_A^{(k)}(\varepsilon) \in \operatorname{cl}\omega_0(A)$ for all sufficiently small $\varepsilon > 0$, where $w_0(A) :=$

pos$\{A^i : i = 0, 1, \ldots\}$. The matrix A in this example also provides a negative answer to the question. This can be seen as follows. Since A is nonnegative, of course, $A \in \pi(\mathbb{R}^3_+)$. According to Theorem 3.6(i), we also have $J_A^{(0)}(\varepsilon) \in \pi(\mathbb{R}^3_+)$ for all sufficiently small $\varepsilon > 0$. (Indeed, $J_A^{(0)}(\varepsilon)$ is a nonnegative matrix, by direct calculation.) If $J_A^{(0)}(\varepsilon) \in \mathrm{cl}\,\omega_0(A)$ for all sufficiently small $\varepsilon > 0$, then we would have $J_A^{(0)}(\varepsilon) \in \pi(K)$ for any proper cone K for which $A \in \pi(K)$ (and for all sufficiently small $\varepsilon > 0$). But for the proper cone K considered in this example, we have $J_A^{(0)}(\varepsilon) \notin \pi(K)$ for any $\varepsilon > 0$. This proves our claim.

4. The spectral cone

Let $A \in \pi(K)$, where K is a proper cone. To study problems related to the K-semipositive Jordan chains for A, it seems worthwhile to look into the set

$$C(A, K) = \{x \in K : (A - \rho(A)I)^i x \in K \text{ for all positive integers } i\}.$$

It is readily checked that $C(A, K)$ is a (closed) subcone of K invariant under $A - \rho(A)I$ (and hence also under A); in fact, it is the largest subset of K invariant under $A - \rho(A)I$. We will refer to $C(A, K)$ as the *spectral cone of* A (for K corresponding to $\rho(A)$). By modifying an argument given in [T-S3, Section 6] we are going to show that $C(A, K)$ is always included in $E(A)$.

Lemma 4.1. *Let $A \in \pi(K)$, where K is a proper cone. Then*

$$\begin{aligned} C(A, K) &= \{x \in E(A) \cap K : (A - \rho(A)I)^i x \in K \textit{ for all positive integers } i\} \\ &= \{x \in K : (A - \rho(A)I)^i x \in K \textit{ for } i = 1, \ldots, \nu_\rho + 1\}. \end{aligned}$$

Proof. To prove the first equality, it suffices to show that $x \in E(A)$ for every $0 \neq x \in C(A, K)$. We first observe that for any such x, we have $\rho_x(A) = \rho(A)$. This is because, the condition $(A - \rho(A)I)x \in K$ implies that $Ax \geq^K \rho(A)x$ and hence $r_A(x) \geq \rho(A)$, where $r_A(x)$ denotes the lower Collatz-Wielandt number of x with respect to K. But, by [T-W, Theorem 2.4(i)], we always have $\rho_x(A) \geq r_A(x)$, where $\rho_x(A)$ denotes the local spectral radius of A at x; so we must have $\rho_x(A) = \rho(A)$. Let $x = x_1 + \cdots + x_k$ denote the representation of x as a sum of generalized eigenvectors of A; say, for $i = 1, \ldots, k$, x_i corresponds to the eigenvalue λ_i, where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A. By the local Perron-Schaefer condition of A at x (see [T-S3, Section 2] or [Tam4, the paragraph following Remark 5.5]), we may assume that $\lambda_1 = \rho(A)$. Say, the order of x_1 is m. Suppose that $(A - \rho(A)I)^m x \neq 0$. Then, by our beginning observation (with $(A-\rho(A)I)^m x$ playing the role of x), the local spectral radius of A at $(A-\rho(A)I)^m x$ is equal to $\rho(A)$. On the other hand, in the representation of $(A-\rho(A)I)^m x$ as a sum of generalized eigenvectors of A, namely,

$$(A-\rho(A)I)^m x = (A-\rho(A)I)^m x_2 + \cdots + (A-\rho(A)I)^m x_k,$$

there is no longer a term which is a generalized eigenvector of A corresponding to $\rho(A)$; so, by the local Perron-Schaefer condition of A at $(A-\rho(A)I)^m x$, the local spectral radius of A at $(A-\rho(A)I)^m x$ must be less than $\rho(A)$. Thus, we arrive at a contradiction. This proves that we must have $(A-\rho(A)I)^m x = 0$, i.e., $x \in E(A)$. This establishes the first equality.

The above argument also shows that if $x \in K$ satisfies $(A-\rho(A)I)^i x \in K$ for $i = 1, \ldots, m$, where $m = \mathrm{ord}_A(x)$, and if $x \notin E(A)$, then necessarily $(A-\rho(A)I)^{m+1} x \notin K$. Hence, we have the second equality. ■

It is possible that the set $\{x \in K : (A-\rho(A)I)^i x \in K \text{ for } i = 1, \ldots, \nu_\rho\}$ strictly includes $C(A,K)$. For instance, if $\nu_\rho = 1$ and A has an eigenvector in $\Phi(\mathcal{N}(A-\rho(A)I) \cap K)$ corresponding to an eigenvalue other than $\rho(A)$, then the set $\{x \in K : (A-\rho(A)I)x \in K\}$ always contains vectors not belonging to $E(A)$ (see the final part of the proof of [T-S3, Corollary 4.10]).

For any $A \in \pi(K)$, clearly we have

$$K \cap (A-\rho(A)I)^{-1}(0) = \mathcal{N}((A-\rho(A)I)) \cap K = \mathcal{N}((A-\rho(A)I) \cap C(A,K),$$

and

$$K \cap (A-\rho(A)I)^{-1}[\mathcal{N}(A-\rho(A)I) \cap C(A,K)] = \mathcal{N}((A-\rho(A)I)^2) \cap C(A,K) \subseteq \mathcal{N}((A-\rho(A)I)^2) \cap K.$$

We contend that the last inclusion can be replaced by an equality.

By the local Perron-Schaefer condition, if $0 \neq x \in K$, then there is a generalized eigenvector y of A corresponding to $\rho_x(A)$ that appears as a term in the representation of x as a sum of generalized eigenvectors of A. Furthermore, $\mathrm{ord}_A(x) = \mathrm{ord}_A(y)$. In fact, by [T-S2, Corollary 4.8], $(A-\rho_x(A)I)^{\mathrm{ord}_A(y)-1} y$, the eigenvector of A associated with y, always belongs to K. With this in mind, we clearly have

$$(A-\rho(A)I)(\mathcal{N}((A-\rho(A)I)^2 \cap K) \subseteq \mathcal{N}(A-\rho(A)I) \cap K,$$

which implies our contention.

Now it is straightforward to verify the following result, which can be used to determine $C(A,K)$ (as we will illustrate in Example 4.4).

Remark 4.2. Let K be a proper cone in $\mathbb{R}^n$ and let $A \in \pi(K)$. Then $\mathcal{N}(A-\rho(A)I) \cap C(A,K) = \mathcal{N}(A-\rho(A)I) \cap K$, $\mathcal{N}((A-\rho(A)I)^2) \cap C(A,K) = \mathcal{N}((A-\rho(A)I)^2) \cap K$ and for any positive integer i, we have

$$\mathcal{N}((A-\rho(A)I)^i) \cap C(A,K) = K \cap (A-\rho(A)I)^{-1}[\mathcal{N}((A-\rho(A)I)^{i-1}) \cap C(A,K)].$$

If m is the maximum length of a K-semipositive Jordan chain for A (for $\rho(A)$), then

$$\begin{aligned} C(A,K) \;=\; & \mathcal{N}((A-\rho(A)I)^m)\cap C(A,K) \supset \mathcal{N}((A-\rho(A)I)^{m-1})\cap C(A,K) \supset \\ & \cdots \supset \mathcal{N}((A-\rho(A)I))\cap C(A,K), \end{aligned}$$

where all inclusions are strict.

Our next result tells us, in particular, that if K is polyhedral, then $C(A,K)$ is always a polyhedral full subcone of $E(A)\cap K$, and hence a proper cone in $E(A)$. In contrast, we would like to add that when K is nonpolyhedral, $C(A,K)$ need not be full in $\mathrm{span}(E(A)\cap K)$ and it may well happen that $\mathrm{span}(E(A)\cap K) = E(A)$ but $C(A,K)$ is only a single ray. See our Example 7.6.

Theorem 4.3. *Let K be a polyhedral proper cone and let $A \in \pi(K)$. Let ε be a given positive number (sufficiently small) such that $J_A^{(i)}(\varepsilon) \in \pi(K)$ for $i = 0,\ldots,\nu_\rho - 1$. Then:*

(i) *$C(A,K)$ is polyhedral.*

(ii) *$J_A^{(0)}(\varepsilon)(E(A)\cap K)$ and $J_A^{(0)}(\varepsilon)K$ are both full subcones of $C(A,K)$.*

(iii) *For $i = 1,\ldots,\nu_\rho$, $\mathcal{N}((A-\rho(A)I)^i)\cap C(A,K)$ is an $(A-\rho(A)I)$-invariant face of $C(A,K)$ and also a full subcone of $\mathcal{N}((A-\rho(A)I)^i)\cap K$. In particular, $C(A,K)$ is a full subcone of $E(A)\cap K$ and hence a proper cone in $E(A)$.*

(iv) *For $i = 0,1,\ldots,\nu_\rho - 1$, $J_A^{(i)}(\varepsilon)K$ and $J_A^{(i)}(\varepsilon)(E(A)\cap K)$ are both A-invariant subcones of $C(A,K)$ and also both full cones in $\mathcal{R}((A-\rho(A)I)^i)\cap E(A)$.*

Proof. (i) First of all, the cone $\mathcal{N}((A-\rho(A)I)^2)\cap K$, being the intersection of two polyhedral cones, is polyhedral. In view of Remark 4.2, $\mathcal{N}((A-\rho(A)I)^3)\cap C(A,K)$ is also polyhedral, as the pre-image of a polyhedral set under a linear map is polyhedral (see [Roc, Theorem 19.3]). Inductively, we can show that $\mathcal{N}((A-\rho(A)I)^i)\cap C(A,K)$ is polyhedral for all positive integers i. Hence, $C(A,K)$, which is $\mathcal{N}((A-\rho(A)I)^{\nu_\rho})\cap C(A,K)$, is also polyhedral.

(ii) Note that $\mathcal{R}(J_A^{(0)}(\varepsilon)) = E(A)$ and the restriction map $J_A^{(0)}(\varepsilon)|_{E(A)}$ is nonsingular. Also, $J_A^{(0)}(\varepsilon)|_{E(A)} \in \pi(E(A)\cap K)$ (and $\mathrm{span}(E(A)\cap K) = E(A)$ as K is polyhedral). Hence, $J_A^{(0)}(\varepsilon)(E(A)\cap K)$ is full subcone of $E(A)\cap K$. On the other hand, we also have the inclusion relations

$$J_A^{(0)}(\varepsilon)(E(A)\cap K) \subseteq J_A^{(0)}(\varepsilon)K \subseteq C(A,K) \subseteq E(A)\cap K,$$

where the second inclusion follows from the fact that, by the relation (2.4), $(A-\rho(A)I)^i J_A^{(0)}(\varepsilon)K = J_A^{(i)}(\varepsilon)K \subseteq K$ for $i = 0,\ldots,\nu_\rho - 1$, and the third inclusion holds by Lemma 4.1. Thus, $J_A^{(0)}(\varepsilon)(E(A)\cap K)$ (also $J_A^{(0)}(\varepsilon)K$) is a full subcone of $C(A,K)$.

(iii) In the above we have shown that $C(A,K)$ is a full subcone of $E(A)\cap K$. For $i = 1,\dots,\nu_{\rho(A)}(A)$, denote the polyhedral cone $\mathcal{N}((A-\rho(A)I)^i)\cap K$ by K_i. By applying what we have done to $A|_{\operatorname{span} K_i}$ ($\in \pi(K_i)$) and noting that $C(A|_{\operatorname{span} K_i}, K_i) = \mathcal{N}((A-\rho(A))^i)\cap C(A,K)$, we infer that $\mathcal{N}((A-\rho(A)I)^i)\cap C(A,K)$ is a full subcone of $\mathcal{N}((A-\rho(A)I)^i)\cap K$. In terms of the concept of spectral pair of a vector, the set $\mathcal{N}((A-\rho(A)I)^i)\cap C(A,K)$ can be written as $\{x \in C(A,K) : \mathrm{sp}_{A-\rho(A)I}(x) \preceq (0,i)\}$. By [T-S2, Corollary 4.10] it follows that the set is an $(A-\rho(A)I)$-invariant face of $C(A,K)$.

(iv) By the relation (2.4), we have,

$$J_A^{(i)}(\varepsilon)K = (A-\rho(A)I)^i J_A^{(0)}(\varepsilon)K \subseteq (A-\rho(A)I)^i C(A,K) \subseteq C(A,K);$$

so $J_A^{(i)}(\varepsilon)K$ is a subcone of $C(A,K)$. The set $J_A^{(i)}(\varepsilon)K$ is invariant under A, because A and $J_A^{(i)}(\varepsilon)$ commute and K is invariant under A. Since $J_A^{(0)}(\varepsilon)K$ is a full cone in $E(A)$, by the relation (2.4) again, $J_A^{(i)}(\varepsilon)K$ is a full cone in $(A-\rho(A))^i E(A)$, which is $\mathcal{R}((A-\rho(A)I)^i)\cap E(A)$. The preceding arguments still hold if we apply them to $J_A^{(i)}(\varepsilon)(E(A)\cap K)$ instead of $J_A^{(i)}(\varepsilon)K$. ■

Our next example illustrates Remark 4.2, Theorem 4.3, and shows that, in general, the cone $J_A^{(0)}(\varepsilon)K$ (for sufficiently small $\varepsilon > 0$) need not be invariant under $A-\rho(A)I$, not even when K is polyhedral. It also shows that $J_A^{(0)}(\varepsilon)K$ varies with ε and is usually strictly included in $C(A,K)$.

Example 4.4. Let K be the polyhedral proper cone in $\mathbb{R}^3$ with extreme vectors e_1, e_1+e_2, e_2+e_3 and e_3, and let $A = J_3(1)$. Then $\rho(A) = 1$ and $\nu_\rho = 3$. As can be readily checked, $A \in \pi(K)$. Since $\mathcal{N}((A-\rho(A)I)^2)\cap K = \operatorname{pos}\{e_1, e_1+e_2\}$, $(A-\rho(A)I)e_2 = e_1$, $(A-\rho(A)I)(e_2+e_3) = e_1+e_2$, $\mathcal{N}(A-\rho(A)I) = \operatorname{span}\{e_1\}$ and the maximum length of a K-semipositive Jordan chain for A is ν_ρ, by Remark 4.2 we have

$$\begin{aligned} C(A,K) &= K \cap (A-\rho(A)I)^{-1}[\mathcal{N}((A-\rho(A)I)^2)\cap K] \\ &= K \cap (\operatorname{pos}\{e_2, e_2+e_3\} + \operatorname{span}\{e_1\}) \\ &= \operatorname{pos}\{e_1, e_1+e_2, e_2+e_3\}. \end{aligned}$$

Now $Z_A^{(0)} = I_3$, $Z_A^{(1)} = J_3(0)$ and $Z_A^{(2)} = J_3(0)^2$. So

$$J_A^{(0)}(\varepsilon) = \begin{bmatrix} 1 & \varepsilon^{-1} & \varepsilon^{-2} \\ 0 & 1 & \varepsilon^{-1} \\ 0 & 0 & 1 \end{bmatrix}.$$

By calculations, we have

$$J_A^{(0)}(\varepsilon)e_1 = e_1, \quad J_A^{(0)}(\varepsilon)(e_1+e_2) = (e_1+e_2) + \varepsilon^{-1}e_1,$$
$$J_A^{(0)}(\varepsilon)(e_2+e_3) = (e_2+e_3) + \varepsilon^{-1}(e_1+e_2) + \varepsilon^{-2}e_3,$$

and

$$J_A^{(0)}(\varepsilon)e_3 = \begin{cases} (\varepsilon^{-2}-\varepsilon^{-1})e_1 + \varepsilon^{-1}(e_1+e_2) + e_3 & \text{for } \varepsilon \le 1 \\ \varepsilon^{-2}(e_1+e_2) + (\varepsilon^{-1}-\varepsilon^{-2})(e_2+e_3) + (1-\varepsilon^{-1}+\varepsilon^{-2})e_3 & \text{for } \varepsilon > 1 \end{cases}.$$

So $J_A^{(0)}(\varepsilon)K \subseteq K$, i.e., $J_A^{(0)}(\varepsilon) \in \pi(K)$ for all $\varepsilon > 0$. Of course, this is something expected (by Theorem 3.3), as $Z_A^{(0)} \in \pi(K)$ in this case. Note also that, for all $\varepsilon > 0$, $J_A^{(0)}(\varepsilon)K$ is strictly included in $C(A,K)$, as $J_A^{(0)}(\varepsilon)(K) \cap \operatorname{span}\{e_1,e_2\} = \operatorname{pos}\{e_1, (e_1+e_2)+\varepsilon^{-1}e_1\} \subset \operatorname{pos}\{e_1, e_1+e_2\} = C(A,K) \cap \operatorname{span}\{e_1,e_2\}$.

Since $Z_A^{(1)}e_3 = e_2 \notin K$, in this case, $Z_A^{(1)} \notin \pi(K)$. By calculations, we also have $J_A^{(1)}(\varepsilon)e_1 = 0$, $J_A^{(1)}(\varepsilon)(e_1+e_2) = e_1$, $J_A^{(1)}(\varepsilon)(e_2+e_3) = \varepsilon^{-1}e_1 + (e_1+e_2)$ and $J_A^{(1)}(\varepsilon)e_3 = (\varepsilon^{-1}-1)e_1 + (e_1+e_2)$. So $J_A^{(1)}(\varepsilon)$ belongs to $\pi(K)$ for all $0 < \varepsilon \le 1$ (but it does not belong to $\pi(K)$ for $\varepsilon > 1$). Since $(A-\rho(A)I)J_A^{(0)}(\varepsilon)e_3 = J_A^{(1)}(\varepsilon)e_3 \notin J_A^{(0)}(\varepsilon)K$, $J_A^{(0)}(\varepsilon)K$ is not invariant under $A-\rho(A)I$.

The following observation will prove to be useful:

Lemma 4.5. *Let K be a proper cone and let $A \in \pi(K)$. For any $x \in E(A)$, we have, $x = y(\varepsilon) - \varepsilon^{-1}(A-\rho(A)I)y(\varepsilon)$, where $y(\varepsilon) := J_A^{(0)}(\varepsilon)x$, $\varepsilon \ne 0$. Moreover, $\operatorname{ht}(y(\varepsilon)) = \operatorname{ht}(x)$. If, in addition, K is polyhedral and $x \in K$ such that $\Phi(x)$ is an A-invariant face of K, then x and $y(\varepsilon)$ generate the same face of K for all sufficiently small $\varepsilon > 0$.*

Proof. The required representation of x follows from the relation

$$J_A^{(0)}(\varepsilon) = Z_A^{(0)} + \varepsilon^{-1}(A-\rho(A)I)J_A^{(0)}(\varepsilon),$$

which, in turn, is a rewriting of the equality $((\rho(A)+\varepsilon)I - A)J_A^{(0)}(\varepsilon) = \varepsilon Z_A^{(0)}$ (cf. (2.3)). Let $m = \operatorname{ht}(x)$. By direct calculation, we have $(A-\rho(A)I)^{m-1}J_A^{(0)}(\varepsilon)x = Z_A^{(m-1)}x \ne 0$ and $(A-\rho(A)I)^m J_A^{(0)}(\varepsilon) = Z_A^{(m)}x = 0$; so $\operatorname{ht}(y(\varepsilon)) = m = \operatorname{ht}(x)$.

Last part. Since $y(\varepsilon) = x + \varepsilon^{-1}(A-\rho(A)I)y(\varepsilon)$ and $y(\varepsilon) \in J_A^{(0)}(\varepsilon)K \subseteq C(A,K)$ for all sufficiently small $\varepsilon > 0$, we have $x \in \Phi(y(\varepsilon))$. On the other hand, if $\Phi(x)$ is invariant under A, then, since $\Phi(x)$ is polyhedral, by applying Theorem 3.6(i) to $A|_{\operatorname{span}\Phi(x)}$ we infer that $\Phi(x)$ is invariant under $J_A^{(0)}(\varepsilon)$. Then, by the definition of $y(\varepsilon)$, we also have $y(\varepsilon) \in \Phi(x)$.

This proves that $\Phi(x) = \Phi(y(\varepsilon))$. ∎

In below we give two further results on the spectral cone. Although these results are not needed in the sequel, for completeness and for possible future use, we also include them here.

Lemma 4.6. *Let $A \in \pi(K)$, where K is a proper cone. Then:*
(i) $\mathcal{N}(A - \rho(A)I) \cap K \subseteq C(A, K) \subseteq E(A) \cap K$.
(ii) The following conditions are equivalent:

- *(a) $\mathcal{N}(A - \rho(A)I) \cap K = C(A, K)$.*
- *(b) A has no distinguished generalized eigenvectors corresponding to $\rho(A)$ of height two.*
- *(c) $(A - \rho(A)I)(E(A) \cap K) = \{0\}$.*

If, in addition, K is polyhedral, then another equivalent condition is that $E(A) \cap K = \mathcal{N}(A - \rho(A)I) \cap K$.

(iii) $C(A, K) = E(A) \cap K$ if and only if $(A - \rho(A)I)(E(A) \cap K) \subseteq E(A) \cap K$. The equivalent conditions are satisfied if $Z_A^{(1)} \in \pi(K)$.

Proof. It suffices to establish part (ii), as parts (i) and (iii) are obvious. By Remark 4.2, if (a) is not satisfied, then the maximum length of a K-semipositive Jordan chain for A exceeds 1, hence A must have a distinguished generalized eigenvector corresponding to $\rho(A)$ of height two. Conversely, if A has a distinguished generalized eigenvector corresponding to $\rho(A)$ of height two, say y, then by Remark 4.2 again, we have, $y \in [\mathcal{N}((A - \rho(A)I)^2) \cap C(A, K)] \backslash \mathcal{N}(A - \rho(A)I)$, and so (a) is not satisfied. This proves the equivalence of (a) and (b). Condition (c) amounts to saying that every distinguished generalized eigenvector of A corresponding to $\rho(A)$ is of height one. So we have (c) $\Longrightarrow$ (b). Since the closed pointed cone $(A - \rho(A)I)(E(A) \cap K)$ is invariant under A, by the cone version of the Perron-Frobenius theorem, if the cone is nonzero, then it must contain an eigenvector of A, necessarily corresponding to $\rho(A)$ (as the cone lies in $E(A)$). But then it will follow that A has a distinguished generalized eigenvector corresponding to $\rho(A)$ of height two. This shows (b) $\Longrightarrow$ (c). When K is polyhedral, another equivalent condition is that $E(A) \cap K = \mathcal{N}(A - \rho(A)I) \cap K$, because then A has a distinguished generalized eigenvector corresponding to $\rho(A)$ of height greater than one if and only if it has one such generalized eigenvector of height two. ∎

Lemma 4.7. *Let $A \in \pi(K)$, where K is a proper cone. For any A-invariant face F of K associated with $\rho(A)$, $C(A, K) \cap F$ equals $C(A|_{\operatorname{span} F}, F)$ and is an $(A - \rho(A)I)$-invariant face of $C(A, K)$.*

Proof. Clearly, we have, $C(A|_{\operatorname{span} F}, F) \subseteq C(A,K) \cap F$. To prove the reverse inclusion, observe that the conditions $x \in F$, F is an A-invariant face and $(A - \rho(A)I)x \in K$ together imply that $(A - \rho(A)I)x \in F$. Now, take any $x \in C(A,K) \cap F$. Making use of the preceding observation and the fact that $(A - \rho(A)I)^i x \in K$ for all positive integers i, inductively, we can show that $(A - \rho(A)I)^i x \in F$ for all positive integers i. Hence, $x \in C(A|_{\operatorname{span} F}, F)$. This proves the reverse inclusion. It is readily checked that $C(A,K) \cap F$ is a face of $C(A,K)$. The set $C(A|_{\operatorname{span} F}, F)$ is invariant under $A - \rho(A)I$, because it is invariant under $A|_{\operatorname{span} F} - \rho(A|_{\operatorname{span} F})I$ and $\rho(A|_{\operatorname{span} F}) = \rho(A)$. ■

Lemma 4.7 provides the following alternative way to derive the first half of Theorem 4.3(iii), assuming that we have already shown that $C(A,K)$ is a full subcone of $E(A) \cap K$ whenever K is polyhedral:

Take $F = \Phi(\mathcal{N}((A-\rho(A)I)^i) \cap K)$. Then $C(A|_{\operatorname{span} F}, F)$ is a full subcone of $E(A|_{\operatorname{span} F}) \cap F$, as F is polyhedral. Apply Lemma 4.7 and observe that $F \cap C(A,K) = \mathcal{N}((A - \rho(A)I)^i) \cap C(A,K)$ and $E(A|_{\operatorname{span} F}) \cap F = \mathcal{N}((A - \rho(A)I)^i) \cap K$.

5. Equality of the height and the level characteristics

Historically, the problem of determining when the spectral height (or Weyr) characteristic and the graph-theoretic level characteristic of a singular M-matrix are equal has been a topic of interest. Richman and Schneider [R-S, Theorem 6.5] first obtained a few equivalent conditions. Further equivalent conditions were found in the subsequent papers [H-S2, 3] by Herhkowitz and Schneider (see also [B-N]), bringing the total number of equivalent conditions to 35. In order to extend these known results to the setting of a cone-preserving map or to provide alternative cone-theoretic proofs for these results, we need to find first the right analogs for the concept of level characteristics of a matrix and that of the level of a vector in this more general setting. Recall that the level characteristic $\lambda(P)$ of a (square) matrix P is defined in terms of the accessibility relation between the classes of P (or in terms of its reduced graph) and the level of a vector is defined in terms of the support of the vector and the classes of P. Clearly, these definitions cannot be carried over directly to the cone-preserving map setting. Fortunately, they have equivalent definitions, given by Remarks 2.2 and 2.3 respectively, that we can count on.

Let $A \in \pi(K)$, where K is a proper cone. For any $0 \neq x \in E(A)$, if $x \in \operatorname{span}(E(A) \cap K)$, we define the *level of* x, denoted by $\operatorname{lev}(x)$, to be the smallest positive integer k (which is at most ν_ρ) such that $x \in \operatorname{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K]$; otherwise, we set $\operatorname{lev}(x)$ to be ∞. Of course, when K is polyhedral, $\operatorname{lev}(x)$ is finite for every $0 \neq x \in E(A)$.

We also define the *level characteristic of* A, denoted by $\lambda(A)$, to be the ν_ρ-tuple $(\lambda_1(A)$, ..., $\lambda_{\nu_\rho}(A))$ given by:

$$\begin{aligned} \lambda_1(A) &= \dim \operatorname{span}[\mathcal{N}(A-\rho(A)I)\cap K] \\ \text{and} & \\ \lambda_k(A) &= \dim \operatorname{span}[\mathcal{N}((A-\rho(A)I)^k \cap K] - \dim \operatorname{span}[\mathcal{N}((A-\rho(A)I)^{k-1})\cap K] \end{aligned}$$

for $k = 2, \ldots, \nu_\rho$. When there is no danger of confusion, we write $\lambda_k(A)$ simply as λ_k.

Note that by the cone version of the Perron-Frobenius theorem $\lambda_1(A)$ is always positive. When K is polyhedral, there is a K-semipositive Jordan chain for A of length ν_ρ (see [Tam2, Theorem 7.5(ii)]); hence $\operatorname{span}[\mathcal{N}((A-\rho(A)I)^{k-1})\cap K]$ is strictly included in $\operatorname{span}[\mathcal{N}((A-\rho(A)I)^k)\cap K]$ and so $\lambda_k(A)$ is also always positive for $k = 2, \ldots, \nu_\rho$. But this is not so for the nonpolyhedral case. See, for instance, our Example 7.6.

Borrowing the definition from the nonnegative matrix case, we call a basis $\mathcal{B}$ for $E(A)$ a *level basis* if the number of vectors in $\mathcal{B}$ of level k is $\lambda_k(A)$ for $k = 1, \ldots, \nu_\rho$. (Clearly, we have $\operatorname{span}(E(A)\cap K) = E(A)$ when such basis exists.) A basis $\mathcal{B}$ for $E(A)$ is called a *height-level basis* if it is a height basis as well as a level basis.

Following [H-S3], we also call a vector $x \in E(A)$ a *peak vector* if $\operatorname{ht}(x) = \operatorname{lev}(x)$.

Lemma 5.1. *Let $A \in \pi(K)$, where K is a proper cone. Then $E(A)$ has a K-semipositive basis if and only if $E(A)$ has a K-semipositive level basis.*

Proof. It suffices to show that when $E(A)$ has a K-semipositive basis, or equivalently when $E(A) = \operatorname{span}(E(A)\cap K)$, it is possible to construct a K-semipositive level basis for $E(A)$. To do that, first choose λ_1 linearly independent vectors $x_1, \ldots, x_{\lambda_1}$ from $\mathcal{N}(A-\rho(A)I)\cap K$, and then inductively, for $k = 2, \ldots, \nu_\rho$, adjoin λ_k linearly independent vectors $x_{\lambda_1+\cdots+\lambda_{k-1}+1}$, ..., $x_{\lambda_1+\cdots+\lambda_k}$ from $\mathcal{N}((A-\rho(A)I)^k)\cap K$ so that

$$\begin{aligned} & \operatorname{span}[\mathcal{N}((A-\rho(A)I)^k)\cap K] \\ =\; & \operatorname{span}[\mathcal{N}((A-\rho(A)I)^{k-1})\cap K] \oplus \operatorname{span}\{x_{\lambda_1+\cdots+\lambda_{k-1}+1}, \ldots, x_{\lambda_1+\cdots+\lambda_k}\}. \qquad \blacksquare \end{aligned}$$

We also define the *peak characteristic* of A ($\in \pi(K)$) to be the ν_ρ-tuple $\xi(A) = (\xi_1(A)$, ..., $\xi_{\nu_\rho}(A))$, where

$$\xi_k(A) = \dim(A-\rho(A)I)^{k-1}(\mathcal{N}((A-\rho(A)I)^k)\cap K),\; k = 1, \ldots, \nu_\rho.$$

In the nonnegative matrix case, our definition reduces to the definition of peak characteristic for a singular M-matrix as introduced by Neumann and Schneider [N-S3, Definition 2].

In general, since $\eta_k(A) = \dim(A - \rho(A)I)^{k-1}\mathcal{N}((A - \rho(A)I)^k)$, we have

$$\xi_k(A) \leq \eta_k(A) \quad \text{for } k = 1, \ldots, \nu.$$

In fact, we also have

$$\xi_k(A) \leq \lambda_k(A) \quad \text{for } k = 1, \ldots, \nu.$$

This is because, in the notation of the proof for Lemma 5.1, $(A - \rho(A)I)^{k-1}\,\text{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K]$ equals $(A - \rho(A)I)^{k-1}\,\text{span}\{x_{\lambda_1+\cdots+\lambda_{k-1}+1}, \ldots, x_{\lambda_1+\cdots+\lambda_k}\}$ and the latter subspace is of dimension at most λ_k.

Remark 5.2. Let $A \in \pi(K)$, where K is a proper cone. For $k = 1, \ldots, \nu_{\rho(A)}(A)$, the number of K-semipositive vectors of height k in any height basis $\mathcal{B}$ for A is at most $\xi_k(A)$.

It suffices to show that if $x_1, \ldots, x_r$ are vectors of height k in a height basis $\mathcal{B}$ for A, then the vectors $A^{k-1}x_1, \ldots, A^{k-1}x_r$ are linearly independent. Suppose not. Then there exist scalars $\alpha_1, \ldots, \alpha_r$ not all zero such that the vector $\alpha_1 x_1 + \cdots + \alpha_r x_r$ belongs to $\mathcal{N}(A^{k-1})$ and hence can be expressed as a linear combination of vectors in $\mathcal{B}$ of height $k-1$ or less. This means that the set of all vectors in $\mathcal{B}$ of height k or less are linearly dependent, which is a contradiction.

Note that the sequence of vectors

$$x, (A - \rho(A)I)x, \ldots, (A - \rho(A)I)^{k-1}x$$

forms a K-semipositive Jordan chain for A of length k if and only if $x \in [\mathcal{N}((A - \rho(A)I)^k) \cap C(A, K)] \backslash [\mathcal{N}((A - \rho(A)I)^{k-1}) \cap C(A, K)]$. Indeed, we also have the following:

Remark 5.3. Let $A \in \pi(K)$ where K is a proper cone. For $k = 1, \ldots, \nu_\rho$, the maximum number of K-semipositive Jordan chains for A of length k whose union is a linearly independent set is equal to the quantity

$$\dim(A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A, K)].$$

Lemma 5.4. *Let $A \in \pi(K)$ where K is a proper cone. For $k = 1, \ldots, \nu_\rho$, we have*

$$\dim(A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A, K)] \leq \xi_k(A) \leq \eta_k(A).$$

If, in addition, K is polyhedral, then for $k = 1, \ldots, \nu_\rho - 1$, the first inequality becomes an equality and for $k = \nu_\rho$, both inequalities become equalities.

Proof. The inequalities are obvious. When K is polyhedral, by Theorem 4.3(iii), for $k = 1, \ldots, \nu_\rho$, $\mathcal{N}((A - \rho(A)I)^k) \cap C(A, K)$ is a full subcone of $\mathcal{N}((A - \rho(A)I)^k) \cap K$, and so the images of these two cones under $(A - \rho(A)I)^{k-1}$ have the same dimension. Hence, the first inequality becomes an equality. Moreover, we also have $\text{span}(E(A) \cap C(A,K)) = \text{span}(E(A) \cap K) = E(A)$; so for $k = \nu_\rho$, the second inequality also becomes an equality. $\blacksquare$

It is straightforward to verify the following:

Remark 5.5. Let $A \in \pi(K)$, where K is a proper cone. For $k = 1, \dots, \nu_\rho$, we have

$$\mathcal{N}(A - \rho(A)I) \cap (A - \rho(A)I)^k C(A,K) = (A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A,K)].$$

Each of the above equalities still holds if we replace $C(A,K)$ on both sides by K. So $\xi_k(A)$ is also equal to $\dim[\mathcal{N}(A - \rho(A)I) \cap (A - \rho(A)I)^k K]$.

In [N-S3] Neumann and Schneider give an algorithm to compute a union of semipositive Jordan chains for a nonnegative matrix P such that the number of vectors in the union at height k equals $\xi_k(P)$ for $k = 1, \dots, \nu_\rho$ (cf. Remark 5.2). They refer to such a set as a *maximal nonnegative union of chains*, and show that any such union can always be extended to a height basis for P. In what follows we show that the latter result can be reinforced and extended to the setting of a cone-preserving map.

Theorem 5.6. *Let $A \in \pi(K)$, where K is a proper cone. For $k = 1, \dots, \nu$, where $\nu = \nu_{\rho(A)}(A)$, let $\gamma_k(A) = \dim(A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A,K)]$. Then:*

(i) $\gamma_1(A) \geq \cdots \geq \gamma_\nu(A)$.

(ii) *There exists a linearly independent subset $\mathcal{C}$ of $E(A)$ which is the union of $\gamma_1(A)$ K-semipositive Jordan chains for A such that in $\mathcal{C}$ the number of vectors of height k equals $\gamma_k(A)$ for $k = 1, \dots, p$, where p is the largest integer j such that $\gamma_j > 0$. Furthermore, there is a height basis for A that includes C.*

If, in addition, K is polyhedral, then:

(iii) $\xi_1(A) \geq \cdots \geq \xi_\nu(A) = \eta_\nu(A)$.

(iv) *There exists a linearly independent subset $\widetilde{\mathcal{C}}$ of $E(A)$ which is the union of $\xi_1(A)$ K-semipositive Jordan chains for A such that in $\widetilde{\mathcal{C}}$ the number of vectors of height k equals $\xi_k(A)$ for $k = 1, \dots, \nu$. Furthermore, there is a height basis for A that includes $\widetilde{\mathcal{C}}$.*

Proof. For $1 \leq k \leq \nu$, denote by Q_k the cone $(A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A,K)]$. By Remark 5.5, Q_k equals $\mathcal{N}((A - \rho(A)I) \cap (A - \rho(A)I)^k C(A,K)$; so we have $Q_1 \supseteq Q_2 \supseteq \cdots \supseteq Q_\nu$, and by the definition of $\gamma_k(A)$, part(i) follows.

Note that p is the largest integer j such that the cone Q_j is nonzero. To construct the desired linearly independent subset $\mathcal{C}$ of $E(A)$, first choose a basis of $\text{span}\, Q_p$ that consists of vectors of Q_p, say, $\{x^1, \dots, x^{\gamma_p}\}$. Then, inductively, extend it to a basis of $\text{span}\, Q_1$, say $\{x^1, \dots, x^{\gamma_1}\}$, such that there are $\gamma_{k-1} - \gamma_k$ vectors chosen from $Q_{k-1} \backslash Q_k$ for $k = 2, \dots, p$.

By the definition of Q_p, each of the vectors $x^1, \ldots, x^{\gamma_p}$ gives rise to a K-semipositive Jordan chain for A of length p. Similarly, for $k = 2, \ldots, p$, each of the vectors $x^{\gamma_k+1}, \ldots, x^{\gamma_{k-1}}$ (which came from $Q_{k-1} \backslash Q_k$) gives rise to a K-semipositive Jordan chain for A of length $k-1$. So, altogether we have γ_1 K-semipositive Jordan chains for A, whose union $\mathcal{C}$ is clearly a linearly independent set, as the eigenvectors $x^1, \ldots, x^{\gamma_1}$ are linearly independent. By our construction, it is clear that the number of vectors of height k in $\mathcal{C}$ equals ξ_k.

Since $\mathcal{C}$ is a Jordan basis and hence a height basis for the restriction of A to the invariant subspace $\operatorname{span} \mathcal{C}$, by the following observation, $\mathcal{C}$ can be extended to a height basis for A. Hence, part(ii) also follows.

Observation. *Let N be a nilpotent operator on a finite-dimensional vector space, and let W be an N-invariant subspace. Then every height basis for $N|_W$ can be extended to a height basis for N.*

Proof of Observation. Let $(\eta_1, \ldots, \eta_p)$ and $(\eta'_1, \ldots, \eta'_q)$ be the height characteristics of N and $N|_W$ respectively. Then $p \geq q$ and $\eta_i \geq \eta'_i$ for $i = 1, \ldots, q$. Consider any height basis for $N|_W$, say $\mathcal{B} = \bigcup_{i=1}^{q} \{x_{ij}\}_{1 \leq j \leq \eta'_i}$, where each x_{ij} is a vector of height i. We want to adjoin to $\mathcal{B}$ vectors $y_{i1}, \ldots, y_{i\delta_i}$, where δ_i equals $\eta_i - \eta'_i$ for $1 \leq i \leq q$ and equals η_i for $q+1 \leq i \leq p$, and each y_{ij} is a vector of height i, such that the resulting set is a height basis for N. To begin with, adjoin vectors $y_{11}, \ldots, y_{1\delta_1}$ to $\{x_{11}, \ldots, x_{1\eta'_1}\}$ so that the resulting set forms a basis for $\mathcal{N}(N)$. Consider any k, $2 \leq k \leq p$. Suppose we have already found the vectors y_{ij} for $1 \leq i \leq k-1$ and $1 \leq j \leq \delta_i$ such that $\mathcal{B}_{k-1} := \bigcup_{i=1}^{k-1} [\{x_{ij}\}_{1 \leq j \leq \eta'_i} \cup \{y_{ij}\}_{1 \leq j \leq \delta_i}]$ forms a basis for $\mathcal{N}(N^{k-1})$. We contend that $\mathcal{B}_{k-1} \cup \{x_{k1}, \ldots, x_{k\eta'_k}\}$ is a linearly independent set. Consider any linear relation of the form

$$\sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \eta'_i}} \alpha_{ij} x_{ij} + \sum_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq \delta_i}} \beta_{ij} y_{ij} = 0.$$

Applying N^{k-1} to both sides, we obtain $\sum_{1 \leq j \leq \eta'_k} \alpha_{kj} N^{k-1} x_{kj} = 0$. But the vectors $N^{k-1} x_{k1}$, $\ldots, N^{k-1} x_{k\eta'_k}$ are linearly independent (as $x_{k1}, \ldots, x_{k\eta'_k}$ are vectors of height k in the height basis $\mathcal{B}$ for $N|_W$), hence we have $\alpha_{kj} = 0$ for $j = 1, \ldots, \eta'_k$. Since $\mathcal{B}_{k-1}$ is a linearly independent set, it follows that we also have $\alpha_{ij} = \beta_{ij} = 0$ for $i = 1, \ldots, k-1$. This proves our contention. Now adjoin vectors $y_{k1}, \ldots, y_{k\delta_k}$ to $\mathcal{B}_{k-1} \cup \{x_{k1}, \ldots, x_{k\eta'_k}\}$ so that the resulting set becomes a basis for $\mathcal{N}(N^k)$. Complete the proof for the Observation by proceeding inductively.

Suppose, in addition, that K is polyhedral. Then by Lemma 5.4 we have $\gamma_k(A) = \xi_k(A)$ for $k = 1, \ldots, \nu$ and moreover $\gamma_\nu(A) = \xi_\nu(A) = \eta_\nu(A) > 0$, i.e., $p = \nu$. So by rewriting (i) and (ii), we obtain (iii) and (iv) respectively. ∎

We would like to add that the inequalities given in part(iii) of Theorem 5.6 are known in the nonnegative matrix (or singular M-matrix) case (see [Her1, Proposition 5.12]). Note, however, that these inequalities need not be valid for a general proper cone K (see, for instance, our Example 7.6). When K is polyhedral, if we denote by A_k the restriction of A to $\mathrm{span}[\mathcal{N}((A-\rho(A)I)^k)\cap K]$ for $k=1,\ldots,\nu_\rho$, then we have $\xi_k(A)=\xi_k(A_k)=\eta_k(A_k)$, as $\rho(A_k)=\rho(A)$ and $\nu_{\rho(A_k)}(A_k)=k$. But this is again something known for the nonnegative matrix case (see [Her1, the paragraph following Definition 4.1]).

Corollary 5.7. *Let $A\in\pi(K)$, where K is a polyhedral proper cone. Let t be a positive integer, not greater than ν_ρ. If $\mathcal{N}((A-\rho(A)I)^k)$ contains a K-semipositive basis for $k=t,t+1,\ldots,\nu_\rho$, then $E(A)$ contains a Jordan basis for A, for which all Jordan chains of length t or greater are K-semipositive.*

Proof. For $k=t,\ldots,\nu$, where $\nu=\nu_\rho$, by the definition of peak characteristic and our hypotheses, we have

$$\begin{aligned}\xi_k &= \dim(A-\rho(A)I)^{k-1}(\mathcal{N}((A-\rho(A)I)^k)\cap K)\\ &= \dim(A-\rho(A)I)^{k-1}\mathcal{N}((A-\rho(A)I)^k)\\ &= \eta_k.\end{aligned}$$

By modifying the proof of Theorem 5.6(iv) slightly, we can construct η_t K-semipositive Jordan chains, whose union is a linearly independent set, with η_ν chains of length ν, $\eta_{\nu-1}-\eta_\nu$ chains of length $\nu-1$, ..., and $\eta_t-\eta_{t+1}$ chains of length t. Then, by a standard argument, we can add $\eta_1-\eta_t$ more (not necessarily K-semipositive) Jordan chains with $\eta_k-\eta_{k+1}$ chains of length k for $k=1,\ldots,t-1$ such that the union of all the chains forms a Jordan basis for A. ■

Corollary 5.8. *Let $A\in\pi(K)$, where K is a polyhedral proper cone. Let t be a positive integer not greater than ν_ρ. If $\eta_k(A)=\lambda_k(A)$ for $k=t+1,\ldots,\nu_\rho$, then $E(A)$ contains a Jordan basis for A, for which all Jordan chains of length t or greater are K-semipositive.*

Proof. When K is polyhedral, we have, $E(A)=\mathrm{span}(E(A)\cap K)$ and so $\eta_1(A)+\cdots+\eta_\nu(A)=\lambda_1(A)+\cdots+\lambda_\nu(A)$, where $\nu=\nu_\rho$. Then, using the conditions $\eta_k(A)=\lambda_k(A)$ for $k=t+1,\ldots,\nu$, we readily obtain $\eta_1(A)+\cdots+\eta_k(A)=\lambda_1(A)+\cdots+\lambda_k(A)$ or $\dim\mathcal{N}((A-\rho(A)I)^k)=\dim\mathrm{span}[\mathcal{N}((A-\rho(A)I)^k)\cap K]$ for $k=t,\ldots,\nu$. In view of Corollary 5.7, our assertion follows. ■

In view of the last part of Remark 2.2, Theorem 6.10 of [H-S2], i.e., the result for a singular M-matrix that corresponds to Corollary 5.8, now follows.

By the Observation given in the proof of Theorem 5.6, every height basis for an invariant subspace of a nilpotent matrix can be extended to a height basis for the whole space. In contrast, it is known that if W is an A-invariant subspace of an $n \times n$ matrix A, then there need not exist a Jordan basis for the restriction map $A|_W$ that can be extended to a Jordan basis for A in $\mathbb{C}^n$ (see [B-R-S]). As expected, parts (ii) and (iv) of Theorem 5.6 are invalid if "height basis" is replaced by "Jordan basis". Indeed, in [N-S3, Example 4] one can find an example of a nilpotent nonnegative matrix for which it is impossible to embed the chains of any maximal nonnegative union of chains into a Jordan basis. [Note, however, that the argument given in the example contains errors — we cannot assume that w^{13} is a multiple of x^{13}, and w^{23} is a linear combination of x^{13} and x^{23}. To remedy, we use the fact that $w^{12}, w^{22} \in \mathcal{R}(d) \cap \mathcal{N}(d^2)$ and also that $\{d(w^{12}), d(w^{22})\} = \{w^{11}, w^{21}\}$ is a linearly independent set to deduce that x^{22} must appear in a representation of w^{12} or w^{22} as a linear combination of x^{11}, x^{12}, x^{21} and x^{22}; hence, either w^{12} or w^{22} is not semipositive, which is a contradiction.]

Theorem 5.9. *Let K be a proper cone, and let $A \in \pi(K)$. Consider the following conditions:*

(a) $\eta(A) = \lambda(A)$.

(b) $\eta(A) = \xi(A)$.

(c) *Every vector in $E(A)$ is a peak vector.*

(d) *For each k, $k = 1, \ldots, \nu_\rho$, $\mathcal{N}((A - \rho(A)I)^k)$ contains a K-semipositive basis.*

(e) *There exists a K-semipositive height basis for A.*

(f) *There exists a K-semipositive height-level basis for A.*

(g) *There exists a K-semipositive Jordan basis for A.*

(h) *For each k, $k = 1, \ldots, \nu_\rho$, $\mathcal{N}((A-\rho(A)I)^k) \cap C(A, K)$ is a full cone in $\mathcal{N}((A-\rho(A)I)^k)$.*

(i) *For each k, $k = 1, \ldots, \nu_\rho$, we have*

$$\eta_k(A) = \dim(A - \rho(A)I)^{k-1}[\mathcal{N}((A - \rho(A)I)^k) \cap C(A, K)].$$

Conditions (a)–(f) *are equivalent and so are conditions* (g)–(i). *Moreover, we always have* (g) $\Longrightarrow$ (a), *and when K is polyhedral, conditions* (a)–(i) *are all equivalent.*

Proof. For any integer k, $1 \le k \le \nu_\rho$, clearly we have

$$\operatorname{span}[(\mathcal{N}((A - \rho(A)I)^k) \cap K] \subseteq \mathcal{N}((A - \rho(A)I)^k).$$

By definition, the dimension of the subspaces on the left and the right side of the inclusion are respectively $\lambda_1(A) + \cdots + \lambda_k(A)$ and $\eta_1(A) + \cdots + \eta_k(A)$. So we have

$$\begin{aligned} & \eta_1(A) + \cdots + \eta_k(A) = \lambda_1(A) + \cdots + \lambda_k(A) \\ \text{iff} \quad & \mathcal{N}((A - \rho(A)I)^k) = \operatorname{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K] \\ \text{iff} \quad & \mathcal{N}((A - \rho(A)I)^k) \text{ contains a } K\text{-semipositive basis.} \end{aligned}$$

On the other hand, the condition $\eta(A) = \lambda(A)$ is clearly equivalent to $\eta_1(A) + \cdots + \eta_k(A) = \lambda_1(A) + \cdots + \lambda_k(A)$ for $k = 1, \ldots, \nu_{\rho(A)}(A)$. So the equivalence of (a) and (d) follows.

If (d) is satisfied, then $\mathcal{N}((A - \rho(A)I)^k) = \text{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K]$ for each k, and hence $\text{ht}(x) = \text{lev}(x)$ for all $x \in E(A)$, i.e., (c) is satisfied. On the other hand, if (d) is not satisfied, then for some k, we can find a vector $x \in \mathcal{N}((A - \rho(A)I)^k)$ such that $x \notin \text{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K]$. For this x, we have $\text{lev}(x) > k \geq \text{ht}(x)$, so x is not a peak vector. This establishes the equivalence of (c) and (d).

(e) $\Longrightarrow$ (d): If $\mathcal{B}$ is a K-semipositive height basis for A, then by definition, for each k, $1 \leq k \leq \nu_p$, $\mathcal{B}$ has precisely $\eta_1 + \cdots + \eta_k$ vectors of height k or less. Clearly, these vectors constitute a K-semipositive basis for $\mathcal{N}((A - \rho(A)I)^k)$.

(a) $\Longrightarrow$ (f): If (a) is satisfied, then we have $E(A) = \text{span}(E(A) \cap K)$, as $\eta_1 + \cdots + \eta_{\nu_\rho} = \lambda_1 + \cdots + \lambda_{\nu_\rho}$. By Lemma 5.1, $E(A)$ must have a K-semipositive level basis. But every K-semipositive vector is a peak vector and also by condition (a) we have $\eta_k = \lambda_k$ for each k, so the latter basis is also a height basis. So $E(A)$ has a height-level basis.

The implication (f) $\Longrightarrow$ (e) is obvious.

(d) $\Longrightarrow$ (b): For each k, $1 \leq k \leq \nu_\rho$, since $\mathcal{N}((A-\rho(A)I)^k) = \text{span}[\mathcal{N}((A-\rho(A)I)^k) \cap K]$, we have

$$
\begin{aligned}
\eta_k &= \dim(A - \rho(A)I)^{k-1}\mathcal{N}((A - \rho(A)I)^k) \\
&= \dim(A - \rho(A)I)^{k-1}\,\text{span}[\mathcal{N}((A - \rho(A)I)^k) \cap K] \\
&= \xi_k.
\end{aligned}
$$

Hence, $\eta(A) = \xi(A)$.

(b) $\Longrightarrow$ (d): Suppose $\xi(A) = \eta(A)$. Then for each k, $1 \leq k \leq \nu_\rho$, we have, $\xi_1 + \cdots + \xi_k = \eta_1 + \cdots + \eta_k$. But for each k, we also have the inequalities

$$\xi_1 + \cdots + \xi_k \leq \lambda_1 + \cdots + \lambda_k \leq \eta_1 + \cdots + \eta_k.$$

It follows that we have $\lambda_1 + \cdots + \lambda_k = \eta_1 + \cdots + \eta_k$ for all k, which is equivalent to (d).

In the above, we have established the equivalence of conditions (a)–(f).

Note that every vector in a K-semipositive Jordan basis for A necessarily belongs to $C(A, K)$. So we have (g) $\Longrightarrow$ (h). Clearly, we have (h) $\Longrightarrow$ (i). By the first half of Theorem 5.6(ii), we also have (i) $\Longrightarrow$ (g). Thus, conditions (g)–(i) are equivalent.

It is clear that we always have (g) $\Longrightarrow$ (d) and hence (g) $\Longrightarrow$ (a).

Condition (d) amounts to saying that, for $k = 1, \ldots, \nu_\rho$, $\mathcal{N}((A-\rho(A)I)^k) \cap K$ is a full cone in $\mathcal{N}((A-\rho(A)I)^k)$. When K is polyhedral, by Theorem 4.3(iii) $\mathcal{N}((A-\rho(A)I)^k) \cap C(A, K)$ is a full subcone of $\mathcal{N}((A - \rho(A)I)^i) \cap K$. Then clearly conditions (d) and (h), and hence conditions (a)–(i), are equivalent. ■

In view of Remark 5.5, to the equivalent conditions (g)–(i) of Theorem 5.9 we may add the following:

$$\dim[\mathcal{N}(A-\rho(A)I)\cap (A-\rho(A)I)^kC(A,K)] = \eta_k \quad for \quad k=1,\ldots,\nu_\rho.$$

Also, the following is equivalent to conditions (a)–(f) of Theorem 5.9:

$$\dim[\mathcal{N}(A-\rho(A)I)\cap (A-\rho(A)I)^kK] = \eta_k \quad for \quad k=1,\ldots,\nu_\rho.$$

If, as in the nonnegative matrix case, we define a *peak basis* for A ($\in \pi(K)$) to be a basis of $E(A)$ consisting of peak vectors, then to the equivalent conditions (a)–(f) of Theorem 5.9 we may add the following: *Every basis for $E(A)$ is a peak basis.* But this is trivial, in view of condition (c) of Theorem 5.9.

Our next result gives several new necessary conditions for the existence of a K-semipositive Jordan basis for A and clarifies the logical relations between them. It also provides an alternative way to find the maximum length of a K-semipositive Jordan chain for A (cf. Remark 4.2).

Theorem 5.10. *Let K be a proper cone, and let $A\in\pi(K)$. Set $C_0 = E(A)\cap K$ and for $k=1,2,\ldots,$ define C_k inductively by $C_k = K\cap (A-\rho(A)I)C_{k-1}$. Let m be the maximum length of a K-semipositive Jordan chain for A. Then*:

(i) $C_0 \supset C_1 \supset \cdots \supset C_{m-1} \supset C_m = \{0\}$*, where the inclusions are all strict.*

(ii) *For $k=1,\ldots,m$, we have*

$$\gamma_k+\cdots+\gamma_m \le \dim C_{k-1} \le \eta_k+\cdots+\eta_\nu,$$

where $\gamma_k = \dim(A-\rho(A)I)^{k-1}[\mathcal{N}((A-\rho(A)I)^k)\cap C(A,K)]$.

(iii) *Consider the following conditions*:

(a) *There exists a K-semipositive Jordan basis for A.*

(b) *For $k=0,\ldots,\nu_\rho-1$, $(A-\rho(A)I)^kC(A,K)$ is a full cone in $\mathcal{R}((A-\rho(A)I)^k)\cap E(A)$.*

(c) *For $k=0,\ldots,\nu_\rho-1$, we have*

$$\dim(A-\rho(A)I)^kC(A,K) = \eta_{k+1}+\cdots+\eta_{\nu_\rho}.$$

(d) *$m=\nu_\rho$ and for $k=0,\ldots,\nu_\rho-1$, C_k is a full cone in $\mathcal{R}((A-\rho(A)I)^k)\cap E(A)$.*

(e) *For $k=0,\ldots,\nu_\rho-1$, $K\cap(A-\rho(A)I)^k(E(A)\cap K)$ is a full cone in $\mathcal{R}((A-\rho(A)I)^k)\cap E(A)$.*

(f) *For $k=0,\ldots,\nu_\rho-1$, $(A-\rho(A)I)^k(E(A)\cap K)$ is a full cone in $\mathcal{R}((A-\rho(A)I)^k)\cap E(A)$.*

(g) *For* $k = 0, \ldots, \nu_\rho - 1$, $\mathcal{R}((A - \rho(A)I)^k) \cap E(A)$ *contains a* K*-semipositive basis.* *We always have* (a) $\Longrightarrow$ (b) $\Longleftrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (e) $\Longrightarrow$ (f) *and* (e) $\Longrightarrow$ (g). *When* K *is polyhedral, condition* (b) *and hence conditions* (c)–(g) *are all satisfied.*

Proof. (i) By induction we readily show that $C_k \supseteq C_{k+1}$ for all nonnegative integers k.

By definition of m, there exists a K-semipositive Jordan chain for A, say $x, (A - \rho(A)I)x, \ldots, (A - \rho(A)I)^{m-1}x$. It is readily seen that $(A - \rho(A)I)^{m-1}x \in C_{m-1}$. Hence $C_{m-1} \neq \{0\}$. If $C_m \neq \{0\}$, choose any nonzero vector $x_m \in C_m$. Then we can find vectors $x_k \in C_k$, $k = 0, \ldots, m-1$, such that $(A - \rho(A)I)x_k = x_{k+1}$ for $k = 0, \ldots, m-1$. By adjoining the nonzero images, if any, of x_m under the action of the positive powers of $A - \rho(A)I$ to the sequence $x_0, x_1, \ldots, x_m$, we obtain a K-semipositive Jordan chain for A of length greater than m, which is a contradiction.

We have the strict inclusion $C_k \supset C_{k+1}$ whenever $C_k \neq \{0\}$, because any vector of C_k with maximum height cannot belong to C_{k+1}.

(ii) It is clear that m equals the largest integer j such that $\gamma_j > 0$. Consider any positive integer $k \leq m$. By Theorem 5.6(ii), there exists a linearly independent subset $\mathcal{C}$ of $E(A)$ which is the union of $\gamma_1(A)$ K-semipositive Jordan chains for A such that there are γ_m of them of length m and $\gamma_j - \gamma_{j+1}$ of them of length j for $j = 1, \ldots, m-1$. Note that the nonzero vectors in the set $(A - \rho(A)I)^{k-1}\mathcal{C}$ all belong to C_{k-1}, and they form a linearly independent set of cardinality $\gamma_k + \cdots + \gamma_m$. So we have $\gamma_k + \cdots + \gamma_m \leq \dim C_{k-1}$.

On the other hand, we also have $C_{k-1} \subseteq \mathcal{R}((A - \rho(A))^{k-1}) \cap E(A)$, and the latter subspace is of dimension $\eta_k + \cdots + \eta_{\nu_\rho}$. So we also have $\dim C_{k-1} \leq \eta_k + \cdots + \eta_{\nu_\rho}$.

(iii) Suppose $\mathcal{B}$ is a K-semipositive Jordan basis for A. Then $\mathcal{B} \subseteq C(A, K)$ and for $k = 0, \ldots, \nu_\rho - 1$, we have

$$(A - \rho(A)I)^k\mathcal{B} \subseteq (A - \rho(A)I)^k C(A, K) \subseteq (A - \rho(A)I)^k E(A).$$

But the nonzero vectors of $(A - \rho(A)I)^k\mathcal{B}$ form a basis for $(A - \rho(A)I)^k E(A) = \mathcal{R}((A - \rho(A)I)^k) \cap E(A)$. This proves the implication (a) $\Longrightarrow$ (b). Since the number of nonzero vectors in $(A - \rho(A)I)^k\mathcal{B}$ is $\eta_{k+1} + \cdots + \eta_{\nu_\rho}$, the equivalence of (b) and (c) also follows.

By induction, we readily establish the inclusions

$$\begin{aligned} (A - \rho(A)I)^k C(A, K) &\subseteq C_k \subseteq K \cap (A - \rho(A)I)^k (E(A) \cap K) \\ &\subseteq (A - \rho(A)I)^k (E(A) \cap K) \subseteq (A - \rho(A)I)^k E(A) \end{aligned}$$

for $k = 0, \ldots, \nu_\rho - 1$, from which the implications (b) $\Longrightarrow$ (d), (d) $\Longrightarrow$ (e) and (e) $\Longrightarrow$ (f) follow.

The implication (e) $\Longrightarrow$ (g) is obvious.

When K is polyhedral, $C(A, K)$ is a full cone in $E(A)$. In this case, $(A - \rho(A)I)^k C(A, K)$ is a full cone in $(A - \rho(A)I)^k E(A)$ and so condition (b) must be satisfied. ■

The following example shows that, for a general proper cone K, conditions (g)–(i) of Theorem 5.9 are not equivalent to conditions (a)–(f).

Example 5.11. Let C be the closed convex set in $\mathbb{R}^2$ with extreme points $(k(k-1)/2, k)^T$, $k = 0, 1, \ldots$ and recession cone $O^+C = \mathrm{pos}\{(1,0)^T\}$. Let C_1 be the proper cone in $\mathbb{R}^3$ obtained from C in the standard way, i.e.,

$$C_1 = \left\{\alpha\begin{pmatrix} x \\ 1\end{pmatrix} : x \in C,\ \alpha \geq 0\right\} \cup \left\{\begin{pmatrix} x \\ 0\end{pmatrix} : x \in O^+C\right\}.$$

Denote by e_i the ith standard unit vector of $\mathbb{R}^4$. Let K be the convex cone in $\mathbb{R}^4$ given by:

$$K = K_1 + K_2, \quad \text{where} \quad K_1 = C \times \{0\} \quad \text{and} \quad K_2 = \mathrm{pos}\{e_2 + e_4, e_4\}.$$

Since $K_1 \cap (-K_2) = \{0\}$, K is a closed cone (see, for instance, [Roc, Corollary 9.1.2]). Indeed, it can be readily shown that K is a proper cone in $\mathbb{R}^4$. Now let $A = J_3(1) \oplus J_1(1)$. Since $AK_1 \subseteq K_1$ and $(A - I)K_2 \subseteq K_1$, clearly $A \in \pi(K)$, $\rho(A) = 1$ and $\nu_\rho = 3$. [Actually, K is also equal to the convex cone generated by e_1, $e_2 + e_4$, e_4, e_3, together with the images of e_3 under the positive powers of A.] Note that we have

$$\begin{aligned} \mathcal{N}(A - \rho(A)I) &= \mathrm{span}\{e_1, e_4\}, \ \ \mathcal{N}((A - \rho(A)I)) \cap K = \mathrm{pos}\{e_1, e_4\}; \\ \mathcal{N}((A - \rho(A)I)^2) &= \mathrm{span}\{e_1, e_2, e_4\}, \ \ \mathcal{N}((A - \rho(A)I)^2) \cap K = \mathrm{pos}\{e_1, e_2 + e_4, e_4\}; \\ \text{and } \ \mathcal{N}((A - \rho(A)I)^3) &= E(A) = \mathbb{R}^4. \end{aligned}$$

So condition (d) and hence the equivalent conditions (a)–(f) of Theorem 5.9 are satisfied. Also, it can be seen that

$$K \cap (A - \rho(A)I)^{-1}\,\mathrm{pos}\{e_1, e_2 + e_4, e_4\} = \mathrm{pos}\{e_1, e_2 + e_4, e_4\}.$$

By Remark 4.2 it follows that $C(A, K) = \mathrm{pos}\{e_1, e_2 + e_4, e_4\}$. Since $\mathcal{N}((A - \rho(A)I)^3) \cap C(A, K) = C(A, K)$ is not a full cone in $\mathbb{R}^4$, condition (h) and hence the equivalent conditions (g)–(i) of Theorem 5.9 are not satisfied. Indeed, since

$$\begin{aligned} \mathcal{N}((A - \rho(A)I)^3) \cap C(A, K) &= C(A, K) \\ &= \mathcal{N}((A - \rho(A)I)^2) \cap C(A, K) \supset \mathcal{N}(A - \rho(A)I) \cap C(A, K), \end{aligned}$$

by the last part of Remark 4.2, the maximum length of a K-semipositive Jordan chain for A is 2, not 3, the index of $\rho(A)$. [Since $\mathcal{R}(A - \rho(A)I) = \mathrm{span}\{e_1, e_2\}$ does not contain a K-semipositive basis, by Theorem 5.10(iii) we can also see why there cannot exist a K-semipositive Jordan basis for A.]

Now replace K by $\widetilde{K}$ given by:

$$\widetilde{K} = \text{pos}\{e_1, e_2 + e_4, e_4, e_3, Ae_3, \ldots, A^{p-1}e_3, (A - I)A^{p-1}e_3\},$$

where p is any fixed positive integer, and take the same A. Then $\widetilde{K}$ is a polyhedral proper cone in $\mathbb{R}^4$, and one can readily check that $A \in \pi(\widetilde{K})$ and there exists a $\widetilde{K}$-semipositive Jordan basis for A, namely, $A^{p-1}e_3$, $A^{p-1}e_2 = (A - I)A^{p-1}e_3$, $e_1 = (A - I)^2 A^{p-1}e_3$ and e_4.

6. A cone-theoretic proof of the preferred-basis theorem

In this section we are going to offer a cone-theoretic proof of the preferred-basis theorem. We will assume the nonnegative-basis theorem and the Rothblum index theorem. The difficult part of the proof of the nonnegative-basis theorem is to show that for each basic class of the given nonnegative matrix there is an associated strongly combinatorial vector. In [Tam2, Theorem 5.2] a proof is given for the existence of the desired strongly combinatorial vector which is based on the Frobenius-Victory theorem (about the distinguished eigenvalues of a nonnegative matrix) and the cone-theoretic result that A ($\in \pi(K)$) has a generalized eigenvector in $\text{int}\, K$ if and only if $\rho(A)$ is the only distinguished eigenvalue of A^T for K^* (see [Tam2, Theorem 5.1]). Later on, in [T-S2, Theorem 3.3] a cone-theoretic proof of the Frobenius-Victory theorem is also given. A cone-theoretic proof of the Rothblum index theorem can be found in [T-S2, Section 5]. Indeed, the result is extended to the setting of a linear map preserving a polyhedral proper cone.

Our cone-theoretic arguments will be given in terms of the concept of a semi-distinguished invariant face. Recall that if $A \in \pi(K)$, then a face F of K is said to be *semi-distinguished A-invariant* (associated with λ) if F is an A-invariant join-irreducible face (i.e., one which cannot be expressed as the join of two A-invariant faces strictly included in it) which contains in its relative interior a generalized eigenvector of A (corresponding to λ) (see [T-S2, Section 4]). We also need the known characterization of invariant faces associated with a nonnegative matrix. In [T-S2, Theorem 3.1], it is shown that for any $n \times n$ nonnegative matrix P, the P-invariant faces of $\mathbb{R}^n_+$ are precisely subsets of $\mathbb{R}^n_+$ of the form F_I for some initial subset I for P, where $F_I = \{x \in \mathbb{R}^n_+ : \text{supp}(x) \subseteq I\}$ and a subset I of $\langle n \rangle$ is called an *initial subset for P* if for every $j \in \langle n \rangle$, I contains j whenever j has access to I. Different types of P-invariant faces are characterized in [T-S2, Theorem 3.6]. In particular, it is proved that F_I is a semi-distinguished P-invariant face associated with $\rho(P)$ if and only if I is the initial subset determined by a basic class, i.e., it is the union of the classes having access to the said basic class. If $x^{(\alpha)}$ is a strongly combinatorial vector associated with a basic class α, then clearly $x^{(\alpha)} \in \text{ri}\, F_I$, where I is the initial subset determined by α and $\text{ri}\, F_I$ denotes the relative interior of F_I. So a vector $x \in E(P)$ is a strongly combinatorial vector if and only if $\Phi(x)$ is a semi-distinguished P-invariant face associated with $\rho(P)$.

We take this opportunity to mention the following (fairly easy) cone version of the Frobenius-Victory theorem, of which the original Frobenius-Victory theorem is a consequence.

Let $A \in \pi(K)$. Recall that a face F is said to be a *distinguished A-invariant face of K* (associated with λ) if F is a nonzero A-invariant face of K such that $\rho(A|_{\operatorname{span} G}) < \rho(A|_{\operatorname{span} F})$ for any nonzero A-invariant face G properly included in F (and $\rho(A|_{\operatorname{span} F}) = \lambda$).

Theorem 6.1. *Let K be a proper cone, and let $A \in \pi(K)$.*

(i) *For any real number λ, λ is a distinguished eigenvalue of A if and only if $\lambda = \rho(A|_{\operatorname{span} F})$ for some distinguished A-invariant face F of K.*

(ii) *If F is a distinguished A-invariant face, then there is a (up to multiples) unique eigenvector x of A corresponding to $\rho(A|_{\operatorname{span} F})$ that lies in F. Furthermore, $x \in \operatorname{ri} F$.*

(iii) *For each distinguished eigenvalue λ of A, the extreme vectors of the cone $\mathcal{N}(\lambda I - A) \cap K$ are precisely all the distinguished eigenvectors of A that lie in the relative interior of certain distinguished A-invariant faces of K associated with λ.*

In passing, we would also like to mention that Hershkowitz and Schneider [H-S3] have introduced the concepts of combinatorial vectors and proper combinatorial vectors in the generalized nullspace of a singular M-matrix. These concepts can also be reformulated in terms of the concept of semi-distinguished invariant faces as follows: For a nonnegative matrix P, a vector $x \in E(P)$ is a combinatorial vector if x lies in the linear span of a semi-distinguished P-invariant face associated with $\rho(P)$; $x \in E(P)$ is a proper combinatorial vector if the smallest P-invariant face containing x is equal to the linear span of a semi-distinguished P-invariant face associated with $\rho(P)$. In [H-S3, Theorem 6.6], many of the equivalent conditions for equality of the height and the level characteristics are expressed in terms of the concepts of a (not necessarily semipositive) combinatorial basis or a proper combinatorial set. We have not explored such possibilities in the setting of a cone-preserving map.

We will make use of the following:

Lemma 6.2. *Let P be an $n \times n$ nonnegative matrix and let α be a basic class of P. If x is a strongly α-combinatorial vector, then $ht(x) = \operatorname{lev}(\alpha)$.*

Proof. Since x is a strongly α-combinatorial vector, $x \in \operatorname{ri} F_\alpha$, where F_α denotes the P-invariant face which consists of all vectors of $\mathbb{R}^n_+$ whose supports are included in the union of all classes having access to α (see [T-S2, Section 3]). Note that $P|_{\operatorname{span} F_\alpha}$ can be identified with the principal submatrix of P with rows and columns indexed by the union of all classes having access to α. So, by [T-S2, Theorem 4.4] we have $\rho(P|_{\operatorname{span} F_\alpha}) = \rho(P)$ and $\operatorname{ht}(x) = \nu_{\rho(P)}(P|_{\operatorname{span} F_\alpha})$. But by the Rothblum index theorem, $\nu_{\rho(P)}(P|_{\operatorname{span} F_\alpha}) = \operatorname{lev}(\alpha)$.

Thus we have $\mathrm{ht}(x) = \mathrm{lev}(\alpha)$. ■

As we point out in the discussion following Theorem 2.1, Theorem 3.1 of [Rot], the main result of his paper, has three parts, its part(2) being the Rothblum index theorem. Upon examination, one can see that of the proofs for the three parts of the theorem the one for part(2) is hardest. Rothblum establishes part(2) by proving the following stronger result, which we now put as a lemma:

Lemma 6.3. *Let P be a nonnegative matrix. Let α be a basic class of P and let $y \in E(P)$ be a strongly α-combinatorial vector. Then for any positive integer k and any class β, we have*
(i) $((P - \rho(P)I)^k y)_\beta = 0$ *if* $d(\beta, \alpha) \leq k$.
(ii) $((P - \rho(P)I)^k y)_\beta \gg 0$ *if* $d(\beta, \alpha) = k + 1$.

As can be easily seen, Lemma 6.3, together with the nonnegative-basis theorem, implies the Rothblum index theorem. [However, it is not clear that Lemma 6.3 can follow from the Rothblum index theorem.] Indeed, once the lemma is obtained, the preferred-basis theorem (and hence also part (3) of [Rot, Theorem 3.1] which is about the existence of what Bru and Neumann [B-N] call a Rothblum basis) is not far away, as we will explain later.

By induction (on the level of α) one can show that Lemma 6.3 follows readily from its special case $k = 1$. Making use of the nonnegative-basis theorem one can also show that this special case of Lemma 6.3 in turn is equivalent to the following:

Lemma 6.4. *Let P be a nonnegative matrix and let $\mathcal{B} = \{x^{(\beta)} : \beta$ is a basic class of $P\}$ be a strongly combinatorial basis for $E(P)$. If x is a strongly combinatorial vector associated with a basic class α, then $(P - \rho(P)I)x = \sum c_\beta x^{(\beta)}$, where $c_\beta \neq 0$ only if $\beta >\!- \alpha$ and moreover $c_\beta > 0$ whenever $d(\beta, \alpha) = 2$.*

To prove Lemma 6.4, we need the following result:

Theorem 6.5. *Let K be a polyhedral proper cone, and let $A \in \pi(K)$. If K is a semi-distinguished A-invariant face of itself, then $\Phi((A - \rho(A)I)C(A, K)) = \Phi(\mathcal{N}((A - \rho(A)I)^{\nu-1}) \cap K)$, where $\nu = \nu_{\rho(A)}(A)$.*

Proof. By the definition of $C(A, K)$, clearly we have $(A - \rho(A)I)C(A, K) \subseteq \mathcal{N}((A - \rho(A)I)^{\nu-1}) \cap K$ and hence $\Phi((A - \rho(A)I)C(A, K)) \subseteq \Phi(\mathcal{N}((A - \rho(A)I)^{\nu-1}) \cap K)$.

Since K is polyhedral, to prove the reverse inclusion, it suffices to show that the dual face of $\Phi((A - \rho(A)I)C(A, K))$ (which consists of all vectors in K^* orthogonal to it) is included in that of $\Phi(\mathcal{N}((A - \rho(A)I)^{\nu-1}) \cap K)$. Let $z \in K^*$ be any nonzero vector orthogonal to

$\Phi((A-\rho(A)I)C(A,K))$. Choose any $x \in \operatorname{int} K$. Then for all sufficiently small $\varepsilon > 0$, we have $J_A^{(0)}(\varepsilon)x \in C(A,K)$ and so

$$\langle z, (A-\rho(A)I)J_A^{(0)}(\varepsilon)x\rangle = 0, \text{ or } \langle J_{A^T}^{(1)}(\varepsilon)z, x\rangle = \langle (A^T-\rho(A)I)(J_A^{(0)}(\varepsilon)^T)z, x\rangle = 0,$$

as $J_A^{(0)}(\varepsilon)^T = J_{A^T}^{(0)}(\varepsilon)$. But $x \in \operatorname{int} K$ and $J_{A^T}^{(1)}(\varepsilon)z \in K^*$ for all sufficiently small $\varepsilon > 0$ (by applying Theorem 3.6(i) to A^T of $\pi(K^*)$), so we must have $J_{A^T}^{(1)}(\varepsilon)z = 0$ for all sufficiently small $\varepsilon > 0$. Hence, we have

$$(\varepsilon^{\nu-2}Z_{A^T}^{(1)} + \varepsilon^{\nu-3}Z_{A^T}^{(2)} + \cdots + Z_{A^T}^{(\nu-1)})z = \varepsilon^{\nu-2}J_{A^T}^{(1)}(\varepsilon)z = 0$$

for all sufficiently small $\varepsilon > 0$. By letting $\varepsilon \to 0^+$, we obtain $Z_{A^T}^{(\nu-1)}z = 0$ and so $(\varepsilon^{\nu-3}Z_{A^T}^{(1)} + \varepsilon^{\nu-4}Z_{A^T}^{(2)} + \cdots + Z_{A^T}^{(\nu-2)})z = 0$. By letting $\varepsilon \to 0^+$ again, we also have $Z_{A^T}^{(\nu-2)}z = 0$. By repeating the argument, after a finite number of steps, we conclude that we have $Z_{A^T}^{(1)}z = 0$.

We always have either $\rho_z(A^T) < \rho(A)$ or $\rho_z(A^T) = \rho(A)$. If the former happens, z is a sum of generalized eigenvectors of A^T corresponding to eigenvalues with modulus strictly less than $\rho(A)$. Then certainly z is orthogonal to $\mathcal{N}((A-\rho(A)I)^{\nu-1})$. If the latter happens, then, since $Z_{A^T}^{(1)}z = 0$, z can be written as $z_1 + z_2$, where z_2 is a sum of generalized eigenvectors of A^T corresponding to eigenvalues other than $\rho(A)$ and z_1 is an eigenvector of A^T corresponding to $\rho(A)$, and indeed, by [T-S2, Corollary 4.8], z_1 is a distinguished eigenvector of A^T (as $\operatorname{ord}_A(z) = 1$). On the other hand, since K is polyhedral and is a semi-distinguished A-invariant face of itself, by the proof of [T-S2, Lemma 5.2], A^T has (up to multiples) a unique distinguished eigenvector corresponding to $\rho(A)$ (and indeed $\rho(A)$ is the only distinguished eigenvalue of A^T) and moreover this unique distinguished eigenvector can be written in the form $(A^T-\rho(A)I)^{\nu-1}w$ for some generalized eigenvector w of A^T corresponding to $\rho(A)$ of height ν. So z_1 is equal to the unique distinguished eigenvector of A^T corresponding to $\rho(A)$ and moreover z_1 is orthogonal to $\mathcal{N}((A-\rho(A)I)^{\nu-1})$. But z_2 is also orthogonal to $\mathcal{N}((A-\rho(A)I)^{\nu-1})$, hence z is orthogonal to $\mathcal{N}((A-\rho(A)I)^{\nu-1})$. In either case, z is orthogonal to $\mathcal{N}((A-\rho(A)I)^{\nu-1}) \cap K$. But $z \in K^*$, it follows that z is orthogonal to $\Phi(\mathcal{N}((A-\rho(A)I)^{\nu-1}) \cap K)$. In the above, we have shown that the dual face of $\Phi((A-\rho(A)I)C(A,K))$ is included in that of $\Phi(\mathcal{N}((A-\rho(A)I)^{\nu-1}) \cap K)$. The proof is complete. ■

For completeness, we also take note of the following, which is not difficult to establish:

Theorem 6.6. *Let P be an $n \times n$ nonnegative matrix. Then*

(i) $\Phi((P-\rho(P)I)C(P,\mathbb{R}^n_+)) = F_I$, *where I is the initial subset for P whose corresponding initial collection of classes $\mathcal{C}$ is determined by the property that a class α is final in $\mathcal{C}$ if and only if α is a basic class such that $\alpha >\!\!- \beta$ for some basic class β.*

(ii) $\Phi(\mathcal{N}((P-\rho(P)I)^{\nu_\rho-1})\cap\mathbb{R}^n_+)=F_J$, *where J is equal to the union of all classes of P that have access to a basic class of level $\nu_\rho-1$ or less.*

(iii) $\Phi((P-\rho(P)I)C(P,\mathbb{R}^n_+))=\Phi(\mathcal{N}((P-\rho(P)I)^{\nu_\rho-1})\cap\mathbb{R}^n_+)$ *if and only if each class final in the collection of all basic classes of P is of level ν_ρ.*

Our next example shows that Theorem 6.5 is no longer valid if we drop the polyhedrality assumption on K.

Example 6.7. Let K be the subset of $\mathbb{R}^3$ given by:

$$\begin{aligned} K &= \{(\xi_1,\xi_2,\xi_3)^T\in\mathbb{R}^3:\xi_3\geq\sqrt{(\xi_1^2+\xi_2^2)},\xi_2\leq 0\} \\ &\quad \cup\{(\xi_1,\xi_2,\xi_3)^T\in\mathbb{R}^3:\xi_3\geq|\xi_1|,\xi_2\geq 0\}. \end{aligned}$$

It is not difficult to see that K is a proper cone in $\mathbb{R}^3$. Indeed, K is the union of the polyhedral cone $\operatorname{pos}\{(-1,0,1)^T,(0,1,0)^T,(1,0,1)^T\}$ and the half-ice-cream cone $\{(\xi_1,\xi_2,\xi_3)^T:\xi_3\geq\sqrt{(\xi_1^2+\xi_2^2)},\xi_2\leq 0\}$.

Let $A=\begin{bmatrix}-1&0&1\\0&0&0\\-1&0&1\end{bmatrix}$.

Note that we have $A((-1,0,1)^T)=2(1,0,1)^T$, $A((0,1,0)^T)=(0,0,0)^T=A((1,0,1)^T)$ and $A((\xi_1,\xi_2,\xi_3)^T)=(\xi_3-\xi_1)(1,0,1)^T$. So $A\in\pi(K)$ and A is a nilpotent matrix with index two. The cone K has three nonzero A-invariant faces, all of which are semi-distinguished A-invariant, namely, $\Phi((1,0,1)^T)$, $\operatorname{pos}\{(1,0,1)^T,(0,1,0)^T\}$ and K itself. In this case, $\Phi((A-\rho(A)I)C(A,K))$ equals the extreme ray $\Phi((1,0,1)^T)$ (as $C(A,K)=K$) and is different from $\Phi(\mathcal{N}((A-\rho(A)I)^{\nu_\rho-1})\cap K)$, which is $\operatorname{pos}\{(1,0,1)^T,(0,1,0)^T\}$.

Proof of Lemma 6.4: By considering the restriction map $P|_{\operatorname{span}\Phi(x)}$ (or the principal submatrix of P with rows and columns indexed by the union of all classes having access to α) instead of P, hereafter we may assume that α is a basic class of P which is also a final class of P.

We proceed by induction on $\operatorname{lev}(\alpha)$. If $\operatorname{lev}(\alpha)=1$, then x must be a distinguished eigenvector of P and clearly our assertion holds. So suppose that $\operatorname{lev}(\alpha)\geq 2$ and assume that our assertion holds whenever the given basic class is of level less than that of α. Hereafter, we assume that ε is a fixed sufficiently small positive number so that, according to Theorem 3.6(i), $J_P^{(0)}(\varepsilon)\in\pi(\mathbb{R}^n_+)$.

By Lemma 4.5 we have $x=y(\varepsilon)-\varepsilon^{-1}(P-\rho(P)I)y(\varepsilon)$, where $y(\varepsilon):=J_P^{(0)}(\varepsilon)x$. By the nonnegative-basis theorem, $(P-\rho(P)I)y(\varepsilon)$ can be written as $\sum_\beta d_\beta x^{(\beta)}$, where the summation is taken over all basic classes β, which necessarily have access to α as α is the only final class of P. We contend that $d_\alpha=0$ and also $d_\beta>0$ whenever β is a basic class such that $d(\beta,\alpha) = 2$. Once this contention is proved, our assertion will follow readily as follows. Using the representation of $y(\varepsilon)$ in terms of the $x^{(\beta)}s$, the relation between x and $y(\varepsilon)$, and the fact that $d_\alpha = 0$, we have

$$(P-\rho(P)I)x = \sum_\beta d_\beta x^{(\beta)} - \varepsilon^{-1}\left(\sum_\beta d_\beta (P-\rho(P)I)x^{(\beta)}\right),$$

where both summations are taken over all basic classes $\beta >\!- \alpha$. But by our induction assumption, for each such β, $(P-\rho(P)I)x^{(\beta)}$ can be represented as a linear combination of only those $x^{(\gamma)}s$ for which $\gamma >\!- \beta$ and hence for which $d(\gamma,\alpha) \geq 3$. It follows that $(P-\rho(P)I)x$ can be represented as a linear combination of $x^{(\beta)}s$ for which $\beta >\!- \alpha$ and moreover if β is a basic class such that $d(\beta,\alpha) = 2$, then the coefficient of $x^{(\beta)}$ in the representation is d_β, which is positive by our contention.

To prove our contention, first note that since x is a strongly α-combinatorial vector and α is the only final class of P, x is a positive vector; so we have $x \in \mathrm{ri}(E(P)\cap\mathbb{R}^n_+)$ and $y(\varepsilon)$ $(= J_P^{(0)}(\varepsilon)x) \in \mathrm{ri}(J_P^{(0)}(\varepsilon)(E(P)\cap\mathbb{R}^n_+)) \subseteq \mathrm{ri}\,C(P,\mathbb{R}^n_+)$, where the last inclusion holds by Theorem 4.3(ii). Hence, by Theorem 6.5, we have $(P-\rho(P)I)y(\varepsilon) \in \mathrm{ri}\,\Phi((\mathcal{N}((P-\rho(P)I)^{\nu-1})\cap\mathbb{R}^n_+)$. Next, note that

$$\mathrm{ht}((P-\rho(P)I)y(\varepsilon)) = \mathrm{ht}(y(\varepsilon)) - 1 = \mathrm{ht}(x) - 1 = \nu - 1,$$

where the second equality follows from Lemma 4.5 and the last equality holds by [T-S2, Lemma 4.3] as $x \in \mathrm{int}\,\mathbb{R}^n_+$. By Lemma 6.2, for any basic class β, $\mathrm{ht}(x^{(\beta)}) = \mathrm{lev}(\beta)$. But α is the only basic class of level ν, in order that $\mathrm{ht}(\sum_\beta d_\beta x^{(\beta)}) = \mathrm{ht}((P-\rho(P)I)y(\varepsilon)) = \nu - 1$, we must have $d_\alpha = 0$. Now consider any basic class γ such that $d(\gamma,\alpha) = 2$. Since $\mathrm{lev}(\gamma) \leq \mathrm{lev}(\alpha) - 1$, we have, $\mathrm{ht}(x^{(\gamma)}) \leq \nu - 1$, and hence $x^{(\gamma)} \in \mathcal{N}((P-\rho(P)I)^{\nu-1})\cap\mathbb{R}^n_+$. But $(P-\rho(P)I)y(\varepsilon) = \sum_\beta d_\beta x^{(\beta)}$ and we have already shown that $(P-\rho(P)I)y(\varepsilon) \in \mathrm{ri}\,\Phi(\mathcal{N}((P-\rho(P)I)^{\nu-1})\cap\mathbb{R}^n_+)$, hence we have $\gamma \subseteq \mathrm{supp}(x^{(\gamma)}) \subseteq \mathrm{supp}((P-\rho(P)I)y(\varepsilon))$; in other words, the γ-subvector of $(P-\rho(P)I)y(\varepsilon)$ is positive. But $(P-\rho(P)I)y(\varepsilon) = \sum_\beta d_\beta x^{(\beta)}$ and the contribution to the γ-subvector of the sum $\sum_\beta d_\beta x^{(\beta)}$ comes from $d_\gamma x^{(\gamma)}$ only, it follows that d_γ must be positive. This proves our contention. The proof is complete. ■

Now we explain how to establish the existence of a preferred-basis for $E(P)$. We need to construct a strongly combinatorial basis $\{x^{(\alpha)} : \alpha$ is a basic class of $P\}$ for $E(P)$ with the property that for each basic class α, the associated strongly combinatorial vector $x^{(\alpha)}$ has the property that if $(P-\rho(P)I)x^{(\alpha)} = \sum_\alpha c_\beta x^{(\beta)}$, then c_β is positive if $\beta >\!- \alpha$ and is zero, otherwise. The nonnegative-basis theorem already guarantees the existence of a strongly combinatorial basis for $E(P)$. Choose any one such basis, say $\{y^{(\alpha)} : \alpha$ is a basic class$\}$.

We are going to construct the desired strongly combinatorial vectors $x^{(\alpha)}s$ by induction on $\text{lev}(\alpha)$. If $\text{lev}(\alpha) = 1$, just take $x^{(\alpha)} = y^{(\alpha)}$. Consider any fixed basic class α with $\text{lev}(\alpha) \geq 2$. Suppose that we have already constructed the desired $x^{(\beta)}s$ for all basic classes β for which $\text{lev}(\beta) < \text{lev}(\alpha)$. Choose $\mathcal{B}$ to be the strongly combinatorial basis $\{y^{(\beta)} : \text{lev}(\beta) \geq \text{lev}(\alpha)\} \cup \{x^{(\beta)} : \text{lev}(\beta) < \text{lev}(\alpha)\}$. Set $x^{(\alpha)} = y^{(\alpha)} + \lambda \sum_{\beta \in \Lambda} x^{(\beta)}$, where λ is a positive number to be chosen and Λ is the collection of all basic classes β such that $d(\beta, \alpha) = 2$. It is clear that $x^{(\alpha)}$ is a strongly α-combinatorial vector whatever choice of λ (as long as it is positive). By our induction assumption, for each $\beta \in \Lambda$, $(P - \rho(P)I)x^{(\beta)}$ can be written as a positive linear combination of all those $x^{(\gamma)}s$ for which $d(\gamma, \beta) \geq 2$ (and hence $d(\gamma, \alpha) \geq 3$). By applying Lemma 6.4 to $y^{(\alpha)}$ (and with the above choice of $\mathcal{B}$), we see that $(P - \rho(P)I)y^{(\alpha)}$ can be represented as a linear combination of those $x^{(\gamma)}s$ for which $\gamma >\!- \alpha$ and moreover if $\gamma \in \Lambda$ then the coefficient of $x^{(\gamma)}$ is positive. So, in the representation of $(P - \rho(P)I)x^{(\alpha)}$ in terms of the basis vectors of $\mathcal{B}$, the coefficients of $x^{(\gamma)}s$ for $\gamma \in \Lambda$ are all positive. Consider any basic class γ such that $d(\gamma, \alpha) \geq 3$. By our induction assumption, for each $\beta \in \Lambda$, the coefficient of $x^{(\gamma)}$ in the representation of $(P - \rho(P)I)x^{(\beta)}$ as a linear combination of the basis vectors of $\mathcal{B}$ is positive if $\gamma >\!- \beta$ and is zero, otherwise. But we have $\gamma >\!- \beta$ for at least one $\beta \in \Lambda$, hence, by choosing $\lambda > 0$ sufficiently large, the coefficient of $x^{(\gamma)}$ in the representation of $(P - \rho(P)I)x^{(\alpha)}$ can be made positive. Since there are only finitely many such basic classes γ, by choosing λ sufficiently large, the vector $x^{(\alpha)}$ has the desired properties. Inductively, the desired preferred-basis of $E(P)$ can be constructed.

7. The majorization relation between the level and the height characteristics

Let $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_q)$ be sequences of nonnegative integers. We append zeros to the shorter sequence to equalize its length to the longer sequence. Let $m = \max\{p, q\}$. We say α is *majorized by* β (or β *majorizes* α), denoted by $\alpha \preceq \beta$, if for every $k \in \langle m \rangle$, $\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i$ and $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i$.

For a finite sequence α of nonnegative integers, we denote by $\hat{\alpha}$ the sequence α reordered in a nonincreasing order.

Here our definition of majorization follows that of [H-S4], which is different from the definition given in [M-O], where β is defined to majorize α if β and α have the same length and $\hat{\beta}$ majorizes $\hat{\alpha}$ in our sense.

The following result is known (see [H-S4]):

Theorem 7.1. *Given two finite sequences of positive integers $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\eta = (\eta_1, \ldots, \eta_q)$, where $\eta_1 \geq \cdots \geq \eta_q$, in order that there exists a nonnegative matrix P such that $\lambda(P) = \lambda$ and $\eta(P) = \eta$, it is necessary and sufficient that $p = q$ and $\hat{\lambda} \preceq \eta$.*

For a proper cone K, if $A \in \pi(K)$ satisfies $\operatorname{span}(E(A) \cap K) = E(A)$, then by the definition of $\lambda(A)$ and $\eta(A)$ it is clear that $\lambda(A)$ and $\eta(A)$ have the same length and $\lambda(A) \preceq \eta(A)$. Below we will strengthen this observation in the case that K is polyhedral, producing an extension of the necessity part of Theorem 7.1.

Theorem 7.2. *Let K be a polyhedral proper cone and let $A \in \pi(K)$. Then $\widehat{\lambda(A)} \preceq \eta(A)$.*

We need two lemmas before we give the proof.

Lemma 7.3. *If B is an $n_2 \times n_2$ nilpotent matrix, X is an $n_2 \times n_1$ matrix and A is the 2×2 block lower triangular nilpotent matrix given by:*

$$A = \begin{bmatrix} 0 & 0 \\ X & B \end{bmatrix},$$

then $(\widehat{n_1, \eta_0(B)}) \preceq \eta_0(A)$, where $\eta_0(C)$ denotes the height characteristic of C with respect to 0.

Proof. Our assertion follows readily by applying the following known result (see [J-S, Lemma]): Under the hypothesis of our lemma, the collection of Jordan blocks of A can be produced from that of B by increasing the size of a select number of Jordan blocks by one, adding copies of $J_1(0)$ and keeping the remaining blocks. ∎

Lemma 7.4. *Let $C = (C_{ij})_{1 \leq i,j \leq p}$ be a $p \times p$ strictly block lower triangular matrix with the ith zero diagonal block of size $v_i \times v_i$. Then $\hat{v} \preceq \eta_0(C)$, where $v = (v_1, \ldots, v_p)$, $\hat{v}$ is the sequence v reordered in nonincreasing order and $\eta_0(C)$ denotes the height characteristic of C with respect to 0.*

Proof. We proceed by induction on p. Our result clearly holds for $p = 1$. Consider $p \geq 2$. Let B denote the nilpotent matrix $(C_{ij})_{2 \leq i,j \leq p}$. Then the given matrix C can be expressed as a 2×2 block lower triangular matrix:

$$C = \begin{bmatrix} 0 & 0 \\ X & B \end{bmatrix},$$

where the $(1,1)$-block is of size $v_1 \times v_1$. By our induction assumption, we have $(\widehat{v_2, \ldots, v_p}) \preceq \eta_0(B)$, and hence $\hat{v} \preceq (v_1, \eta_0(B))$. On the other hand, by Lemma 7.3, $(v_1, \eta_0(B)) \preceq \eta_0(C)$. So we have $\hat{v} \preceq \eta_0(C)$. ∎

Proof of Theorem 7.2: First, note that

$$\mathrm{span}[\mathcal{N}(A-\rho(A)I)\cap K] \subseteq \cdots \subseteq \mathrm{span}[\mathcal{N}((A-\rho(A)I)^{\nu})\cap K] = E(A)$$

is a chain of ν $(= \nu_\rho)$ A-invariant subspaces such that the ith member of the chain is a subspace of dimension $\lambda_1(A)+\cdots+\lambda_i(A)$. For $i = 2,\ldots,\nu_\rho$, it is clear that we have

$$(A-\rho(A)I)[\mathcal{N}((A-\rho(A)I)^{i})\cap C(A,K)] \subseteq \mathcal{N}((A-\rho(A)I)^{i-1})\cap C(A,K).$$

But by Theorem 4.3(iii), for each i, we have

$$\mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i})\cap C(A,K)] = \mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i})\cap K];$$

hence we have

$$(A-\rho(A)I)\,\mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i})\cap K] \subseteq \mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i-1})\cap K].$$

In other words, $A-\rho(A)I$ maps the ith member of the above chain of subspaces into its $(i-1)$th member. Now it is clear that we can choose a basis for $E(A)$ such that the representative matrix of the restriction of $A-\rho(A)I$ to $E(A)$ is a $p\times p$ strictly block lower triangular matrix with the ith diagonal block of size $\lambda_i\times\lambda_i$. Note that $\eta(A)=\eta_0(A-\rho(A)I)$. In view of Lemma 7.4 it follows that we have $\widehat{\lambda(A)} \preceq \eta(A)$. ■

Our argument, in fact, also establishes the following:

Remark 7.5. Let K be a proper cone and let $A\in\pi(K)$. If $E(A)=\mathrm{span}(E(A)\cap K)$ and

$$(A-\rho(A)I)\,\mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i})\cap K] \subseteq \mathrm{span}[\mathcal{N}((A-\rho(A)I)^{i-1})\cap K]$$

for $i=1,\ldots,\nu_\rho$, then $\widehat{\lambda(A)} \preceq \eta(A)$.

Our next example, which is borrowed from the proof of [T-S1, Theorem 7.13], shows that Theorem 7.2 is invalid if the polyhedrality assumption on K is omitted.

Example 7.6. Let n denote an odd integer greater than 1. For any integer k and positive integer r, let $\binom{k}{r}$ denote the number $k(k-1)\cdots(k-r+1)/r!$. Let C be the unbounded convex set in $\mathbb{R}^{n-1}$ with extreme points

$$x_k = \left(\binom{k}{n-1},\binom{k}{n-2},\ldots,\binom{k}{1}\right)^T, \quad k = 0,\pm1,\pm2,\ldots,$$

and recession cone $O^+C = \mathrm{ray}(e_1^{(n-1)})$, where $e_i^{(m)}$ denote the ith standard unit vector of $\mathbb{R}^m$. Let K be the proper cone in $\mathbb{R}^n$ given by: $K = \{\alpha\binom{x}{1} : \alpha\geq 0,\ x\in C\}\cup(O^+C\times\{0\})$.

Take $A = J_n(1)$. Noting that for each integer k, $\binom{x_k}{1} = A^k e_n^{(n)}$, we readily check that $A \in \pi(K)$ (and in fact we have $AK = K$). It is also readily seen that all nonzero vectors of K are generalized eigenvectors of A of height n, except for those that lie on the ray$(e_1^{(n)})$, which are eigenvectors. In this case, we have $\lambda_1(A) = 1$, $\lambda_i(A) = 0$ for $2 \le i \le n-1$ and $\lambda_n(A) = n-1$. On the other hand, we have $\eta_i(A) = 1$ for $i = 1, \ldots, n$. So we do not have the relation $\widehat{\lambda(A)} \preceq \eta(A)$. (But we still have $\lambda(A) \preceq \eta(A)$.) Note also that in this case, $C(A, K)$ equals ray$(e_1^{(n)})$, though span$(E(A) \cap K) = E(A)$.

To complete our cone-theoretic approach, we now explain how we prove Remark 2.2. Once this is done, the necessity part of Theorem 7.1 will follow from Theorem 7.2.

Of course, here lev(x) is defined in terms of supp(x), in the usual way for the nonnegative matrix case. It is clear that we have $\mathrm{span}\{x^{(\alpha_j)} : 1 \le j \le m,\ \mathrm{lev}(\alpha_j) \le k\} \subseteq \{x \in E(P) : \mathrm{lev}(x) \le k\}$. Conversely, consider any $x \in E(P)$ with $\mathrm{lev}(x) \le k$. By the nonnegative-basis theorem, we can write x in the form $\sum_{1 \le j \le m} c_j x^{(\alpha_j)}$. By the combinatorial properties of the strongly combinatorial vectors, it is readily seen that if $h = \max\{\mathrm{lev}(\alpha_j) : c_j \ne 0\}$, then $h = \mathrm{lev}(x)$. Hence, x can be expressed as a linear combination of those $x^{(\alpha_j)}s$ with $\mathrm{lev}(\alpha_j) \le k$. This shows that $\mathrm{span}\{x^{(\alpha_j)} : 1 \le j \le m,\ \mathrm{lev}(\alpha_j) \le k\} = \{x \in E(P) : \mathrm{lev}(x) \le k\}$.

By Lemma 6.2, we have $\mathrm{ht}(x^{(\alpha_j)}) = \mathrm{lev}(\alpha_j)$. But each $x^{(\alpha_j)}$ is semipositive, so we have

$$\mathrm{span}\{x^{(\alpha_j)} : 1 \le j \le m,\ \mathrm{lev}(\alpha_j) \le k\} \subseteq \mathrm{span}[\mathcal{N}((P - \rho(P)I)^k) \cap \mathbb{R}^n_+].$$

To complete the proof, it remains to show that if $0 \ne w \in \mathcal{N}((P - \rho(P)I)^k) \cap \mathbb{R}^n_+$, then $\mathrm{lev}(x) \le k$. Again by the nonnegative-basis theorem we can write $w = \sum_{1 \le j \le m} a_j x^{(\alpha_j)}$. Suppose that $\mathrm{lev}(w) = p > k$. Then, as noted above, we have $p = \max\{\mathrm{lev}(\alpha_j) : a_j \ne 0\}$. Since w is semipositive, it is clear that $a_j > 0$ whenever $\mathrm{lev}(\alpha_j) = p$. Now $(P - \rho(A)I)w = \sum_{1 \le j \le m} a_j (P - \rho(P)I) x^{(\alpha_j)}$. For each j for which $a_j \ne 0$, we write $(P - \rho(P)I)x^{(\alpha_j)}$ as a linear combination of $x^{(\alpha_1)}, \ldots, x^{(\alpha_m)}$ and apply Lemma 6.4. Suppose $(P - \rho(P)I)w = \sum_{1 \le j \le m} b_j x^{(\alpha_j)}$. It is not difficult to see that $\max\{\mathrm{lev}(\alpha_j) : b_j \ne 0\} = p - 1$ and also $b_j > 0$ whenever $\mathrm{lev}(\alpha_j) = p - 1$. Then we consider $(P - \rho(P)I)^2 w$ and repeat. Finally, we can conclude that $(P - \rho(P)I)^k w$ is a nonzero vector, which is a contradiction. This proves that we must have $\mathrm{lev}(w) \le k$. The proof is complete.

Before we end this section, we would like to point out that the study of the relation between the level characteristic and the height characteristic of a nonnegative matrix (or, of a singular M-matrix) has been extended first to general essentially triangular matrices and then to general matrices. It was suggested that the level characteristic was not really the right concept to work with and two different but related sequences were defined and used to obtain improvements and generalizations of the original majorization result. For the details, we refer the interested reader to the excellent survey paper [Her2] by Hershkowitz and the references therein.

8. Principal components of a nonnegative matrix

In [N-S1, Lemma 2 and Theorem 1] Neumann and Schneider obtain the fundamental result given below about the nonnegativity of the elements in the principal components of a nonnegative matrix. In their paper, they apply the result to show that a strongly combinatorial (semipositive) basis for the Perron generalized eigenspace of a nonnegative matrix P can be extracted from the columns of the transform principal component $J_P^{(0)}(\varepsilon)$ (for sufficiently small $\varepsilon > 0$), strengthening the work of Hartwig, Neumann and Rose [H-N-R] in this direction. They also use the result to obtain a necessary condition (which leads to a characterization) for the nonnegativity of the principal eigenprojection of a nonnegative matrix and also to offer a simple alternative proof of a result on the asymptotic behavior of powers of a nonnegative matrix ([F-S, Theorem 5.10]). In the subsequent paper [M-N-S], McDonald, Neumann and Schneider also use the result to establish a theorem about the growth rate of the individual elements in the resolvent of the minus M-matrix associated with a nonnegative matrix.

Theorem 8.1. *Let P be an $n \times n$ nonnegative matrix. Let $i, j \in \langle n \rangle$ and let $d = d_{i,j}$ be the singular distance from i to j. Suppose that $d > 0$. Then:*

(i) $(Z_P^{(k)})_{i,j} = 0$ *for $d \le k \le \nu_\rho - 1$; and*

(ii) $(Z_P^{(d-1)})_{i,j} > 0$.

The proof given in [N-S1] for Theorem 8.1(ii), the difficult part of the theorem, is matrix combinatorial. It is done by induction on the number of blocks in the Frobenius normal form of the matrix, and the argument distinguishes four cases, depending on whether the classes that contain i, j are basic or nonbasic. In this section, we are going to offer a more conceptual proof for the result, one which depends on the nonnegativity of the 0th transform principal component and on the preferred-basis theorem (or rather on an equivalent formulation of the apparently weaker Lemma 6.3). We are able to extract from our argument a result about a cone-preserving map on a polyhedral proper cone.

For any $A \in \mathcal{M}_n(\mathbb{C})$ and $x \in \mathbb{C}^n$, we denote by W_x the A-invariant subspace generated by x, that is, $W_x = \operatorname{span}\{x, Ax, A^2x, \ldots\}$.

Lemma 8.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $\lambda \in \sigma(A)$. For any $0 \ne x \in \mathbb{C}^n$, we have*

$$W_{F_\lambda^{(0)}(\varepsilon)x} = W_{E_\lambda^{(0)}x} = \mathcal{N}((A - \lambda I)^n) \cap W_x,$$

where $E_\lambda^{(0)}$ *(respectively,* $F_\lambda^{(0)}(\varepsilon)$*) denotes the* 0*th (respectively,* 0*th transform) component of* A *corresponding to* λ*, provided that* ε *is nonzero with sufficiently small modulus. If* A*,* λ *and* x *are all real, then the subspaces under consideration can all be taken to be real.*

Proof. Since $E_\lambda^{(0)}$ commutes with every polynomial in A, we have

$$\begin{aligned} W_{E_\lambda^{(0)}x} &= \{p(A)E_\lambda^{(0)}x : p \text{ is a complex polynomial}\} \\ &= E_\lambda^{(0)}(\{p(A)x : p \text{ is a complex polynomial}\}) \\ &= E_\lambda^{(0)}W_x. \end{aligned}$$

(In the case that A, λ and x are all real, we may take $E_\lambda^{(0)}$, $W_{E_\lambda^{(0)}x}$, W_x and p all to be real.)

For a similar reason, we also have $W_{F_\lambda^{(0)}(\varepsilon)x} = F_\lambda^{(0)}(\varepsilon)W_x$. Now

$$E_\lambda^{(0)}W_x = E_\lambda^{(0)}(A|_{W_x})W_x = \mathcal{N}((A|_{W_x} - \lambda I)^n) = \mathcal{N}((A-\lambda I)^n) \cap W_x.$$

When ε is nonzero, sufficiently small in modulus, we also have

$$\begin{aligned} F_\lambda^{(0)}(\varepsilon)W_x &= \varepsilon((\lambda+\varepsilon)I - A)^{-1}E_\lambda^{(0)}W_x \\ &= \varepsilon((\lambda+\varepsilon)I - A)^{-1}[\mathcal{N}((A-\lambda I)^n) \cap W_x] \\ &= \mathcal{N}((A-\lambda I)^n) \cap W_x. \end{aligned}$$

So our assertion follows. ∎

If $A \in \pi(K)$ and F is a face of K, then we denote by $\widehat{F}$ the smallest A-invariant face of K including F. Note that for any $x \in K$, we have $\widehat{\Phi(x)} = \Phi((A+I)^{n-1}x) = \Phi(W_x \cap K)$ and $\operatorname{span}\widehat{\Phi(x)} = W_x$ (see [T-S2, Lemma 2.1]).

Theorem 8.3. *Let* K *be a polyhedral proper cone and let* $A \in \pi(K)$*. Let* $\varepsilon > 0$ *be such that* $J_A^{(i)}(\varepsilon) \in \pi(K)$ *for* $i = 0, \dots, \nu_\rho - 1$*. Then for any* $0 \neq x \in K$*, we have*

(i) $\Phi(\widehat{J_A^{(0)}(\varepsilon)x}) = \Phi(E(A) \cap \widehat{\Phi(x)})$; *and*

(ii) $\operatorname{span}\Phi(\widehat{J_A^{(0)}(\varepsilon)x}) = W_{Z_A^{(0)}x}$.

Proof. We may assume that $\rho_x(A) = \rho(A)$; otherwise, (i) and (ii) both hold trivially.

(i) Denote by B the restriction of A to W_x ($= \operatorname{span}\widehat{\Phi(x)}$). Then $B \in \pi(\widehat{\Phi(x)})$ and $\rho(B) = \rho(A)$. For $i = 0, \dots, \nu_\rho - 1$, since K and W_x are both invariant under $J_A^{(i)}(\varepsilon)$, so is $W \cap K$;

hence $\widehat{\Phi(x)}$, which is $\Phi(W_x \cap K)$, is also invariant under $J_A^{(i)}(\varepsilon)$. But $J_B^{(i)}(\varepsilon) = J_A^{(i)}(\varepsilon)|_{W_x}$, so we have $J_B^{(i)}(\varepsilon) \in \pi(\widehat{\Phi(x)})$ for each i. By applying Theorem 4.3 (ii) and (iii) to B, we infer that $J_B^{(0)}(\varepsilon)\widehat{\Phi(x)}$ is a full subcone of $E(B) \cap \widehat{\Phi(x)}$. Thus

$$(A+I)^{n-1}J_A^{(0)}(\varepsilon)x = J_A^{(0)}(\varepsilon)(A+I)^{n-1}x \in J_A^{(0)}(\varepsilon)\,\mathrm{ri}\,\widehat{\Phi(x)} = \mathrm{ri}(E(A) \cap \widehat{\Phi(x)}).$$

Hence, we have

$$\Phi(\widehat{J_A^{(0)}(\varepsilon)x}) = \Phi((A+I)^{n-1}J_A^{(0)}(\varepsilon)x) = \Phi(E(A) \cap \widehat{\Phi(x)}).$$

(ii) follows from Lemma 8.2 (by taking $\lambda = \rho(A)$) and the fact that $\mathrm{span}\,\Phi(\widehat{J_A^{(0)}(\varepsilon)x}) = W_{J_A^{(0)}(\varepsilon)x}$. ■

Proof of Theorem 8.1: Let α, β be the classes of P containing j and i respectively. Choose any strongly combinatorial basis $\mathcal{B}$ for $E(P)$, say, $\mathcal{B} = \{x^{(\gamma)} : \gamma \text{ basic class}\}$.

(i) By Theorem 8.3(ii) and (i) we have

$$Z_P^{(0)}e_j \in \mathrm{span}\,\Phi(\widehat{J_P^{(0)}(\varepsilon)e_j}) = \mathrm{span}\,\Phi(E(P) \cap \widehat{\Phi(e_j)}) \subseteq \mathrm{span}\,\Phi(\widehat{\Phi(e_j)}) = \mathrm{span}\,\widehat{\Phi(e_j)}.$$

Let I denote the initial subset for P determined by α, i.e., the union of all classes having access to α. Since $\mathrm{supp}(e_j) = \{j\} \subseteq \alpha$, I is the smallest initial subset that includes $\mathrm{supp}(e_j)$; so by [T-S2, Theorem 3.1] $\widehat{\Phi(e_j)}$ equals the P-invariant face F_I. By applying the nonnegative-basis theorem to the restriction of P to $\mathrm{span}\,\widehat{\Phi(e_j)}$, we see that $Z_P^{(0)}e_j$ can be expressed in the form $\sum_{\gamma\in\Gamma} c_\gamma x^{(\gamma)}$, where we denote by Γ the collection of all basic classes having access to α. By a reformulation of Lemma 6.3 (cf. Lemma 6.4), for each $\gamma \in \Gamma$, $(P-\rho(P)I)^k x^{(\gamma)}$ can be expressed as a linear combination of those $x^{(\omega)}$s for which $d(\omega,\gamma) \geq k+1$ and hence for which $d(\omega,\alpha) \geq k+1$. Thus $Z_P^{(k)}e_j$, which is $(P-\rho(P)I)^k Z_P^{(0)}e_j$, can be expressed as a linear combination of those $x^{(\omega)}$s for which $d(\omega,\alpha) \geq k+1$. Since $k \geq d_{i,j} = d(\beta,\alpha)$, the β-subvector of $Z_P^{(k)}e_j$ must be zero. Hence, we have $(Z^{(k)})_{i,j} = 0$.

(ii) By Theorem 8.3(i), $J_P^{(0)}(\varepsilon)e_j$ and $E(P) \cap \widehat{\Phi(e_j)}$ generate the same P-invariant face; so the union of all classes having access to $\mathrm{supp}(J_P^{(0)}(\varepsilon)e_j)$ is equal to the union of all basic classes having access to α. By the nonnegative-basis theorem, $J_P^{(0)}(\varepsilon)e_j$ can be expressed in the form $\sum_{\tau\in\Gamma} d_\tau x^{(\tau)}$, where Γ has the same meaning as before. Furthermore, d_τ is positive whenever τ is a basic class such that $d(\tau,\alpha) = 1$, because $J_P^{(0)}(\varepsilon)e_j$ is semipositive and all such τ are precisely all the classes final in Γ. (If α is a basic class, then there is only one such τ, namely, α.) Now by the definition of $J_P^{(0)}(\varepsilon)$, we have

$$J_P^{(0)}(\varepsilon)e_j = Z_P^{(0)}e_j + \varepsilon^{-1}Z_P^{(1)}e_j + \cdots + \varepsilon^{\nu_\rho - 1}Z_P^{(\nu_\rho - 1)}e_j.$$

Express each $Z_P^{(r)}e_r$ as a linear combination of the vectors in the basis $\mathcal{B}$ and note that, by our proof of (i), for $r = 1, \ldots, \nu_\rho - 1$, in the representation for $Z_P^{(r)}e_j$, the coefficient of $x^{(\tau)}$ is zero whenever $d(\tau, \alpha) = 1$. It follows that we have $c_\tau = d_\tau > 0$, whenever τ is a basic class for which $d(\tau, \alpha) = 1$. Since $Z_P^{(d-1)}e_j = \sum_{\gamma \in \Gamma} c_\gamma (P - \rho(P)I)^{d-1}x^{(\gamma)}$, using a reformulation of Lemma 6.3, we see that $Z_P^{(d-1)}e_j$ can be expressed as a linear combination of those $x^{(\gamma)}s$ for which $d(\gamma, \alpha) \geq d$ and moreover the coefficient of $x^{(\gamma)}$ is positive whenever $d(\gamma, \alpha) = d$. But β is a class (basic or nonbasic) such that $d(\beta, \alpha) = d_{i,j} = d$; so there must exist a basic class γ such that $d(\gamma, \alpha) = d$ and $\beta >= \gamma$. Now it is not difficult to see that the β-subvector of $Z_P^{(d-1)}e_j$ must be strictly positive. Therefore, we have $(Z_P^{(d-1)})_{i,j} > 0$. ■

In passing, we would like to point out that our argument also establishes the following known fact (see [N-S1, Lemma 2(ii)] and [N-S2]): *If α, β are classes of a nonnegative matrix P such that $d = d(\alpha, \beta) > 0$, then the principal submatrix of $Z_P^{(d-1)}$ with rows and columns indexed by the union of all classes that have access from α and also have access to β is semipositive.*

Using (2.1) and Theorem 8.1, one can readily deduce the following known result on the growth rate (as $\varepsilon \to 0^+$) of the individual elements in the resolvent of a minus (singular) M-matrix (associated with a nonnegative matrix) [M-N-S, Theorem 3.1(iii)]. For completeness, we would also like to mention that the decay rate (as $\varepsilon \to \infty$) of the individual elements in the resolvent is also given in [M-N-S, Theorem 3.4(ii)] (in terms of the shortest path length). These results are used in [M-N-S] to develop properties of splittings of an M-matrix.

Corollary 8.4. *Let P be an $n \times n$ nonnegative matrix. Let $i, j \in \langle n \rangle$ and let $d = d_{i,j}$ be the singular distance from i to j. If $d > 0$, then*

$$\lim_{\varepsilon \to 0^+} (\varepsilon^d)(((\rho(P) + \varepsilon)I - P)^{-1})_{i,j} = (Z_P^{(d-1)})_{i,j} > 0.$$

9. Final remarks

In this section, we conclude the paper with a few remarks.

In the paper we deal with the Perron generalized eigenspace only. Our treatment actually also covers the corresponding situation for distinguished eigenvalues, because for a distinguished eigenvalue λ, we can apply our results to the restriction map $A|_{\operatorname{span} F_\lambda}$, where F_λ is the A-invariant face $\{x \in K : \rho_x(A) \le \lambda\}$, and define the spectral cone of A for K corresponding to λ by

$$C(A; K, \lambda) = \{x \in \mathcal{N}((A - \lambda I)^n) \cap K : (A - \lambda I)^i x \in K \text{ for } i = 1, 2, \ldots\}.$$

In [Tam3, p.69] this author points out that until today very little is known about the elements of $\pi(K)$ which are not nonnegative linear combinations of its rank-one elements. So, the fact that $J_A^{(0)}(\varepsilon)$ belongs to $\pi(K)$ (for K polyhedral and sufficiently small $\varepsilon > 0$) can be regarded as a nice result, difficult to come by. It is nice, because whereas the spectral cone plays a prevalent role in the unified approach of this paper, it is the K-semipositivity of the 0th transform principal component that makes this possible (for the polyhedral case). Another example of this kind is the K-nonnegativity of the principal component $Z_A^{(\nu_\rho - 1)}$ (for a proper cone K). The latter result has been used several times in our sequence of papers. In particular, it is needed in introducing the concept of the spectral pair of a face and in showing that every cone-preserving map satisfies the local Perron-Schaefer condition (see [T-S2, Lemma 4.2, Theorem 4.7] and [Tam4, Subsection 5.3.3]). Still another example of this kind is provided by the K-nonnegativity of $\sum_\lambda E_\lambda^{(\nu_\rho - 1)}$, where the summation is taken over all eigenvalues λ in the peripheral spectrum of A that have the same index as that of $\rho(A)$. The result is crucial for establishing the relations between the Perron-Schaefer condition and the peripheral spectrum of a cone-preserving map (see [T-S1]). All these examples suggest that the investigation of the cone $\pi(K)$ may be beneficial to the study of the spectral properties of its individual elements. (See [Tam4, Section 6.5] for further remarks.)

Despite this work, we have not yet obtained a cone version of the preferred-basis theorem and, indeed, not even of the nonnegative-basis theorem. We know that if A preserves a polyhedral proper cone, then $E(A)$ contains a K-semipositive basis. But the nonnegative-basis theorem guarantees the existence of a strongly combinatorial basis, not merely a semipositive basis. In view of the correspondence between the semi-distinguished classes and the semi-distinguished invariant faces of a nonnegative matrix as given in [T-S2, Theorem 3.6], we would like to make the following conjecture:

Conjecture 9.1. *Let K be a polyhedral proper cone. Let $A \in \pi(K)$ and let m be the algebraic multiplicity of $\rho(A)$. Then there exist m semi-distinguished A-invariant faces of K associated with $\rho(A)$, say $F_1, \ldots, F_m$, such that for any choice of vectors $x_1, \ldots, x_m$ with $x_i \in E(A) \cap \operatorname{ri} F_i$ for $i = 1, \ldots, m$, $\{x_1, \ldots, x_m\}$ is a linearly independent set (and hence forms a basis of $E(A)$).*

In the nonnegative matrix case, the concepts of level characteristic and semi-distinguished classes (associated with the spectral radius) are closely related; indeed, the former is defined in terms of the latter. For a cone-preserving map, the way we define its level characteristic makes unclear its connections with the semi-distinguished invariant faces. It seems that there is something worth exploring in this direction.

We conclude with an example which is in agreement with the conjecture and also shows that the following natural cone version extension of the second part of the preferred-basis theorem does not hold:

It is possible to choose the faces $F_1, \ldots, F_m$ and the vectors $x_1, \ldots, x_m$ in such a way that in addition to the property described in the conjecture the following is also satisfied: For $i = 1, \ldots, m$, we have $(A - \rho(A)I)x_i = \sum_{k=1}^{m} c_{ik}x_k$, where c_{ik} is positive if F_k is strictly included in F_i and equals 0 otherwise.

Example 9.2. Let e_i denote the ith standard unit vector of $\mathbb{R}^5$. Let K denote the polyhedral proper cone in $\mathbb{R}^5$ given by: $K = K_1 \oplus K_2$, where $K_1 = \text{pos}\{e_1, e_1 + e_2, e_3, e_2 + e_3\}$ and $K_2 = \text{pos}\{e_4, e_5\}$. Let A be the 5×5 real matrix defined by:

$$Ae_1 = Ae_2 = Ae_3 = 0, \ Ae_4 = 2e_1 + e_2 \ \text{ and } \ Ae_5 = e_2 + 2e_3.$$

Clearly, $A \in \pi(K)$. It can be verified that the faces $\Phi(2e_1+e_2)\oplus\Phi(e_4)$, $\Phi(2e_3+e_2)\oplus\Phi(e_5)$ and the extreme rays of K_1 are precisely all the nonzero join-irreducible A-invariant, and hence semi-distinguished A-invariant, faces of K (as A is nilpotent). In this case, the algebraic multiplicity of $\rho(A)$ equals 5, and A has precisely six semi-distinguished A-invariant faces, all associated with $\rho(A)$. It is readily seen that for any choice of vectors $x_1, \ldots, x_5$, with x_1 from $\text{ri}(\Phi(2e_1 + e_2) \oplus \Phi(e_4))$, x_2 from $\text{ri}(\Phi(2e_3 + e_2) \oplus \Phi(e_5))$, and x_3, x_4, x_5 each from the relative interior of any three of the four extreme rays of K_1, the set $\{x_1, \ldots, x_5\}$ is always linearly independent. So this example is in agreement with our conjecture.

We now attempt to choose semi-distinguished A-invariant faces $F_1, \ldots, F_5$ and vectors $x_1, \ldots, x_5$ respectively from the relative interior of these faces so that the requirement given in the natural cone version of the second part of the preferred-basis theorem is met, in addition to its first part. In order that the vectors $x_1, \ldots, x_5$ form a linearly independent set, clearly we must choose the faces $\Phi(2e_1 + e_2) \oplus \Phi(e_4)$, $\Phi(2e_3 + e_2) \oplus \Phi(e_5)$, together with three of the four extreme rays of K_1; say $F_4 = \Phi(2e_1 + e_2) \oplus \Phi(e_4)$ and $F_5 = \Phi(2e_3 + e_2) \oplus \Phi(e_5)$. It is readily seen that, for any choice of x_4, x_5 from $\text{ri}\, F_4$ and $\text{ri}\, F_5$ respectively, Ax_4 (respectively, Ax_5) is a positive multiple of Ae_4 (respectively, Ae_5) and hence lies in the relative interior of F_4 (respectively, F_5). So for any choice of the remaining faces F_1, F_2, F_3 (from the four extreme rays of K_1) and the remaining vectors x_1, x_2, x_3, either Ax_4 or Ax_5 cannot be written as a positive linear combination of vectors chosen from $\{x_1, \ldots, x_5\}$ in the desired manner. This shows that the natural cone version extension of the second part of the preferred-basis theorem is invalid.

This author would like to thank Hans Schneider and the referee for some helpful comments.

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ON LOCAL PERRON-FROBENIUS THEORY

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Abstract. For $A \in \mathbb{R}^{nn}$, $x \in \mathbb{R}^n$, and any nonnegative integer k, we denote by $w_k(A,x)$ the convex cone generated by $A^k x, A^{k+1}x, \ldots$ in $\mathbb{R}^n$ and refer to it as an A-cyclic cone. We obtain equivalent conditions for $w_k(A,x)$ or its closure to satisfy one or a combination of the following properties: (i) nonzero, (ii) pointed, or (iii) not a linear subspace. In particular, we show that $\mathrm{cl}\, w_k(A,x)$ is pointed if and only if A satisfies the local Perron-Schaefer condition at x, i.e., in the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvector y corresponding to $\rho_x(A)$, the local spectral radius of A at x, and furthermore the order of y is not less than that of any other generalized eigenvector that appears in the representation and corresponds to an eigenvalue with modulus $\rho_x(A)$. In case A satisfies the Perron-Schaefer condition, we show how one can construct A-invariant proper cones by taking the sum of the closures of finitely many A-cyclic cones. By considering the linear map L_A on $\mathbb{R}^{nn}$ given by $\mathrm{L}_A(X) = AX$, we recover the intrinsic Perron-Frobenius theorems obtained by H. Schneider. We also prove similar results for the convex cone generated by $\{\exp(tA)x : t \geq 0\}$ (instead of $\{A^i : i \geq 0\}$); thus we also obtain local versions of the known Perron-Frobenius type theorems for cross-positive matrices.

Work to be completed

* Research of this author partially supported by the National Science Council of the Republic of China
2000 *Mathematics Subject Classification*: 15A06, 15A48.
Key words and phrases: Cone-preserving map, Perron-Frobenius theory, local spectral radius, local Perron-Schaefer condition, A-cyclic cone, cross-positive matrix
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1. Introduction

This is the seventh of a sequence of papers (namely, [T–W], [Tam 2], [T–S 1], [T–S 2], [T–S 3], [Tam 4] and the current paper) on a newly developed subject, the geometric spectral theory of positive linear operators (in finite dimensions), which is concerned with the study of the classical Perron-Frobenius theory of a (square, entrywise) nonnegative matrix and its generalizations from the geometric cone-theoretic viewpoint. (For a survey on the subject, see [Tam 3].)

In the previous papers of the sequence, we usually fix a proper (closed, pointed, full convex) cone K and a matrix A such that K is invariant under A (i.e. $AK \subseteq K$; then we write $A \in \pi(K)$) and consider different aspects of A in each paper. In this work we change our viewpoint somewhat. Given a real (square) matrix A, we consider the problem of contructing closed, pointed (or, proper, if possible) cones K that A preserves. In particular, we consider the question of the existence and construction of an A-invariant closed, pointed cone C that contains a given vector x, or. more generally, that includes a given finite set of vectors.

The answer to the question of when a given real matrix leaves invariant a proper cone is known and is provided by the following:

Theorem. *Let A be an $n \times n$ real matrix. Then there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ if and only if A satisfies the following set of conditions:*

(a) *$\rho(A)$, the spectral radius of A, is an eigenvalue of A.*

(b) *For each eigenvalue λ of A with modulus $\rho(A)$, $\nu_\lambda(A) \leq nu_{\rho(A)}(A)$, where $\nu_\lambda(A)$ denotes the index of λ as an eigenvalue of A.*

Parts (a) and (b) of the direct ("only if") part of the above theorem is due to Perron and Schaefer respectively, whilst its converse ("if") part is due to Vandergraft and Elsner independently. The conditions (a) and (b) together is now usually referred to as the $Perron - Schaefercondition$. (Actually, (b) implies (a).)

As pointed out by Schneider [Sch 1], for a matrix A that satisfies the Perron-Schaefer condition, the known construction of a proper cone and the proof of its invariance (under A) are elementary, but rather involved. In order to characterize the Perron-Schaefer condition on A by a geometric property directly associated with A, for each nonnegative integer k, he introduced the $intrinsicconew_k(A)$ of A. which consists of all nonnegative linear combinations of $A^k, A^{k+1}, \ldots$. (In fact, in [Sch 1] Schneider formulated his results in terms of a complex matrix. But since "cone" is a real concept, for conceptual clarity we prefer to use real matrices.) The following result is obtained in [Sch 1, Theorem 1.4]:

Theorem. *Let $A \in \mathcal{M}_n(\mathbb{R})$, and let k be a nonnegative integer. Then the cone $\mathrm{cl}\, w_k(A)$ is pointed if and only if A satisfies the Perron-Schaefer condition.*

Schneider [Sch 1] mentioned that when K is a proper cone and $A \in \pi(K)$, we have $\operatorname{cl} w_0(A) \subseteq \pi(K)$, and hence $\operatorname{cl} w_0(A)$ is pointed, as $\pi(K)$ is a proper cone in the underlying matrix space. Then he noted that, using this observation, one can readily show that the direct part of Theorem A and the direct part of Theorem B (for $k = 0$) are "equivalent", and the converse part of Theorem A implies that of Theorem B. He also asked whether there is a simple argument to derive the converse part of Theorem A from the converse part of Theorem B; or, in other words, a simple way to construct an invariant proper cone for a matrix that satisfies the Perron-Schaefer condition. [In passing, we would like to point out that the cone-theoretic and analytic approach adopted by Schneider in [Sch 1] actually establishes, without assuming the Perron-Frobenius theorem (for a cone-preserving map), that if $A \in \pi(K)$, then we have the conclusion of the Perron-Frobenius theorem, the Perron-Schaefer condition, and also the K-nonnegativity of the $(\nu_{\rho(A)}(A) - 1$th component of A corresponding to its spectral radius (a result which first appeared explicitly in [Sch 1, Theorem 5.2] but which had appeared in some form in [Kar]).

In this paper, we show that there is a natural simple way to construct examples of invariant proper cones for a matrix that satisfies the Perron-Schaefer condition if we use a "local version" of Theorem B and the concept of A-cyclic cone.

For any $A \in \mathcal{M}_n(\mathbb{R}), x \in \mathbb{R}^n$ and nonnegative integer k, we denote by $w_k(A, x)$ the convex cone $\operatorname{pos}\{A^k x, A^{k+1}x, \ldots\}$ and refer to it as an $A-cycliccone$. It is easy to see that $\operatorname{cl} w_0(A, x)$ is an A-invariant closed cone containing x, and also that there is a closed, pointed A-invariant cone containing x if and only if the cone $\operatorname{cl} w_0(A, x)$ is pointed (then the latter cone is the smallest A-invariant cone containing x). A fundamental question to ask is, when the cone $\operatorname{cl} w_0(A, x)$ is pointed. As we will show in Section 3, the cone $\operatorname{cl} w_0(A, x)$ is pointed if and only if A satisfies the following condition, which we first introduced in the paper [T–S 2] and now refer to as the *local Perron-Schaefer condition*: in the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvector y that corresponds to the local spectral radius $rho_x(A)$ of A at x, and furthermore the order of the generalized eigenvector y is not less than that of any other generalized eigenvector that appears in the representation and corresponds to an eigenvalue with modulus $\rho_x(A)$. By considering the linear map L_A on $\mathcal{M}_n(\mathbb{R}$ defined by $L_A(X) = AX$, we also recover Theorem B.

In Section 4 we show that for a (real) matrix A that satisfies the Perron-Schaefer condition it is always possible to find a proper cone K invariant under A, which can be expressed as the sum of the closure of finitely many A-cyclic cones. For a matrix A, that need not satisfy the Perron-Schaefer condition and any given finite set of vectors at which A satisfies the local Perron-Schaefer condition, we also find a necessary and sufficient condition for the existence of an A-invariant proper cone that includes the given finite set of vectors.

Besides the above-mentioned Theorem B, Schneider [Sch 1] also provided other results that relate the algebraic properties of a matrix A (such as, having a positive (or negative) eigenvalue) to its geometric properties (such as, $\operatorname{cl} w_k(A)$ is not a real subspace, $w_k(A)$ is a pointed nonzero cone). In Section 5 we show that there are corresponding local versions of these results, which can also be used to recover Schneider's original results.

In Section 6 automorphisms In Section 7, analgous results for cross-positive matrices Section 8, cone reachability

2. Preliminaries

We denote by $J_k(\lambda)$ the $k \times k$ upper triangular elementary Jordan block associated with the eigenvalue λ.

If $A \in \mathcal{M}_n$ and x is a nonzero vector of $\mathbb{C}^n$ such that $(\lambda I - A)^k x = \mathbf{0}$ for some $\lambda \in \mathbb{C}$ and some positive integer k, then x is called a *generalized eigenvector of* A corresponding to (the eigenvalue) λ. The least such integer k is called the *order of* x as a generalized eigenvector of A, and is denoted by $\mathrm{ord}_A(x)$.

If $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$, we can write $x = x_1 + \cdots + x_m$, where $x_1, \ldots, x_m$ are generalized eigenvectors of A corresponding respectively to the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then we define $\rho_x(A)$, *local spectral radius of* A *at* x, to be the quantity $\max_{1 \leq i \leq m} |\lambda_i|$. We also set $\mathrm{ord}_A(x) = \max\{\mathrm{ord}_A(x_i) : |\lambda_i| = \rho_x(A)\}$, where $\mathrm{ord}_A(x_i)$ denotes the order of the generalized eigenvector x_i, and refer to $\mathrm{ord}_A(x)$ as the *order of* x *relative to* A. [We reserve the term "height" for the order of a generalized eigenvector that corresponds to the spectral radius.] The ordered pair $(\rho_x(A), \mathrm{ord}_A(x))$, denoted by $\mathrm{sp}_A(x)$, is called the *spectral pair of* x *relative to* A. It was first introduced in [T-S2] and has proved to be a useful concept. We also need the following equivalent definitions for the local spectral radius (see [T–W, Theorem 2.3]):

(i) $\rho_x(A) = \lim\ \sup_{m \to \infty} \|A^m x\|^{1/m}$, where $\|\cdot\|$ is any norm of $\mathbb{C}^n$.

(ii) $\rho_x(A) = \rho(A|_{W_x})$, where W_x is the cyclic space relative to A generated by x, i.e. the linear subspace $\mathrm{span}\{A^i x :\ i \geq 0\}$.

Let $A \in \mathcal{M}_n(\mathbb{R})$. It is known that a necessary and sufficient condition for the existence of a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ is that the following set of conditions is satisfied:

(a) $\rho(A)$ is an eigenvalue of A; and

(b) If λ is an eigenvalue in the peripheral spectrum of A, then $\nu_\lambda(A) \leq \nu_{\rho(A)}(A)$.

The above set of conditions is now referred to as the *Perron-Schaefer condition* (see [Sch 2, the paragraph following Theorem 1.1], and also [T–S 1, Section 7] for our recent work involving the condition).

According to [T–S 2, Theorem 4.7], if $A \in \pi(K)$, then for any $0 \neq x \in K$, the following condition is always satisfied:

There is a generalized eigenvector y of A corresponding to $\rho_x(A)$ that appears as a term in the representation of x as a sum of generalized eigenvectors of A, and moreover we have

$\mathrm{ord}_A(x) = \mathrm{ord}_A(y)$.

By analogy, we refer to the preceding condition as the *local Perron-Schaefer condition at* x.

In [T–S 2, Theorems 4.6 and 4.7] it is first shown that if $A \in \pi(K)$ and F is a face of K, then the spectral pair $\mathrm{sp}_A(x)$ is independent of the choice of x from the relative interior of F, and then as a by-product we obtain the the property that if $A \in \pi(K)$, then A satisfies the local Perron-Schaefer condition at x for every $x \in K$. Actually, the latter property can be proved directly. The point is, then $W_x \cap K$ is a proper cone in W_x and $A|_{W_x} \in \pi(W_x \cap K)$, so $A|_{W_x}$ satisfies the Perron-Schaefer condition, but the latter condition is equivalent to the condition that A satisfies the local Perron-Schaefer condition at x, in view of the following useful fact:

Remark 2.1. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$. If $x = x_1 + \cdots + x_k$ is the representation of x as a sum of generalized eigenvectors of A corresponding respectively to the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ and if $\mathrm{ord}_A(x_j) = n_j$ for $j = 1, \ldots, k$, then relative to the ordered basis $\bigcup_{j=1}^k \{(A - \lambda_j I)^{n_j - 1} x_j, (A - \lambda_j I)^{n_j - 2} x_j, \ldots, (A - \lambda_j I) x_j, x_j\}$ for W_x, $A|_{W_x}$ is represented by the matrix $J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$

Remark 2.2. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Then $\bigcup_{i=1}^k \{E_{\lambda_i}^{(j)} : 0 \leq j \leq \nu_{\lambda_i}(A) - 1\}$ forms a Jordan basis for $L_A|_{\mathrm{span}\, w_0(A)}$, where L_A is the linear operator on $\mathcal{M}_n(\mathbb{C})$ given by $L_A(X) = AX$ and $E_{\lambda_i}^{(j)}$ denotes the jth component of A corresponding to λ.

In this paper we formulate our results in the real setting, because “cone” is a real concept.

3. Local Perron-Schaefer conditions

Following [Sch 2], for any $A \in \mathcal{M}_n(\mathbb{C})$ and any nonnegative integer k, we denote by $w_k(A)$ the convex cone $\mathrm{pos}\{A^i : i = k, k+1, \ldots\}$. When x is a vector of $\mathbb{C}^n$, we also denote by $w_k(A, x)$ the convex cone $\mathrm{pos}\{A^i x : i = k, k+1, \ldots\}$.

Theorem 3.1. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ *and let* $0 \neq x \in \mathbb{R}^n$. *The following conditions are equivalent*:

(a) *A satisfies the local Perron-Schaefer condition at x.*

(b) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*

(c1) *For every nonnegative integer k, the convex cone* $\mathrm{cl}\, w_k(A, x)$ *is pointed.*

(c2) *For some nonnegative integer k, the convex cone* $\mathrm{cl}\, w_k(A, x)$ *is pointed.*

(d1) *For every nonnegative integerk, the convex cone* $\operatorname{cl} w_k(A|_{W_x})$ *is pointed.*
(d2) *For some nonnegative integer k, the convex cone* $\operatorname{cl} w_k(A|_{W_x})$ *is pointed.*
(e1) *There is a closed, pointed, convex cone C containing x such that $AC \subseteq C$.*
(e2) *There is a closed, pointed, convex cone C full in W_x such that $AC \subseteq C$.*

Proof. The equivalence of (a) and (b) was done.

Consider also the following condition:

(c1)' The convex cone $\operatorname{cl} w_0(A, x)$ is pointed.

Clearly, we have (c1) $\Longrightarrow$ (c1)' $\Longrightarrow$ (c2). The implication (c1)' $\Longrightarrow$ (c1) clearly also holds, as $w_0(A, x) \supseteq w_k(A, x)$ for all positive integers k.

Since $\operatorname{cl} w_0(A, x)$ is always a closed full convex cone in W_x which contains x and is invariant under A, the implications (c1)' $\Longrightarrow$ (e1) and (e1) $\Longrightarrow$ (e2) are both clear.

When (e2) is satisfied, we have $A|_{W_x} \in \pi(C)$; so the implication (e2) $\Longrightarrow$ (b) follows.

When A satisfies the local Perron-Schaefer condition at x, by applying Vandergraft's or Elsner's construction (see [Van] or [Els 1]) to $A|_{W_x}$, we can find a closed, pointed cone C which contains x and is invariant under A. So we also have the implication (a) $\Longrightarrow$ (e1).

So far, we have established the equivalence of the conditions (a), (b), (c1), (c1)', (e1) and (e2).

[In the above, instead of applying Vandergraft's or Elsner's construction to show (a) $\Longrightarrow$ (e1), we may proceed by proving the implication (a) $\Longrightarrow$ (c1)' directly as follows:

Let the representation of x as a sum of generalized eigenvectors of A be $x_1 + \ldots + x_k$, where x_i is a generalized eigenvector corresponding to λ_i and is of order ν_i. We may suppose that $\lambda_1 = \rho_x(A)$ and $\nu_1 = \operatorname{ord}_A(x)$. To show the pointedness of $\operatorname{cl} w_0(A, x)$, suppose y and $-y$ both belong to $\operatorname{cl} w_0(A, x)$. Then there exist polynomials $p_m(t)$ and $q_m(t), m = 1, 2, \ldots$, all with nonnegative coefficients such that $y = \lim_{m\to\infty} p_m(A)x$ and $-y = \lim_{m\to\infty} q_m(A)x$. We want to show that $y = 0$. For $i = 1, \ldots, k$, we denote x_i by $x_{i\nu_i}$, $(A - \lambda_i I)x_i$ by $x_{i,\nu_i-1}, \ldots$, and $(A - \lambda_i I)^{\nu_i-1}x_i$ by x_{i1}. Then relative to the ordered basis $\{x_{11}, x_{12}, \ldots, x_{1\nu_1}, x_{21}, \ldots, x_{2\nu_2}, \ldots, x_{k1}, \ldots, x_{k\nu_i}\}$, the Jordan form of $A|_{W_x}$ is given by $\bigoplus_{i=1}^k J_{\nu_i}(\lambda_i)$. Also, for positive integers m we have

$$p_m(A)x = \sum_{i=1}^{k} \left(\frac{p_m^{(\nu_i-1)}(\lambda_i)}{(\nu_i - 1)!} x_i 1 + \ldots + p_m'(\lambda_i) x_{i,\nu_i-1} + p_m(\lambda_i) x_{i,\nu_i} \right),$$

and with a similar expression for $q_m(A)x$. Since $\lim_{m\to\infty}(p_m(A) + q_m(A))x = 0$ and the vectors $x_{ij_{1\leq i\leq k, 1\leq j\leq \nu_i}}$ are linearly independent, we have $(p_m^{(j)} + q_m^{(j)})(\lambda_i) \to 0$ as $m \to \infty$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \nu_i$. In particular, we have, $\lim_{m\to\infty}(p_m(\rho_x(A)) + q_m(\rho_x(A))) = 0$. Since $p_m(t)$ and $q_m(t)$ $(m = 1, 2, \ldots)$ are polynomials with nonnegative coefficients, for $i = 2, \ldots, k$, we have $|p_m(\lambda_i)| \leq p_m(\rho_x(A)) + q_m(\rho_x(A))$; hence we also have $\lim_{m\to\infty} p_m(\lambda_i) = 0$ for $i = 1, \ldots, k$. Indeed, by Cauchy's integral formula for derivatives (or by Cauchy's estimate), we also have, $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \nu_j - 1$. Hence, we have $y = \lim_{m\to\infty} p_m(A)x = 0$, as desired.]

Now suppose condition (c2) holds. Then for some nonnegative integer k, the convex cone $\mathrm{cl}\, w_k(A, x)$ is pointed. Note that $w_k(A, x) = w_0(A, A^k x)$. By the equivalence of (c1)' and (a)(but with x replaced by $A^k x$), we infer that A satisfies the local Perron-Schaefer condition at $A^k x$. But it is easy to check from the definition that the latter condition is equivalent to that A satisfies the local Perron-Schaefer condition at x, i.e, condition (a) holds. So (c2) is also another equivalent condition.

To complete the proof, it remains to show that (d1) and (d2) are additional equivalent conditions. For the purpose, it suffices to show that for any nonnegative integer k, $\mathrm{cl}\, w_k(A|_{W_x})$ is pointed if and only if $\mathrm{cl}\, w_k(A, x)$ is pointed.

For any nonnegative integer k, the cone $\mathrm{cl}\, w_k(A|_{W_x})$ is not pointed if and only if there exist polynomials $p_j(t), q_j(t)$ for $j = 1, 2, \ldots$, all with nonnegative coefficients, such that $\lim\limits_{j\to\infty} A^k p_j(A)$ and $\lim\limits_{j\to\infty} A^k q_j(A)$ both exist, are nonzero, and satisfy $\lim\limits_{j\to\infty} A^k p_j(A) = -\lim\limits_{j\to\infty} A^k q_j(A)$. For the cone $\mathrm{cl}\, w_k(A, x)$ not to be pointed, there is also a similar equivalent condition. On the other hand, it is not difficult to show each of the following: (1) For any polynomial $p(t)$, $p(A|_{W_x}) = O$ if and only if $p(A)x = 0$; and (2) for any sequence of polynomials $(r_j(t))_{j\geq 1}$, $\lim_{j\to\infty} r_j(A|_{W_x})$ exists if and only if $\lim_{j\to\infty} r_j(A)x$ exists. It follows that the cone $\mathrm{cl}\, w_k(A|_{W_x})$ is not pointed if and only if the cone $\mathrm{cl}\, w_k(A, x)$ is not pointed. The proof is complete. ∎

That conditions (c1) and (c2) of Theorem 3.1 are equivalent can also be shown directly as follows. It suffices to show that for any nonnegative integer j, $\mathrm{cl}\, w_j(A, x)$ is pointed if and only if $\mathrm{cl}\, w_{\nu_0}(A, x)$ is pointed. When $j > \nu_0$, we have, $A^{j-\nu_0} w_{\nu_0}(A, x) = w_j(A, x)$, and since the restriction of A to $\mathrm{span}\{A^i x : i = \nu_0, \nu_0 + 1, \ldots\}$ is an isomorphism, our assertion clearly holds in this case. The case when $j = 0, 1, \ldots, \nu_0 - 1$ can also be settled by applying the following observation and also the fact that we always have $\mathrm{cl}\,(K_1) \oplus K_2) = \mathrm{cl}\, K_1 \oplus \mathrm{cl}\, K_2$ whenever K_1, K_2 are cones whose linear spans meet only at the zero vector.

Recall that a closed, pointed polyhedral cone is said to be *simplicial* if its number of extreme rays is equal to the dimension of its linear span.

Remark 3.2. Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $0 \neq x \in \mathbb{R}^n$. Suppose that in the representation of x as a sum of generalized eigenvectors of A, there is a generalized eigenvector corresponding to 0 and its order is ν_0. Then for $j = 0, 1, \ldots, \nu_0 - 1$, $w_j(A, x)$ is the direct sum of the simplicial cone $\mathrm{pos}\{A^j x, A^{j+1} x, \ldots, A^{\nu_0 - 1} x\}$ and the cone $w_{\nu_0}(A, x)$.

To show the above remark, we write x as $y + z$, where y is a generalized eigenvector of A corresponding to 0 of order ν_0 and z belongs to the direct sum of generalized eigenspaces of A corresponding to nonzero eigenvalues of A, and then make use of the fact that the vectors $y, Ay, \ldots, A^{\nu_0 - 1} y$ are linearly independent and also that the generalized eigenspace of A

corresponding to 0 meets the direct sum of the generalized eigenspaces of A corresponding to nonzero eigenvalues only at the zero vector.

Theorem 3.1 can also be used to deduce the following *intrinsic Perron-Frobenius theorem*, which has appeared in [Sch 2, Theorem 1.4] in its complex version.

Corollary 3.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then A satisfies the Perron-Schaefer condition if and only if for any (or, for some) nonnegative integer k, the cone $\mathrm{cl}\, w_k(A)$ is pointed.*

We need a lemma before we come to the proof of Corollary 3.3.

Lemma 3.4. *Let $0 \neq A \in \mathcal{M}_n(\mathbb{R})$. Let L_A be the linear operator on $\mathcal{M}_n(\mathbb{R})$ given by $L_A(X) = AX$, and let W_A denote the cyclic space relative to L_A generated by A. Also let m be the degree of the minimal polynomial of A.*

(i) *If A is nonsingular, then $\{I_n, A, \ldots, A^{m-1}\}$ forms a basis for W_A, and $L_A|_{W_A}$ and A have the same minimal polynomial.*

(ii) *If A is singular, then $\{A, \ldots, A^{m-1}\}$ forms a basis for W_A, and the minimal polynomial of $L_A|_{W_A}$ is equal to the minimal polynomial of A divided by the polynomial t. In this case, the subspaces $\mathrm{span}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $\mathrm{span}\{A^i : i = \nu_0(A), \nu_0(A)+1, \ldots\}$ are independent.*

Proof. Let $q(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ be the minimal polynomial of A.

(i) Since A is nonsingular, we have $a_0 \neq 0$ and from the relation $q(A) = 0$, we readily see that $I_n \in W_A$. The relation $q(A) = 0$ also implies that $\{I_n, A, \ldots, A^{m-1}\}$ is a spanning set for W_A, and indeed it forms a basis, else we would obtain a nonzero annihilating polynomial for A of degress less than m, which is a contradiction. Straightforward calculation shows that, relative to the basis $\{I_n, A, \ldots, A^{m-1}\}$, $L_A|_{W_A}$ is represented by the companion matrix of $q(t)$, that is, the matrix C_q given by

$$C_q = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{m-2} \\ & & & 1 & -a_{m-1} \end{bmatrix}.$$

But $q(t)$ is the minimal polynomial of C_q, hence it is also the common minimal polynomial for A and $L_A|_{W_A}$.

(ii) When A is singular, using the fact that $q(t)$ is a nonzero annihilating polynomial for A of least degree, we readily show that the set $\{A, A^2, \ldots, A^{m-1}\}$ forms a basis for W_A. By direct calculation, we find that relative to this basis $L_A|_{W_A}$ is represented by the companion matrix of the polynomial $q(t)/t$. So the minimal polynomial of $L_A|_{W_A}$ is equal to the minimal polynomial of A divided by t.

If the subspaces $\mathrm{span}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $\mathrm{span}\{A^i : i = \nu_0(A), \nu_0(A)+1, \ldots\}$ meet at a nonzero vector, then we would find a nonzero annihilating polynomial $u(t)$ for A of the form $\sum_{j=r}^{s} b_j t^j$ with $b_s \neq 0$ and $b_r \neq 0$ such that $0 \leq r \leq \nu_0(A) - 1$ and $s \geq \nu_0(A)$. But then $u(t)$ is not divisible by $q(t)$, as it does not contain $t^{\nu_0(A)}$ as a factor. So we arrive at a contradiction. ■

Alternative method Note that the spectral resolution of A in terms of its components, i.e.,

$$A = \sum_{\lambda \in \sigma(A)} (\lambda E_\lambda^{(0)} + E_\lambda^{(1)})$$

is in fact the representation of A as a sum of generalization eigenvectors of L_A, $E_\lambda^{(0)}$, and hence also $\lambda E_\lambda^{(0)} + E_\lambda^{(1)}$ being a generalized eigenvector corresponding to λ of order ν_λ. Note also that if 0 is an eigenvalue of A of order ν_0, then in the above representation the generalized eigenvector of L_A corresponding to 0 is $E_0^{(1)}$, which is of order $\nu_0 - 1$, not ν_0. (If $\nu_0 = 1$, then in the above representation there is no term correpsonding to the eigenvalue 0.)

Proof of Corollary 3.3. We need only consider the case when A is non-nilpotent. By Lemma 3.4, the minimal polynomial of $L_A|_{W_A}$ is equal to that of A or that of A divided by t, depending on whether A is nonsingular or singular. So the nonzero part of the spectrum of A and that of $L_A|_{W_A}$ are the same, and for each nonzero eigenvalue λ, the indices $\nu_\lambda(A)$ and $\nu_\lambda(L_A|_{W_A})$ are also equal. It follows that A satisfies the Perron-Schaefer condition if and only if $L_A|_{W_A}$ satisfies the Perron-Schaefer condition. By Theorem 3.1(with L_A and A playing respectively the role of A and x), the condition that $L_A|_{W_A}$ satisfies the Perron-Schaefer condition is equivalent to that for some (or, for every) nonnegative integer k, the cone $\mathrm{cl}\, w_k(L_A, A)$ is pointed. The latter condition, in turn, is equivalent to the condition that for some (or, for every)positive integer k, the cone $\mathrm{cl}\, w_k(A)$ is pointed, as we have $w_k(L_A, A) = w_{k+1}(A)$ for every nonnegative integer k. It remains to show that $\mathrm{cl}\, w_0(A)$ is pointed if and only if $\mathrm{cl}\, w_{\nu_0}(A)$ (or $\mathrm{cl}\, w_1(A)$) is pointed. If A is nonsingular, then L_A is also nonsingular. In this case, L_A maps $w_0(A)$ one-to-one, onto $w_1(A)$, hence also maps $\mathrm{cl}\, w_0(A)$ one-to-one, onto $\mathrm{cl}\, w_1(A)$. Then clearly $\mathrm{cl}\, w_0(A)$ is pointed if and only if $\mathrm{cl}\, w_1(A)$ is pointed. If A is singular, then by the last part of Lemma 3.4(ii), $w_0(A)$ is the direct sum of the simplical cone $\mathrm{pos}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $w_{\nu_0(A)}(A)$. Hence, $\mathrm{cl}\, w_0(A)$ is also the direct sum of the simplicial cone $\mathrm{pos}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $\mathrm{cl}\, w_{\nu_0(A)}(A)$. One readily shows that the latter equality relation implies that $\mathrm{cl}\, w_0(A)$ is pointed if and only if $\mathrm{cl}\, w_{\nu_0}(A)$ is pointed. The proof is complete. ■

The complex version of Corollary 3.3 also holds, as the above proof can be carried over. Or, one may also deduce the complex version from the real version. The point is, we can treat an $n \times n$ complex matrix A as a real linear transformation acting on $\mathbb{C}^n$ (as a real vector space of dimension $2n$). As such, A is similar to the matrix $\operatorname{diag}(A, \bar{A})$, where $\bar{A}$ denotes the conjugate matrix of A (see [T–S 1, Lemma 8.1]). Hence, a complex matrix A satisfies the Perron-Schaefer condition if and only if A, treated as a real linear transformation, satisfies the Perron-Schaefer condition. Similarly, the complex version of Theorem 3.1 also holds.

Analogous to Theorem 3.1 we have the following result:

Theorem 3.5. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $x \neq 0 \in \mathbb{R}^n$. Let ν_0 be the order of the generalized eigenvector corresponding to* 0 *that appears in the representation of x as a sum of generalized eigenvectors of A. (If such vector does not exist, we take ν_0 to be* 0.) *Then the following conditions are equivalent:*

(a1) *In the representation of x as a sum of generalized eigenvectors of A, there is a generalized eigenvector corresponding to a positive eigenvalue.*

(a2) *The linear map $A|_{W_x}$ has a positive eigenvalue.*

(b1) *For every integer $k \geq \nu_0$, the convex cone $w_k(A, x)$ is nonzero, pointed.*

(b2) *For some integer $k \geq \nu_0$, the convex cone $w_k(A, x)$ is nonzero, pointed.*

(c1) *For every integer $k \geq \nu_0$, the convex cone $w_k(A, x)$ is not a vector subspace of $\mathbb{R}^n$.*

(c2) *For some integer $k \geq \nu_0$, the convex cone $w_k(A, x)$ is not a vector subspace of $\mathbb{R}^n$.*

(d) *For every nonnegative integer k, the convex cone $w_k(A, x)$ is nonzero, pointed.*

Proof. The equivalence of (a1) and (a2) follows from the connection between the representation of x as a sum of generalized eigenvectors of A and the Jordan form of $A|_{W_x}$ as mentioned at the end of Section 2. Since the restriction of A to span $\{A^i x : i = \nu_0, \nu_0 + 1, \ldots\}$ is nonsingular and $w_k(A, x) = A^{k-\nu_0} w_{\nu_0}(A, x)$ for all positive integers $k > \nu_0$, the equivalence of (b1) and (b2) and also of (c1) and (c2) are clear.

(a) $\Longrightarrow$ (d): Condition (a) clearly implies that x is not a generalized eigenvector of A corresponding to 0. So, for every nonnegative integer k, $w_k(A, x)$ is a nonzero cone. Since $w_0(A, x) \supseteq w_k(A, x)$ for all positive integers k, it remains to show that the cone $w_0(A, x)$ is pointed. Assume that the contrary holds. Then there exist nonzero polynomials $p(t), q(t)$ with nonnegative coefficients such that the vectors $p(A)x, q(A)x$ are both nonzero and are negative to each other. Then we also have $p(A)A^i x = -q(A)A^i x$ for every nonnegative integer i, and hence $p(A)y = -q(A)y$ for every vector $y \in W_x$. By condition (a) $A|_{W_x}$ has a positive eigenvalue, say α. Choose y to be an eigenvector of $A|_{W_x}$ corresponding to α. Then we have $p(\alpha)y = -q(\alpha)y$, and hence $p(\alpha) + q(\alpha) = 0$. But $p(t) + q(t)$ is a nonzero polynomial with nonnegative coefficients, so we arrive at a contradiction.

Clearly we have the implications (d) $\Longrightarrow$ (b1) and (b1) $\Longrightarrow$ (c1).

(c2) $\Longrightarrow$ (a): Suppose there is an integer $k \geq \nu_0$ such that the set $w_k(A, x)$ is not a real vector subspace. If $A|_{W_x}$ is nilpotent, then $w_k(A, x)$ would be the zero subspace, which is a contradiction. So $A|_{W_x}$ is non-nilpotent.

Assume to the contrary that condition (a) does not hold. Then $A|_{W_x}$ can be represented by a matrix of the form $B \oplus N$, where B is a nonsingular matrix, each of whose eigenvalues is either complex non-real or negative real, and N is a nilpotent matrix with index of nilpotency ν_0. Note that the summand B must exist, as $A|_{W_x}$ is non-nilpotent. However, the summand N may not exist. Since B does not have a nonnegative eigenvalue, we can find a polynomial $u(t)$ of positive degree and with nonnegative coefficients such that $u(0) = 1$ and $u(\lambda) = 0$ for all eigenvalues λ of B (see [Sch 2, Lemma 4.2] or [R–R, Lemma 4]). Then $u(B)$ is a nilpotent matrix and we can find a positive integer p such that $u(B)^p = 0$. Let $v(t)$ to be the polynomial $u(t)^p$. Then $v(t)$ is a polynomial of positive degree and with nonnegative coefficients such that $v(0) = 1$ and $v(B) = 0$. Now it should be clear that we have $(A|_{W_x})^{\nu_0} v(A|_{W_x}) = 0$ and so $A^{\nu_0} v(A)x = 0$. The latter condition, in turn, implies that for every integer $l \geq k (\geq \nu_0)$, we have $A^l v(A)x = 0$, and hence that $-A^l x \in w_{l+1}(A, x) \subseteq w_k(A, x)$, as $v(0) = 1$ and $v(t)$ has nonnegative coefficients, at least two of which are positive. Since the negative of each vector in the spanning set $\{A^l x : l \geq k\}$ for $w_k(A, x)$ belongs to $w_k(A, x)$, it follows that $w_k(A, x)$ is a real vector subspace, which contradicts our assumption on k.

The proof is complete. ■

Remark 3.6. To the list of equivalent conditions in Theorem 3.5, we can add the following:

(b)$_j$ For every integer $k \geq j$, the convex cone $w_k(A, x)$is nonzero, pointed (where j stands for any nonnegative integer).

(e1) For every integer $k \geq \nu_0$, the convex cone $w_k(A|_{W_x})$ is nonzero, pointed.

(e2) For some integer $k \geq \nu_0$, the convex cone $w_k(A|_{W_x})$ is nonzero, pointed.

(f1) For every integer $k \geq \nu_0$, the convex cone $w_k(A|_{W_x})$ is not a vector subspace of $L(W_x)$.

(f2) For some integer $k \geq \nu_0$, the convex cone $w_k(A|_{W_x})$ is not a vector subspace of $L(W_x)$.

(In (f1) and (f2) we use $L(W_x)$ to denote the (real) vector space of linear mappings on W_x)

That (b)$_j$ is another equivalent condition clearly follows from the equivalence of conditions (b1) and (d) of Theorem 3.5. To show that conditions (e1) and (e2) are additional equivalent conditons, we can apply the following two easily proved facts: For every nonnegative integer k, we have

1. $W_k(A|_{W_x})$ is nonzero if and only if $w_k(A, x)$ is nonzero;

2. $W_k(A|_{W_x})$ is pointed if and only if $w_k(A, x)$ is pointed.

As for conditions (f1) and (f2), we use the assertion that for every nonnegative integer k, $w_k(A, x)$ is a vector subspace of $\mathbb{R}^n$ if and only if $w_k(A|_{W_x})$ is a vector subspace of $L(W_x)$. Noting first that a convex cone C is a real vector subspace if and only if $-C \subseteq C$, we infer the above assertion by the following argument:

$w_k(A,x)$ is a vector subspace of $\mathbb{R}^n \iff -A^k x \in w_k(A,x) \iff -A^k|_{W_x} \in w_k(A|_{W_x}) \iff w_k(A|_{W_x})$ is a vector subspace of $L(W_x)$.

Note, however, that the condition "$w_0(A,x)$ is a nonzero, pointed cone" does not imply the equivalent conditions of Theorem 3.5. This is because, when x is a generalized eigenvector of A corresponding to 0, $w_0(A,x)$ is a simplicial cone, whereas $w_k(A,x)$ is the zero cone for all positive integers $k \geq \nu_0$.

As a corollary of Theorem 3.5 we obtain the following known result ([Sch 2, Theorem 1.6 and Theorem 6.4]):

Corollary 3.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *A has a positive eigenvalue.*

(b) *For every (or, for some) integer $k \geq \nu_0(A)$, the cone $w_k(A)$ is nonzero, pointed.*

(c) *For every (or, for some) integer $k \geq \nu_0(A)$, the set $w_k(A)$ is not a real vector subspace of $\mathcal{M}_n(\mathbb{C})$.*

If, in addition, A is non-nilpotent, the following is also an equivalent condition:

(d) *For every (or, for some) nonnegative integer k, the cone $w_k(A)$ is nonzero, pointed.*

Proof. Denote by L_A the linear operator on $\mathcal{M}_n(\mathbb{C})$ defined by: $L_A(X) = AX$. By Lemma 3.4, A and $L_A|_{W_A}$ the same nonzero eigenvalues; so it is clear that condition (a) is equivalent to that $L_A|_{W_A}$ has a positive eigenvalue. Applying Theorem 3.5 to L_A at A, we see that the latter condition is equivalent to the following:

(b)$'$ For every (or, for some) integer $k \geq \nu_0(L_A|_{W_A})$, $W_k(L_A, A)$ is a nonzero pointed cone.

Note that we always have $w_k(L_A|_{W_A}, A) = w_{k+1}(A)$. If A is singular, then by Lemma 3.4 again we have $\nu_0(L_A|_{W_A}) = \nu_0(A) - 1$, and so condition (b)' is equivalent to condition (b). If A is nonsingular, then so is L_A and hence $L_A|_{W_A}$. Then condition (b)' becomes: $w_k(A)$ is a nonzero pointed cone for every (or, for some) positive integer k. But, in this case, certainly, $w_0(A)$ is also nonzero and pointed, being linearly isomorphic with $w_k(A)$ for any positive integer k. So, in fact, we have condition (b). In the above, we have established the equivalence of conditions (a) and (b). In a similar way, we can also show that (a) and (c) are equivalent.

When A is non-nilpotent, $w_{\nu_0}(A,x)$ is always nonzero, and in view of Remark 3.2, it is readily seen that condition (d) is another equivalent condition. ∎

Condition (c) of Corollary 3.7 can be replaced by the following (which is the form it appears in [Sch 2, Theorem 1.6]):

(c)$'$ For every (or, for some) integer $k \geq \nu_0(A)$, the set $\operatorname{cl} w_k(A)$ is not a real subspace of $\mathcal{M}_n(\mathbb{C})$.

The reason is, a convex set C in a finite dimensional vector space is a real subspace if and only if its closure is a real subspace. The latter is a consequence of the fact that for a convex set C in a finite-dimensional vector space we have $\operatorname{relint}(\operatorname{cl} C) = \operatorname{relint} C$.

Again it is not difficult to see that the real version of Theorem 3.7 is also valid.

Theorem 3.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$, and let $0 \neq x \in \mathbb{C}^n$. The following conditions are equivalent*:

(a) *In the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvector corresponding to a nonnegative eigenvalue.*

(b) *The cone $w_0(A, x)$ is not a real subspace.*

Proof. (a) $\Longrightarrow$ (b): If $w_0(A, x)$ is a real subspace, then we must have $-x \in w_0(A, x)$ and hence $-x = p(A)x$ for some polynomial $p(t)$ with nonnegative coefficients. Then clearly $-A^j x = p(A)A^j x$ for all nonnegative integers j and hence $-Ay = p(A)y$ for all vectors $y \in W_x$. By condition (a) $A|_{W_x}$ has a nonnegative eigenvalue, say, α. Choose y to be a corresponding eigenvector. They we have $-y = p(A)y = p(\alpha)y$ and so $-1 = p(\alpha)$, which is a contradiction.

(b) $\Longrightarrow$ (a): Suppose that condition (a) does not hold. Then $A|_{W_x}$ is nonsingular and each of its eigenvalues is either complex non-real or negative real. Then we can find a polynomial $v(t)$ of positive degree m and with nonnegative coefficients such that $v(0) = 1$ and $v(A|W_x) = 0$. Then we have $v(A)x = 0$, from which we readily obtain $-x \in \operatorname{pos}\{Ax, \ldots, A^m x\}$, and hence $-A^j x \in w_0(A, x)$ for each nonnegative integer j. It follows that $w_0(A, x)$ is a real subspace. ■

In view of Lemma 3.4, an application of Theorem 3.8 (with A and x replaced by L_A and A respectively) yields the following:

Corollary 3.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *Either A has a positive eigenvalue, or 0 is an eigenvalue of A with index ≥ 2.*

(b) *$w_1(A)$ is not a real subspace.*

The following result has appeared in [Sch 2, Theorem 6.3]. We give another proof here.

Theorem 3.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *A has a nonnegative eigenvalue.*

(b) *$w_0(A)$ is not a real subspace.*

Proof (a) $\Longrightarrow$ (b): Suppose that condition (a) holds. If A is nonsingular, then necessarily A has a positive eigenvalue and by Corollary 3.7, (a) $\Longrightarrow$ (c), condition (b) follows. If A is singular then, by part(ii) of Lemma 3.4, $w_0(A)$ is the direct sum of the cones

$\mathrm{pos}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $w_{\nu_0(A)}(A)$. But the former cone is pointed, so $w_0(A)$ cannot be a real subspace.

(b) $\Longrightarrow$ (a): If A is singular, then 0 is an eigenvalue of A and condition (a) clearly holds. If A is nonsingular, by Corollary 3.7, (c)$\Longrightarrow$(a), A has a positive eigenvalue. ∎

4. Local Perron-Frobenius type theorems for cross-positive matrices

The following result is obtained by Elsner ([Els 1], Satz 4.1):

Theorem. *For an $n \times n$ real matrix A, let $\xi(A)$ denote the spectral abscissa of A. The following conditions are equivalent:*

(a) *$\xi(A) \in \sigma(A)$ and $\nu_{\xi(A)}(A) \geq \nu_\lambda(A)$ for all $\lambda \in \sigma(A)$ with $\Re\lambda = \xi(A)$.*

(b) *There exists a proper cone K in $\mathbb{R}^n$ such that A is exponentially K-nonnegative.*

We refer to condition (a) of the preceding theorem as the ESV-condition.

We say $A \in \mathcal{M}_n(\mathbb{R})$ satisfies the *local ESV condition* at x if in the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvectors y corresponding to $\xi_x(A) :=$ the spectral abscissa of $A|_{W_x}$ and moreover the order of y is not less than that of any generalized eigenvector in the representation that corresponds to an eigenvalue with real part equal to $\xi_x(A)$.

Theorem. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $O \neq x \in \mathbb{R}^n$. The following conditions are equivalent:*

(a) *A satisfies the local ESV condition.*

(b) *$A|_{W_x}$ satisfies the ESV condition.*

(c) *The cone $\mathrm{cl}(pos\{e^{tA}x : t \geq 0\})$ is pointed.*

(d) *There exists a closed, pointed cone C containing x full in W_x such that $A|_{W_x}$ is cross-positive on C.*

(e) *There exists a closed, pointed cone C full in W_x such that $A|_{W_x}$ is cross-positive on C.*

Proof. The proof of the equivalence of (a) and (b) is similar to that for the equivalence of conditions (a) and (b) in Theorem ?.

The implication (e) $\longrightarrow$ (b) follows from a known property of a cross-positive matrix, whereas the implication (d) $\longrightarrow$ (e) is obvious. The implication (a) $\longrightarrow$ (d) follows from a modification of Elsner's construction as given in ...

(c) $\longrightarrow$ (d): Take C to be the cone $\mathrm{cl}(\mathrm{pos}\{e^{tA}x : t \geq 0\})$, and noting that the latter cone contains x, is full in W_x and invariant under e^{sA} for all nonnegative integers s.

(d) $\longrightarrow$ (c): Since A is cross-positive on C, we have $\exp(tA)C \subset C$ for all $t \geq 0$. But C also contains x, so the cone $\text{pos}\{e^{tA}x : t \geq 0\}$) and hence its closure, being included in C, is pointed. ■

[Q] The following condition (e)' is clearly stronger than condition (e) of the preceding theorem. Is it in fact another equivalent condition ? If not, is there a geometric characterization for it ? (e) There exists a closed, pointed cone C full in W_x such that $A|_{W_x} \in \pi_1(C)$.

Remark 4.1. Let K be a proper cone in $\mathbb{R}^n$ and let $A \in \mathcal{M}_n(\mathbb{R})$. Then $A \in \Sigma(K)$ if and only if $Ax \in \text{cl}\,\text{cone}(x, K)$ for all $x \in K$.

This follows from the definition of cross-positive matrices and the fact that for all $x \in K$, we have $d_K(\Phi(x))^* = \text{cone}(x, K)^*$ (see [Tam 1]).

Note that $K \subseteq \text{cone}(x, K)$, and also that $\text{cone}(x, K) \subseteq \text{cone}(y, K)$ whenever $\Phi(x) \subseteq \Phi(y)$. When $x \in \text{int}\, K, \text{cone}(x, K) = \mathbb{R}^n$.

5. Construction of closed, pointed invariant cones

In below we give some easily-proved properties on A-cyclic cones or their closures.

Lemma 5.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$, let $x, y \in \mathbb{R}^n$, and let k be any nonnegative integer. Then:*
(i) $w_k(A, x) = w_0(A, A^k x)$.
(ii) $y \in w_k(A, x)$ *if and only if* $w_0(A, y) \subseteq w_k(A, x)$.
(iii) $y \in \text{cl}\, w_k(A, x)$ *if and only if* $\text{cl}\, w_0(A, y) \subseteq \text{cl}\, w_k(A, x)$.

Lemma 5.2. *Let $A \in \mathcal{M}|_n(\mathbb{R})$, let $x \in \mathbb{R}^n$ and let k be any nonnegative integer. Suppose that A satisfies the local Perron-Schaefer condition at x. Then:*
(i) *The cone* $\text{cl}\, w_0(A, x)$ *is pointed.*
(ii) $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)} x \in \bigcap_{k=0}^{\infty} \text{cl}\, w_k(A, x)$.
(iii) $(\sum_j E_\lambda^{(\nu_{\rho_x(A)}-1)})x \in \bigcap_{k=0}^{\infty} \text{cl}\, w_k(A, x)$, *where the summation runs through all eigenvalues λ in the peripheral spectrum of $A|_{W_x}$ with the same index as that of $\rho_x(A)$.*
(iv) $core_{\text{cl}\, w_0(A,x)}(A|_{W_x}) = \bigcap_{k=0}^{\infty} \text{cl}\, w_k(A, x)$.
(v) *For any* $y \in w_0(A, x), \text{sp}_A(y) = \text{sp}_A(x)$.
(vi) *For any* $0 \neq y \in \text{cl}\, w_0(A, x), \rho_y(A) = \rho_x(A)$.
(vii) $\rho_x(A)$ *is the only distinguished eigenvalue of A for* $\text{cl}\, w_0(A, x)$ *and (up to multiples),* $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)} x$ *is the only eigenvector of A in* $\text{cl}\, w_0(A, x)$.

Proof. (i) By modifying an argument given in [Tam 4, proof of Theorem 3.1], we have the following one-line-proof for the fact that the eigenvector $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}x$ of A belongs to $\mathrm{cl}\, w_k(A,x)$ for every nonnegative integer k: Denoting $\rho_x(A)$ by ρ and $\nu_{\rho_x(A)}(A)$ by ν, we have

$$E_\rho^{(\nu-1)}(x) = \lim_{m\to\infty} [(\nu-1)!]m^{-(\nu-1)}\rho^{-k}(\rho+1)^{-(m-\nu+1)}(A+I)^m A^k x,$$

and hence $E_\rho^{(\nu-1)}x \in \mathrm{cl}\, w_k(A,x)$.

Part(iii) can be proved by modifying an argument given in [T–S1, proof of Theorem 7.1(i)].

(iv) Denote $\nu_0(A|_{W_x})$ by ν_0. Making use of Remark 3.2 and the fact that the restriction map $A|_{W_x}$ is nonsingular on $\mathrm{span}\, w_{\nu_0}(A,x)$, we have the following:

$$\begin{aligned}
\$core_{\mathrm{cl}\, w_0(A,x)}(A|_{W_x}) &= \bigcap_{i=\nu_0}^{\infty} A^i \,\mathrm{cl}\, w_0(A,x) \\
&= \bigcap_{j=0}^{\infty} A^{\nu_0+j} \,\mathrm{cl}\, w_{\nu_0}(A,x) \\
&= \bigcap_{j=0}^{\infty} \mathrm{cl}\, A^{\nu_0+j} w_{\nu_0}(A,x) \\
&= \bigcap_{i=2\nu_0+j}^{\infty} \mathrm{cl}\, w_i(A,x) \\
&= \bigcap_{i=0}^{\infty} \mathrm{cl}\, w_i(A,x) \qquad (5.1)
\end{aligned}$$

(v) Obvious.

(vi) Consider any $y \in \mathrm{cl}\, w_0(A,x)$ for which $\rho_y(A) < \rho_x(A)$. We can find polynomials $p_m(t)$ all with nonnegative coefficients such that $\lim_{m\to\infty} p_m(A)x = y$. We make use of the argument given in the proof for part(iii). Since $\rho_y(A) < \rho_x(A)$, necessarily we have $\lim_{m\to\infty} p_m(\rho(A)) = 0$, which implies $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $i = 1, \ldots, k$ and $j = 0, \ldots, \nu_j - 1$, and hence $y = 0$.

(vii) By part(vi) and the fact that each distinguished eigenvalues of $A|_{W_x}$ for $\mathrm{cl}\, w_0(A,x)$ is equal to $\rho_y(A)$ for some nonzero $y \in \mathrm{cl}\, w_0(A,x)$ (see, for instance, [Tam 2, Theorem 2.4]), $\rho_x(A)$ is clearly the only distinguished eigenvalue of $A|_{W_x}$. But $A|_{W_x}$ is a cyclic operator, so each of its eigenvalue, and in particular $\rho_x(A)$, is of geometric multiplicity one. Hence, by part(i) our assertio follows. ■

Theorem 5.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Let $x, y \in \mathbb{R}^n$. The following conditions are equivalent:*

(a) *There exists a proper cone K in $\mathbb{R}^n$ containing both x and y such that $A \in \pi(K)$.*

(b) *There exists a closed pointed cone C containing both x and y such that $AC \subseteq C$.*

(c) *A satisfies the local Perron-Schaefer conditions at x and at y, and we have $\operatorname{cl} w_0(A,x) \cap (-\operatorname{cl} w_0(A,y)) = \{0\}$.*

(d) *A satisfies the local Perron-Schaefer conditions at x and at y, and the vectors $E_{\rho_x}^{\nu(\rho_x-1)}x$ and $E_{\rho y}^{\nu(\rho_y-1)}y$ are not positive multiples of each other.*

Proof. First, it is easy to show that for any pointed cones C_1, C_2 in the same real vector space, the cone $C_1 + C_2$ is pointed if and only if $C_1 \cap (-C_2) = \{0\}$.

(a)$\Longrightarrow$(b): Obvious.

(b)$\Longrightarrow$(c): Clearly A satisfies the local Perron-Schaefer conditions at x and at y. Since $\operatorname{cl} w_0(A,x) \subseteq C$, $\operatorname{cl} w_0(A,y) \subseteq C$and C is pointed, it is clear that we have $\operatorname{cl} w_0(A,x) \cap \operatorname{cl}(-w_0(A,y)) = \{0\}$.

(c)$\Longrightarrow$(b): First of all, the cones $\operatorname{cl} w_0(A,x)$ and $\operatorname{cl} w_0(A,y)$ are both pointed, because A satisfies the local Perron-Schaefer conditions at x and at y. As $\operatorname{cl} w_0(A,x) \cap \operatorname{cl}(-w_0(A,y)) = \{0\}$, by our beginning remark, the cone $\operatorname{cl} w_0(A,x) + \operatorname{cl} w_0(A,y)$ is also pointed. But by [Roc,Corollary 9.1.2] the latter cone is also closed, so it is the desired closed pointed A-invariant cone that contains both x and y.

(c)$\Longrightarrow$(d): By Lemma 5.2(ii) $E_{\rho_x}^{(\nu_{\rho_x}-1)}x \in \operatorname{cl} w_0(A,x)$ and $E_{\rho_y}^{(\nu_{\rho_y}-1)}y \in \operatorname{cl} w_0(A,y)$. If these vectors are negative multiples of each other, then $E_{\rho_x}^{(\nu_{\rho_x}-1)}x$ is a nonzero vector of $\operatorname{cl} w_0(A,x) \cap (-\operatorname{cl} w_0(A,y))$, which is a contradiction.

(d)$\Longrightarrow$(c): If $\operatorname{cl} w_0(A,x)$ and $-\operatorname{cl} -w_0(A,y)$ have a common nonzero vector, say, z, then by Lemma 5.1(iii) their intersection would include the nonzero pointed A-invariant cone $\operatorname{cl} w(A,z)$. But then by Lemma 5.2(iii)the vector $E_{\rho_z}^{\nu(\rho_z-1)}z$, which is the unique (up to multiples) eigenvector of A in $\operatorname{cl} w_0(A,z)$, must be a positive multiple of $E_{\rho_x}^{\nu(\rho_x-1)}x$, which is the unique eigenvector of A in $\operatorname{cl} w_0(A,x)$. Similarly,$E_{\rho_z}^{(\nu_{\rho_z}-1)}z$ and $-E_{\rho_y}^{(\nu_{\rho_y}-1)}y$ are also positive multiples of each other. Hence, $E_{\rho_x}^{(\nu_{\rho_x}-1)}x$ and $E_{\rho_y}^{(\nu_{\rho_y}-1)}y$ are negative multiples of each other, which is a contradiction. ■

The following condition is necessary but not sufficient for the equivalent conditions of Theorem 5.3. However, it is a sufficient condition if, in addition, we also have $\operatorname{sp}_A(x) = \operatorname{sp}_A(y)$.

(e) A satisfies the local Perron-Schaefer conditions at $\alpha x+\beta y$ for every pair of nonnegative real numbers α, β, not both zero, , and $\operatorname{sp}_A(\alpha x + \beta y) = \max\{\operatorname{sp}_A(x), \operatorname{sp}_A(y)\}$ for every pair of positive real numbers α, β, where the maximum is taken in the sense of lexicographic ordering.

The point is, when condition (a)of Theorem 5.3 holds, we have $A \in \pi(K)$ and $x, y \in K$. So clearly A satisfies the local Perron-Schaefer condition at x and at y, and by the basic properties for spectral pairs of vectors (see [T–S, Theorem 4.9]), the second half of condition (e) also holds. On the other hand, note that for any vectors x, y, we have, $\mathrm{sp}_A(\alpha x + \beta y) = \max\{\mathrm{sp}_A(x), \mathrm{sp}_A(y)\}$ whenever $\mathrm{sp}_A(x) \neq \mathrm{sp}_A(y)$. So, in particular, if A satisfies the local Perron-Schaefer conditions at x and at y and if we also have $\rho_x(A) = \rho_y(A)$ and $\nu_{\rho_x(A)} > \nu_{\rho_y(A)}$, then condition (e) is met. However, in this case condition (d) is not necessarily satisfied, as we may have $E_{\rho_x}^{(\nu_{\rho_x}-1)}x = -E_{\rho_y}^{(\nu_{\rho_y}-1)}y$. Now, suppose, in addition that $\mathrm{sp}_A(x) = \mathrm{sp}_A(y)$. Assume that $E_{\rho_x}^{(\nu_{\rho_x}-1)}x = -\gamma E_{\rho_y}^{(\nu_{\rho_y}-1)}y$ for some $\gamma > 0$. Then $x + \gamma y$ is a nonzero vector (as $\mathrm{sp}_A(x+\gamma y) = \mathrm{sp}_A(x) \neq (0,0)$) and in its representation as a sum of generalized eigenvectors of A, either there is no generalized eigenvector corresponding to $\rho_{x+\gamma y}(A) (= \rho_x(A))$, or if there is one such generalized eigenvector then it must be of order less than $\mathrm{ord}_A(x+\gamma y)$. So A does not satisfy the local Perron-Schaefer condition at $x + \gamma y$, which is a contradiction.

Lemma 5.4. *Let C_1, C_2 be closed, pointed cones in the same finite-dimensional real vector space. Then $C_1 + C_2$ is a closed, pointed cone if and only if $C_1 \cap (-C_2) = 0$.*

Lemma 5.5. *Let C_1, C_2 be closed, pointed cones in $\mathbb{R}^n$. The following conditions are equivalent:*

(a) $C_1 \cap (-C_2) = \{0\}$.

(b) $C_1 + C_2$ *is pointed.*

(c) $C_1 + C_2$ *is closed, pointed.*

Suppose, in addition, that C_1, C_2 are invariant under a given matrix $A \in \mathcal{M}_n(\mathbb{R})$. For $i = 1, 2$, denote by D_i the cone generated by the distinguished eigenvectors of A for K_i. Then the following are other equivalent conditions:

(d) $C_1 \cap (-C_2)$ *does not contain an eigenvector of A.*

(e) *The cone $D_1 + D_2$ is pointed.*

Lemma 5.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Let K_1, K_2 be closed, pointed cones in $\mathbb{R}^n$ invariant under A. If $K_1 + K_2$ is pointed, then for each distinguished eigenvalue λ of A for $K_1 + K_2$, we have*
$(K_1 + K_2) \cap \mathcal{N}(A - \lambda I) = (K_1 \cap \mathcal{N}(A - \lambda I)) + (K_1 \cap \mathcal{N}(A - \lambda I))$.

Proof. Let $x \in K_1 + K_2$ be an eigenvector of A corresponding to the eigenvalue λ. Then there exist $x_i \in K_i, i = 1, 2$ such that $x = x_1 + x_2$. For $i = 1, 2$, by considering the representations of x_i as a sum of generalized eigenvectors of A, we readily see that we have $x_i = u_i + v_i$, where u_i is the zero vector or is a generalized eigenvector of A corresponding to λ and v_i is the zero vector or is the sum of some generalized eigenvectors of A corresponding to eigenvalues other than λ, and moreover $v_2 = -v_1$. We contend that u_1, u_2 are each an eigenvector or the zero vector. Suppose not. Then necessarily u_1, u_2 are generalized eigenvectors of A corresponding to λ of the same order, say, ν, and moreover, $E_\lambda^{(\nu-1)}u_1 = -E_\lambda^{(\nu-1)}u_2$. But by [T-S 2, Corollary 4.8] $E_\lambda^{(\nu-1)}u_i \in K_i$, so the cone $K_1 + K_2$ is not pointed, which is a contradiction. This proves our contention and by [T-S 2, Corollary

4.8] again, we have $u_i \in (K_1 \cap \mathcal{N}(A - \lambda I))$ for $i = 1, 2$. This establishes the inclusion $(K_1 + K_2) \cap \mathcal{N}(A - \lambda I) \subset (K_1 \cap \mathcal{N}(A - \lambda I)) + (K_1 \cap \mathcal{N}(A - \lambda I))$ and hence also the equality as the reverse inclusion is obvious. ■

Corollary 5.7. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $x_1, \dots x_k, k \geq 2$ be vectors of $\mathbb{R}^n$ and suppose that A satisfies the local Perron-Schaefer condition at $x_1, \dots, x_k$. Then there exists a closed, pointed cone (or a proper cone) in $\mathbb{R}^n$ that contains $x_1, \dots, x_k$ and is invariant under A if and only if the cone $\operatorname{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)} x_i : i = 1, \dots, k\}$ is pointed.*

Lemma 5.8. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let C_1, C_2 be closed, pointed cones invariant under A. Then $C_1 + C_2$ is a closed, pointed cone invariant under A if and only if $C_1 \cap (-C_2)$ does not contain an eigenvector of A.*

Proof. "Only if" part: Since $C_1 + C_2$ is pointed, we must have $C_1 \cap (-C_2) = \{0\}$ and so $C_1 \cap (-C_2)$ cannot contain an eigenvector of A.

"If" part: If $C_1 \cap (-C_2)$ is nonzero, then it is a nonzero closed pointed cone invariant under A and by the Perron-Frobenius theorem it contains an eigenvector of A, which is a contradiction. So $C_1 \cap (-C_2) = \{0\}$ and by Lemma 5.4 $C_1 + C_2$ is a closed, pointed cone, which is clearly invariant under A. ■

Lemma 5.9. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $x \in \mathbb{R}^n$. Let C be a closed pointed A-invariant cone. If A satisfies the local Perron-Schaefer condition at x and $E_{\rho_x}^{(\nu_{\rho_x}-1)} x \notin -C$, then $\operatorname{cl} w_0(A, x) + C$ is a closed, pointed cone invariant under A.*

Proof. Since A satisfies the local Perron-Schaefer condition at x, by Lemma 5.2 $\operatorname{cl} w_0(A, x)$ is a closed pointed cone invariant under A and $E_{\rho_x}^{(\nu_{\rho_x}-1)} x$ is the only eigenvector of A in $\operatorname{cl} w_0(A, x)$. If $E_{\rho_x}^{(\nu_{\rho_x}-1)} x \notin -C$, then $\operatorname{cl} w_0(A, x) \cap (-C)$ contains no eigenvector of A and by Lemma 5.5 $\operatorname{cl} w_0(A, x) + C$R is a closed, pointed cone invariant under A. ■

Since $\operatorname{cl} w_0(A, x) \cap (-\operatorname{cl} w_0(A, y)) = \{0\}$, $\operatorname{cl} w_0(A, x) + \operatorname{cl} w_0(A, y)$ is a closed, pointed cone invariant under A, which clearly contains both x and y. If the cone is full in $\mathbb{R}^n$, then it is the desired proper cone. Otherwise, we can find a nonzero vector $z \in \mathbb{R}^n$ which lies outside $\operatorname{span}(\operatorname{cl} w_0(A, x) + \operatorname{cl} w_0(A, y)$ such that A satisfies the local Perron-Schaefer condition at z. Replacing z by $-z$ if necessary, we may assume that $-E_{\rho_z}^{(\nu_{\rho_x}-1)} z \notin \operatorname{cl} w_0(A, x) + \operatorname{cl} w_0(A, y)$. So $\operatorname{span}(\operatorname{cl} w_0(A, x) + \operatorname{cl} w_0(A, y) + \operatorname{cl} w_0(A, z)$ is a closed, pointed cone invariant under A. If the latter cone is full in $\mathbb{R}^n$, then we are again done. Otherwise, repeat the process. In this way, we obtain an increasing sequence of A-invariant closed, pointed cones, each containing both x and y such that the dimension of each cone in the sequence is at least 1 greater than that of the preceding cone. So it is clear that after a finite number of steps we obtain a proper cone invariant under A that contains both x and y.

Theorem 5.10. *Let $A \in \mathbf{M}_n(\mathbb{R})$ satisfy the Perron-Schaefer condition. Let m be the largest geometric multiplicity of the eigenvalues of A. Then there exists a proper cone K in $\mathbb{R}^n$ invariant under A which is the sum of the closures of m A-cyclic cones, but there does not exist an A-invariant proper cone which is the sum of the closures of less than m A-cyclic cones.*

Not every A-invariant proper cone can be written as the sum of the closures of finitely many A-cyclic cones. For instance, take A to be the identity matrix and consider a nonpolyhedral proper cone. But if infinite sum is allowed, then it is always possible.

$A \in \pi(K)$ if and only if $w_0(A) \in \pi(K)$.

A is eventually K-nonnegative if and only if there exists $k \in \mathbb{Z}_+$ such that $w_k(A) \subseteq \pi(K)$.

$w_0(A, x)$ is equal to the conical hull of the orbit of x under A.

What are the invariant faces, the core, and the spectral cone for $A|_{W_x}$ $(\in \pi(\operatorname{cl} w(A, x)))$?

Observation. *Suppose A satisfies the local Perron-Schaefer condition at x. Then for any vector $y \in \mathbb{R}^n$, $z \in (\operatorname{cl} w_k(A, x))^*$ if and only if $x \in (\operatorname{cl} w_k(A^T, z))^*$.*

Theorem 5.11. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$. The following conditions are necessary for the existence of a proper cone K in $\mathbb{R}^n$ such that $A, B \in \pi(K)$.*

(a) *The cone* $\operatorname{cl} \operatorname{pos}\{p(A, B) : p(r, s)$ *is a monomial in the noncommuting indeterminates* $r, s\}$ *is pointed.*

(b1) $\operatorname{cl} w_0(A) \cap (-\operatorname{cl} w_0(B)) = \{0\}$.

(b2) *Either $\rho(A) \neq \rho(B)$, or $\rho(A) = \rho(B) = \rho$, say, and ?*

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DIGRAPHS FOR CONE-PRESERVING MAPS REVISITED

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Abstract. Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$ and let $\pi(K)$ denote the proper cone in $\mathcal{M}_n(\mathbb{R})$ which consists of all real matrices A that satisfy $AK \subseteq K$. For any $A \in \pi(K)$, Barker and Tam have introduced four digraphs associated with A, one of which is the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ with vertex set $\mathcal{E}$ consisting of the extreme rays of K such that (E_1, E_2) is an arc if and only if $E_2 \subseteq \Phi(AE_1)$, where $\Phi(S)$ denotes the face of K generated by S. In this work it is shown that once the inclusion relation between the faces of K is given, the latter digraph determines the remaining three digraphs; moreover, when K is polyhedral, the latter digraph equals the usual digraph of B^T, for any nonnegative matrix B with maximum number of positive entries that satisfies $AP = PB$, where P is a minimal generating matrix for K. The relations between the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and the position of A in the cone $\pi(K)$ are studied. In particular, it is shown that for any $A, B \in \pi(K)$, the digraphs $(\mathcal{E}, \mathcal{P}(A, K))$ and $(\mathcal{E}, \mathcal{P}(B, K))$ are equal if and only if the collection of simple faces containing A and the corresponding collection for B are the same, where by a simple face of $\pi(K)$ we mean a face of the form $\pi_{F,G} := \{A \in \pi(K) : AF \subseteq G\}$ for some faces F, G of K. It is also proved that the K-irreducibility or K-primitivity of A is completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ together with a knowledge of whether $E_1 \vee \cdots \vee E_j = K$ for every given finite collection $\{E_1, \ldots, E_j\}$ of extreme rays of K. Some open questions are also posed.

Work to be completed

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

We assume a basic knowledge of cones. For references, see [Bar 2].

Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. Let $\pi(K)$ denote the set of all n-by-n real matrices A that satisfy $AK \subseteq K$. In [B–T] and [T–B], for any $A \in \pi(K)$, four directed graphs associated with A are introduced by Barker and Tam as follow: Let $\mathcal{E}$ denote the set of all extreme rays of K, and let $\mathcal{F}'$ denote the set of all nontrivial faces of K. If $F, G \in \mathcal{F}'$, we say there is a $\mathcal{P}$-arc from F to G if $\Phi(AF) \supseteq G$, and an $\mathcal{I}$-arc from F to G if $\Phi((I + A)F) \supseteq G$, where $\Phi(S)$ denotes the face of K generated by the subset S. Let $\mathcal{P}$ and $\mathcal{I}$ denote the set of $\mathcal{P}$-arcs and $\mathcal{I}$-arcs respectively. Then $(\mathcal{F}', \mathcal{P})$ (respectively $(\mathcal{F}', \mathcal{I})$) denotes the directed graph with vertex set $\mathcal{F}'$ and arc set $\mathcal{P}$ (respectively, $\mathcal{I}$). If necessary, we write $\mathcal{F}'(K)$, $\mathcal{P}(A)$, $\mathcal{P}(A, K)$ etc. to indicate the dependence on K, on A, and on A and K respectively. The digraphs $(\mathcal{E}, \mathcal{P})$ and $(\mathcal{E}, \mathcal{I})$ are defined in a similar manner. If K equals the nonnegative orthant $\mathbb{R}^n_+$, then $\mathcal{E} = \{\Phi(e_1), \ldots, \Phi(e_n)\}$, where $e_1, \ldots, e_n$ are the standard unit vectors of $\mathbb{R}^n$. In this case, $(\Phi(e_i), \Phi(e_j))$ is a $\mathcal{P}$-arc if and only if $a_{ji} > 0$; hence, we can identify $(\mathcal{E}, \mathcal{P})$ with $G(A^T)$, where $G(A)$ is the usual digraph associated with the square matrix A.

A matrix $A \in \pi(K)$ is said to be K-*irreducible* if A leaves no nontrivial face of K invariant, A is K-*positive* if $A(K \backslash \{0\}) \subseteq \text{int } K$ and is K-*primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$.

It is well-known (see, for instance, [B–R]) that for a nonnegative matrix A, A is irreducible if and only if its digraph $G(A)$ is strongly connected; A is primitive if and only if $G(A)$ is strongly connected and the greatest common divisor of the lengths of its circuits equals 1; and A is irreducible (respectively, primitive) if and only if A^T is irreducible (respectively, primitive). Since the digraph $(\mathcal{E}, \mathcal{P})$ becomes $G(A^T)$ in the nonnegative matrix case, one may expect that the K-irreducibility or K-primitivity of a cone-preserving map A is determined by the digraph $(\mathcal{E}, \mathcal{P})$. However, this is not true in general. In fact, in [B–T] it is proved that, for any $A \in \pi(K)$, the K-irreducibility (respectively, K-primitivity) of A is equivalent to the strong connectedness of the digraph $(\mathcal{F}', \mathcal{I})$ (respectively, $(\mathcal{F}', \mathcal{P})$). The following diagram summarizes the connections between the strong connectedness of the four digraphs:

$$\begin{array}{ccc}
(\mathcal{E}, \mathcal{P}) \text{ is strongly connected} & & (\mathcal{F}', \mathcal{P}) \text{ is strongly connected} \\
\Downarrow & & \Downarrow \\
(\mathcal{E}, \mathcal{I}) \text{ is strongly connected} & \Longrightarrow & (\mathcal{F}', \mathcal{I}) \text{ is strongly connected}
\end{array}$$

Indeed, two of the above implications are trivial; they follow from the fact that $(\mathcal{E}, \mathcal{P})$

(respectively, $(\mathcal{F}', \mathcal{P})$) is a spanning subdigraph of $(\mathcal{E}, \mathcal{I})$ (respectively, $(\mathcal{F}', \mathcal{I})$). (However, the proof of the remaining implication is nontrivial.)

In the subsequent paper [T–B], it is proved that, loosely speaking, the phenomenon of irreducibility of operators being determined by the extreme rays alone is characteristic of simplicial cones (i.e., polyhedral cones whose number of extreme rays equal the dimension of their linear spans). More specifically, the following is obtained:

Theorem A. *K is simplicial if for any $A \in \pi(K)$, $(\mathcal{E}, \mathcal{P}(A))$ is strongly connected whenever A is K-irreducible.*

Theorem A amounts to saying that if K is non-simplicial, then we can always find a K-irreducible matrix A for which the graph $(\mathcal{E}, \mathcal{I}(A))$ is not strongly connected. Unfortunately, there is a gap in the proof, which to date is still not fixed.

Similarly, the following result is also given in [T–B]:

Theorem B. *Suppose that K is a 2-neighborly proper cone. Then K is simplicial if, for any $A \in \pi(K)$, $(\mathcal{E}, \mathcal{I}(A))$ is strongly connected whenever A is K-irreducible.*

Here we call a proper cone K 2-*neighborly* if $x_1 + x_2 \in \partial K$ for any two extreme vectors $x_1, x_2 \in K$. Actually, Theorem B can be strengthened slightly by replacing "K-irreducible" by "K-primitive"; the proof is just a minor modification of the original proof for Theorem B as given in [T–B].

As is well-known, if K is a proper cone [respectively, polyhedral (proper) cone] in $\mathbb{R}^n$, then $\pi(K)$ is a proper cone [respectively, polyhedral cone] in the space of n-by-n real matrices (see [S–V] or [Tam 3]).

It is clear that each of the following are equivalent conditions on two n-by-n nonnegative matrices A, B: (a) A, B have the same zero pattern; (b) A, B have the same digraph; and (c) A, B generate the same face in the cone of n-by-n nonnegative matrices. When the equivalent conditions are satisfied and one of A, B (and hence both) is primitive, the exponents of A and B are also equal.

In [Niu] Niu obtained the following corresponding result for matrices preserving a polyhedral cone:

Theorem C. *Let K be a polyhedral cone, and let $A, B \in \pi(K)$.*
(i) $(\mathcal{E}, \mathcal{P}(A)) = (\mathcal{E}, \mathcal{P}(B))$ *if and only if* $\Phi(A) = \Phi(B)$.

(ii) *If* $(\mathcal{E}, \mathcal{P}(A))$ *is a subdigraph of* $(\mathcal{E}, \mathcal{P}(B))$ *and if* A *is* K*-primitive, then so is* B *and we have* $\gamma(B) \leq \gamma(A)$.

(iii) *If* $\Phi(A) = \Phi(B)$, *then* A *and* B *are both* K*-primitive or both not* K*-primitive, and if they are, then* $\gamma(A) = \gamma(B)$.

When K is a general nonpolyhedral cone, Theorem C no longer holds. Then the situation is more complicated. The nonpolyhedral case is different from the polyhedral case in that the faces of a nonpolyhedral cone may or may not be exposed. We call a face F of a proper cone K *exposed* if it equals K or is the intersection of K with a supporting hypersubspace; or equivalently, $F = \mathrm{cl}_K(F)$, where cl_K denotes the composite map $d_{K^*} \circ d_K$, with K^* being the dual cone of K and d_K being the duality operator of K given by $d_K(F) = K^* \cap (\mathrm{span}\, F)^\perp$. For the necessary background knowledge of the duality operators (of K and of $\pi(K)$), we refer the reader to [Tam 2,3].

For a proper cone K and any $A \in \pi(K)$, we have the following three natural faces of $\pi(K)$ that contain A, arranged in ascending order with respect to the inclusion relation: $\Phi(A), \{B \in \pi(K) : Bx \in \Phi(Ax) \text{ for all } x \in K\}$, and $\mathrm{cl}_{\pi(\mathrm{K})}(\Phi(\mathrm{A}))$ (the smallest exposed face that contans A). The middle face is also "natural" in that it is equal to the intersection of all simple faces of $\pi(K)$ that contains A ([Tam 3, Theorem 4.6]). By a *simple face* of $\pi(K)$ we mean a face of the form $\pi_{F,G}$ for some faces F, G of K, where $\pi_{F,G} = \{A \in \pi(K) : AF \subseteq G\}$ (see [Tam 3]).

It turns out that, when K is a general proper cone, for any $A, B \in \pi(K)$, the digraphs $(\mathcal{E}, \mathcal{P}(A))$ and $(\mathcal{E}, \mathcal{P}(B))$ are equal if and only if the intersection of simple faces containing A is equal to the intersection of simple faces containing B. More specifically, we have the following results which extend and refine part(i) of Theorem C.

We always use $\mathrm{Ext} K$ to denote the set of nonzero extreme vectors of K.

Theorem 1. *Let* K *be a proper cone, and let* $A, B \in \pi(K)$. *Consider the following conditions*:

(a) $\Phi(A) \subseteq \Phi(B)$.

(b) $(\mathcal{E}, \mathcal{P}(A))$ *is a subdigraph of* $(\mathcal{E}, \mathcal{P}(B))$.

(c) *For all* $x \in \mathrm{Ext}\, K$, $\Phi(Ax) \subseteq \Phi(Bx)$.

(d) *For all* $x \in K$, $\Phi(Ax) \subseteq \Phi(Bx)$.

(e) *A belongs to the intersection of all simple faces of* $\pi(K)$ *that contain* B.

(f) $(\mathcal{F}', \mathcal{P}(A))$ *is a subdigraph of* $(\mathcal{F}', \mathcal{P}(B))$.

(g) $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$.

Conditions (b)–(f) *are equivalent and they always imply condition* (g) *and are implied by condition* (a).

By Theorem 1 we immediately obtain

Corollary 1. *Let K be a proper cone, and let $A, B \in \pi(K)$. Consider the following conditions*:

(a) $\Phi(A) = \Phi(B)$

(b) $(\mathcal{E}, \mathcal{P}(A)) = (\mathcal{E}, \mathcal{P}(B))$.

(c) *For all $x \in \mathrm{Ext}\, K$, $\Phi(Ax) = \Phi(Bx)$.*

(d) *For all $x \in K$, $\Phi(Ax) = \Phi(Bx)$.*

(e) *Simple faces of $\pi(K)$ that contain A and those that contain B are the same.*

(f) $(\mathcal{F}', \mathcal{P}(A)) = (\mathcal{F}', \mathcal{P}(B))$.

(g) $\mathrm{cl}_{\pi(K)}(\Phi(A)) = \mathrm{cl}_{\pi(K)}(\Phi(B))$.

Conditions (b)–(f) *are equivalent and they always imply condition* (g) *and are implied by condition* (a).

Among the four digraphs associated with A in $\pi(K)$, the one we would like most to work with is the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, because it has the fewest vertices and arcs and also it reduces to the usual digraph (but with the direction of arcs reversed) in the nonnegative matrix case. The equivalence of conditions (b) and (f) of Corollary 1 suggests that it might be possible to capture the other three digraphs from $(\mathcal{E}, \mathcal{P}(A, K))$ This is indeed true, and in fact we can say more.

Theorem 2. *Suppose the digraph $(\mathcal{F}', \mathcal{P}(I, K))$, or equivalently the inclusion relation between the faces of K, is given. Then from the digraph $(\mathcal{F}', \mathcal{P}(A, K))$ one can determine the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and conversely. When either one of the two aforementioned digraphs is known, one can also determine the digraphs $(\mathcal{E}, \mathcal{I}(A, K))$ and $(\mathcal{F}', \mathcal{I}(A, K))$, but not conversely.*

As for parts(ii) and (iii) of Theorem C we have the following extension and refinement to the setting of a proper cone, which essentially says that for any $A \in \pi(K)$, whether A is K-primitive or not and, when it is what the value of $\gamma(A)$ is, is completely determined by the smallest exposed face of $\pi(K)$ that contains A:

Theorem 3. *Let K be a proper cone, and let $A, B \in \pi(K)$.*

(i) *If* $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$ *and A is K-primitive, then B is also K-primitive and we have $\gamma(B) \leq \gamma(A)$.*

(ii) *If* $\mathrm{cl}_{\pi(K)}(\Phi(A)) = \mathrm{cl}_{\pi(K)}(\Phi(B))$*, then A and B are both K-primitive or both not K-primitive, and if they both are, then $\gamma(A) = \gamma(B)$.*

[Mention the face-semiring ?]

Note that a sufficient condition for the three faces of $\pi(K)$ containing A to be equal is that the duality operator $d_{\pi(K)}$ of $\pi(K)$ is injective, which always holds if K is polyhedral. The relation between the injectivity or surjectivity of the duality operator of K and that of $\pi(K)$ is not fully understood. In [Tam 3] a few open questions were posed and to date they remain unsolved. Now to the known equivalent conditions for the injectivity of $d_{\pi(K)}$ we can add two more. We have the following:

Theorem 4. *For a proper cone K, the following conditions are equivalent:*

(a) $d_{\pi(K)}$ *is injective.*

(b) *For any $A, B \in \pi(K)$, conditions* (a)-(g) *of Theorem 1 are equivalent.*

(c) *For any $A, B \in \pi(K)$, conditions* (a)-(g) *of Corollary 1 are equivalent.*

(d) *d_K is injective (or bijective), and each face of $\pi(K)$ can be written as an intersection of simple faces.*

(e) *Each face of $\pi(K)$ other than $\pi(K)$ itself can be written as an intersection of maximal faces.*

For any two square (not necessarily nonnegative) matrices A, B of the same size, it is clear that A and B have the same (usual) digraph if and only if so do A^T and B^T. The same is not true for cone-preserving maps in general (see Example 4). What we have is the following:

Theorem 5. *Let K be a proper cone. In order that for any $A, B \in \pi(K)$, we have*

$$(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K)) \quad \textit{iff} \quad (\mathcal{E}, \mathcal{P}(A^T, K^*)) = (\mathcal{E}, \mathcal{P}(B^T, K^*))$$

it is necessary and sufficient that the duality operator d_K be bijective (which is the case if K is polyhedral).

In terms of the digraph $(\mathcal{E}, \mathcal{P})$ we can characterize the K-irreducibility and the K-primitivity of A.

For simplicity, we call a strongly connected component of a digraph a *strong component*. We call a strong component *final* if there is no arc that issues from the strong component and enters another strong component. Given two vertices x, y of a digraph, we say vertex x *has access to vertex* y (or, y *has access from* x) if either $x = y$ or there is a directed path in the digraph from x to y. Accessibility relation between a vertex and a subset of vertices of a digraph can be defined in a similar way.

We have the following results:

Theorem 6. *Let* K *be a proper cone and let* $A \in \pi(K)$. *The following are equivalent statements*:

(a) A *is* K*-irreducible.*

(b) *The following conditions are both satisfied:*

(i) *For any final strong component* $\mathcal{C}$ *of* $(\mathcal{E}, \mathcal{P})$, *the join of all extreme rays which form the vertices of* $\mathcal{C}$ *is* K.

(ii) *For any* $x \in \operatorname{Ext} K$, *if the vertex* $\Phi(x)$ *has no access to a final strong component of* $(\mathcal{E}, \mathcal{P})$, *then the cone generated by all vertices of* $(\mathcal{E}, \mathcal{P})$ *which have access from* $\Phi(x)$ *intersects* int K.

(c) *For any* $x \in \operatorname{Ext} K$, *the join of all extreme rays which have access from* $\Phi(x)$ *equals* K.

The preceding theorem clearly implies the known result that if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is strongly connected then A is K-irreducible.

Notice that condition (b)(ii) of Theorem 5 is intended for the nonpolyhedral case only; when K is polyhedral, the condition is vacuously satisfied.

By the *index of imprimitivity* of a strongly connected digraph we mean the greatest common divisor of the its circuit lengths. We call a strongly connected digraph *primitive* if its index of imprimitivity is one. It is known that if a strongly connected digraph has index of imprimitivity $m > 1$, then it is cyclically m-partite, i.e., its vertex set can be partitioned into m disjoint subsets, referred to as its *sets of imprimitivity*, say, $V_1, \ldots, V_m$, such that (r, s) is an arc only if there exists i, $1leilem$, such that $r \in V_i$ and $s \in V_{i+1}$, where V_{m+1} is taken to be V_1. (For references, see [B-R].)

Theorem 7. *Let* K *be a polyhedral cone and let* $A \in \pi(K)$. *In order that* A *is* K*-primitive, it is necessary and sufficient that for any final strong component* $\mathcal{C}$ *of* $(\mathcal{E}, \mathcal{P}(A, K))$, *either* $\mathcal{C}$ *is a primitive digraph and the join of all extreme rays which* *form the vertices of $\mathcal{C}$ is K, or $\mathcal{C}$ has index of imprimitivity greater than 1 and the join of all extreme rays in one (or, each) of the sets of imprimitivity of $\mathcal{C}$ is K.*

By Theorems 6 and 7 the K-irreducibility or K-primitivity of A is completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ together with a knowledge of when a given finite collection $\{E_1, \ldots, E_j\}$ of extreme rays satisfies $E_1 \vee \cdots \vee E_j = K$.

When K is simplicial, the join of a collection of extreme rays of K equals K if and only if the collection consists of all extreme rays of K. With this in mind, one can readily see that Theorems 6 and 7 reduce to the classical results for nonnegative matrices when we take K to be the nonnegative orthant $\mathbb{R}^n_+$.

In Theorems 6 and 7, besides given the digraph $(\mathcal{E}, \mathcal{P}(A))$, we also assume that it is possible to determine whether any given finite collection of extreme vectors $x_1, \ldots, x_p$ satisfy $x_1 + \cdots + x_p \in \operatorname{int} K$. Certainly, if K is given up to combinatorial equivalence, then the latter is assumed known. In fact, we have the following:

Theorem 8. *Let K_1, K_2 be proper cones each with a bijective duality operator. Then K_1 and K_2 are combinatorial equivalent if and only if there exists a bijection $\varphi : \mathcal{E}(K_1) \longrightarrow \mathcal{E}(K_2)$ such that for any positive integer p and any $E_1, \ldots, E_p \in \mathcal{E}(K_1)$, $E_1 \vee \cdots \vee E_p = K_1$ if and only if $\varphi(E_1) \vee \cdots \vee \varphi(E_p) = K_2$.*

Theorem 9. *Let G be a strongly connected finite digraph. In order that there exist a polyhedral cone K and a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A, K)) = G$ it is necessary and sufficient that either G is primitive or it has index of imprimitivity greater than 1 and none of its sets of imprimitivity is a singleton.*

An earlier version of this work was reported by this author at the Com^2MaC International Conference on Combinatorial Matrix Theory held at Pohang, Korea on January 14-17, 2002.

2. PROOFS

Proof of Theorem 1. (b) $\Longrightarrow$ (c): Consider any $x \in \operatorname{Ext} K$. If $Ax = 0$, then clearly $\Phi(Ax) = \{0\} \subseteq \Phi(Bx)$. Otherwise, we can write $Ax = y_1 + \cdots + y_r$, where $y_1, \cdots, y_r \in \operatorname{Ext} K$. Then, for $j = 1, \ldots, r$, $(\Phi(x), \Phi(y_j))$ are $\mathcal{P}(A)$-arcs, and by condition (b) they are also $\mathcal{P}(B)$-arcs, i.e., $y_1, \ldots, y_r \in \Phi(Bx)$. So we have $Ax \in \Phi(Bx)$, or equivalently, $\Phi(Ax) \subseteq \Phi(Bx)$.

(c) $\Longrightarrow$ (d): Consider any $0 \neq x \in K$. We can write $x = x_1 + \cdots + x_s$, where $x_1, \ldots, x_s \in \operatorname{Ext} K$. By condition (c), for each $i = 1, \ldots, s$, there exists $\alpha_i > 0$ such that $Ax_i \overset{K}{\leq} \alpha_i B x_i$, where $\overset{K}{\leq}$ denotes the partial ordering of $\mathbb{R}^n$ induced by K. Let $\alpha = \max\{\alpha_1, \ldots, \alpha_s\}$. Then $\alpha > 0$ and we have $Ax \overset{K}{\leq} \alpha Bx$, so $\Phi(Ax) \subseteq \Phi(Bx)$.

(d) $\Longrightarrow$ (e): Suppose (F, G) is a $\mathcal{P}(A)$-arc. Take any x from $\operatorname{ri} F$, the relative interior of F. Then $F = \Phi(x)$ and we have

$$G \subseteq \Phi(AF) = \Phi(Ax) \subseteq \Phi(Bx) = \Phi(BF).$$

where the inclusion holds by condition (c). So (F, G) is also a $\mathcal{P}(B)$-arc.

(e) $\Longrightarrow$ (b): Obvious.

This establishes the equivalence of conditions (b)–(e).

The implications (a) $\Longrightarrow$ (d) $\Longrightarrow$ (f) are known and are not difficult to show (see [Tam 3]). ∎

Proof of Theorem 2. Assume that we know the inclusion relation between the faces of K. Then clearly we can determine $\mathcal{E}$, the set of extreme rays of K: it consists of precisely all those elements $E \in \mathcal{F}'$ for which there is no $F \in \mathcal{F}'$, $F \neq E$, such that (E, F) is an arc of $(\mathcal{F}', \mathcal{P}(I, K))$. Moreover, we can also determine $\Phi\mathcal{S}$, the smallest face including a given collection $\mathcal{S}$ of extreme rays.

Suppose we are given the digraph $(\mathcal{E}, \mathcal{P}(A, K))$. To obtain $(\mathcal{F}', \mathcal{P}(I, K))$, it suffices to determine for $F \in \mathcal{F}'$ what $\Phi(AF)$ is. (This is because, then for any $G \in \mathcal{F}'$, we can decide whether we have $Phi(F) \supseteq G$, as the inclusion relation between the faces is known.) Now we have $\Phi(AF) = \Phi(S)$, where $S = \bigcup \operatorname{Ext} \Phi(Ax)$, the union being taken over all $x \in \operatorname{Ext} F$. But since the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given, for each vector $x \in \operatorname{Ext} F$, we can determine $\operatorname{Ext} \Phi(Ax)$. So $\Phi(AF)$ is determined.

Similarly, from $(\mathcal{E}, \mathcal{P}(A, K))$ we can also determine the digraphs $(\mathcal{E}, \mathcal{I}(A, K))$ and $(\mathcal{F}', \mathcal{I}(A, K))$. (Okay ?)

If we are given the digraph $(\mathcal{F}', \mathcal{P}(A, K))$, then the arcs of the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ are determined by: for any $E_1, E_2 \in \mathcal{E}$, (E_1, E_2) is an arc in $(\mathcal{E}, \mathcal{P}(A, K))$ if and only if (E_1, E_2) is an arc in $(\mathcal{F}', \mathcal{P}(A, K))$. Once the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is determined, so are the digraphs $(\mathcal{E}, \mathcal{I}(A, K))$ and $(\mathcal{F}', \mathcal{I}(A, K))$. ∎

Proof of Theorem 3. (i) First, we contend that for any positive integer j, we have $A^j \in \operatorname{cl}_{\pi(K)}(\Phi(B^j))$. We proceed by induction on j. For $j = 1$, this holds by our assumption. Consider any $j \geq 2$ and suppose that we already have

$A^{j-1} \in \mathrm{cl}_{\pi(K)}(\Phi(B^{j-1}))$. By [Tam 3, Theorem 3.2], the latter condition implies that we have $A^{j-1}x \in \mathrm{cl}_K(\Phi(B^{j-1}x))$ for all $x \in K$. Then, by [Tam 3, Theorem 3.3(b) and Theorem 3.2], we have $A^j x \in \mathrm{cl}_K(\Phi(AB^{j-1}x)) \subseteq \mathrm{cl}_K(\Phi(B^j x))$ for all $x \in K$, and by [Tam 3, Theorem 3.2] again, we have $A^j \in \mathrm{cl}_{\pi(K)}(\Phi(B))$. This proves our contention.

Now let γ be the exponent of A. Then A^γ is K-positive. But by our contention we have $A^\gamma \in \mathrm{cl}_{\pi(K)}(B^\gamma)$, so necessarily B^γ is also K-positive. This proves that B is K-primitive and $\gamma(B) \leq \gamma(A)$.

Part (ii) clearly follows from part(i). ■

According to [Tam 3, Corollary 3.4], if $A, B \in \pi(K)$ satisfy $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$ and if A is K-irreducible, then so is B. The proof of part(i) of Theorem 3 shows that the corresponding result for K-primitivity also holds.

The proof of Theorem 4 relies on the following general result on cones:

Lemma 1. *The following are equivalent conditions on a proper cone K:*
(a) *d_K is injective.*
(b) *For any faces F, G of K, $\mathrm{cl}_K(F) \subseteq \mathrm{cl}_K(G)$ implies $F \subseteq G$.*
(c) *For any faces F, G of K, $\mathrm{cl}_K(F) = \mathrm{cl}_K(G)$ implies $F = G$.*

Proof of Theorem 4.
Proof of Theorem 5.

[Before coming to the proofs of Theorems 6 and 7, I would like to point out that now I have some reservations on the validity of these results for a proper cone K. (If K is polyhedral, I have no doubt that the results are correct.) In the proofs, we assume that if $\Phi(x)$ is any vertex in $(\mathcal{E}, \mathcal{P})$, then there is a path from $\Phi(x)$ to some vertex that belongs to a final strong component. If $(\mathcal{E}, \mathcal{P})$ is a finite graph, which is the case if K is polyhedral, certainly there is no problem. But for a general proper cone K, $(\mathcal{E}, \mathcal{P})$ may have infinitely many extreme rays. Maybe the assertion is still correct, because K is of finite dimension. But it requires a proof. Next, it is well-known that every finite strongly connected digraph is either primitive or cyclically m-partite for some integer $m \geq 2$. I am not sure whether the corresponding result still holds for an infinite strongly connected digraph, or in particular for a strong component of $(\mathcal{E}, \mathcal{P})$. In (the statement and the proof of) Theorem 5 I have already assumed that the answer to the latter question is in the affirmative. Also, in parts of the proofs, I need to consider the cone C generated by extreme vectors that belong to the vertices of some final strong component. It is true that the cone C is invariant under A. But I am not sure whether C has to be closed.]

Proof of Theorem 6. "Only if": Suppose that there exists a final strong component $\mathcal{C}$ such that the join of all extreme rays which form the vertices of $\mathcal{C}$ is not equal to K. Let C denote the cone generated by the extreme rays that form the vertices of $\mathcal{C}$. Consider any $x \in \operatorname{Ext} C$. Since A is K-irreducible, Ax is a nonzero vector of K and so it can be written as a positive linear combination of certain extreme vectors of K. Notice that if one of the extreme vectors which appears in this representation lies outside C, then in the digraph $(\mathcal{E}, \mathcal{P}(A))$ there must exist an arc from $\Phi(x)$ to some vertex not belonging to $\mathcal{C}$, which contradicts the assumption that $\mathcal{C}$ is a final strong component. Hence, each extreme vectors that appear in the representation of Ax belongs to C; so Ax itself also belongs to C. Since this is true for each extreme vector x of C, we must have $AC \subseteq C$.

Since the join of all vertices of $\mathcal{C}$ is not equal to K, we have $C \subseteq \partial K$. But C is invariant under A, so $\Phi(C)$ is a nontrivial A-invariant face of K. This contradicts the assumption that A is K-irreducible.

"If" part: To prove that A is K-irreducible, it suffices to show that $I + A$ is K-primitive, or equivalently, to show that for any $x \in \operatorname{Ext} K$, there exists a positive integer m such that $(I + A)^m x \in \operatorname{int} K$.

Consider any $\Phi(x) \in \mathcal{E}$. Clearly there exists a path in $(\mathcal{E}, \mathcal{P})$ from $\Phi(x)$ to some vertex $\Phi(y)$ that belongs to a final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$. Hence, there exists a positive integer p such that $y \in \Phi(A^p x)$. Let C denote the cone generated by the extreme rays that form the vertices of C. As done in the proof of the "only if" part, we have $AC \subseteq C$. Indeed, $A|_{\text{span } C}$ is irreducible with respect to C, as $(\mathcal{E}(C), \mathcal{P}(A|_{\text{span}\, C}, C))$ equals $\mathcal{C}$ and $\mathcal{C}$ is strongly connected. So, there exists a positive integer q such that $(I + A)^q y \in \text{ri } C \subseteq \text{int } K$, where the last inclusion holds by our assumption on the strong components of $(\mathcal{E}, \mathcal{P})$. Thus, we have $(I + A)^{p+q} x \in \text{int } K$, as desired. This completes the proof. ■

By abuse of language, for any $\mathcal{T} \subseteq \mathcal{E}(K)$, we shall use $\Phi(\mathcal{T})$ to denote $\Phi(\bigcup E)$, where the union is taken over all extreme rays $E \in \mathcal{T}$.

The proof of Theorem 7 depends on the following:

Lemma 2. *Let K be a proper cone, and let $A \in \pi(K)$. If $F \in \mathcal{F}'$ and $\mathcal{T} \subseteq \mathcal{E}$ are such that $F = \Phi(\mathcal{T})$, then $\Phi(AF) = \Phi(\Delta(\mathcal{T}))$, where $\Delta(\mathcal{T})$ has the same meaning as defined in Theorem 6.*

Proof. For any $E \in \Delta(\mathcal{T})$, by definition, there exists $E' \in \mathcal{T}$ such that (E', E)

is a $\mathcal{P}(A)$-arc, i.e., $E \subseteq \Phi(AE')$. But $\Phi(\mathcal{T}) = F$, so we have $E \subseteq \Phi(AF)$. This establishes the inclusion $\Delta(\mathcal{T}) \subseteq \mathcal{E}(\Phi(AF))$, and hence also $\Phi(\Delta(\mathcal{T})) \subseteq \Phi(AF)$.

To prove the reverse inclusion, let $E_1, \ldots, E_k \in \mathcal{T}$ be such that $F = E_1 \vee \cdots \vee E_k$. Then

$$\begin{aligned} \Phi(AF) &= \textstyle\bigvee_{i=1}^{k} \Phi(AE_i) \\ &= \textstyle\bigvee_{i=1}^{k} \Phi(\Delta(\{E_i\})) \\ &= \Phi(\Delta(\{E_1, \ldots, E_k\})) \\ &\subseteq \Phi(\Delta(\mathcal{T})), \end{aligned}$$

where the second equality holds as it is clear that for any $E \in \mathcal{E}$, $\Delta(E) = \mathcal{E}(\Phi(AE))$. This completes the proof. ∎

Proof of Theorem 7. "If" part: To prove that A is K-primitive, it suffices to show that for any $x \in \operatorname{Ext} K$ there exists a positive integer l such that $A^l x \in \operatorname{int} K$. Consider any $x \in \operatorname{Ext} K$. As shown in the proof for the "if" part of Theorem 4, there exists a positive integer p such that $\Phi(A^p x) \supseteq \Phi(y)$ for some extreme ray $\Phi(y)$ which is a vertex of some final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$. As in the proof of Theorem 4 we use C to denote the cone generated by the extreme vectors that belong to extreme rays in $\mathcal{C}$. As already done in the proof of Theorem 4, $A|_{\text{span } C}$ must be C-irreducible. If the digraph $\mathcal{C}$ is primitive, then clearly $A|_{\text{span } C}$ is C-primitive, and so there exists a positive integer q such that $A^q y \in \text{ri } C \subseteq \text{int } K$, where the last inclusion follows from our assumption on the final strong components of $(\mathcal{E}, \mathcal{P})$. But then we have $\Phi(A^{p+q}x) \supseteq \Phi(A^q y) = K$, i.e., $A^{p+q}x \in \text{int } K$, as desired. So it remains to consider the case when $\mathcal{C}$ is cyclically m-partite with $m > 1$. By our hypothesis, there is an ordered partition $\mathcal{E}_1, \ldots, \mathcal{E}_m$ for $\mathcal{E}(\mathcal{C})$ such that for some j, $1 \leq j \leq m$, the join of all extreme rays in $\mathcal{E}_j$ is K. Since $\mathcal{C}$ is strongly connected, there is a path in $(\mathcal{E}, \mathcal{P})$ from $\Phi(y)$ to $\Phi(u)$ for some vertex $\Phi(u) \in \mathcal{E}_j$. Hence, there exists a positive integer q such that $\Phi(A^q y) \supseteq \Phi(u)$. From the known theory of strongly connected digraph (see [B–R, Lemma 3.4.4]), there exists $r_0 \in \mathbb{Z}_+$ such that for all $r \in \mathbb{Z}_+$, $r \geq r_0$, there exists a directed walk of length rm in $\mathcal{C}$ from $\Phi(u)$ to any vertex of $\mathcal{E}_j$. But the join of all extreme rays in $\mathcal{E}_j$ is K, so this means that $\Phi(A^{r_0 m}u) = K$. Thus, we have $A^{p+q+r_0 m}x \in \text{int } K$, as desired.

"Only if" part: Assume to the contrary that there exists a final strong component $\mathcal{C}$ of $(\mathcal{E}, \mathcal{P})$ which is m-cyclic and with the ordered partition $\mathcal{E}_1, \ldots, \mathcal{E}_m$ for $\mathcal{E}(\mathcal{C})$ such that the join of all extreme rays in some $\mathcal{E}_j$ is not equal to K. (If $\mathcal{C}$ is primitive, then $m = 1$ and our argument still covers this case.) Take any $\Phi(x)$ from $\mathcal{E}_j$. Since

$\mathcal{C}$ is a final strong component and is m-cyclic (and for any positive integer k and any $\mathcal{S} \subseteq \mathcal{E}$, $\Delta^k(\mathcal{S})$ consists of precisely all those $E \in \mathcal{E}$ for which there is a directed walk of length k from some vertex in $\mathcal{S}$ to E), clearly we have $\Delta^{rm}(\Phi(x)) \subseteq \mathcal{E}_j$ for all positive integers r. By Lemma 2, this implies that $A^{rm}x \in \Phi(\mathcal{E}_j) \subseteq \partial K$ for all positive integers r, where the inclusion follows from our assumption on $\mathcal{E}_j$. This contradicts the K-primitivity of A. ■

Proof of Theorem 8. "Only if" part: Obvious.

"If" part: Observe that if M is a subset of $\mathcal{E}(K_1)$ maximal with respect to the property that $\bigvee\{E : E \in M\} \neq K$, then the positive hull of the extreme rays in M is a maximal face of K_1. Indeed, every maximal face of K_1 can be obtained in this way. The same can also be said for K_2. The given bijection φ between $\mathcal{E}(K_1)$ and $\mathcal{E}(K_2)$ clearly induces a bijection between the maximal faces of K_1 and those of K_2. Now since the duality operator d_{K_1} (also d_{K_2}) is bijective, each face of K_1 (also, of K_2) is an intersection of maximal faces. (For a proof, use [Tam 3, Lemma 5.12].) In other words, each face of K_1 (respectively, of K_2) is the positive hull of the extreme rays belonging to an intersection of subsets of $\mathcal{E}(K_1)$ (respectively, of $\mathcal{E}(K_2)$), each maximal with respect to the property that the join of its extreme rays is not equal to the whole cone. Hence, φ also induces a bijection between the faces of K_1 and those of K_2. And, moreover, it is easy to see that the latter bijection also preserves the inclusion relation. Therefore, the cones K_1 and K_2 are combinatorially equivalent. ■

Proof of Theorem 8. "Necessity" part: If G has index of imprimitivity greater than 1 and one of its sets of imprimitivity is a singleton, then the join of all extreme rays in that set of imprimitivity cannot be K. So by Theorem 7, A is not K-primitive, which is a contradiction.

"Sufficiency" part: If G is a primitive digraph, we can take a nonnegative primitive matrix A. So suppose that G has index of imprimitivity $l > 1$, and $V_1, \ldots, V_l$ are its sets of imprimitivity each with cardinality greater than 1. To be specific, for $j = 1, \ldots, l$, let $V_j = \{\Phi(x_{j1}), \ldots, \Phi(x_{jd_j})\}$. Let K be the polyhedral cone with extreme vectors $x_{11}, \ldots, x_{1j_1}, \ldots, x_{l1}, \ldots, x_{ld_l}$ that satisfy the relations:

$$x_{11} + \ldots + x_{1d_1} = x_{21} + \ldots + x_{2d_2} = \cdots = x_{l1} + \ldots + x_{ld_l}.$$

Note that, by our construction, the join of all extreme rays in each V_j is equal to K.

Now let A be the linear map determined by:

$$Ax_{jp} = \sum_{q=1}^{d_{j+1}} a^j_{pq} x_{j+1,q}, \text{for} j = 1, \ldots, m, p = 1, \ldots, d_j (\text{with} m + 1 \text{taken as} 1),$$

where a^j_{pq} equals $\frac{1}{d^-(\Phi(x_{j+1,q}))}$ if $(\Phi(x_{jp}), \Phi(x_{j+1,q}))$ is an arc ($d^-(\Phi(x_{j+1,q}))$ being the in-degree of the vertex $\Phi(x_{j+1,q})$) and equals 0 otherwise. Note that A preserves the relations of K, so it is a well-defined linear map. Also, it is clear that A belongs to $\pi(K)$ and we have $(\mathcal{E}, \mathcal{P}(A, K)) = G$. By Theorem 7 it follows that A is K-primitive. ■

3. EXAMPLES, REMARKS AND OPEN QUESTIONS

The following example shows that if $A \in \pi(K)$, then the K-irreducibility or K-primitivity of A is not completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$; it also depends on K.

By a *minimal cone* we mean a polyhedral cone whose number of extreme rays equals the dimension of the cone plus one. We will need properties of a minimal cone, which we describe below. For the details, we refer the reader to [Tam 1].

It is known that if K is an indecomposable minimal cone with m extreme vectors, then after normalization we may assume that the extreme vectors of K are $x_1, ..., x_m$ and (up to multiples) they satisfy the unique linear relation $\sum_{i \in I} = \sum_{j \in J}$, where $< m >$ is the disjoint union of the nonempty subsets I and J. In this case, the maximal faces of K are precisely subcones of K of the form $M_{i,j} := \text{pos}\{x_k : k \in < m >, k \neq i, j\}$ for some $i \in I$ and $j \in J$. So the maximal faces of K, and hence also the nonzero nontrivial faces of K, are all simplicial. It follows that if x is a nonzero vector lying in the boundary of K and if x can be written as the positive linear combination of certain, say q, extreme vectors of K, then the extreme vectors in $\Phi(x)$ are precisely these q extreme vectors.

Example 1. Let K be a minimal cone in $\mathbb{R}^5$ generated by extreme vectors $x_1, \ldots, x_6$ that satisfy $x_1 + x_2 + x_3 = x_4 + x_5 + x_6$. Let A be the 5-by-5 matrix given by:

$$\begin{aligned} Ax_1 &= x_2 + x_3, \ Ax_2 = x_3 + x_1, \ Ax_3 = x_1 + x_2, \\ Ax_4 &= x_5 + x_6, \ \text{ and } \ Ax_5 = x_6 + x_4. \end{aligned}$$

Then $Ax_6 = A(x_1 + x_2 + x_3) - A(x_4 + x_5) = x_4 + x_5$. So $A \in \pi(K)$. Since $\Phi(Ax_1)$ equals $x_2 + x_3$ and is a boundary vector of $K, \Phi(Ax_1)$ contains precisely the extreme vectors $x_2,\ x_3$. Hence, in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$, there are arcs $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ but no other arcs with initial vertex $\Phi(x_1)$. In the same manner, we can determine all other arcs of $(\mathcal{E}, \mathcal{P}(A, K))$. It turns out that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by the following diagram:

$$\begin{tikzcd}[arrows=-] & \Phi(x_1) \arrow[dl] \arrow[dr] & & & \Phi(x_4) \arrow[dl] \arrow[dr] & \\ \Phi(x_2) \arrow[rr] & & \Phi(x_3) & \Phi(x_5) \arrow[rr] & & \Phi(x_6) \end{tikzcd} \quad ,$$

where we use $\Phi(x_i)$ — $\Phi(x_j)$ to denote the presence of the pair of arcs $(\Phi(x_i), \Phi(x_j))$ and $(\Phi(x_j), \Phi(x_i))$. One can also readily check that A is K-primitive with $\gamma(A) = 2$. [This example also shows that, when A is K-primitive, the undirected graph of $(\mathcal{E}, \mathcal{P}(A, K))$ need not be connected.]

On the other hand, it is clear that one can also find a 6-by-6 nonnegative matrix, whose digraph is the same as the above one. Then any such nonnegative matrix is not even irreducible, not to say, primitive.

In [B–T, the paragraph following Proposition 1], an example is provided to show that, in general, the K-primitivity of A does not imply the strong connectedness of $(\mathcal{E}, \mathcal{P})$, indeed not even that of $(\mathcal{E}, \mathcal{I})$. Below we are going to borrow the said example (but rewriting it and putting it in a more general form):

Example 2. Let K be a minimal proper cone in $\mathbb{R}^4$ generated by the distinct extreme vectors $x_1, \dots, x_5$ that satisfy the relation

$$2(x_1 + x_2 + x_3) = 3(x_4 + x_5).$$

Let A be the 4-by-4 matrix given by:

$$\begin{aligned} & Ax_1 = (x_1 + x_2)/2,\ Ax_2 = (x_2 + x_3)/2,\ Ax_3 = (x_3 + x_1)/2, \\ \text{and} \quad & Ax_4 = (x_4 + x_5)/2. \end{aligned}$$

After a little calculation, we obtain $Ax_5 = (x_4 + x_5)/2$; so $A \in \pi(K)$. Indeed, it is easy to check that A is K-primitive and $\gamma(A) = 2$. Note that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. In fact, the digraph $(\mathcal{E}, \mathcal{P}(A))$ has two strongly connected components with vertex sets $V_1 = \{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$ and $V_2 = \{\Phi(x_4), \Phi(x_5)\}$.

The induced subdigraph on V_1 is composed of loops at each of the vertices together with the 3-circuit $(\Phi(x_1), \Phi(x_2))$, $(\Phi(x_2), \Phi(x_3))$, $(\Phi(x_3), \Phi(x_1))$. The induced subdigraph on V_2 is complete. There is no arcs from V_1 to V_2, but there are arcs from each vertex of V_2 to all vertices of V_1.

Note that the matrix A is singular. If we take $B = A + \varepsilon I$, then for $\varepsilon > 0$ sufficiently small, B is nonsingular and K-primitive. Furthermore, we have $(\mathcal{E}, \mathcal{P}(B)) = (\mathcal{E}, \mathcal{P}(A))$ and $\gamma(B) = \gamma(A)$.

Now let C be the 4-by-4 matrix given by:

$$\begin{aligned} & Cx_1 = (x_1 + x_2)/2,\ Cx_2 = (x_2 + x_3)/2,\ Cx_3 = (x_3 + x_1)/2, \\ \text{and} \quad & Cx_4 = \tfrac{2}{3}(x_2 + x_3). \end{aligned}$$

Then, after a little calculation, we have $Cx_5 = \frac{2}{3}x_1$. So $C \in \pi(K)$. In fact, it is ready to see that C is K-primitive and $\gamma(C) = 3$. Also, the digraph $(\mathcal{E}, \mathcal{P}(C))$ has three strongly connected components with vertex sets $\{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$, $\{\Phi(x_4)\}$ and $\{\Phi(x_5)\}$ respectively. The subdigraph on $\{\Phi(x_1), \Phi(x_2), \Phi(x_3)\}$ are the same as before. The remaining $\mathcal{P}(C)$-arcs are $(\Phi(x_4), \Phi(x_2))$, $(\Phi(x_4), \Phi(x_5))$ and $(\Phi(x_5), \Phi(x_1))$.

It is clear that if G is any digraph with m vertices, then we can always find an m-by-m nonnegative matrix whose digraph is G. On the other hand, if K is a given non-simplicial polyhedral cone with m extreme rays, we need not be able to find some $A \in \pi(K)$ such that $(\mathcal{E}, \mathcal{P}(A, K))$ is the prescribed digraph G. There are certain constraints that have to be met in order that G is of the form $(\mathcal{E}, \mathcal{P}(A, K))$. For instance, if $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ are any vectors of K that satisfy $\Phi(x_1 + \cdots + x_r) = \Phi(y_1 + \cdots + y_s)$, then for any $A \in \pi(K)$ we must have $\Phi(Ax_1 + \cdots + Ax_r) = \Phi(Ay_1 + \cdots + Ay_s)$. Rewriting this in terms of digraphs, we have the following:

Remark 1. Suppose the digraph $(\mathcal{F}', \mathcal{P}(I, K))$ is given. Then for any $A \in \pi(K)$, the digraph of $(\mathcal{E}, \mathcal{P}(A, K))$ necessarily satisfies the following condition:

For any $\Phi(x_1), \ldots, \Phi(x_r), \Phi(y_1), \ldots, \Phi(y_s) \in \mathcal{E}$, if the smallest element F of $\mathcal{F}'$ ["smallest" in the sense of inclusion, which can be determined from the digraph $(\mathcal{F}', \mathcal{P}(I, K))$] with the property that $(F, \Phi(x_i))$ is a $\mathcal{P}(I, K)$-arc for $i = 1, \ldots, r$ is the same as the smallest element G of $\mathcal{F}'$ with the property that $(G, \Phi(y_i))$ is a $\mathcal{P}(I, K)$-arc for $i = 1, \ldots, s$, then the smallest element $\tilde{F}$ of $\mathcal{F}'$ with the property that $(F, \Phi(w))$ is a $\mathcal{P}(I, K)$ arc for all $\Phi(w) \in \mathcal{E}$ such that $(\Phi(x_i), \Phi(w))$ is a $\mathcal{P}(A, K)$ arc for some $i = 1, \ldots, r$ is the same as the smallest element $\tilde{G}$ with the corresponding property, but with $y_1, \ldots, y_s$ in place of $x_1, \ldots, x_r$.

Question 1. Given a polyhedral cone K, determine all digraphs G which are of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some $A \in \pi(K)$.

Question 1 can be considered as an "allow" question with K fixed. One may also ask a similar question for which K is not fixed. But, as mentioned above, any (finite) digraph is the digraph associated with some nonnegative matrix. So we should exclude the simplicial cones.

Question 2. Determine all (finite) digraphs G which are of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some non-simplicial (polyhedral) cone K and some $A \in \pi(K)$ (also, K-irreducible or K-primitive A).

Certainly, there are digraphs which are not of the form $(\mathcal{E}, \mathcal{P}(A, K))$ for some non-simplicial polyhedral cone K and some $A \in \pi(K)$, for instance, any digraphs with three or less vertices. Below is a less trivial example:

Example 3. Consider the digraph G which consists of two 3-circuits with one vertex in common. Note that G is a strongly connected digraph with index of imprimitivity 3 and one of its sets of imprimitivity is a singleton. So G cannot be of the form $\mathcal{E}, \mathcal{P}(A, K))$, where A is K-primitive. We are going to show that G is not the form $\mathcal{E}, \mathcal{P}(A, K))$ for some non-simplicial polyhedral cone K and some $A \in \pi(K)$. We assume the contrary. Then the only possible dimensions for the non-simplicial cone K is 3 or 4. We treat both cases together. To be specific, suppose $(\mathcal{E}, \mathcal{P}(A, K))$ is represented by the following diagram:

$$\begin{tikzcd}
& \Phi(x_4) \arrow[dl] & & \Phi(x_2) \arrow[dr] & \\
\Phi(x_5) \arrow[rr] & & \Phi(x_1) \arrow[ul] \arrow[ur] & & \Phi(x_3) \arrow[ll]
\end{tikzcd}$$

Since $(\Phi(x_4), \Phi(x_5))$ is the only arc in $(\mathcal{E}, \mathcal{P}(A, K))$ with initial vertex $\Phi(x_4)$, $\Phi(Ax_4)$ contains (up to multiples) exactly one extreme vector of K, namely x_5; so we must have $A\Phi(x_4) = \Phi(x_5)$. Similarly, we have $A\Phi(x_2) = \Phi(x_3)$, $A\Phi(x_5) = A\Phi(x_3) = \Phi(x_1)$ and also $A\Phi(x_1) = \Phi(x_4) \vee \Phi(x_2)$, which is a 2-dimensional simplicial face. Note that the condition $A\Phi(x_5) = A\Phi(x_3) = \Phi(x_1)$ implies that A maps $\Phi(x_5)$ and $\Phi(x_3)$ each onto the extreme ray $\Phi(x_1)$; hence A must be singular. Certainly, AK is the polyhedral cone generated by the images of the extreme vectors of K under A. So from the above information on A, we have, $AK = \text{pos}\,\{x_1, x_3, x_5, \alpha_2 x_2 + \alpha_4 x_4\}$, assuming $Ax_1 = \alpha_2 x_2 + \alpha_4 x_4$, where $\alpha_2, \alpha_4 > 0$; hence, rank $A = \dim AK \geq 3$. But

A is singular, it follows that K must be a 4-dimensional minimal cone and we have $\alpha_2 x_2 + \alpha_4 x_4 \in \text{span}\{x_1, x_3, x_5\}$; say,

$$\alpha_2 x_2 + \alpha_4 x_4 = \alpha_1 x_1 + \alpha_3 x_3 + \alpha_5 x_5. \tag{3.1}$$

Then the latter is the unique (up to multiples) linear relation for the extreme vectors $x_1, \ldots, x_5$ of the minimal cone K. Applying A to both side of (3.1), we obtain

$$\alpha_2 \lambda_2 x_3 + \alpha_4 \lambda_4 x_5 = \alpha_1 \alpha_2 x_2 + \alpha_1 \alpha_4 x_4 + (\alpha_3 \lambda_3 + \alpha_5 \lambda_5) x_1, \tag{3.2}$$

where $\lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5 > 0$. From (3.1) and (3.2), (α_3, α_5) is a nonzero multiple of $(\alpha_2 \lambda_2, \alpha_4 \lambda_4)$; so $\alpha_3,\ \alpha_5$ must be both positive or both negative. If $\alpha_3,\ \alpha_5$ are both negative, then we would arrive at a contradiction — namely, x_1 is not an extreme vector of K if $\alpha_1 > 0$, or K is not pointed if $\alpha_1 \leq 0$. So they are both positive. Then $\alpha_3 \lambda_3 + \alpha_5 \lambda_5 > 0$, and again by comparing (3.1) and (3.2), we obtain $\alpha_1 < 0$. Then from (3.2), it follows that x_1 is a positive linear combination of $x_2,\ x_3,\ x_4$ and x_5, which contradicts the assumption that x_1 is an extreme vector.

The following is another fundamental question:

Question 3. Let K_1 and K_2 be proper cones which are combinatorially equivalent. Is it true that for any digraph G, if there exists $A_1 \in \pi(K_1)$ such that $(\mathcal{E}, \mathcal{P}(A_1, K_1)) = G$, then there always exists $A_2 \in \pi(K_2)$ such that $(\mathcal{E}, \mathcal{P}(A_2, K_2)) = G$?

Recall that for a proper cone K and any matrix A, $A \in \pi(K)$ if and only if $A^T \in \pi(K^*)$. The following example illustrates that for $A, B \in \pi(K)$, $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ does not imply $(\mathcal{E}, \mathcal{P}(A^T, K^*)) = (\mathcal{E}, \mathcal{P}(B^T, K^*))$ It also shows that in general $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ does not imply $\Phi(A) = \Phi(B)$.

Example 4. Let K be a proper cone whose dual cone K^* is not facially exposed. Choose a non-exposed face $\Phi(z)$ of K^*. Let $w \in K^*$ be such that $\Phi(w)$ equals $\text{cl}_{K^*}(\Phi(z))$, the exposed face of K^* generated by z. Choose any $x \in \text{int}\ K$. Take $A = xz^T$ and $B = xw^T$. Clearly $A, B \in \pi(K)$. By our choices of w and z, for any $y \in K$, we have $z^T y = 0$ if and only if $w^T y = 0$. Hence, we have $\Phi(Ay) = \Phi(By)$ for any $y \in K$. By Corollary 1 this means that $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$. If $B \in \Phi(A)$, then there exists $\alpha > 0$ such that $A - \alpha B = x(z - \alpha w)^T \in \pi(K)$. But the face $\Phi(w)$ properly includes $\Phi(z)$, so $w \notin \Phi(z)$ and hence $z - \alpha w \notin K^*$.

Take $y \in K$ such that $(z - \alpha w)^T y < 0$. Then $(A - \alpha B)y$ is a negative multiple of x, which is a contradiction. So we must have $B \notin \Phi(A)$. Finally, take any $u \in \text{int } K^*$. Then $B^T u = (x^T u)w \notin \Phi(z) = \Phi(A^T u)$. By Corollary 1 again, we have $(\mathcal{E}, \mathcal{P}(A^T, K^*)) \neq (\mathcal{E}, \mathcal{P}(B^T, K^*))$.

We call two cones K_1, K_2 *linearly isomorphic* if there exists a nonsingular linear transformation T from span K_1 to span K_2, which maps K_1 onto K_2. We call the cones K_1, K_2 *combinatorially equivalent*, if their face lattices $\mathfrak{F}(K_1)$ and $\mathfrak{F}(K_2)$ are isomorphic (as lattices), or equivalently, the digraphs $(\mathcal{F}', \mathcal{P}(I, K))$ and $(\mathcal{F}', \mathcal{P}(I, K_2))$ are equal (up to graph isomorphism). [I don't know whether it is true that if K_1, K_2 are combinatorially equivalent cones, then so are $\pi(K_1)$ and $\pi(K_2)$.]

It is clear that linearly isomorphic cones are combinatorially equivalent, but the converse is not true. Here is an example:

Example 5. Let e_j, $j = 1, 2, 3$, denote the standard unit vectors of $\mathbb{R}^3$. Let K_1 be the polyhedral cone in $\mathbb{R}^3$ generated by the extreme vectors $e_1,\ e_2,\ e_3,\ 2e_1+e_2-e_3$ and $e_1 + 2e_2 - e_3$. We are going to show that there exists a vector $u \in \mathbb{R}^3$ such that the polyhedral cone K_2 generated by the extreme vectors $e_1,\ e_2,\ e_3,\ 2e_1+e_2-e_3$ and u is not linearly isomorphic with K_1. Clearly, any 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent. In particular, the cones K_1, K_2 are combinatorially equivalent. We want to construct u in such a way that the extreme ray $\Phi(u)$ is neighborly to $\Phi(2e_1 + e_2 - e_3)$ and $\Phi(e_2)$. If T is a linear isomorphism which maps K_2 onto K_1, then T must carry extreme rays to extreme rays. Certainly, T maps neighborly extreme rays to neighborly extreme rays. So, there are eight choices for the action of T on the extreme rays of K_2: the images of $\Phi(e_2),\ \Phi(e_3),\ \Phi(e_1),\ \Phi(2e_1 + e_2 - e_3)$ under T, in this order, can be $\Phi(e_2),\ \Phi(e_3),\ \Phi(e_1),\ \Phi(2e_1 + e_2 - e_3)$, or $\Phi(2e_1 + e_2 - e_3),\ \Phi(e_1),\ \Phi(e_3),\ \Phi(e_2)$, or $\Phi(e_3),\ \Phi(e_1),\ \Phi(2e_1 + e_2 - e_3),\ \Phi(e_1 + 2e_2 - e_3)$, and so forth. It is not difficult to show that once the action of T on $\Phi(e_2),\ \Phi(e_3),\ \Phi(e_1)$ and $\Phi(2e_1 + e_2 - e_3)$ are known, T is uniquely determined up to multiples. We choose u in such a way that, for each of the eight choices, Tu does not lie on the remaining extreme ray of K_1; that is the one which is different from $T\Phi(e_2)$, $T\Phi(e_3)$, $T\Phi(e_1)$ and $T\Phi(2e_1+e_2-e_3)$; then $TK_2 \neq K_1$. Clearly, such u exists. Therefore, the cones K_1 and K_2 are not linearly isomorphic.

Putting it in another way, Theorem A tells us that if K is non-simplicial, then there always exists $A \in \pi(K)$ such that A is K-irreducible and $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. One may wonder whether in this case there is also a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected. The answer turns out to be "no".

A proper cone is said to be *strictly convex* if every boundary vector is extreme.

To establish the strong connectedness of $(\mathcal{E}, \mathcal{P}(A))$, we are going to show that there is a path in $(\mathcal{E}, \mathcal{P})$ from any given vertex $\Phi(x)$ of $\mathcal{E}$ to any other vertex of $\mathcal{E}$. Let p denote the least positive integer such that $A^p x \in \text{int } K$. Then $x, Ax, \ldots, A^{p-1}x$ all belong to ∂K and, in fact, are (nonzero) extreme vectors of K, as K is strictly convex. So there is a path in $(\mathcal{E}, \mathcal{P})$ passing through the vertices $\Phi(x), \Phi(Ax), \ldots, \Phi(A^{p-1}x)$ (and in this order). Since $A(A^{p-1}x) \in \text{int } K$, there is also a $\mathcal{P}$-arc from $\Phi(A^{p-1}x)$ to any other vertex of $\mathcal{E}$. It follows that there is a path from $\Phi(x)$ to any other vertex of $\mathcal{E}$.

Question 4. If K is a non-simplicial polyhedral cone, then does there always exist a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A))$ is not strongly connected ?

According to [Tam 1, Proposition 3.1], if M is a finite nonempty subset of $\text{Ext}\, K$, where K is a non-simplicial cone, and if $\text{pos}\, K$ meets $\text{int}\, K$, then there exists a proper diagonal D of K such that $\text{Ext}\, D \subseteq M$. In view of this observation and Theorem 7, if A is K-primitive, $(\mathcal{E}, \mathcal{P}(A, K))$ is strongly connected and has index of imprimitivity $l > 1$, then K has at least l proper diagonals whose sets of extreme rays are mutually disjoint. Since an indecomposable minimal cone has precisely two proper diagonals, it follows that for any indecomposable minimal cone K there does not exist a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A, K))$ is a strongly connected digraph with index of imprimitivity greater than two.

Question 5. Characterize digraphs that require K-irreducibility but does not allow K-primitivity.

(Cf. Theorem 8) Certainly, a circuit is one such example. Another example is provided by the following.

Example 6. Let G be a digraph with vertex set $\{1, \ldots, m, m+1, \ldots, m+p\}$ where $m \geq 3$ and $p \geq 1$, and arcs $(i, i+1), i = 1, 2, \ldots, m = 1, (m, 1), (1, m+j), (m+j, 3), j = 1, \ldots, p\}$. As G is strongly connected, G requires K-irreducibility. (Can it be the digraph $(\mathcal{E}, \mathcal{A})$ for some K-nonnegative matrix ?) However, G does not allow for K-primitivity. This is because G is a primitive digraph with index of imprimitivity m, its sets of imprimitivity being $\{1\}, \{2, m+1, m+2, \ldots, m+p\}, \{3\}, \ldots, \{m\}$. Hence, at least one of the sets of imprimitivity is a singleton, and so the equivalent conditions for K-primitivity as given by Theorem 7 for any K-primitive matrix A for which $(\mathcal{E}, \mathcal{P}(A, K)) = G$.

Given a compact convex body C in $\mathbb{R}^n$ that contains the origin. We call an $n \times n$ real matrix A C-*primitive* if there exists a positive integer k such that $A^k C \subseteq C$. If A is C-primitive the smallest k such that $AK \subseteq K$ is referred to as the *exponent* of A (relative to C) and is denoted by $\gamma_C(A)$ or simply by $\gamma(A)$ if there is no danger of confusion. We also denote by $\gamma(C)$ the quantity $\max\{\gamma(A) : A \text{ is C-primitive}\}$.

Problem. Given a compact convex body C, determine $\max \gamma(C - p)$, where the maximum is taken over all $p \in \operatorname{int} C$.

(Consider projectively equivalent compact convex sets ?)

Finally, we would like to point out that the tool of a minimal generating matrix, which is used by some people in the study of polyhedral cones (see, for instance, [B–F–H]), also has some connections with our study. Namely, if K is a polyhedral cone and $A \in \pi(K)$, and if B is the nonnegative matrix with the maximum number of positive entries that satisfies $AP = PB$, where P is the minimal generating matrix for K (i.e., its column vectors form a set of distinct representatives of the extreme rays of K), then the usual digraph of B^T is equal to our digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$. (We also know that then A and $B^T|_{\mathcal{R}(P^T)}$ are similar, and hence we must have $m_A \leq m_B$ and $\sigma(A) \subseteq \sigma(B)$. Furthermore, if B is primitive, then A is K-primitive and $\gamma(B) \geq \gamma(A)$.) Since the strong connectedness of $(\mathcal{E}, \mathcal{P}(A, K))$ implies the K-irreducibility of A but not conversely, we recover the known fact that if A is K-irreducible, there need not exist an irreducible nonnegative matrix B that satisfies $AP = PB$ (quote [B-F-H]?).

The following are some other ideas I have not yet pursued:

1. I guess primitive (respectively, strongly connected) digraphs are precisely the digraphs G which require the property that for every proper non-simplicial cone K and every $A \in \pi(K)$, A is K-primitive (respectively, K-irreducible) whenever $(\mathcal{E}(K), \mathcal{P}(A, K)) = G$.

Are there primitive graphs G which are not of the form $\mathcal{E}(K), \mathcal{P}(A, K))$ for some non-simplicial proper cone K and some $A \in \pi(K)$?

2. In my long survey paper "A cone-theoretic approach to the spectral theory of positive linear operators ...", there appears in Theorem 6.3 a result which says that if $A \in \pi(K)$, where K is a polyhedral cone with m maximal faces,

then there exists an m-by-m nonnegative matrix B and some B-invariant subspace W of $\mathbb{R}^m$, $W \bigcap \text{int } \mathbb{R}^m_+ \neq \emptyset$, such that the cone-preserving maps $A \in \pi(K)$ and $B|_W \in \pi(W \bigcap \mathbb{R}^m_+)$ are equivalent. This result may also have some connection with our work, but I have not yet explored it.

3. We may also consider $A \in \pi(K_1, K_2)$, where K_1, K_2 are proper cones, possibly in different euclidean spaces. For each such A, we can associate with it two bipartite graphs: The first bipartite graph has bipartition $\{\mathcal{E}(K_1), \mathcal{E}(K_2)\}$ for which there is an edge joining E_1, E_2, where $E_1 \in \mathcal{E}(K_1)$ and $E_2 \in \mathcal{E}(K_2)$ if and only if $E_2 \subseteq \Phi(AE_1)$. Similarly, we can define a bipartite graph with bipartition $\{\mathcal{F}'(K_1), \mathcal{F}'(K_2)\}$. I think Theorem 1 and Corollary 1 also have corresponding results in this setting, because their proofs rely on [Tam 3], but the latter paper is done in this general setting. Of course, in this case, we do not have the concept of K-irreducibility or K-primitivity.

4. We may even work in the setting of non-linear cone-preserving maps, say, in the class of monotone homogeneous maps on a fixed proper cone, or even in a broader class. I think we can always associate A with a digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$ defined in the same way as in this paper as long as A is a map which preserves K and possesses the following property: for all $x, y \in K$, if $\Phi(x) = \Phi(y)$ then $\Phi(Ax) = \Phi(Ay)$. (Is the latter property equivalent to, for all $x, y \in K$, if $\Phi(x) \subseteq \Phi(y)$, then $\Phi(Ax) \subseteq \Phi(Ay)$?), We can still have the concepts of K-irreducibility and K-primitivity and, I believe, many of our results can be carried over to this more general setting.

5. In the nonnegative matrix case, we can describe the A-invariant faces completely. (See my paper with Hans "On the invariant faces associated with a cone-preserving map.) If the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ for A ($\in \pi(K)$) is given, I am wondering to what extent we can describe all the A-invariant faces ?

Can we show the existence of gaps in the values of $\gamma(A)$, where A is K-primitive ? Consider the minimal cone case.

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EXPONENTS OF K-PRIMITIVE MATRICES

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August 5, 2006

Abstract. Let K be a proper (i.e., closed, pointed, full convex) cone in $\mathbb{R}^n$. An $n \times n$ matrix A is said to be K-primitive if there exists a positive integer k such that $A^k(K \setminus \{0\}) \subseteq \operatorname{int} K$; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. For a polyhedral proper cone K, the maximum value of $\gamma(A)$, taken over all K-primitive matrices A, is denoted by $\gamma(K)$. We treat the problem of determining the maximum value of $\gamma(K)$ as K runs through all n-dimensional polyhedral cones with m extreme rays and solve for it two special cases, namely, when $m = n + 1$ and when $n = 3$. In each case, we determine also the cones K in the relevant class (and the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attain the maximum value. We also obtain some partial results for the general case and pose some open questions.

Work to be completed

AMS classification: 15A48; 05C50; 47A06.
Key words: Cone-preserving map; K-primitive matrix; Exponents; Polyhedral cone; Minimal cone; Indecomposable summand.

1. Introduction

If K is a polyhedral (proper) cone in $\mathbb{R}^n$ with m extreme rays, what is the maximum value of the exponents of K-primitive matrices ? This question was posed by Steve Kirkland in an open problem session at the 8th ILAS conference held in Barcelona in July, 1999. Here by a K-primitive matrix we mean a real square matrix A for which there exists a positive integer k such that A^k maps every nonzero vector of K into the interior of K; the least such k is referred to as the exponent of A and is denoted by $\gamma(A)$. In view of Wielandt's sharp bound for exponents of (nonnegative) primitive matrices of a given order, Kirkland conjectured that $m^2 - 2m + 2$ is an upper bound for the maximum value considered in his question. This work is an outcome of our attempt to answer Kirkland's question.

In the classical nonnegative matrix case, the determination of upper bounds for the exponents of primitive matrices under various assumptions has been treated mainly by a graph-theoretic approach. In this work we will make use of one of the four digraphs, namely, $(\mathcal{E}, \mathcal{P}(A, K))$, introduced by Barker and Tam ([B–T], [T–B]). (Formal definitions will be given later.) Based on the same digraph, Niu[Niu] has started an initial study on the exponents of K-primitive matrices over a polyhedral cone K. His work has motivated partly the work of [Tam 4] and our present work.

When the polyhedral cone K is a nonnegative orthant (or a simplicial cone), the study of K-primitive matrices is reduced to the classical nonnegative matrix case. The general polyhedral cone case differs from the nonnegative matrix case in at least the following two respects. First, in the nonnegative matrix case the (distinct) extreme vectors of the underlying cone are linearly independent, whereas in the general polyhedral cone case the extreme vectors of of the underlying cone satisfy certain nonzero linear relations. Second, in the nonnegative matrix case, given any (finite) digraph, it is always possible to find a nonnegative matrix with the given digraph as its usual associated digraph. On the other hand, in the general polyhedral cone case, we are confronted with the realization problem. Usually it is not easy to tell whether there is a polyhedral cone K for which there is a K-nonnegative matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by a prescribed digraph. As expected, and also illustrated by this work, the study of the polyhedral cone case is more difficult than the classical nonnegative matrix case.

We now describe the contents of this paper in some detail.

Section 2 contains most of the definitions which we need in the paper. In particular, we give the definition for a minimal cone and collect the relevant known results.

In Section 3 we obtain a Sedláček-Dulmage-Mendelsohn type upper bound for the local exponents, and hence also an upper bound for the exponent, of a K-primitive matrix A in terms of the lengths of circuits in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ and the degree of the minimal polynomial of A. We also give some applications of the result. The results of this section may suggest that for a K-primitive matrix A, the larger is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A, K))$ the larger is the value of $\gamma(A)$. And we note that the length of the shortest circuit is at most $m - 1$.

In Section 4 we first identify those digraphs on m vertices, with the length of the shortest circuit equal to $m - 1$, that may be realized as $(\mathcal{E}, \mathcal{P}(A, K))$ for some K-primitive matrix A, where K is a polyhedral cone with m extreme rays. Up to graph isomorphism, there are two of them, represented by Figure 1 and Figure 2 respectively. (It turns out that they are also precisely the primitive digraphs on m vertices with the length of the shortest circuit equal to $m - 1$.) We also find that, for a polyhedral cone K with m extreme rays, if there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2 then K is either indecomposable or is the direct sum of a ray and an indecomposable minimal cone of a special kind and in the latter case the digraph is given by Figure 2. (Here by a minimal cone we mean a polyhedral cone whose number of extreme rays equals the dimension of the cone plus 1; equivalently, it is a polyhedral cone whose extreme vectors satisfy, up to multiples, a unique linear relation.) Next, we prove that if K is a polyhedral cone with m extreme rays, then for any K-primitive matrix A, $\gamma(A) \leq (m_A - 1)(m - 1) + 1$, where m_A denotes the degree of the minimal polynomial of A; moreover, the equality holds only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. Consequently, if K is an n-dimensional polyhedral cone with m extreme rays then its exponent $\gamma(K)$, which is defined to be $\max\{\gamma(A) : A \text{ is } K\text{-primitive}\}$, does not exceed $(n - 1)(m - 1) + 1$. As a by-product, we answer in the affirmative the conjecture posed by Kirkland mentioned at the beginning of this section.

In Section 5 we prove that the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even. We also determine (up to linear isomorphism) the minimal cones K (and also the corresponding K-primitive matrices A) such that $\gamma(K)$ (and $\gamma(A)$) attains the maximum value. It is found that in the (essentially) unique linear relation on the extreme vectors of an optimal minimal cone the number of vectors on the two sides of the relation differ by at most 1, and furthermore if K is an optimal minimal cone and A is a K-primitive matrix such that $\gamma(A)$ attains the maximum value then necessarily the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2.

In Section 6 we prove that the maximum value of $\gamma(K)$ as K runs through all

3-dimensional polyhedral cones with m extreme rays is $2m-1$. In fact, we show that if K is a 3-dimensional polyhedral cone with m extreme rays and A is a K-primitive matrix, then $\gamma(A) = 2m - 1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. For every positive integer $m \geq 5$ we also demonstrate the existence of a 3-dimensional polyhedral cone K with m extreme rays for which there does not exist a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since every two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, this means that the exponents of combinatorially equivalent cones may be different.

In Section 7 we show that for every pair of positive integers m, n with $3 \leq n \leq m$ there exists an n-dimensional polyhedral cone K with m extreme rays for which there is a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. We also pose a conjecture that tells for which pair m, n it is true that the upper bound $(n-1)(m-1)+1$ for $\gamma(K)$ is attained.

In Section 8, the final section, we give a few open questions, together with some remarks and examples.

Finally, we would like to point out that the proofs given in this paper do not depend on known results for primitive matrices. In fact, our approach covers the simplicial cone case and so it also provides an alternative treatment for the classical nonnegative matrix case.

2. Preliminaries

We take for granted standard properties of nonnegative matrices, complex matrices and graphs that can be found in textbooks. A familiarity with elementary properties of finite-dimensional convex sets, convex cones and cone-preserving maps is also assumed. To fix notation and terminology, we give some definitions.

Let K be a nonemtpy subset of a finite-dimensional real vector space V. K is called a *convex cone* if $\alpha x + \beta y \in K$ for all $x, y \in K$ and $\alpha, \beta \geq 0$; K is *pointed* if $K \cap (-K) = \{0\}$; K is *full* if its interior $\operatorname{int} K$ (in the usual topology of V) is nonempty; equivalently, $K - K = V$. If K is closed and satisfies all of the above properties, K is called a *proper cone.*

In this paper, unless specified otherwise, we always use K to denote a proper in the n-dimensional Euclidean space $\mathbb{R}^n$.

We denote by $\geq^K$ the partial ordering of $\mathbb{R}^n$ induced by K, i.e., $x \geq^K y$ if and only if $x - y \in K$. Sometimes we also write $x \gg^K 0$ (respectively, $x >^K 0$) for $x \in \operatorname{int} K$ (respectively, $x \geq^K 0$ and $x \neq 0$) and call the vector x *K-strictly positive* (respectively, *K-semipositive*).

A subcone F of K is called a *face* of K if $0^K \leq y^K \leq x$ and $x \in F$ imply $y \in F$. If $S \subseteq K$, we denote by $\Phi(S)$ the *face of K generated by S*, that is, the intersection of all faces of K including S. If $x \in K$, we write $\Phi(\{x\})$ simply as $\Phi(x)$. It is known that for any vector $x \in K$ and any face F of K, $x \in \operatorname{ri} F$ if and only if $\Phi(x) = F$. (Here we denote by $\operatorname{ri} F$ the relative interior of F.) A vector $x \in K$ is called an *extreme vector* if either x is the zero vector or x is nonzero and $\Phi(x) = \{\lambda x : \lambda \geq 0\}$; in the latter case, the face $\Phi(x)$ is called an *extreme ray*. We also use $\operatorname{Ext} K$ to denote the set of all nonzero extreme vectors of K. Two nonzero extreme vectors are said to be *distinct* if they are not multiples of each other. The cone K itself and the set $\{0\}$ are always faces of K, known as *trivial faces*. Other faces of K are said to be *nontrivial*.

If S is a nonempty subset of a vector space, we denote by $\operatorname{pos} S$ the *positive hull* of S, i.e., the set of all possible nonnegative linear combinations of vectors taken from S.

A closed pointed cone K is said to be the *direct sum* of its subcones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$ if every vector of K can be expressed uniquely as $x_1 + x_2 + \cdots + x_p$, where $x_i \in K_i, 1 \leq i \leq p$. K is called *decomposable* if it is the direct sum of two nonzero subcones; otherwise, it is said to be indecomposable. It is well-known that every closed pointed cone K can be written as

$$K = K_1 \oplus \cdots \oplus K_p,$$

where each K_j is an indecomposable cone ($1 \leq j \leq p$). Except for the ordering of the summands, the above decomposition is unique. We will refer to the K_js as *indecomposable summands* of K.

By a *polyhedral cone* we mean a proper cone which has finitely many extreme rays. By the dimension of a proper cone we mean the dimension of its linear span. A polyhedral cone is said to be *simplicial* if its number of extreme rays is equal to its dimension. The nonnegative orthant $\mathbb{R}^n_+ := \{(\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \xi \geq 0 \forall i$ is a typical example of a simplicial cone.

We denote by $\pi(K)$ the set of all real $n \times n$ matrices A (identified with linear mappings on $\mathbb{R}^n$) such that $AK \subseteq K$. Members of $\pi(K)$ are said to be *K-nonnegative* and are often referred to as *cone-preserving maps*. It is clear that $\pi(\mathbb{R}^n_+$ consists of all $n \times n$ nonnegative matrices.

A matrix $A \in \pi(K)$ is said to be *K-irreducible* if A leaves no nontrivial face of K invariant, A is *K-positive* if $A(K \setminus \{0\}) \subseteq \text{int } K$ and is *K-primitive* if there is a positive integer p such that A^p is K-positive. If A is K-primitive, then the smallest positive integer p for which A^p is K-positive is called the *exponent* of A and is denoted by $\gamma(A)$. For convenience, sometimes we write $A \gg^K 0$ to mean A being K-positive.

An equivalent condition for A to be K-irreducible is that A has no eigenvector in ∂K (see, for instance, [S–V]). Hence, the positive powers of a K-primitive matrix are all K-irreducible.

It is known that the set $\pi(K)$ forms a proper cone in the space of $n \times n$ real matrices, the interior of $\pi(K)$ being the subset consisting of K-positive matrices. Also, $\pi(K)$ is polyhedral if and only if K is polyhedral. It is also known that for proper cones K_1, K_2 in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively, the set $\pi(K_1, K_2)$ which consists of all $n_2 \times n_1$ matrices A such that $AK_1 \subseteq K_2$ is a proper cone in the space of $n_2 \times n_1$ real matrices. (See [Tam 2], [S–V] or [Bar 1].)

It is clear that if K is a simplicial cone with n extreme rays then K is linearly isomorphic with $\mathbb{R}^n_+$. The simplicial cones may be considered as the simplest kind of cones. The next simplest kind of cones, and also the one which we will deal with considerably in this work, are the minimal cones. Minimal cones were first introduced and studied by Fiedler and Pták [F–P]. We call an n-dimensional polyhedral cone *minimal* if it has precisely $n+1$ extreme rays. Clearly, if K is a minimal cone with (distinct) extreme vectors $x_1, \ldots, x_{n+1}$, then (up to multiples) these vectors satisfy a unique linear relation. Also, a minimal cone is indecomposable if and only if the linear relation for its extreme vectors is full, i.e., in the relation the coefficient of each extreme vector is nonzero (see [F–P, Theorem 2.25]). It is readily shown that every decomposable minimal cone is the direct sum of an indecomposable minimal cone and a simplicial cone.

It is easy to construct examples of minimal cones. Given any positive integers $p, q \geq 2$ with $p + q = n + 1$, we can construct as follows an n-dimensional indecomposable minimal cone K such that in the linear relation for its extreme vectors the number of (nonzero) terms on its two sides are respectively p and q. Choose any basis for $\mathbb{R}^n$, say, $\{x_1, \ldots, x_n\}$, and let K be the polyhedral cone $\text{pos}\{x_1, \ldots, x_n, x_{n+1}\}$, where $x_{n+1} = (x_1 + \cdots + x_p) - (x_{p+1} + \cdots + x_n)$. Then

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}$$

is the linear relation for the vectors $x_1, \ldots, x_{n+1}$. In view of the relation and its (essential) uniqueness, none of the vectors $x_1, \ldots, x_{n+1}$ can be written as a nonnegative linear combination of the remaining vectors. So they are precisely the extreme vectors of K. Therefore, K is the desired indecomposable minimal cone.

It is not difficult to show that if K_1, K_2 are minimal cones then they are linearly isomorphic if and only if they have the same number of indecomposable summands that are single rays, and the number of vectors on the two sides of their linear relations on extreme vectors are the same.

We need the following known characterization of maximal faces of an indecomposable minimal cone ([Tam 1, Theorem 4.1]):

Theorem A. *Let K be an indecomposable minimal cone generated by extreme vectors $x_1, \ldots, x_{n+1}$ that satisfy*

$$x_1 + \cdots + x_p = x_{p+1} + \cdots + x_{n+1}.$$

Then for each pair $(i, j), 1 \leq i \leq p$ and $p+1 \leq j \leq n+1, \operatorname{pos} M_{ij}$ is a maximal face of K, where $M_{ij} = \{x_1, \ldots, x_{n+1}\} \setminus \{x_i, x_j\}$. Moreover, each maximal face of K is of this form.

Note that by the preceding theorem every maximal face, and hence every nontrivial face, of a minimal cone is a simplicial cone in its own right.

In dealing with (nonzero, linear) relations on (nonzero) extreme vectors of a polyhedral cone, we find it convenient to write such relations in the form $\sum_{i=1}^{p} \alpha_i x_i = \sum_{j=1}^{q} \beta_j y_j$, where the extreme vectors $x_1, \ldots x_p, y_1, \ldots, y_q$ that appear on the two sides are distinct and the coefficients α_i, β_j are all positive. In this case, clearly we have $p, q \geq 2$.

Let R be a relation on the extreme vectors of K. Suppose that the vectors that appear in R come from $p(\geq 2)$ different indecomposable summands of K, say, $K_1, \ldots, K_p$. To be specific, let R be given by: $\sum_{i \in M} \alpha_i x_i = \sum_{j \in N} \beta_j y_j$, where M, N are finite index sets, each with at least two elements and the α_is, β_js are all positive real numbers. For each $r = 1, \ldots, p$, let $M_r = \{i \in M : x_i \in K_r\}$ and $N_r = \{j \in N : y_j \in K_r\}$. Then for each fixed r, rewriting relation R, we obtain

$$\sum_{i \in M_r} \alpha_i x_i - \sum_{j \in M_r} \beta_j y_j = \sum_{j \in N \setminus N_r} \beta_j y_i - \sum_{i \in M \setminus M_r} \alpha_i x_i.$$

Now the vector on the left side of the above relation belongs to $\operatorname{span} K_r$, whereas the one on the right side belongs to $\sum_{s \neq r} \operatorname{span} K_s$. But $\operatorname{span} K_r \cap \sum_{s \neq r} \operatorname{span} K_s = \{0\}$ (as $K_1, \ldots, K_p$ are distinct indecomposable summands of K), so we obtain the relation

$$\sum_{i \in M_r} \alpha_i x_i = \sum_{j \in M_r} \beta_j y_j,$$

which we denote by R_r. This is true for each r. It is clear that relation R can be obtained by adding up relations $R_1, \ldots, R_p$. In this case, we say relation R *splits into the subrelations* $R_1, \ldots, R_p$. Note that each R_r has at least four (nonzero) terms. So when we pass from the relation R to one of its subrelations R_r, the number of terms involved in the relation decreases by at least four.

It is not difficult to show that a (nonzero) extreme vector of K generates a 1-dimensional indecomposable summand of K if and only if the vector is not involved in any relation on $\mathrm{Ext}K$.

Let $A \in \pi(K)$. In this work we need the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ associated with A defined in the following way: Its vertex set is $\mathcal{E}$, the set of all extreme rays of K; $(\Phi(x), \Phi(y))$ is an arc whenever $\Phi(y) \subseteq \Phi(Ax)$. If there is no danger of confusion, we write $(\mathcal{E}, \mathcal{P}(A, K))$ simply as $(\mathcal{E}, \mathcal{P})$. It is readily checked that if K is the nonnegative orthant $\mathbb{R}^n_+$ then $(\mathcal{E}, \mathcal{A})$ equals the usual diagraph associated with A^T, the transpose of A. The digraph $(\mathcal{E}, \mathcal{P})$ is one of the four digraphs introduced in [B–T] for A. In [Tam 4] it is shown that when the inclusion relation between the faces of K is known (for instance, when K is given), then from $(\mathcal{E}, \mathcal{P})$ one can determine the other three digraphs.

It is not difficult to show that for any $A, B \in \pi(K)$, if $\Phi(A) = \Phi(B)$, then either A, B are both K-primitive or they are both not K-primitive, and if they are both K-primitive then $\gamma(A) = \gamma(B)$. In [Niu] it is proved that if K is a polyhedral cone then for any $A, B \in \pi(K)$, we have $(\mathcal{E}, \mathcal{P}(A, K)) = (\mathcal{E}, \mathcal{P}(B, K))$ if and only if $\Phi(A) = \Phi(B)$. So it is not a surprise to find that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ plays a role in determining a bound for $\gamma(A)$. (When K is nonpolyhedral, the situation is more subtle. We refer the interested readers to [Tam4] for the details.)

Let K be a polyhedral cone. Then $\pi(K)$ is also a polyhedral cone and hence has finitely many faces. As noted above, if A, B generate the same face of $\pi(K)$ or, equivalently, if they belong to the relative interior of the same face of $\pi(K)$, then either A, B are both K-primitive or they are both not K-primitive, and if they are both K-primitive then $\gamma(A) = \gamma(B)$. It follows that there are only finitely many values that can be attained by the exponents of K-primitive matrices.

For a proper cone K, we say K *has finite exponent* if the set of exponents of K-primitive matrices is bounded above; then we denote the maximum exponent by $\gamma(K)$ and refer to it as the *exponent of* K. From the above discussion, it is clear that every polyhedral cone has finite exponent. In Section 8, the final section of this paper, borrowing an example given in [B–L, p.66] of a norm in $\mathbb{R}^2$ for which the critical exponent does not exist, we provide an example of a proper cone in $\mathbb{R}^n$ which does not have finite exponent.

We will make use of the concept of a *primitive digraph*, which can be defined as a digraph for which there is a positive integer k such that for every pair of vertices i, j there is a directed walk of length k from i to j; the least such k is referred to as the exponent of the digraph. It is clear that a nonnegative matrix is primitive if and only if its usual digraph is primitive. It is also well-known that primitive digraphs are precisely strongly connected digraphs with the greatest common divisor of the lengths of their circuits equal to 1.

If A is K-nonnegative and the digraph $(\mathcal{E}, \mathcal{P})$ is primitive, then necessarily A is K-primitive and furthermore the exponent of A (as a K-primitive matrix) cannot exceed the exponent of the primitive digraph $(\mathcal{E}, \mathcal{P}(A, K))$. This can be readily seen as follows: Let K be a polyhedral cone K in $\mathbb{R}^n$ with m extreme extreme rays. Let P be the minimal generating matrix for K (i.e., its column vectors form a set of distinct representatives of the extreme rays of K). Let $A \in \pi(K)$. Choose a nonnegative matrix B with the maximum number of positive entries that satisfies $AP = PB$. Then, as pointed out in [Tam 4], the usual digraph of B^T is equal to the digraph $(\mathcal{E}(K), \mathcal{P}(A, K))$. Let k denote the exponent of the latter digraph. Then k is also the exponent of B^T and hence of B. Now $A^k P = PB^k$ and B^k is a positive matrix. So A^k maps each extreme vector of K into $\operatorname{int} K$. It follows that A is a K-primitive matrix whose exponent is at most the exponent of the primitive digraph $(\mathcal{E}, \mathcal{P}(A, K))$.

In the study of the exponents of a primitive matrix, the concept of local exponent has been introduced. If A is an $n \times n$ primitive matrix, then for any $i, j \in \{1, \ldots, n\}$, we denote by $\gamma(A : i, j)$ the smallest positive integer k such that the (i, j) entry of A^p is positive (or, equivalently, there is a directed walk in the usual digraph of A of length p from vertex i to vertex j) for all integers $p \geq k$. Since $\gamma(A) = \max\{\gamma(A : i, j) : 1 \leq i, j \leq n\}$, finding upper bounds for $\gamma(A)$ is equivalent to finding upper bounds for $\gamma(A : i, j), 1 \leq i, j \leq n$. (For reference, see [B–R, Section 3.5].)

Likewise, to study the exponents of K-primitive matrices, we make use of the concept of local exponent defined in the following way. For any K-nonnegative matrix A (not necessarily K-primitive or K-irreducible) and any $0 \neq x \in K$, by the *local exponent of* A *at* x, denoted by $\gamma(A, x)$, we mean the smallest nonnegative integer k such that $A^k x \in \operatorname{int} K$. If no such k exists, we set $\gamma(A, x)$ equal ∞. (If A is a primitive matrix and e_j is the jth standard unit vector, then $\gamma(A, e_j)$ equals $\gamma(A^T : j)$, which is defined to be the smallest integer k such that all elements in column j of A^k are nonzero.) Clearly, A is K-primitive if and only if the set of local exponents of A is bounded above; in this case, $\gamma(A)$ is equal to $\max\{\gamma(A, x) : 0 \neq x \in K\}$, which is also the same as the maximum taken over all nonzero extreme vectors of

K. By a compactness argument Barker [Bar 1] has shown that the K-primitivity of A is equivalent to the apparently weaker condition (which is also the definition adopted by him for K-primitivity) that all local exponents of A are finite.

It is known that for a K-nonnegative matrix A, if some positive power of A is K-positive, then so are all larger powers of A. For the action of a K-nonnegative matrix A on a vector $x \in K$, we have a similar property — if $A^i x$ belongs to $\text{int}\, K$, then so does $A^j x$ for all positive integers $j > i$. This assertion can be established inductively. The point is, if $A^i x \in \text{int}\, K$ but $A^{i+1} x \in \partial K$, then since A maps an interior vector of K (namely, $A^i x$) into ∂K, necessarily A maps K into ∂K, which is a contradiction.

By the definition of $(\mathcal{E}, \mathcal{P}(A, K))$, we have

Fact 2.1. If there is a path in $(\mathcal{E}, \mathcal{P}(A, K))$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\Phi(A^k x) \supseteq \Phi(y)$.

Making use of the preceding fact, we can readily show the following:

Fact 2.2. Let $A \in \pi(K)$ and let $x, y \in \text{Ext}\, K$. Suppose that $\gamma(A, y)$ is finite. If there is a path in $(\mathcal{E}, \mathcal{P})$ of length k from $\Phi(x)$ to $\Phi(y)$, then $\gamma(A, x)$ is also finite and we have $\gamma(A, x) \leq k + \gamma(A, y)$.

In many of our examples, our construction of a matrix (considered as a linear mapping) relies on the following easy result in linear algebra:

Lemma 2.3. *Let $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ be two families of vectors in finite-dimensional vector spaces V_1 and V_2 respectively. In order that there exists a linear mapping $T : V_1 \to V_2$ satisfying $T(x_i) = y_i$ for $i = 1, \ldots, k$, it is necessary and sufficient condition if $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$ is a relation for $x_1, \ldots, x_k$, then the corresponding relation $\alpha_1 y_1 + \cdots + \alpha_k y_k = 0$ also holds.*

Two proper cones K_1, K_2 are said to be *linearly isomorphic* if there exists a nonsingular linear transformation ϕ from span K_1 to span K_2, which maps K_1 onto K_2. If K_1 and K_2 are linearly isomorphic cones, then either K_1, K_2 both have finite exponent or they both do not have finite exponent, and if they have, then $\gamma(K_1) = \gamma(K_2)$. This is because, if ϕ is a linear isomorphism that takes K_1 onto K_2 then the linear mapping $A \mapsto \phi A \phi^{-1}$ takes $\pi(K_1)$ onto $\pi(K_2)$ and matches K_1-primitive matrices with K_2-primitive matrices having the same exponent. (Here $\phi A \phi^{-1}$ is interpreted as a composition of linear maps.)

Under inclusion as the partial order, the set of all faces of K, denoted by $\mathcal{F}(K)$, forms a lattice with meet and join given respectively by: $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$. Two proper cones K_1, K_2 are said to be *combinatorially equivalent*, if their face lattices $\mathfrak{F}(K_1)$ and $\mathfrak{F}(K_2)$ are isomorphic (as lattices).

In Tam [4] it is proved that the K-irreducibility or K-primitivity of a K-nonnegative matrix A is completely determined by the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ together with a knowledge of whether $E_1 \vee \cdots \vee E_j = K$ for every given finite collection $\{E_1, \ldots, E_j\}$ of extreme rays of K. In other words, for a K-nonnegative matrix A, if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given and K is known up to combinatorial equivalence, then the K-irreducibility or K-primitivity of A is completely determined.

3. Upper bounds for exponents

The following theorem of Sedláček [Sed] and Dulmage and Mendelsohn [D–M] (see, for instance, [B–R, Theorem 3.5.4]) gives an upper bound for the exponent of a primitive matrix A in terms of the lengths of circuits in its digraph.

Theorem B. *Let A be a primitive matrix of order n. If s is the length of the shortest circuit in the digraph of A, then $\gamma(A) \leq n + s(n-2)$.*

By setting $s = n - 1$ in Theorem B, one recovers the sharp general upper bound $(n-1)^2 + 1$, due to Wielandt [Wie] for exponents of primitive matrices of order n.

Below we give an analogous result of Theorem B on the local exponents of a cone-preserving map, which is essential to our work. Note that we state our result in a general form in which the underlying cone K need not be polyhedral and the cone-preserving map need not be K-primitive.

If D is a digraph, v is a vertex of D and W is a nonempty subset of vertices of D, then by the *distance* from v to W we mean the length (i.e., the number of edges) of the shortest path from v to a vertex of W. If v belongs to W, the distance is taken to be zero.

For a square matrix C, we denote by m_C the degree of the minimal polynomial of C.

Lemma 3.1. *Let K be a proper cone and let $A \in \pi(K)$. Let $\Phi(x)$ be a vertex of $(\mathcal{E}, \mathcal{P})$ which is at a distance $w (\geq 0)$ to a circuit $\mathcal{C}$ of length l. Suppose that A^l is*

K-irreducible. Then $\gamma(A,x)$ is finite and

$$\gamma(A,x) \leq w + (m_{A^l} - 1)l \leq w + (m_A - 1)l \leq w + (n-1)l.$$

Proof. Let $\mathcal{C} : \Phi(x_1) \to \Phi(x_2) \to \cdots \to \Phi(x_l) \to \Phi(x_1)$ be the circuit under consideration. (Here, for convenience, we represent the arc $(\Phi(x), \Phi(y))$ by $\Phi(x) \to \Phi(y)$.) Without loss of generality, we may assume that the distance from $\Phi(x)$ to $\Phi(x_1)$ is w. Since there is a path of length l from $\Phi(x_1)$ to itself, we have $\Phi(A^l x_1) \supseteq \Phi(x_1)$, which implies the following chain of inclusions:

$$\Phi(x_1) \subseteq \Phi(A^l x_1) \subseteq \Phi(A^{2l} x_1) \subseteq \cdots \subseteq \Phi(A^{jl} x_1) \subseteq \Phi(A^{(j+1)l} x_1) \subseteq \cdots.$$

Let p denote the dimension of the subspace $\mathrm{span}\{(A^l)^j x_1 : j = 0, 1, \ldots\}$. By the above chain of inclusions, clearly the face $\Phi((A^l)^{p-1} x_1)$ contains the vectors $x_1, A^l x_1, \ldots, (A^l)^{p-1}$ and hence includes $\Phi(\mathrm{span}\{(A^l)^j x_1 : j = 0, 1, \ldots\} \cap K)$, which is the smallest A^l-invariant face of K that contains x_1. Since A^l is K-irreducible, the latter face is K. So $(A^l)^{p-1} x_1) \in \mathrm{int}\, K$ and by Fact 2.2 again, we have

$$\gamma(A,x) \leq w + \gamma(A, x_1) \leq w + (p-1)l.$$

It is clear that $p \leq m_{A^l}$. But we also have $m_{A^l} \leq m_A$, so the desired inequalities follow. ■

For possible future use, we also include the following closely related result:

Lemma 3.2. *In Lemma 3.1 if we replace the assumption "A^l is K-irreducible" by "the circuit $\mathcal{C}$ contains a vertex $\Phi(u)$ for which $\gamma(A,u)$ is finite", the result is still valid.*

Proof. The argument given in the proof of Lemma 3.1 works here, except that we have to show that the smallest A^l-invariant face of K that contains x_1 is K. This can be seen as follows.

Since $\gamma(A,u)$ is finite for some vertex $\Phi(u)$ of $\mathcal{C}$, by Fact 2.2, $\gamma(A, x_1)$ is also finite. So $A^j x_1\, \mathrm{int}\, K$ for all positive integers j sufficiently large, which implies that the smallest A^l-invariant face of K that contains x_1 is K. ■

Using Lemma 3.1, one can readily deduce the following result.

Corollary 3.3. *Let K be a polyhedral cone in $\mathbb{R}^n$ ($n \geq 3$) with m extreme rays. Let $A \in \pi(K)$. Suppose that the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected. Let s be the shortest circuit length of the digraph. If A^s is K-irreducible, then A is K-primitive and $\gamma(A) \leq m + s(m_A - 2)$.*

Clearly, the following result of Niu [Niu] is a consequence of Corollary 3.3:

Theorem C. *Let K be a polyhedral cone with m extreme rays, and let A be K-primitive. If the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected and s is the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$, then $\gamma(A) \leq m + s(m-2)$.*

In Theorem C, by choosing $K = \mathbb{R}^n_+$ we recover Theorem B.

It is known (see [S–V]) that if A is K-irreducible, then $(I+A)^{n-1} \gg^K O$ (where n is the dimension of K). Hartwig and Neumann [H–N] have shown that in the nonnegative matrix case the result can be strengthened by replacing n by m_A, the degree of minimal polynomial of A. Now we can show that the latter improvement is also valid for a cone-preserving map on a proper cone.

Corollary 3.4. *If $A \in \pi(K)$ is K-irreducible, then $(I+A)^{m_A-1} \gg^K O$.*

Proof. If A is K-irreducible, then clearly $I+A$ is also K-irreducible and in the digraph $(\mathcal{E}, \mathcal{P}(I+A, K))$ there is a loop at each vertex. In view of Lemma 3.1, $(I+A)^{m_A-1}x \in \operatorname{int} K$ for each nonzero extreme vector x of K. Hence, we have $(I+A)^{m_A-1} \gg^K O$. ∎

We denote by $\mathcal{N}(A)$ the nullspace of A. It is easy to show that for any $A \in \pi(K), \mathcal{N}(A) \cap K = \{0\}$ if and only if the outdegree of each vertex of $(\mathcal{E}, \mathcal{P}(A, K))$ is positive. As a consequence, for any K-primitive matrix A, the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ has at least one circuit.

As yet another application of Lemma 3.1, we obtain the following result, which is an extension of the corresponding result for a symmetric primitive matrix (cf. [B–R, Theorem 3.5.3]).

Corollary 3.5. *Let $A \in \pi(K)$. If A is K-primitive and the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is symmetric, then*

$$\gamma(A) \leq 2(m_{A^2} - 1) \leq 2(m_A - 1).$$

Proof. Since A is K-primitive, $\mathcal{N}(A) \cap K = \{0\}$; hence the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ has an outgoing edge (possibly a loop) at each vertex. As the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is symmetric, it follows that $(\mathcal{E}, \mathcal{P}(A^2, K))$ has a loop at each vertex. By Lemma 3.1 we have $\gamma(A^2) \leq m_{A^2} - 1$ and hence $\gamma(A) \leq 2(m_{A^2} - 1) \leq 2(m_A - 1)$. ∎

It is clear that for any K-primitive matrix $A, m_A \geq 2$. When $m_A = 2$, more can be said.

Lemma 3.6. *Let A be K-primitive. If $m_A = 2$ then $\gamma(A) = 1$ or 2.*

Proof. Since $m_A = 2$ (and A is a real matrix), there exist real numbers a, b such that $A^2 + aA + bI = 0$. Clearly, a, b cannot be both zero, as A is not nilpotent. By the pointedness of the cone $\pi(K)$, at least one of a, b is negative. If $b < 0$ and $a \geq 0$, then A^2 belongs to the face $\Phi(I)$ (of $\pi(K)$) and so it must be K-reducible, which is a contradiction. If $a < 0$ and $b \geq 0$, then $A^2 \in \Phi(A)$ or $A^2 \leq \alpha A$ for some $\alpha > 0$, which implies that all positive powers of A lie in $\Phi(A)$. But A^p is K-positive, or equivalently, belongs to $\operatorname{int} \pi(K)$, for p sufficiently large. It follows that in this case we must have $\Phi(A) = \pi(K)$, or in other words, $\gamma(A) = 1$. In the remaining case when a, b are both negative, A^2 is a positive linear combination of A and I and hence lies in the relative interior of the face $\Phi(A + I)$. Then one readily shows that all positive powers of A also lie in $\operatorname{ri} \Phi(A+I)$. By the K-primitivity of A, A^p belongs to $\operatorname{int} \pi(K)$ for p sufficiently large. This implies that $\Phi(A + I) = K$ and hence A^2 also belongs to $\operatorname{int} \pi(K)$; so we have $\gamma(A) \leq 2$. This completes the proof. ■

The results of Lemma 3.1 or Corollary 3.3 may suggest that for a K-primitive matrix A the longer is the shortest circuit in $(\mathcal{E}, \mathcal{P}(A, K))$ the larger is the value of $\gamma(A)$. Note that if K is a polyhedral cone with m extreme rays, then for any K-primitive matrix A, the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is at most $m - 1$. This is because, if the length of the shortest circuit is m, then the digraph must be a circuit of length m and so A^m is K-reducible, which is a contradiction. In our next section we will identify all those digraphs on m vertices with the length of the shortest circuit equal to $m - 1$ which possibly can be realized as $(\mathcal{E}, \mathcal{P}(A, K))$ for some polyhedral cone K with m extreme rays and some K-primitive matrix A. It turns out that up to graph isomorphism there are two such digraphs.

4. Special digraphs for K-primitive matrices

Lemma 4.1. *Let K be a polyhedral cone with m (≥ 3) extreme rays. Let A be a K-primitive matrix. Then the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ equals $m - 1$ if and only if (up to graph isomorphism) the digraph*

$(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 1 *or Figure* 2 *:*

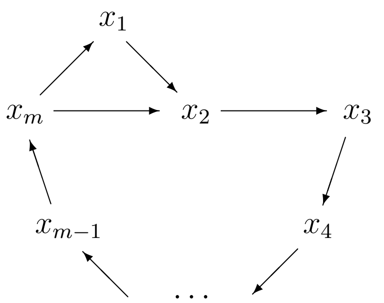

Figure 1.

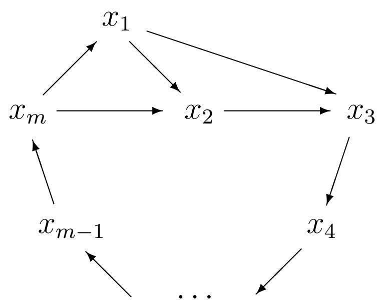

Figure 2.

(*For simplicity, we label the vertex* $\Phi(x_i)$ *simply by* x_i*.*).

Proof. The "if" part is obvious. To prove the "only if" part, let $x_1, \ldots, x_m$ denote the distinct extreme vectors of K. Let A be a K-primitive matrix such that the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is $m-1$. Without loss of generality, we may assume that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ contains the circuit $\mathcal{C}$ (of length $m-1$) that is made up of the arcs $(\Phi(x_m), \Phi(x_2))$ and $(\Phi(x_j), \Phi(x_{j+1}))$ for $j = 2, 3, \ldots, m-1$. Being a circuit of shortest length, $\mathcal{C}$ cannot contain any chord, nor can it have loops at its vertices. If there is no arc from a vertex of $\mathcal{C}$ to the remaining vertex $\Phi(x_1)$, then we have $A\Phi(x_m) = \Phi(x_2)$ and $A\Phi(x_j) = \Phi(x_{j+1})$ for $j = 2, 3, \ldots, m-1$ and it will follow that $A^{m-1}x_m$ is a positive multiple of x_m, hence A^{m-1} is K-reducible, which contradicts the assumption that A is K-primitive. So there is at least one arc from a vertex of $\mathcal{C}$ to $\Phi(x_1)$. Suppose $(\Phi(x_m), \Phi(x_1))$ is one such arc. Since $\mathcal{C}$ is a circuit of shortest length, none of the arcs $(\Phi(x_2), \Phi(x_1)), (\Phi(x_3), \Phi(x_1)), \ldots, (\Phi(x_{m-2}), \Phi(x_1))$ can be present. But we do not rule out the possibility that $(\Phi(x_{m-1}), \Phi(x_1))$ is an arc.

Similarly, there is also an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$. Since the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is $m-1$, there cannot be an arc of the form $(\Phi(x_1), \Phi(x_j))$ with $4 \leq j \leq m$. So $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ are the only possible arcs with initial vertex $\Phi(x_1)$, and at least one of them must be present. Note, however, that if $(\Phi(x_1), \Phi(x_3))$ is an arc but $(\Phi(x_1), \Phi(x_2))$ is not, then $A^{m-1}x_m$ is a positive multiple of x_m, which is a contradiction. So the arc $(\Phi(x_1), \Phi(x_2))$ must be present, but the arc $(\Phi(x_1), \Phi(x_3))$ may or may not be present.

Summing up, the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ contains precisely the following arcs: those that constitute the circuit $\mathcal{C}$, the arcs $(x_m, x_1), (x_1, x_2)$ and also possibly the arcs (x_1, x_3) or (x_{m-1}, x_1). But we should rule out also the case when the arcs (x_1, x_3) and (x_{m-1}, x_1) are both present. We can now conclude that up to graph isomorphism the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2, noting that the digraph that consists of the circuit $\mathcal{C}$ and the arcs $(x_m, x_1), (x_1, x_2), (x_{m-1}, x_1)$ is isomorphic with the one given by Figure 2. ■

Note that the digraphs given by Figure 1 and Figure 2 are precisely the (two known) primitive digraphs on m vertices with the length of the shortest circuit equal to $m-1$. Moreover, Figure 1 is the same as the (unique) digraph associated with an $m \times m$ primitive matrix whose exponent attains Wielandt's bound $m^2 - 2m + 2$.

Lemma 4.2. *Let K be a polyhedral cone with m (≥ 4) extreme rays. Let A be a K-nonnegative matrix. Suppose that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2. Then:*

(i) *A is K-primitive.*

(ii) *A is nonsingular.*

(iii) *$\gamma(A)$ equals $\gamma(A, x_1)$ or $\gamma(A, x_2)$ depending on whether the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2.*

(iv) *Assume, in addition, that K is non-simplicial. If $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 then K must be indecomposable. If the digraph is given by Figure 2 then either K is indecomposable or m is odd and K is the direct sum of a ray and an indecomposable minimal cone for which the unique linear relation on its extreme vectors has the same number of terms on its two sides.*

Proof. (i) Follows from the fact that the digraphs given by Figure 1 and Figure 2 are both primitive.

(ii) We treat the case when the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, the argument for the other case being similar. Then for $j = 1, \ldots, m-1, Ax_j$ is a positive multiple of x_{j+1}. So $x_2, x_3, \ldots, x_m$ all belong to $\mathcal{R}(A)$, the range space of A. On the other hand, since $x_2 \in \mathcal{R}(A)$ and Ax_m is a positive linear combination of x_1 and x_2, we also have $x_1 \in \mathcal{R}(A)$. Therefore, the linear map A is onto and hence is nonsingular.

(iii) Note that for any $0 \neq x \in K$ and $j = 0, 1, \ldots, \gamma(A, x) - 1, \gamma(A, x) = \gamma(A, A^j x) + j$.

First, consider the case when the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. For $j = 2, \ldots, m$, since $A^{j-1}x_1$ is a positive multiple of x_j, we have

$$\gamma(A, x_j) = \gamma(A, A^{j-1}x_1) = \gamma(A, x_1) - j + 1;$$

hence

$$\gamma(A) = \max_{1 \leq j \leq m} \gamma(A, x_j) = \gamma(A, x_1).$$

Now suppose the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2. For $j = 3, \ldots, m$, we have $\gamma(A, x_2) = \gamma(A, x_j) + j - 2$; hence $\gamma(A, x_2) > \gamma(A, x_j)$ for each such j. A little calculation shows that $A^m x_2$ is a positive linear combination of x_2 and x_3. But Ax_1 is also a positive linear combination of x_2 and x_3, hence $\Phi(Ax_1) = \Phi(A^m x_2)$. So we have

$$\gamma(A, x_2) = \gamma(A, A^m x_2) + m = \gamma(A, Ax_1) + m = \gamma(A, x_1) - 1 + m,$$

which implies $\gamma(A, x_2) > \gamma(A, x_1)$. Therefore, we have $\gamma(A) = \gamma(A, x_2)$.

(iv) First, we show that each of the extreme vectors $x_1, \ldots, x_m$, except possibly x_2, is involved in at least one relation on $\operatorname{Ext} K$. For the purpose, it suffices to show that x_3 is involved in one such relation; because, by applying A or its positive powers to a relation on $\operatorname{Ext} K$ involving x_3, we can obtain for each of the vectors $x_4, \ldots, x_{m-1}, x_m, x_1$ a relation that involves the vector. Suppose that x_3 is not involved in any (nonzero) relation on $\operatorname{Ext} K$. Take any relation S on $\operatorname{Ext} K$; as K is non-simplicial, such relation certainly exists. Note that, since x_3 does not appear in S, x_4 (and also x_3) cannot appear in the (necessarily nonzero) relation obtained from S by applying A. Similarly, the vectors x_3, x_4, x_5 all do not appear in the relation obtained from S by applying A^2. Continuing the argument, we can show that the only vectors that can appear in the nonzero relation obtained from S by applying A^{m-3} are x_1, x_2. This contradicts the hypothesis that x_1, x_2 are distinct nonzero extreme vectors of K.

Next, we note that if $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 and x_2 is not involved in any relation on $\operatorname{Ext} K$ then, by applying A repeatedly to a nonzero relation on $\operatorname{Ext} K$ sufficiently many times, we would obtain a nonzero relation on ExtK that involves less than four vectors, which is a contradiction. So if x_2 is not involved in any relation on $\operatorname{Ext} K$ then $(\mathcal{E}, \mathcal{P})$ must be given by Figure 2.

Now we contend that K is either indecomposable or is the direct sum of a ray and an indecomposable cone. By what we have done above, each of the extreme vectors $x_1, \ldots, x_m$, except possibly x_2, belongs to an indecomposable summand of

K that is not a ray. Let K_1 be the indecomposable summand of K that contains x_m. To establish our contention, it suffices to show that there is no relation on $\operatorname{Ext} K$ that involves vectors not belonging to K_1. Assume to the contrary that there are such relations. Let T_0 be one such shortest relation (i.e., one having the minimum number of terms). Note that, since x_m is not involved in T_0, the relation obtained from T_0 by applying A has the same number of terms as T_0, unless x_1 appears in T_0 (but x_2 does not) and $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, in which case the said relation may have one term more than T_0. Suppose that the extreme vectors that appear in T_0 are $x_{k_1}, x_{k_2}, \ldots, x_{k_s}$, where $1 \leq k_1 < k_2 < \cdots < k_s \leq m-1$. It is readily seen that the relations obtained from T_0 by applying A^i for $i = 1, \ldots, m-k_s$ all have the same number of terms.

Let q denote the least positive integer such that $x_{m-q} \notin K_1$. Certainly, we have $k_s \leq m-q$. If $k_s < m-q$, we replace T_0 by the relation obtained from T_0 by applying A^{m-q-k_s}. Hereafter, we assume that $k_s = m-q$. Because of the replacement, T_0 is now either a shortest relation (on extreme vectors not belonging to K_1) or has one term more than a shortest relation.

For $j = 1, 2, \ldots$, let T_j denote the relation obtained from T_0 by applying A^j. Note that relation T_1 has at most one term more than T_0 and it involves x_{m-q+1}, which belongs to K_1. If T_1 involves also extreme vectors not belonging to K_1, then T_1 splits and we would obtain a relation on extreme vectors not belonging to K_1, shorter than the shortest relation, which is a contradiction. So T_1 is a relation on $\operatorname{Ext} K_1$ and $x_{k_1+1}, x_{k_2+1}, \ldots, x_{k_s+1}$ all belong to K_1. By the same argument we may assume that for $j = 1, \ldots, q, T_j$ is a relation on $\operatorname{Ext} K_1$; or, in other words, we have $x_{k_j+r} \in K_1$ for $r = 1, \ldots, q$ and $j = 1, \ldots, s$. Note that x_m is involved in T_q but x_1 is not (as x_m is not involved in T_{q-1}). So x_1, x_2 are both involved in T_{q+1} and lie on the same side of it. As a consequence, x_2, x_3 are both involved in T_{q+2}, x_3, x_4 are both involved in T_{q+3}, and so forth. Clearly, T_{q+1} has one term more than T_q and hence at most two terms more than a shortest relation.

If T_{q+1} involves vectors not belonging to the same indecomposable summand of K, in particular, if x_1, x_2 belong to different indecomposable summands, then the relation splits and the minimality of T_0 is violated. So we assume that x_1, x_2 belong to the same indecomposable summand of K, say K_2 (which may or may not be the same as K_1), and T_{q+1} is a relation on $\operatorname{Ext} K_2$. For $j = 1, \ldots, s-1$, let l_j be the smallest positive integer, greater than $q+1$, such that $x_{k_j+l_j} \notin K_2$. Also, let l_s be the smallest positive integer, greater than $q+1$, such that $x_{l_s-q+1} \notin K_2$ and set $l = \min_{1 \leq j \leq s} l_j$. Note that $k_{s-1} + q + 1 \leq k_s$ as $k_{s-1} < k_s, x_{k_{s-1}+j} \in K_1$ for $j = 1, \ldots, q$ and $x_{k_s} \notin K_1$. If $k_{s-1} + q + 1 = k_s$, then since $x_{k_s} \notin K_1$, it follows that

$K_1 \neq K_2$ and $l_{s-1} = q + 2$; hence $k_{s-1} + l_{s-1} = k_s + 1$. If $k_{s-1} + q + 1 < k_s$, then since one of the alternatives $x_{k_s} \notin K_2$ or $x_{k_s+1} \notin K_2$ must occur, we have $k_{s-1} + l_{s-1} \leq k_s + 1$. In any case, the latter inequality always holds and hence we must have

$$k_{s-1} + l \leq k_s + 1 \leq m.$$

In view of the preceding inequality, the relations T_l and T_{q+1} have the same number of terms. By definition of l, T_l involves at least one vector not in K_2, but it also involves at least one vector in K_2, namely, x_{l-q}. So the relation splits and we obtain a relation shorter than a shortest relation, which is a contradiction.

It remains to show that if $(\mathcal{E}, \mathcal{P})$ is given by Figure 2 and K is the direct sum of the ray $\mathrm{pos}\{x_2\}$ and the indecomposable polyhedral cone generated by the remaining extreme vectors $x_1, x_3, x_4, \ldots, x_m$, then the latter cone is an indecomposable minimal cone such that the (essentially) unique linear relation on its extreme vectors has the same number of terms on its two sides (and, as a consequence, m must be odd). Note that the assumption that x_2 does not appear in any relation on $\mathrm{Ext}\, K$ guarantees that every relation obtained from a shortest relation on $\mathrm{Ext}\, K$ by applying A or its positive powers is still a shortest relation. We contend every shortest relation involves each of the vectors $x_1, x_3, \ldots, x_m$. Assume that the contrary holds. Take a shortest relation R. Since R has at least four terms, one of the vectors $x_3, x_4, \ldots, x_m$ must appear in R. On the other hand, R cannot involve all of these vectors; otherwise, x_1 does not appear in R, and so the relation obtained from R by applying A involves the vector x_2, which is a contradiction. Thus we can find an $i, 4 \leq i \leq m$, such that x_i appears in R and x_{i-1} does not or the other way round. Then the relation obtained from R by applying A^{m-i+1} involves one of the vectors x_m, x_1 but not both, and so the relation obtained from R by applying A^{m-i+2} must involve the vector x_2, which is a contradiction. This proves our contention. Since every shortest relation on $\{x_1, x_3, \ldots, x_m\}$ is a full relation, it is clear that any two relations on the latter set are multiples of each other; else, by subtracting an appropriate multiple of one relation from another we would obtain a shorter nonzero relation. This proves that the cone $\mathrm{pos}\{x_1, x_3, \ldots, x_m\}$ is minimal. Let R denote the linear relation on $\mathrm{Ext}\, K$. Since x_2 does not appear in any relation on $\mathrm{Ext}\, K, x_m, x_1$ must appear on opposite sides R. So x_1, x_3 also appear on opposite sides of the relation obtained from R by applying A and hence on opposite sides of relation R. Continuing the argument, we infer that for $j = 3, \ldots, m-1, x_j$ and x_{j+1} lie on opposite sides of R. It follows that m is odd and R has the same number of terms on its two sides. ■

Remark 4.3. If A is K-primitive and $m_A \geq 3$, then we have

$$(m_A - 1)(l - 2) + 2 \leq (m_A - 1)(l - 1)$$

for any real number l, where the inequality becomes equality if and only if $m_A = 3$.

Theorem 4.4. *Let K be an n-dimensional polyhedral cone with $m \geq 4$ extreme rays and let A be K-primitive. Then:*

(i) $\gamma(A) \leq (m_A - 1)(m - 1) + 1$, *where the equality holds only if the digraph* $(\mathcal{E}, \mathcal{P}(\mathcal{A}, \mathcal{K}))$ *is given by Figure* 1.

(ii) $\gamma(A) = (m_A - 1)(m - 1)$ *only if either* $(\mathcal{E}, \mathcal{P}(A, K))$ *is given by Figure* 1 *or Figure* 2*, or* $m_A = 3$ (*and* $\gamma(A) = (m_A - 1)(m - 2) + 2$).

(iii) $\gamma(A) = (m_A - 1)(m - 2) + 2$ *only if either* $(\mathcal{E}, \mathcal{P}(\mathcal{A}, \mathcal{K}))$ *is given by Figure* 1*, Figure* 2*, Figure* 3*, Figure* 4 *or Figure* 5 (*and* $m = 5$)*, or* $(\mathcal{E}, \mathcal{P}(\mathcal{A}, \mathcal{K}))$ *is obtained from Figure* 4 *by deleting any one or two of the three arcs* $(\Phi(x_{m-1}), \Phi(x_1))$, $(\Phi(x_m), \Phi(x_1))$ *and* $(\Phi(x_m), \Phi(x_2))$*, or from Figure* 3 *with* $m = 4$ *by adding the arc* $(\Phi(x_3), \Phi(x_1))$ *or substituting it for the arc* $(\Phi(x_4), \Phi(x_1))$.

(iv) *If* $(\mathcal{E}, \mathcal{P}(\mathcal{A}, \mathcal{K}))$ *is not given by Figures* 1– 5*, nor is derived from Figure* 4 *or from Figure* 3 (*with* $m = 4$) *in the way as described in part*(iii)*, then*

$$\gamma(A) \leq (m_A - 1)(m - 2) + 1.$$

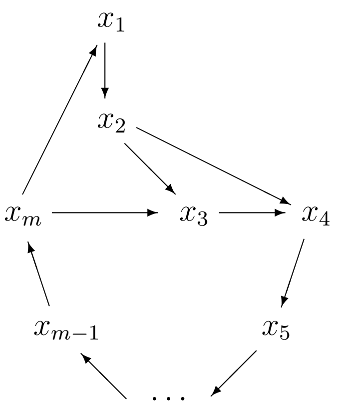

Figure 3.

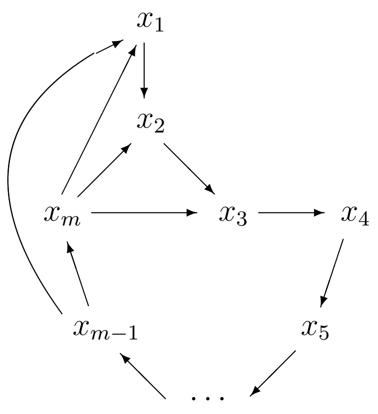

Figure 4.

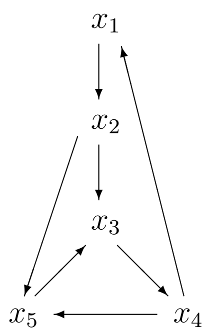

Figure 5.

Proof. When $m_A = 2$, by Lemma 3.6 we have $\gamma(A) \le 2$. As $m \ge 4$, in this case, the inequality $\gamma(A) \le (m_A - 1)(m-2) + 1$ is clearly satisfied and none of the equalities $\gamma(A) = (m_A - 1)(m-1)$ or $\gamma(A) = (m_A - 1)(m-2) + 2$ can be attained. Hereafter, we assume that $m_A \ge 3$.

If the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is m, then the digraph is necessarily an m-circuit, which is impossible as A is K-primitive. So the length of the shortest circuit is at most $m-1$.

(i) Since A is K-primitive, A is non-nilpotent. So the outdegree of each vertex of $(\mathcal{E}, \mathcal{P})$ is positive. Consider any vertex $\Phi(x)$ of the digraph $(\mathcal{E}, \mathcal{P})$. It is clear that $\Phi(x)$ lies on or has access to a circuit of length $l \le m-1$. By Lemma 3.1 we have

$$\begin{aligned} \gamma(A, x) &\le m - l + (m_{A^l} - 1)l = (m_{A^l} - 2)l + m \le (m_A - 2)l + m \\ &\le (m_A - 2)(m-1) + m = (m_A - 1)(m-1) + 1. \end{aligned} \tag{1}$$

Since this is true for every nonzero extreme vector x of K, the inequality $\gamma(A) \le (m_A - 1)(m-1) + 1$ follows.

Consider the case when the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is less than or equal to $m-2$. We contend that every vertex of the digraph lies on or has access to a circuit of length less than or equal to $m-2$. Take any vertex $\Phi(x)$ of the digraph. As we have explained before, $\Phi(x)$ lies on or has access to some circuit, say $\mathcal{C}$. If $\mathcal{C}$ is of length less than or equal to $m-2$, we are done. If $\mathcal{C}$ is of length m, then $\Phi(x)$ lies on $\mathcal{C}$. Furthermore, in order that A is K-primitive, $\mathcal{C}$ must contain a chord or has a loop at one of its vertices. Hence $\Phi(x)$ must lie on or has access to a circuit of length at most $m-2$. So suppose that $\mathcal{C}$ is of length $m-1$. We may assume that $\mathcal{C}$ does not contain a chord, nor does it have a loop at one of its vertices, else we are done. By the K-primitivity of A, we readily show that there is at least one arc from a vertex of $\mathcal{C}$ to the remaining vertex and vice versa (cf. the proof of Lemma 4.1). Hence, the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected. It follows that $\Phi(x)$ (and indeed every vertex of the digraph) lies on or has access to a circuit of shortest length (which is less than or equal to $m-2$ by our assumption). This proves our contention.

In this case, again by Lemma 3.1 and in view of the above contention and Remark 4.3, we have

$$\begin{aligned} \gamma(A) &\le (m_A - 2)(m-2) + m \\ &= (m_A - 1)(m-2) + 2 \\ &\le (m_A - 1)(m-1). \end{aligned} \tag{2}$$

When the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is $m-1$, by Lemma 4.1 the digraph is given by Figure 1 or Figure 2. If the digraph is given by Figure 2, then each vertex lies on a circuit of length $m-1$ and by Lemma 3.1 we obtain $\gamma(A) \leq (m_A - 1)(m-1)$.

Now we can conclude that in order that $\gamma(A) = (m_A - 1)(m-1) + 1$ it is necessary that the digraph is given by Figure 1.

(ii) Suppose that $\gamma(A) = (m_A - 1)(m-1)$. If the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is equal to $m-1$, then by Lemma 4.1 the digraph is given by Figure 1 or Figure 2. If the length of the shortest circuit is less than or equal to $m-2$ then since the second inequality in (2) becomes equality, by Remark 4.3 (with $l = m$) we have $m_A = 3$.

(iii) Suppose that $\gamma(A) = (m_A - 1)(m-2) + 2$. Note that in this case it is not possible that every vertex of $(\mathcal{E}, \mathcal{P})$ lies on or has access to a circuit of length $\leq m-3$ or is at a distance at most 1 to a circuit of length $m-2$, because then by (1) (with $l = m-3$) or by Lemma 3.1 (with $w = 1$ and $l = m-2$) it will follow that $\gamma(A) \leq (m_A - 1)(m-2) + 1$. On the other hand, if $(\mathcal{E}, \mathcal{P})$ has a vertex which lies on or has access to a circuit of length $m-1$ or m but does not lie on or has access to a circuit of length $m-2$ or less, then by the argument given in the proof for part(i) it will follow that the digraph $(\mathcal{E}, \mathcal{P})$ is strongly connected and the length of its shortest circuit is $m-1$. Then according to Lemma 4.1 $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. It remains to show that if $(\mathcal{E}, \mathcal{P})$ has a vertex which is at a distance 2 to a circuit of length $m-2$ and which does not lie on or has access to a circuit of length $m-3$ or less, nor is it at a distance at most 1 to another circuit of length $m-2$, then the digraph is given by Figure 3 or Figure 4, or is derived from them in the manner as described in the theorem.

To treat the remaining case we assume that the digraph $(\mathcal{E}, \mathcal{P})$ contains the circuit $\mathcal{C} : \Phi(x_3) \rightarrow \Phi(x_4) \rightarrow \cdots \rightarrow \Phi(x_{m-1}) \rightarrow \Phi(x_m) \rightarrow \Phi(x_3)$ and also the path $\Phi(x_1) \rightarrow \Phi(x_2) \rightarrow \Phi(x_3)$. For $i = 2, 3, \ldots, m$, since $\Phi(x_i)$ is at a distance at most 1 to the circuit $\mathcal{C}$, which is of length $m-2$, by Lemma 3.1 we have $\gamma(A, x_i) \leq (m_A - 1)(m-2) + 1$. This forces $\gamma(A, x_1) = (m_A - 1)(m-2) + 2$, which, in turn, implies that $\Phi(x_1)$ does not lie on or has access to a circuit of length $m-3$ or less, nor is $\Phi(x_1)$ at a distance at most 1 to a circuit of length $m-2$. Besides the arcs on the circuit $\mathcal{C}$ and the above-mentioned path, $(\mathcal{E}, \mathcal{P})$ certainly has other arcs. We want to find out what possible additional arcs there can be.

Note that there is at least one arc from a vertex of $\mathcal{C}$ to either one of the vertices $\Phi(x_1)$ or $\Phi(x_2)$; else, A^{m-2} maps the extreme ray $\Phi(x_3)$ of K onto itself, which contradicts the K-primitivity of A. Since x_1 is not allowed to lie on a circuit of length $m-2$ or less, none of the arcs $(\Phi(x_j), \Phi(x_1))$, for $j = 2, \ldots, m-2$, can be present. Similarly, since $\Phi(x_1)$ is not allowed to be at a distance 1 to a circuit of length $m-2$ or less, the arcs $(\Phi(x_j), \Phi(x_2))$, for $j = 1, \ldots, m-1$, also cannot be present. So $(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ are the only possible arcs from a vertex of $\mathcal{C}$ to either $\Phi(x_1)$ or $\Phi(x_2)$, and also we know that at least one of them is present.

There cannot exist an arc from $\Phi(x_1)$ to a vertex of $\mathcal{C}$, because in the presence of any such arc the distance from $\Phi(x_1)$ to the circuit $\mathcal{C}$ becomes 1. Similarly, for $m \geq 6$,the arcs $(\Phi(x_2), \Phi(x_j))$, for $j = 5, \ldots, m$, also cannot exist. So $(\Phi(x_2), \Phi(x_4))$ is the only possible arc from $\Phi(x_1)$ or $\Phi(x_2)$ to a vertex of $\mathcal{C}$ when $m \geq 6$. But when $m = 5$, the arc $(\Phi(x_2), \Phi(x_5))$ may be present. It is readily seen that in this case the arcs $(\Phi(x_5), \Phi(x_1))$ and $(\Phi(x_4), \Phi(x_2))$ both cannot be present. So the arc $(\Phi(x_4), \Phi(x_1))$ must be present and the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 5.

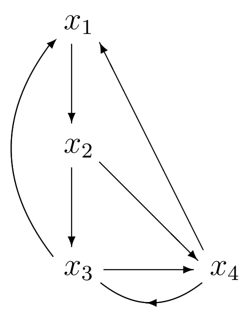

Figure $3'$

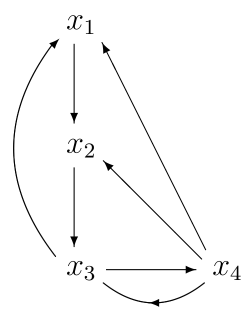

Figure $4'$

Consider the case when the arc $(\Phi(x_2), \Phi(x_4))$ is present. If $m \geq 5$, then neither of the arcs $(\Phi(x_m), \Phi(x_2))$ or $(\Phi(x_{m-1}), \Phi(x_1))$ can be present, else $\Phi(x_1)$ is at a distance at most 1 to a circuit of length $m-2$, which is not allowed. So in this case the arc $(\Phi(x_m), \Phi(x_1))$ must be present and the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 3. If $m = 4$, we find that the arc $(\Phi(x_4), \Phi(x_2))$ cannot be present, but the arcs $(\Phi(x_3), \Phi(x_1))$ and $(\Phi(x_4), \Phi(x_1))$ may be present and, indeed, at least one of them must be present. So the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure $3'$ or is obtained from it by deleting one of the arcs $(\Phi(x_3), \Phi(x_1)), (\Phi(x_4), \Phi(x_1))$. In other words, the digraph is given by Figure 3 (with $m = 4$) or is derived from it in the manner as described in the theorem.

Now consider the remaining case when the arc $(\Phi(x_2), \Phi(x_4))$ is absent. The presence of any one, two or three of the arcs $(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1))$ and

$(\Phi(x_m), \Phi(x_2))$ will produce only circuits of length at least $m-1$, but that causes no problem. Then the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 4 (which becomes Figure $4'$ when $m=4$) or is obtained from it by deleting any one or two of the three arcs $(\Phi(x_{m-1}), \Phi(x_1)), (\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$.

(iv) Done in the course of proving (iii). ■

In passing we would like to point out that the proof of Theorem 4.4(i) also shows that $(m_A - 1)(m-2) + 2$ is an upper bound for $\gamma(A)$ if the length of the shortest circuit in the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is less than or equal to $m-2$ (or, equivalently, if the digraph is not given by Figure 1 or Figure 2). In fact, we can state a better result, after we obtain the following general result for a digraph:

Remark 4.5. Let D be a digraph on m vertices, each of which has positive out-degree. If the length of the shortest circuit in D is greater than $[\frac{m-1}{2}]$, then every vertex of D lies on or has access to a circuit of D of shortest length.

Here is the proof: Since each vertex of D has positive out-degree, each vertex lies on or has access to a circuit. Denote by $s(D)$ the length of the shortest circuit in D. If there is a vertex that does not lie on or has access to a circuit of length $s(D)$, then such vertex must lie on or has access to a circuit, say $\mathcal{C}$, of length $s(D)+1$ or more. It is clear that the circuit $\mathcal{C}$ is vertex disjoint from every circuit of shortest length. Consequently, we have $m \geq s(D) + (s(D)+1)$ or $[\frac{m-1}{2}] \geq s(D)$, which is a contradiction.

Remark 4.6. Let K be an n-dimensional polyhedral cone with m extreme rays. Let A be a K-primitive matrix. Let s be the length of the shortest circuit in $(\mathcal{E}, \mathcal{P}(A, K))$. If $s > [\frac{m-1}{2}]$, then

$$\gamma(A) \leq s(m_A - 1) + 2 \leq s(n-1) + 2.$$

It is interesting to note that the digraphs given by Figure 3 and Figure 4 and those given by Figure 1 and Figure 2 share at least two common properties: they are all primitive, and if $(\mathcal{E}, \mathcal{P}(A, K))$ is given by one of them then necessarily A is nonsingular (cf. Lemma 4.2(i) and (ii)). However, the digraph obtained from Figure $4'$ (i.e., Figure 4 with $m=4$) by removing the arcs (x_4, x_2) and (x_3, x_1) is strongly connected but not primitive, whereas the one obtained from Figure $3'$ (i.e., Figure 3 with $m=4$) by removing the arcs (x_4, x_1) and (x_3, x_1) is not even strongly connected.

Corollary 4.7. *If K is an n-dimensional polyhedral cone with $m = n + k \geq 4$ extreme rays, then $\gamma(K) \leq (n-1)(m-1)+1 = m^2-(k+2)m+k+2$. The equality holds only if there exists a K-primitive matrix A with $m_A = n$ such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1.

Proof. Follows from part(i) of Theorem 4.4, as the maximum value of m_A taken over all K-primitive matrices A is n. ■

We can now give an affirmative answer to Kirkland's conjecture mentioned at the beginning of Section 1.

Corollary 4.8. *For any positive integer $m \geq 4$,*

$$\max\{\gamma(K) : K \text{ is a polyhedral cone with } m \text{ extreme rays }\} = m^2 - 2m + 2.$$

Proof. Let K be an n-dimensional polyhedral cone with m extreme rays. Since $n \leq m$, by Corollary 4.7, $\gamma(K) \leq (m-1)^2 + 1$. So we have

$$\max\{\gamma(K) : K \text{ is a polyhedral cone with } m \text{ extreme rays}\} \leq m^2 - 2m + 2.$$

On the other hand, by Wielandt's bound we also have $\gamma(\mathbb{R}^m_+) = m^2-2m+2$. Hence, the desired equality follows. ■

We would like to emphasize that in Corollary 4.8 the number of extreme rays (i.e., m) for the polyhedral cones K under consideration is fixed but there is no restriction on their dimensions (i.e., n). As for the quantity

$$\max\{\gamma(K) : K \text{ is an } n\text{-dimensional polyhedral cone with } m \text{ extreme rays}\}$$

by Corollary 4.7 it is always less than or equal to $(n-1)(m-1)+1$, but for general m, n we do not know its exact value. In the next section, we are going to determine the exact value of the said quantity when $m = n + 1$, i.e., when the cones under consideration are minimal.

5. Minimal cones case

For a polyhedral cone K with m extreme rays and a K-primitive matrix A, Lemma 4.2 gives us information on K and A when the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or Figure 2. The results of this section will show that if, in addition, K is a minimal cone then much more can be said.

Lemma 5.1. *Let K be a polyhedral cone with m (≥ 4) extreme rays. Let A be a K-nonnegative matrix.*

(i) *Suppose K is (indecomposable) minimal and $(\mathcal{E},\mathcal{P})$ is given by Figure 1. If n is odd, then after normalization the linear relation on* $\operatorname{Ext} K$ *and the matrix A are given respectively by:*

$$x_1 + x_3 + \cdots + x_{m-3} + x_{m-1} = x_2 + x_4 + \cdots + x_{m-2} + x_m, \tag{3}$$

and

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2, \text{ where } \alpha > 0, \\ Ax_i &= x_{i+1} \text{ for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2. \end{aligned} \tag{4}$$

If n is even, then after normalization the linear relation on $\operatorname{Ext} K$ *and the matrix A are given respectively by :*

$$x_1 + x_2 + x_4 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m, \tag{5}$$

and

$$\begin{aligned} Ax_1 &= \alpha x_2, \text{ where } \alpha > 0, \\ Ax_i &= x_{i+1} \text{ for } i = 2, 3, \ldots, m-1, \\ Ax_m &= x_1 + (1+\alpha)x_2. \end{aligned} \tag{6}$$

(ii) *Suppose K is indecomposable, minimal and $(\mathcal{E},\mathcal{P})$ is given by Figure 2. If n is even, then, after normalization, the linear relation on* $\operatorname{Ext} K$ *and the matrix A are given respectively by (5) and (7), or by (8) and (9):*

$$\begin{aligned} Ax_1 &= \alpha x_2 + (1-\beta)x_3, \text{ where } \alpha > 0, 0 < \beta < 1, \\ Ax_2 &= \beta x_3, \\ Ax_i &= x_{i+1}, \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + (1+\alpha)x_2. \end{aligned} \tag{7}$$

or

$$x_2 + x_3 + x_5 + \cdots + x_{m-2} + x_m = x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1}, \tag{8}$$

and

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2 + (1+\beta)x_3, \text{ where } \alpha, \beta > 0 \\ Ax_2 &= \beta x_3, \\ Ax_i &= x_{i+1} \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2. \end{aligned} \tag{9}$$

If n is odd, then after normalization, the relation on Ext K *is given by relation* (3) *and the matrix* A *is given by equation* (10):

$$\begin{aligned} Ax_1 &= (1+\alpha)x_2 + \beta x_3, \text{ where } \alpha, \beta > 0, \\ Ax_2 &= (1+\beta)x_3, \\ Ax_i &= x_{i+1}, i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2. \end{aligned} \tag{10}$$

(iii) *Suppose* K *is the direct sum of a ray and an indecomposable minimal cone for which the unique linear relation on its extreme vectors has the same number of terms on its two sides and* $(\mathcal{E}, \mathcal{P})$ *is given by Figure 2. Then after normalization the unique linear relation on* Ext K *and the matrix* A *are given respectively by*:

$$x_1 + x_4 + x_6 + \cdots + x_{m-3} + x_{m-1} = x_3 + x_5 + \cdots + x_{m-2} + x_m \tag{11}$$

and

$$\begin{aligned} Ax_1 &= \alpha x_2 + x_3, \text{ where } \alpha > 0, \\ Ax_2 &= \beta x_3, \text{ where } \beta > 0, \\ Ax_i &= x_{i+1} \text{ for } i = 3, \ldots, m-1, \\ Ax_m &= x_1 + \alpha x_2. \end{aligned} \tag{12}$$

Proof. (i) Let R denote the unique linear relation on Ext K. First, we contend that x_m, x_1 lie on different sides of R. Suppose not. Then x_1, x_2 lie on the same side of the relation obtained from R by applying A. But the latter relation is just a multiple of R, so x_1, x_2 also lie on the same side of relation R. By continuing the argument we can then obtain that $x_1, x_2, \ldots, x_m$ all lie on the same side of R, which is a contradiction. This proves our contention. The same argument, in fact, also shows that for $j = 2, 3, \ldots, m-1$, x_j, x_{j+1} lie on different sides of R. Now a simple parity count shows that x_2, x_m lie on the same side or opposite sides of R, depending on whether n is odd or even. So when n is odd (i.e., m is even), after normalizing the extreme vectors of K we may assume that relation R is given (3).

As the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, we have

$$Ax_1 = \beta x_2, Ax_i = \lambda_{i+1} x_{i+1} \text{ for } i = 2, \ldots, m-1 \text{ and } Ax_m = \lambda_1 x_1 + \alpha x_2,$$

where α, β and $\lambda_1, \lambda_3, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Substituting the values of the Ax_is into the relation obtained from (3) by applying A, we obtain the relation:

$$\beta x_2 + \lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_m x_m = \lambda_3 x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-1} x_{m-1} + (\lambda_1 x_1 + \alpha x_2).$$

But relation (3) and the above relation are positive multiples of each other, it follows that we have $\lambda_1 = \lambda_3 = \lambda_4 = \cdots = \lambda_m$ and $\beta = \lambda + \alpha$, where we use λ to denote the common value of the λ_js. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by (4).

When n is even, we can show in a similar way that the unique linear relation and the matrix A are given by (5) and (6) respectively.

(ii) We consider the case when n is even first. By the same kind of argument that we have used for part (i) we can show that for $j = 3, \ldots, m$, the vectors x_j, x_{j+1} lie on different sides of the relation on $\operatorname{Ext} K$ (where x_{m+1} is taken to be x_1). Hence, the vectors $x_3, x_5, \ldots, x_m$ lie on one side of the relation and the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on the other side. As for the vector x_2 it can be on either side. If x_2 is on the same side as x_1 then, after normalizing the extreme vectors of K, we may assume that the relation on $\operatorname{Ext} K$ is given by (5); if x_2 lies on the side opposite to x_1, we may assume that the relation is given by (8).

We treat the subcase when the relation is given by (5), the argument for the remaining subcase being similar. Since the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2, we have

$$Ax_1 = \alpha x_2 + \gamma x_3, Ax_2 = \beta x_3, Ax_i = \lambda_{i+1} x_{i+1} \text{ for } i = 3, \ldots, m-1 \text{ and } Ax_m = \lambda_1 x_1 + \delta x_2,$$

where $\alpha, \beta, \delta, \gamma$ and $\lambda_1, \lambda_3, \lambda_4, \ldots, \lambda_m$ are some positive numbers. Applying A to relation (5), we obtain the relation:

$$\lambda_4 x_4 + \lambda_6 x_6 + \cdots + \lambda_{m-1} x_{m-1} + (\lambda_1 x_1 + \delta x_2) = (\alpha x_2 + \gamma x_3) + \beta x_3 + \lambda_5 x_5 + \cdots + \lambda_{m-2} x_{m-2} + \lambda_m x_m.$$

But relation (5) and the above relation are positive multiples of each other, it follows that we have $\lambda_4 = \lambda_5 = \cdots = \lambda_m = \lambda$, say, and $\lambda_1 = \lambda, \delta = \lambda + \alpha$ and $\gamma + \beta = \lambda$. Replacing A by a positive multiple, we may assume that $\lambda = 1$. Then A is given by equation (7).

Now we consider the case when n is odd. Again, we can show that for $j = 3, \ldots, m$, the vectors x_j, x_{j+1} lie on different sides of the relation on $\operatorname{Ext} K$. Hence, $x_1, x_3, x_5, \ldots, x_{m-3}, x_{m-1}$ lie on one side of the unique relation and $x_4, x_6, \ldots, x_{m-2}, x_m$ lie on the other side. If x_1, x_2 lie on the same side of the relation then, since x_1, x_3 also lie on the same side, the same is true for the pair x_2, x_3. Then by applying A we find that x_3, x_4 also lie on the same side of the relation, which contradicts what we have observed above. So x_2 lies on the same side as $x_4, x_6, \ldots, x_m$, and after normalization we may assume that the unique relation is given by (3). In a similar way as before we can also show that after normalization A is given by (10).

(iii) Suppose K is the direct sum of the ray $\operatorname{pos}\{x_2\}$ and the indecomposable minimal cone $\operatorname{pos}\{x_1, x_3, \ldots, x_m\}$. In this case, by part(iv) of Lemma 4.2, m is odd. From the last part of the proof for Lemma 4.2(iv) we also know that in the linear relation on $\operatorname{Ext}(K)$ the vectors $x_1, x_4, x_6, \ldots, x_{m-1}$ lie on one side and the vectors $x_3, x_5, \ldots, x_m$ lie on the other side. After normalizing the extreme vectors of K, we may assume that the relation on $\operatorname{Ext} K$ is given by relation (11). By the same kind of argument as before, we can also show that A, after normalization, is given by equation (12). ■

Lemma 5.2. *Let K be an n-dimensional minimal cone with extreme vectors $x_1, \ldots, x_m$ (where $m = n + 1$). Let A be an $n \times n$ real matrix. Then:*

(i) *A is K-primitive and $\gamma(A) = n^2 - n + 1$ if the linear relation on* $\operatorname{Ext} K$ *and the matrix A are given by relation (3) and equation (4) respectively.*

(ii) *A is K-primitive and $\gamma(A) = n^2 - n$ if the linear relation on* $\operatorname{Ext} K$ *and the matrix A are given respectively by relation (11) and equation (12), or relation (5) and equation (6), or relation (8) and equation (9), or relation (3) and equation (10).*

(iii) *A is K-primitive and $\gamma(A) = n^2 - n - 1$ if the linear relation on* $\operatorname{Ext} K$ *and the matrix A are given by relation (5) and equation (7) respectively.*

Proof. (i) First, note that A is well-defined, as it preserves the linear relation on $\operatorname{Ext} K$. Also, we can see as follows that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 and as a consequence A is K-primitive. It is clear that for $j = 1, \ldots, m-1$, $(\Phi(x_i), \Phi(x_{i+1}))$

is the only outgoing arc from vertex $\Phi(x_i)$. By definition $Ax_m = x_1+\alpha x_2$, so $\Phi(Ax_m)$ equals $\Phi(x_1 + x_2)$, which is the smallest face of K containing x_1, x_2. By Theorem A $\Phi(x_1 + x_2)$ is a nontrivial face. Since every nontrivial face of a minimal cone is simplicial, x_1, x_2 are the only extreme vectors of $\Phi(x_1 + x_2)$. It follows that $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$ are the only outgoing arcs from vertex $\Phi(x_m)$.

A straightforward calculation yields the following: $A^{m-1}x_1 = (1+\alpha)x_m$; $A^m x_1 = (1 + \alpha)(x_1 + \alpha x_2)$, i.e., $\Phi(A^m x_1) = \Phi(x_1 + x_2)$; and $\Phi(A^{j(m-1)}x_1) = \Phi(x_{m-j+1} + x_{m-j+2} + \cdots + x_{m-1} + x_m)$ for $j = 1, \ldots, m-2$. Hence, $A^{(m-2)(m-1)}x_1$ is a positive linear combination of $x_3, x_4, \ldots, x_m$ and so by Theorem A it belongs to the relative interior of a maximal face of K. On the other hand, $A^{(m-2)(m-1)+1}x_1)$ belongs to $\operatorname{int} K$ as it can be written as a positive linear combination of all x_i except x_3. So $\gamma(A, x_1) = (m-2)(m-1) + 1 = n^2 - n + 1$. But $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, by Lemma4.2(iii), $\gamma(A) = \gamma(A, x_1) = n^2 - n + 1$.

(ii) Suppose the linear relation on $\operatorname{Ext} K$ and the matrix A are given by relation (11) and equation (12) respectively. We readily check that A is well-defined and $(\mathcal{E}, \mathcal{P})$ is given by Figure 2; so A is K-primitive. A straightforward calculation yields the following: $A^{m-2}x_2 = \beta x_m$; $A^{m-1}x_2 = \beta(x_1 + \alpha x_2)$, i.e., $\Phi(A^{m-1}x_2) = \Phi(x_1 + x_2)$; $\Phi(A^{j(m-1)-1}x_2) = \Phi(x_{m-j+1} + x_{m-j+2} + \cdots + x_m)$ for $j = 1, \ldots, m-2$ and, in particular, $A^{(m-1)(m-2)-1}x_2$ is a positive linear combination of $x_3, x_4, \ldots, x_m$. So $A^{(m-1)(m-2)-1}x_2$ belongs to the direct summand $\operatorname{pos}\{x_1, x_j, 3 \leq j \leq m\}$ of K and hence lies in ∂K. On the other hand, $A^{(m-1)(m-2)}x_2$ belongs to $\operatorname{int} K$, as it can be written a positive linear combination of all x_is except x_3. This shows that $\gamma(A, x_2) = (m-1)(m-2) = n(n-1)$. In view of Lemma 4.2(iii), we have $\gamma(A) = \gamma(A, x_2) = n(n-1)$.

When the linear relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5) and equation(6) respectively, a straightforward calculation shows that $A^{m^2-3m+1}x_1$, being a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$, belongs to ∂K, and $A^{m^2-3m+2}x_1$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to $\operatorname{int} K$. Since $(\mathcal{E}, \mathcal{P})$ is given by Figure 1, we have $\gamma(A) = \gamma(A, x_1) = m^2 - 3m + 2 = n^2 - n$.

When the linear relation on $\operatorname{Ext} K$ and the matrix A are given by relation (8) and equation(9) respectively, a straightforward calculation shows that $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to ∂K, whereas $A^{(n-1)(m-1)}x_2$ belongs to $\operatorname{int} K$, as it can be written as a positive linear combination of all the x_is except x_3. Since $(\mathcal{E}, \mathcal{P})$ is given by Figure 2, we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) = n^2 - n$.

Similarly, we can show that A is K-primitive and $\gamma(A) = n^2 - n$ when the linear relation and the matrix A are given by relation (5) and equation (10) respectively.

(iii) When the linear relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5) and equation (7) respectively, we find that $A^{(n-1)(m-1)-2}x_2$ is a positive linear combination of $x_2, x_3, \ldots, x_{m-1}$ and so belongs to ∂K, whereas $A^{(n-1)(m-1)-1}x_2$, being a positive linear combination of $x_3, x_4, \ldots, x_m$, belongs to $\operatorname{int} K$. It follows that we have $\gamma(A) = \gamma(A, x_2) = (n-1)(m-1) - 1 = n^2 - n - 1$. ■

Theorem 5.3. *Let $n \geq 3$ be a given positive integer.*

(I) *The maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cones is $n^2 - n + 1$ if n is odd, and is $n^2 - n$ if n is even.*

(II) *Suppose n is odd.*

(i) *For any n-dimensional minimal cone K, $\gamma(K) = n^2 - n + 1$ if and only if K is indecomposable and the linear relation on its extreme vectors has the same number of terms on its two sides.*

(ii) *Let K be an n-dimensional minimal cone for which the linear relation on its extreme vectors has the same number of terms on its two sides. For any K-primitive matrix $A, \gamma(A) = n^2 - n + 1$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1.

(III) *Suppose n is even.*

(i) *For any n-dimensional minimal cone $K, \gamma(K) = n^2 - n$ if and only if either K is indecomposable and in the linear relation on its extreme vectors the number of terms on its two sides differ by* 1*, or K is the direct sum of a ray and an indecomposable minimal cone for which the linear relation on its extreme vectors has the same number of terms on its two sides.*

(ii) *Let K be an indecomposable minimal cone such that in the linear relation on its extreme vectors the number of terms on its two sides differ by* 1*. For any K-primitive matrix $A, \gamma(A) = n^2 - n$ if and only if either the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 1*, or the digraph is given by Figure* 2 *and x_1, x_2 appear on opposite sides of the linear relation on the extreme vectors of K.*

(iii) *Let K be the direct sum of a ray and an indecomposable minimal cone for which the linear relation on its extreme vectors has the same number of terms on its two sides. For any K-primitive matrix $A, \gamma(A) = n^2 - n$ if and only if the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure* 2.

Proof. We first observe that when n is even, there is no minimal cone K such that $\gamma(K) = n^2 - n + 1$. Assume to the contrary that there is one such K. Choose a K-primitive matrix A that satisfies $\gamma(A) = n^2 - n + 1$. By Corollary 4.7 $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. Since n is even, by the second half of Lemma 5.1(i), the linear relation on $\operatorname{Ext} K$ and the matrix A are given by relation (5) and equation (6) respectively. So by Lemma 5.2(ii), we have $\gamma(A) = n^2 - n$, which contradicts our choice of A.

For any positive integer n, by Corollary4.7, $\gamma(K) \leq n^2 - n + 1$ for every n-dimensional minimal cone K.

Let n be odd. Take any n-dimensional indecomposable cone K for which the linear relation on its extreme vectors has the same number of terms on its two sides. After re-indexing and normalizing the extreme vectors $x_1, \ldots, x_m$ of K (where $m = n + 1$), we may assume that the linear relation on $\operatorname{Ext} K$ is given by relation (3). Let A be the $n \times n$ real matrix given by equation (4). By Lemma 5.2(i) A is K-primitive and $\gamma(A) = n^2 - n + 1$. So we have $\gamma(K) = n^2 - n + 1$. This establishes (I) for odd n as well as the "if" part of (II)(i).

Now let n be even. In view of the above observations, the maximum value of $\gamma(K)$ as K runs through all n-dimensional minimal cone is at most $n^2 - n$. We are going to show that the value $n^2 - n$ can be attained.

Take any indecomposable minimal cone K such that in the linear relation on $\operatorname{Ext} K$ the number of vectors on its two sides differs by 1. Normalizing the extreme vectors of K, we may assume that the linear relation is given by (5). Let A be the matrix given by equation (6). By Lemma 5.2(ii) we have $\gamma(A) = n^2 - n$. For this K, certainly we have $\gamma(K) = n^2 - n$. This establishes (I) for even n and completes the proof for (I).

If K is the direct sum of a ray and an indecomposable minimal cone for which the linear relation on its extreme vectors has same number of terms on its two sides, then after normalization we may assume that the linear relation is given by (11). By Lemma 5.2(ii) again, the matrix A defined by equation (12) satisfies $\gamma(A) = n^2 - n$.

So we have also established the "if" part of (III)(i).

To prove the "only if" part of (II)(i), let K be an n-dimensional minimal cone that satisfies $\gamma(K) = n^2 - n + 1$. By Corollary 4.7 there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. By part(iv) of Lemma 4.2, K is indecomposable, and by part(i) of Lemma 5.1, after normalizing and re-indexing, the relation on $\operatorname{Ext} K$ is given by relation (3). So the relation has the same number of terms on its two sides.

The "only if" part of (II)(i) follows from part(i) of Theorem 4.4 (by taking

$m = n+1$ and $m_A = n$), whereas its "if" part is a consequence of Lemma 5.1(i) and Lemma 5.2(i).

The proof of part(II) is completed.

To prove the "only if" part of (III)(i), let K be an n-dimensional minimal cone such that $\gamma(K) = n^2 - n$. Choose a K-primitive matrix A such that $\gamma(A) = \gamma(K)$. By part(ii) of Theorem 4.4, in this case we have $m_A = n$ and either $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2, or $m_A = 3$. The case $m_A = 3$ cannot happen; otherwise, $n = 3$, contradicting the assumption that n is even. Then, by part(iv) of Lemma 4.2, K is either indecomposable or is the direct sum of a ray and an indecomposable minimal cone for which the linear relation on its extreme vectors has the same number of terms on its two sides. In the latter case, we are done. In the former case, $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. If the digraph is given by Figure 1 then, since K is indecomposable minimal, by part(i) of Lemma 5.1, after normalization, the linear relation on $\operatorname{Ext} K$ is given by relation(5). If the digraph is given by Figure 2, then by part(ii) of the same lemma, after normalization, the linear relation on $\operatorname{Ext} K$ is given by relation(5) or relation(8). In any case, in the relation on ExtK the number of terms on its two sides differ by 1.

Note that the argument given in the above proof for the "only if" part of (III)(i) almost also establishes the "only if" part of (III)(ii), except that we need to explain why when the digraph is given by Figure 2, the vectors x_1, x_2 appear on opposite sides of the linear relation on $\operatorname{Ext} K$. To complete the argument, it suffices to rule out the possibility that the relation is given by (5). Suppose the relation is given by (5). Then according to part(i) of Lemma 5.1, after normalization, the matrix A is given by equation (7). So by part(iii) of Lemma 5.2, we have $\gamma(A) = n^2 - n - 1$, which is a contradiction.

To prove the "if" part of (III)(ii), first suppose $(\mathcal{E}, \mathcal{P})$ is given by Figure 1. In this case by Lemma 5.1(i), after normalization, the linear relation and the matrix A are given by relation(5) and equation(6) respectively. Then by Lemma 5.2(ii), $\gamma(A) = n^2 - n$. Now consider the case when the digraph is given by Figure2 and moreover x_1, x_2 appear on opposite sides of the linear relation on $\operatorname{Ext} K$. By part(ii) of Lemma 5.1, after normalization, the linear relation on $\operatorname{Ext} K$ and the matrix A are given respectively by relation (5) and equation (7), or by relation (8) and equation (9). Since it is assumed that x_1, x_2 appear on opposite sides of the relation, we rule out the former possibility. Then by Lemma 5.2(ii), $\gamma(A) = n^2 - n$.

The "if" part of (III)(iii) follows from Lemma 5.1(iii) and Lemma 5.2(ii). To prove the "only if" part, let A be a K-primitive matrix such that $\gamma(A) = n^2 - n$. By the argument given in the proof for the "only if" part of (III)(i), we can show that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. According to part(iv) of Lemma 4.2, if the digraph is given by Figure 1, then K is necessarily indecomposable. But now it is given that K is decomposable, so the digraph is given by Figure 2.

The proof is complete. ∎

By Corollary 4.7 and Theorem 5.3(I) we readily deduce the following

Corollary 5.4. *For any positive integer $m \geq 4$, the maximum value of $\gamma(K)$ as K runs through all non-simplicial polyhedral cones K with m extreme rays (of all possible dimensions) is $m^2 - 3m + 3$ if m is even, and is $m^2 - 3m + 2$ if m is odd.*

6. The 3-dimensional case

Two distinct extreme rays $\Phi(x), \Phi(y)$ (or, distinct extreme vectors x, y) of K are said to be *neighborly* if $x + y \in \partial K$.

Lemma 6.1. *Let K be a 3-dimensional polyhedral cone with extreme vectors $x_1, \ldots, x_m$. Let $A \in \pi(K)$ and suppose that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 or Figure 2. Then:*

(i) *For $i = 1, \ldots, m, \Phi(x_i)$ and $\Phi(x_{i+1})$ (where $\Phi(x_{m+1})$ is taken to be $\Phi(x_1)$) are neighborly extreme rays of K.*

(ii) *$\gamma(A)$ equals $2m-1$ or $2m-2$ depending on whether $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 or by Figure 2.*

Proof. First, consider the case when the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. Since $(\Phi(x_m), \Phi(x_1)), (\Phi(x_m), \Phi(x_2))$ are arcs of $(\mathcal{E}, \mathcal{P})$, we have $x_1, x_2 \in \Phi(Ax_m)$ and so $x_1 + x_2 \in \Phi(Ax_m)$. If $x_1 + x_2 \in \operatorname{int} K$, then $\Phi(Ax_m) = K$ and hence $(\Phi(x_m), \Phi(x_j))$ is an arc of $(\mathcal{E}, \mathcal{P})$ for each $j, j = 1, \ldots, m$, which is a contradiction. So $\Phi(x_1), \Phi(x_2)$ are neighborly extreme rays of K and we have $Ax_m = \alpha_1 x_1 + \alpha_2 x_2$ for some positive numbers α_1, α_2.

It is not difficult to establish the following:

Assertion. Let C be a convex polygon in $\mathbb{R}^2$ with extreme points $w_1, \ldots, w_m$ and edges $\overline{w_i w_{i+1}}, i = 1, \ldots, m$, where $\overline{w_i w_{i+1}}$ denotes the line segment joining the points w_i, w_{i+1} and w_{m+1} is taken to be w_1. Let $\tilde{C}$ be the polygon with extreme points $w_1', w_2, w_3, \ldots, w_m$, where $w_1' = (1-\lambda)w_1 + \lambda w_2$ for some $\lambda, 0 < \lambda < 1$. Then the edges of $\tilde{C}$ are $\overline{w_1'w_2}, \overline{w_iw_{i+1}}, i = 2, \ldots, m-1$ and $\overline{w_mw_1'}$.

By Lemma 4.2(ii) A is nonsingular; so the cones K and AK are linearly isomorphic under A. Note that AK is generated by the (distinct) extreme vectors $x_1', x_2, x_3, \ldots, x_m$, where we denote by x_1' the vector Ax_m. Using an equivalent formulation of the above assertion in terms of 3-dimensional polyhedral cones, we readily show that for all $i, j, 2 \le i, j \le m, x_i, x_j$ are neighborly extreme vectors of AK if and only if they are neighborly extreme vectors of K.

Now x_1, x_2 are neighborly extreme vectors of K and A is a linear isomorphism which takes K onto AK, so Ax_1, Ax_2 are neighborly extreme vectors of AK. But Ax_1, Ax_2 are respectively positive multiples of x_2 and x_3, so x_2, x_3 are neighborly extreme vectors of AK and, in view of what we have done above, x_2, x_3 are also neighborly extreme vectors of K. By repeating the argument, we can show that for $i = 2, 3, \ldots, m-1$, x_i and x_{i+1} (also, x_i and x_{i-1}) are neighborly extreme vectors of K. Finally, it is also clear that x_m, x_1 are neighborly extreme vectors of K.

By direct calculation, $A^{2(m-1)}x_1$ is a positive linear combination of x_{m-1} and x_m, and as x_{m-1} and x_m are neighborly extreme vectors, $A^{2(m-1)}x_1 \in \partial K$. On the other hand, $A^{2m-1}x_1$ is a positive linear combination of x_m, x_1 and x_2, so it belongs to $\operatorname{int} K$. This shows that $\gamma(A, x_1) = 2m-1$. In view of Lemma 4.2(iii), we have $\gamma(A) = \gamma(A, x_1) = 2m-1$.

When the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2, we employ a similar argument. For convenience, we denote the extreme vectors of AK by $y_1, \ldots, y_m$, where $y_1 = Ax_m, y_2 = Ax_1$ and $y_i = x_i$ for $i = 3, \ldots, m$. In view of Figure 2, y_1 (respectively, y_2) is a positive linear combination of x_1 and x_2 (respectively, x_2 and x_3). In this case, K and AK are still linearly isomorphic under A. By a similar argument as before, we can show that for $i, j = 1, \ldots, m, y_i, y_j$ are neighborly extreme vectors of AK if and only if x_i, x_j are neighborly extreme vectors of K. Note that x_2, x_3 are neighborly extreme vectors of K, as $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ are the only arcs with initial vertex $\Phi(x_1)$. Similarly, x_1, x_2 are also neighborly extreme vectors of K. Using the argument given for the previous case, inductively we can then show that x_i, x_{i+1} are neighborly extreme vectors of K for $i = 3, \ldots, m-1$. Finally, we can also conclude that x_m, x_1 are also neighborly extreme vectors of K.

By direct calculation, $A^{2m-3}x_2$ is a positive linear combination of x_{m-1} and x_m, whereas $A^{2m-2}x_2$ is a positive linear combination of x_m, x_1 and x_2; so $A^{2m-3}x_2 \in \partial K$ and $A^{2m-2}x_2 \in \operatorname{int} K$. By Lemma 4.2(iii) we have $\gamma(A) = \gamma(A, x_2) = 2m-2$. ∎

Theorem 6.2. *For every positive integer $m \geq 3, \max_K \gamma(K) = 2m-1$, where the maximum is taken over all 3-dimensional polyhedral cones with m extreme rays.*

Proof. By Corollary 4.7 (with $n = 3$) we have $\gamma(K) \leq 2m-1$ for every 3-dimensional polyhedral cone K with m extreme rays. In view of Lemma 6.1(ii), it remains to construct a polyhedral cone K in $\mathbb{R}^3$ with m extreme rays for which there exists a K-primitive matrix A such that $(\mathcal{E}, \mathcal{P})$ is given by Figure 1.

Consider the m points $y_1, \ldots, y_m$ in $\mathbb{R}^2$ given by:

$$y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } y_j = \begin{pmatrix} r^{j-1}\cos(j-1)\theta \\ r^{j-1}\sin(j-1)\theta \end{pmatrix} \text{ for } j = 2, \ldots, m.$$

Take B to be the 2×2 matrix $r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Note that $y_j = B^{j-1}y_1$ for $j = 2, \ldots, m$. We want to show that it is possible to choose the real parameters r, θ in such a way that $y_1, \ldots, y_m$ are precisely the extreme points of the convex polygon $C := \mathrm{conv}\{y_1, \ldots, y_m\}$, arranged in order in the anti-clockwise sense, and moreover $B^m y_1$ lies on the relative interior of the line segment joining y_1 and y_2, i.e.,$B^m y_1 = \mu_1 y_1 + \mu_2 y_2$ for some $\mu_1, \mu_2 > 0$ with $\mu_1 + \mu_2 = 1$. When these requirements are met, we choose K to be the polyhedral cone in $\mathbb{R}^3$ with extreme vectors $x_j = \begin{pmatrix} y_j \\ 1 \end{pmatrix}$ for $j = 1, \ldots, m$ and take A to be the 3×3 matrix $B \oplus [1]$. Then A is K-nonnegative, as B maps C into itself. By the definition of A it is clear that $(\Phi(x_i), \Phi(x_{i+1}))$ are arcs of $(\mathcal{E}, \mathcal{P})$ for $i = 2, \ldots, m-1$, and so are $(\Phi(x_m), \Phi(x_1))$ and $(\Phi(x_m), \Phi(x_2))$. Since the line segment $\overline{y_1 y_2}$ forms one side of the polygon $C, x_1 + x_2 \in \partial K$; so, besides the above-mentioned two arcs, there are no other arcs with initial vertex $\Phi(x_m)$. Thus, the said digraph is given by Figure 1 and we are done.

After a little calculation, the condition we impose on $B^m y_1$ can be rewritten as:

$$\begin{aligned} \mu_1 + \mu_2 &= 1 && (13) \\ \mu_1 + \mu_2 r\cos\theta &= r^m \cos m\theta && (14) \\ \mu_2 r \sin\theta &= r^m \sin m\theta && (15) \end{aligned}$$

We contend that for any real number $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, we can find a (unique) positive real number r_θ (which is less than 1) so that equations (13)–(15) are all satisfied. From equations (14) and (15), we obtain $\mu_1 = -r^m \frac{\sin(m-1)\theta}{\sin\theta}$ and $\mu_2 = r^{m-1}\frac{\sin m\theta}{\sin\theta}$. For any θ in the said range, we have $0 < \theta < \pi, 2\pi < m\theta < 2\pi + \theta$ and $2\pi - \theta <$

$(m-1)\theta < 2\pi$; hence $\sin\theta > 0, \sin m\theta > 0, \sin(m-1)\theta < 0$, and so μ_1, μ_2 are both positive.

Substituting the values of μ_1 and μ_2 into equation (13), we obtain

$$r^{m-1}\frac{\sin m\theta}{\sin\theta} - r^m \frac{\sin(m-1)\theta}{\sin\theta} = 1.$$

Let θ be any fixed real number in $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and consider the polynomial function g_θ defined on $\mathbb{R}$ by:

$$g_\theta(t) = \frac{\sin(m-1)\theta}{\sin\theta}t^m - \frac{\sin m\theta}{\sin\theta}t^{m-1} + 1.$$

We have $g_\theta(0) = 1 > 0$ and

$$g_\theta(1) = \frac{\sin(m-1)\theta - \sin m\theta}{\sin\theta} + 1 = -\frac{\cos(m-\frac{1}{2})\theta}{\cos\frac{\theta}{2}} + 1 < 0,$$

where the second equality follows from the trigonometric identity $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ and the inequality holds as $2\pi - \frac{1}{2}\theta < (m-\frac{1}{2})\theta < 2\pi + \frac{\theta}{2}$ and $0 < \frac{\theta}{2} < \frac{\pi}{2}$. In addition, we also have

$$g_\theta'(t) = \frac{t^{m-2}}{\sin\theta}\left[mt\sin(m-1)\theta - (m-1)\sin m\theta\right] < 0$$

for all $t \in (0,\infty)$, as $\sin m\theta > 0$ and $\sin(m-1)\theta < 0$. It follows that the polynomial g_θ has a unique positive real root, which is less than 1. We denote the root by r_θ. This proves our contention.

To complete our argument, it remains to show that — as a consequence of the condition we impose on $B^m y_1$ — $y_1, \ldots, y_m$ are precisely the extreme points of the polygon C, arranged in order in the anti-clockwise sense (relative to the origin).

Clearly the extreme points of C are among $y_1, \ldots, y_m$. Since the Euclidean norm of y_1 is 1, whereas that of y_j, for $j = 2, \ldots, m$, is less than 1 , y_1 is certainly an extreme point of C. If y_2 is not an extreme point of C, then y_2 lies in the relative interior of some line segment that joins two distinct points of C. But then y_3, which is By_2, also lies in the relative interior of a line segment joining two distinct points of C (as B is nonsingular and maps C into itself) and hence is not an extreme point of C. Continuing the argument, we conclude that $y_2, \ldots, y_m$ are all not extreme points of C; so y_1 is the only extreme point of C, which is impossible. This proves that y_2 is also an extreme point of C.

To proceed further, we need to apply the following:

Assertion. Let M be a compact convex set in $\mathbb{R}^2$ which contains the origin as an interior point. Let u_1, u_2 be two distinct extreme points of M which are not negative multiples of each other. If the line segment $\overline{u_1 u_2}$ does not form a face of M, then M has an extreme point of the form $\alpha_1 u_1 + \alpha_2 u_2$, where $\alpha_1, \alpha_2 > 0$.

Proof. Since $\overline{u_1 u_2}$ does not form a face of C, any point in the relative interior of $\overline{u_1 u_2}$, in particular the point $\frac{u_1+u_2}{2}$, belongs to $\operatorname{int} M$. So for β greater than and sufficiently close to $\frac{1}{2}$, $\beta(u_1 + u_2) \in M$. The set of all such β is clearly bounded above. Let β_0 denote its supremum. Then $\beta_0(u_1 + u_2) \in \partial M$. Suppose that the latter point is not an extreme point of M — otherwise we are done. Then there exist two distinct extreme points, say u_3, u_4, such that $\beta(u_1+u_2) = (1-\mu)u_3 + \mu u_4$ for some $\mu, 0 < \mu < 1$. Express u_3, u_4 as a linear combination of u_1, u_2; say, we have $u_3 = \gamma_1 u_1 + \gamma_2 u_2$ and $u_4 = \delta_1 u_1 + \delta_2 u_2$. Substituting into the preceding relation and equating the coefficients of u_1 and u_2 on the two sides of the resulting relation, we obtain

$$(1-\mu)\gamma_1 + \mu\delta_1 = \beta_0 = (1-\mu)\gamma_2 + \mu\delta_2.$$

Note that the latter implies that $(1-\mu)(\gamma_1+\gamma_2) + \mu(\delta_1+\delta_2) = 2\beta_0 > 1$. So at least one of the sums $\gamma_1+\gamma_2, \delta_1+\delta_2$ exceeds 1. By symmetry there is no loss of generality in assuming that $\gamma_1 + \gamma_2 > 1$. If γ_1, γ_2 are both positive, then the extreme point u_3 can be expressed as a positive linear combination of u_1 and u_2, and we are done. It remains to show that the cases $\gamma_1 > 0, \gamma_2 \leq 0$ and $\gamma_1 \leq 0, \gamma_2 > 0$ both cannot occur. We treat only the former case, as the argument for the latter case is similar.

We are going to show that the line segment joining u_3 and $\beta_0(u_1+u_2)$ meets the ray generated by u_1 at a point of the form λu_1, where $\lambda > 1$. Once this is established, it will follow that u_1 is not an extreme point of M, which is a contradiction. Now for any real number ω, we have

$$(1-\omega)\beta_0(u_1+u_2) + \omega u_3 = [(1-\omega)\beta_0 + \omega\gamma_1]\, u_1 + [(1-\omega)\beta_0 + \omega\gamma_2]\, u_2,$$

and moreover the sum of the coefficients of u_1, u_2 for the latter point equals $(1-\omega)(2\beta_0) + \omega(\gamma_1+\gamma_2)$ and is greater than 1 whenever $0 \leq \omega \leq 1$, as $\beta_0 > \frac{1}{2}$ and $\gamma_1 + \gamma_2 > 1$. In particular, if we set $\omega = \frac{\beta}{\beta-\gamma_2}$, then $0 < \omega \leq 1$ (as $\gamma_2 \leq 0$) and the point $(1-\omega)\beta(u_1+u_2) + \omega u_3$ becomes a multiple of u_1 with coefficient greater than 1. This completes our argument.

Next, in view of the definition of the y_is, it is clear that none of the points $y_3, \ldots, y_m$ can be written as a positive linear combination of y_1 and y_2. (This can be seen, for instance, by considering the arguments of the complex numbers corresponding to these points.) So, by the above Assertion, the line segment $\overline{y_1 y_2}$ forms one side of the polygon C.

Assume to the contrary that $y_1, y_2, \ldots, y_m$ do not constitute the set of extreme points of C. By the argument we have used for y_2, we can readily show that for each $j, j = 3, \ldots, m-1$ if y_j is not an extreme point of C then $y_{j+1}, \ldots, y_m$ are also each not an extreme point of C. So in this case the set of extreme points of C must be $\{y_1, y_2, \ldots, y_s\}$ for some $s \le m-1$. Moreover, it is clear that the point y_s must belong to the open lower half-plane (i.e., with negative second component). Since none of the extreme points of C can be expressed as a positive linear combination of y_1 and y_s, by the above Assertion, the line segment $\overline{y_s y_1}$ forms one side of the polygon C. (Similarly, the line segments $\overline{y_1 y_2}, \overline{y_2 y_3}, \ldots, \overline{y_{s-1} y_s}$ are each a side of the polygon C.) As the point y_{s+1} can be expressed as a positive linear combination of y_1 and y_s, it follows that the point must lie in either the interior of the triangle $\text{conv}\{y_1, y_s, 0\}$ or in the relative interior of the line segment $\overline{y_1 y_s}$. In the former case, $y_{s+1} \in \operatorname{int} C$. As B is nonsingular and maps C into itself, necessarily B takes interior points of C to interior points of C. It follows that By_m, which is $B^{m-s} y_{s+1}$, belongs to $\operatorname{int} C$, in contrary to the assumption that $B^m y_1$, which is the same as By_m, lies in the relative interior of the side $\overline{y_1 y_2}$ of the polygon C. In the latter case, y_{s+1} is a convex combination of y_1, y_s with both coefficients positive. Then a little calculation shows that By_{s+1} is a convex combination of y_1, y_2, y_s with all coefficients positive and hence belongs to $\operatorname{int} C$. But then since $By_m = B^{m-s-1}(By_{s+1})$, as in the previous case, we can show that $By_m \in \operatorname{int} C$, which is again a contradiction.

This proves that $y_1, \ldots, y_m$ are precisely the extreme points of C. It is also clear that the points follow this order when arranged in order in the anti-clockwise sense (relative to the origin). The proof is complete. ■

The proof of Theorem 6.2 also establishes the following:

Remark 6.3. For every positive integer $m \ge 3$ there is a 3-dimensional polyhedral cone K with m extreme rays for which there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.

In contrast to Remark 6.3, we have the following:

Theorem 6.4. *For every positive integer $m \geq 5$ there exists a 3-dimensional polyhedral cone K with m extreme rays for which there is no K-primitive matrix A with digraph $(\mathcal{E}, \mathcal{P}(A, K))$ given by Figure 1.*

Proof. Let K be the polyhedral cone in R^3 with extreme vectors

$$y_j = (\cos\frac{2j\pi}{m}, \sin\frac{2j\pi}{m}, 1)^T, j = 1, \ldots, m.$$

We contend that there is no K-primitive matrix A for which $(\mathcal{E}, \mathcal{P}(\mathcal{A}, \mathcal{K}))$ is given by Figure 1, where $x_1, x_2, \ldots, x_m$ is a rearrangement of $y_1, \ldots, y_m$.

We assume to the contrary that there is one such A. By Lemma 6.1, for $i = 1, \ldots, m, x_i$ and x_{i+1} are neighborly extreme rays of K (where x_{m+1} is taken to be x_1). Note that for each j, the extreme vectors neighborly to y_j are y_{j+1} and y_{j-1}. [We adopt the convention that for each integer j, y_j equals y_k where k is the unique integer that satisfies $1 \leq k \leq m, k \equiv j \pmod{m}$.] Suppose $x_1 = y_{j_1}$, where $1 \leq j_1 \leq m$. Then since x_2 is neighborly to x_1, x_2 must be either y_{j_1+1} or y_{j_1-1}. Consider first the case when $x_2 = y_{j_1+1}$. Since x_3 is neighborly to x_2, it is equal to either y_{j_1+2} or y_{j_1}. But we already have $x_1 = y_{j_1}$, so x_3 must be y_{j_1+2}. Continuing the argument, we can show that $x_j = y_{j_1+j-1}$ for $j = 1, \ldots, m$. Then we take $\hat{A}$ to be $\begin{bmatrix} \cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix} \oplus [1]$. If $x_2 = y_{j_1-1}$, we can show in a similar way that $x_j = j_{j_1-j+1}$ for $j = 1, \ldots, m$. In this case, we take $\hat{A}$ to be $\begin{bmatrix} \cos\frac{2\pi}{m} & \sin\frac{2\pi}{m} \\ -\sin\frac{2\pi}{m} & \cos\frac{2\pi}{m} \end{bmatrix} \oplus [1]$. We are going to show that, in either case, $\hat{A}$ and A are positive multiples of each other.

Let K_1 denote the 3-dimensional polyhedral cone $\mathrm{pos}\{x_1, x_2, x_3, x_4\}$. It is clear that $A \in \pi(K_1, K)$. As the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1 and $m \geq 5, Ax_i$ is a positive multiple of x_{i+1} for $i = 1, 2, 3, 4$. So A maps every extreme vector of K_1 to an extreme vector of K. But K_1 is indecomposable and A is nonsingular, by a variant of a sufficient condition for a cone-preserving map to be extreme due to Loewy and Schneider [L–T, Theorem 3.3] (see [Loe, Theorem 1] or [Tam 3, Theorem 5.2]) it follows that A is an extreme matrix of the proper cone $\pi(K_1, K)$. Our definition of $\hat{A}$ guarantees that $\hat{A}x_i$ is a positive multiple of x_{i+1} for $i = 1, \ldots, 4$; hence $\hat{A}x_i$ is a positive multiple of Ax_i for $i = 1, \ldots, 4$. But $\{x_1, \ldots, x_4\}$ is the set of extreme vectors of K_1, it follows that $\hat{A}$ belongs to the face of $\pi(K_1, K)$ generated by A; thus A and $\hat{A}$ are positive multiples of each other.

It is clear that the digraph $(\mathcal{E}, \mathcal{P}(\hat{A}, K))$ is not given by Figure 1 (and also $\hat{A}$ is not K-primitive). So we arrive at a contradiction. ■

Let K_1, K_2 be linearly isomorphic proper cones. If D is a digraph that can be realized as $(\mathcal{E}, \mathcal{P}(A_1, K_1))$ for some K_1-nonnegative matrix A_1, then clearly D can also be realized as $(\mathcal{E}, \mathcal{P}(A_2, K_2))$ for some K_2-primitive matrix A_2. On the other hand, if K_1, K_2 are assumed to be combinatorially equivalent only then the same cannot be said.

Remark 6.5. Let K_1, K_2 be combinatorially equivalent proper cones. Then:

(i) If G is a digraph such that $G = (\mathcal{E}(K_1), \mathcal{P}(A_1, K_1))$ for some K_1-primitive matrix A_1, then there need not exist a K_2-primitive matrix A_2 such that $(\mathcal{E}(K_2), \mathcal{P}(A_2, K_2)) = G$.

(ii) The values of $\gamma(K_1), \gamma(K_2)$ need not be the same.

Since any two 3-dimensional polyhedral cones with the same number of extreme rays are combinatorially equivalent, the preceding remark follows from Remark 6.3 and Theorem 6.4.

7. The higher-dimensional case

Theorem 7.1. *Let m be a given positive integer, $m \geq 4$. For every positive integer $n, 3 \leq n \leq m$, there is an n-dimensional polyhedral cone K with m extreme rays for which there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.*

Proof. We may assume that $n \geq 4$, as the case $n = 3$ is covered by Theorem 6.2. Our construction of the desired polyhedral cone K and K-primitive matrix A will make use of the roots of the polynomial

$$h(t) := t^m - ct - (1 - c),$$

where c is a real number to be chosen from the open interval $(0, 1)$. First, we need to understand the behavior of the roots of $h(t)$.

As the roots of $h'(t)$ are precisely the $(m-1)$th roots of $\frac{c}{m}$, $h(t)$ and $h'(t)$ have a common root if and only if one of the $(m-1)$th roots of $\frac{c}{m}$ is a root of $h(t)$. After some manipulations, the latter condition yields

$$\alpha_m c^m = (c-1)^{m-1}, \tag{16}$$

where $\alpha_m = \frac{(m-1)^{m-1}}{m^m}$. It is clear that $0 < \alpha_m < 1$. When m is even, for any real number $c \le 1$, equation (16) cannot be met as its left side and right side have different parity. When m is odd, by considering the polynomial $f_m(t) = \alpha_m t^m - (1-t)^{m-1}$ and its derivative, we readily show that there exists a unique $c \in (0,1)$, which we denote by c_m, that satisfies (16). We can now conclude that for every $c \in [0,1]$, the roots of $h(t)$ are always distinct when m is even, and when m is odd, the roots of $h(t)$ are also distinct, provided that $c \neq c_m$.

Note that the polynomial $h(t)$ can be rewritten as $(t-1)(p(t)-c)$, where $p(t)$ is the polynomial given by

$$p(t) = t^{m-1} + t^{m-2} + \cdots + t + 1.$$

So 1 is always a root of $h(t)$, and for any complex number $w \neq 1$, w is a root of $h(t)$ if and only if w is a root of the equation $p(t) = c$. Also, for $0 \le c \le 1$, a little calculation shows that a complex number z_0 is a common root of $h(t)$ and $h'(t)$ if and only if $p(z_0) = c$ and $p'(z_0) = 0$.

When m is even, it is clear that $p'(t) > 0$ for $t \ge 0$. In view of the relation $p'(t) = \frac{(m-1)t^m - mt^{m-1}+1}{(t-1)^2}$ (for $t \neq 1$), we also have $p'(t) > 0$ for $t < 0$. So $p(t)$ is a strictly increasing function and the equation $p(t) = c$ has exactly one real root, which belongs to $(-1,0)$ for $c \in (0,1)$.

When m is odd, $p(t)$ is a strictly convex function (as its second derivative always takes positive values); so it attains an absolute minimum at some point t_m. In view of the connection between $h(t), p(t)$ and their derivatives mentioned above, $p(t_m) = c_m$ and t_m must be $-(\frac{c}{m})^{\frac{1}{m-1}}$, as $(\frac{c}{m})^{\frac{1}{m-1}}$, the other real $(m-1)$th root of $\frac{c}{m}$, is not a root of $h(t)$. So, in this case, the equation $p(t) = c$ has two distinct real roots if $c > c_m$, one real root (which is a double root) if $c = c_m$, and no real roots if $c < c_m$. In any case, each real root of $p(t) = c_m$ or, in other words, each real root of $h(t)$ other than 1 belongs to $(-1,0)$.

Now come to the construction of K and A. We deal with the case when m is odd first. For any real number θ, not an integral multiple of π, let $g_\theta(t)$ be the polynomial given by:

$$g_\theta(t) = \frac{\sin(m-1)\theta}{\sin\theta} t^m - \frac{\sin m\theta}{\sin\theta} t^{m-1} + 1.$$

According to the proof of Theorem 6.2, for any $\theta \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, g_θ has a unique positive real root, denoted by r_θ, which is always less than 1. Since r_θ is a continuous function of θ and $r_{\frac{2\pi}{m-1}} = 1$ (as $g_{\frac{2\pi}{m-1}}(t) = -t^{m-1} + 1$), we have $r_\theta^{m-1} \frac{\sin m\theta}{\sin \theta} \to 1$ as $\theta \to \frac{2\pi}{m-1}^{-}$. Now choose $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, sufficiently close to $\frac{2\pi}{m-1}$, so that $r_{\theta_1}^{m-1} \frac{\sin m\theta_1}{\sin \theta_1} > c_m$, and take c to be $r_{\theta_1}^{m-1} \frac{\sin m\theta_1}{\sin \theta_1}$. Then by the proof of Theorem 6.2, $0 < c < 1$. Also, by what we have done before, the polynomial $h(t)$ has distinct roots, three of which are real, namely, 1 and say a_1, a_2 and the remaining $2k$ (where $m - 3 = 2k$) roots are non-real complex and occur in conjugate pairs. Furthermore, from the proof of Theorem 6.2 (or by direct verification) we have

$$\begin{aligned} (1-c) + cr_{\theta_1} \cos \theta_1 &= r_{\theta_1}^m \cos m\theta_1, \\ cr_{\theta_1} \sin \theta_1 &= r_{\theta_1}^m \sin m\theta_1, \end{aligned}$$

which imply that $r_{\theta_1}(\cos \theta_1 \pm \sin \theta_1)$ are roots of $h(t)$. Denote the other non-real complex roots of $h(t)$ by $r_j(\cos \theta_j \pm i \sin \theta_j), j = 2, \ldots, k$. For simplicity, we write r_{θ_1} as r_1. Now write n as $2 + 2p$ or $3 + 2p$, depending on whether n is even or odd. In either case we have $1 \le p \le k$.

Let K be the polyhedral cone in $\mathbb{R}^n$ given by:

$$K = \text{pos}\{x_1, \ldots, x_m\},$$

where for $j = 1, \ldots, m$,

$$x_j = \begin{bmatrix} r_1^{j-1} \cos(j-1)\theta_1 \\ r_1^{j-1} \sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1} \cos(j-1)\theta_p \\ r_p^{j-1} \sin(j-1)\theta_p \\ a_1^{j-1} \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} r_1^{j-1} \cos(j-1)\theta_1 \\ r_1^{j-1} \sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1} \cos(j-1)\theta_p \\ r_p^{j-1} \sin(j-1)\theta_p \\ a_1^{j-1} \\ a_2^{j-1} \\ 1 \end{bmatrix},$$

depending on whether n is even or odd. It is clear that K is a pointed cone. A sufficient condition for K to be a full cone is that the $n \times n$ matrix whose jth column is x_j, for $j = 1, \ldots, n$, is nonsingular. Upon pre-multiplying the latter matrix by one of the following two $n \times n$ matrices depending on whether n is even or odd:

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus I_2 \quad \text{or} \quad \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \oplus I_3,$$

we obtain the Vandermonde matrix

$$\begin{bmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ 1 & \bar{z}_1 & \cdots & \bar{z}_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_p & \cdots & z_p^{n-1} \\ 1 & \bar{z}_p & \cdots & \bar{z}_p^{n-1} \\ 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & z_1 & \cdots & z_1^{n-1} \\ 1 & \bar{z}_1 & \cdots & \bar{z}_1^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & z_p & \cdots & z_p^{n-1} \\ 1 & \bar{z}_p & \cdots & \bar{z}_p^{n-1} \\ 1 & a_1 & \cdots & a_1^{n-1} \\ 1 & a_2 & \cdots & a_2^{n-1} \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $z_j = r_j e^{i\theta_j}$ for $j = 1, \ldots, k$, which is always nonsingular, as $1, a_1, a_2, z_1, \bar{z}_1, \ldots, z_p, \bar{z}_p$ are distinct, being roots of the polynomial $h(t)$. So K is a full cone.

Now let A be the matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a_1] \oplus [1]$$

or

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [a_1] \oplus [a_2] \oplus [1],$$

again depending on whether n is even or odd.

As can be readily checked, $A^j x_1 = x_{j+1}$ for $j = 1, \ldots, m-1$. If we also have $Ax_m = \mu_1 x_1 + \mu_2 x_2$ for some positive real numbers μ_1, μ_2, then it will follow that $A \in \pi(K)$. When n is even, the latter condition amounts to saying that there exist positive real numbers μ_1, μ_2 such that

$$\begin{bmatrix} r_1^m \cos m\theta_1 \\ r_1^m \sin m\theta_1 \\ \vdots \\ r_p^m \cos m\theta_p \\ r_p^m \sin m\theta_p \\ a_1^m \\ 1 \end{bmatrix} = \mu_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} r_1 \cos\theta_1 \\ r_1 \sin\theta_1 \\ \vdots \\ r_p \cos\theta_p \\ r_p \sin\theta_p \\ a_1 \\ 1 \end{bmatrix}.$$

Or, in terms of the complex numbers $z_j, 1 \le j \le k$, introduced above, the condition becomes

$$\begin{aligned} z_j^m &= \mu_1 + \mu_2 z_j, \\ \bar{z}_j^{\,m} &= \mu_1 + \mu_2 \bar{z}_j, \end{aligned}$$

for $j = 1, \ldots, p$, and

$$\begin{aligned} a_1^m &= \mu_1 + \mu_2 a_1, \\ 1 &= \mu_1 + \mu_2, \end{aligned}$$

where $\mu_1, \mu_2 > 0$. Since $z_1, \bar{z}_1, \ldots, z_p, \bar{z}_p, a_1$ and 1 are roots of the polynomial $h(t)$, the above set of conditions is satisfied if we take $\mu_2 = c$ (and $\mu_1 = 1 - \mu_2$). This proves that $A \in \pi(K)$. When n is odd, a similar argument also works.

It remains to show that $x_1, \ldots, x_m$ are precisely the extreme vectors of K (the polyhedral cone generated by them) and the face $\Phi(x_1 + x_2)$ contains (up to multiples) only x_1, x_2 as its extreme vectors. Once this is done, it will follow that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1.

For $j = 1, \ldots, m$, denote by u_j the subvector of x_j formed by its 1st, 2nd and last components. Since $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and $r_1 = r_{\theta_1}$, by the proof of Theorem 6.2 $u_1, \ldots, u_m$ are precisely the extreme vectors of the polyhedral cone $\mathrm{pos}\{u_1, \ldots, u_m\}$. So it is clear that each x_j cannot be written as a nonnegative linear combination of the remaining x_ls or, in other words, $x_1, \ldots, x_m$ are precisely the extreme vectors of K. The proof of Theorem 6.2 also guarantees that u_1, u_2 are neighborly extreme vectors of the 3-dimensional polyhedral cone $\mathrm{pos}\{u_1, \ldots, u_m\}$, which means that there is no representation of $u_1 + u_2$ as a positive linear combination of $u_1, \ldots, u_m$, in which at least one of the vectors $u_3, \ldots, u_m$ is involved. As a consequence, there is also no representation of $x_1 + x_2$ as a positive linear combination of $x_1, \ldots, x_m$, in which at least one of the vectors $x_3, \ldots, x_m$ is involved. Hence, the face of K generated by $x_1 + x_2$ is 2-dimensional, as desired. This completes the proof for the case when m is odd.

When m is even, we choose any $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$ and take c to be $r_{\theta_1}^{m-1} \frac{\sin m\theta_1}{\sin \theta_1}$, where r_θ denotes the unique positive real root of the polynomial $g_\theta(t)$ defined as before. In this case, the polynomial $h(t)$ has two real roots, namely, 1 and, say a. We write m as $2+2k$ and let the non-real complex roots of $h(t)$ be $r_j(\cos\theta_j \pm i \sin\theta_j)$ for $j = 1, \ldots, k$ (where θ_1 is the real number just chosen and $r_1 = r_{\theta_1}$). Now write n as $2 + 2p$ or $1 + 2p$ (with $1 \le p \le k$), depending on whether n is even or odd. Let K be the polyhedral cone in $\mathbb{R}^n$ given by:

$$K = \text{pos}\{x_1, \ldots, x_m\},$$

where for even n, the x_js are given in the same way as for the odd m case (but with a_1 replaced by a) and for odd n, they are given by:

$$x_j = \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ 1 \end{bmatrix}.$$

When n is even, we take A to be the same matrix as before. When n is odd, we take A to be the matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1].$$

The subsequent arguments are similar to those for the previous case. We omit the details. ■

In Theorem 7.1, when m and n are both odd, we have the following alternative construction for K and A. Write $m = 2k+3, n = 2p+1$ with $p \leq k$. Let K to be the polyhedral cone in $\mathbb{R}^n$ given by $K = \text{pos}\{x_1, \ldots, x_m\}$, where for $j = 1, \ldots, m$,

$$x_j = \begin{bmatrix} r_1^{j-1}\cos(j-1)\theta_1 \\ r_1^{j-1}\sin(j-1)\theta_1 \\ \vdots \\ r_p^{j-1}\cos(j-1)\theta_p \\ r_p^{j-1}\sin(j-1)\theta_p \\ 1 \end{bmatrix},$$

and take A to be the $n \times n$ matrix

$$r_1 \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \oplus \cdots \oplus r_p \begin{bmatrix} \cos\theta_p & -\sin\theta_p \\ \sin\theta_p & \cos\theta_p \end{bmatrix} \oplus [1],$$

where again $r_j(\cos\theta_j \pm \sin\theta_j), j = 1, \ldots, p$ (and 1) are roots of the polynomial $h(t)$. In the proof of Theorem 7.1, the constant c that appears in the formula for $h(t)$ is taken to be $r_{\theta_1}^{m-1}\frac{\sin m\theta_1}{\sin\theta_1}$, where $\theta_1 \in (\frac{2\pi}{m}, \frac{2\pi}{m-1})$, sufficiently close to $\frac{2\pi}{m-1}$. Here c is defined by the same formula (in terms of θ_1) but we may take θ_1 to be any number in the open interval $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$. The point is, in this case, no real root of $h(t)$ other than 1 is involved in the construction of K and A. Since $\lim_{\theta\to\frac{2\pi}{m}} r_\theta^{m-1}\frac{\sin m\theta}{\sin\theta} = 0$, we can make c as small as we please (so that $h(t)$ is close to the polynomial $t^m - 1$) by choosing θ_1 sufficiently close to $\frac{2\pi}{m}$. This observation may have some nice implications.

Consider the polyhedral cone K_0 in $\mathbb{R}^n$ given by $K_0 = \text{pos}\{y_1, \ldots, y_m\}$, where for $j = 1, \ldots, m$,

$$y_j = \begin{bmatrix} \cos(j-1)\frac{2\pi}{m} \\ \sin(j-1)\frac{2\pi}{m} \\ \cos(j-1)\frac{4\pi}{m} \\ \sin(j-1)\frac{4\pi}{m} \\ \vdots \\ \cos(j-1)\frac{2p\pi}{m} \\ \sin(j-1)\frac{2p\pi}{m} \\ 1 \end{bmatrix}.$$

We *claim* that $y_m + y_{m-1} + \ldots + y_{m-n+2} \in \partial K_0$. If the claim holds, then by choosing $h(t)$ with $c > 0$ sufficiently small, the cone K constructed will have the property that $x_m + x_{m-1} + \ldots + x_{m-n+2} \in \partial K$. Then it will follow that $\gamma(A) = (n-1)(m-1)+1$.

In the above, we are dealing with the case when m and n both odd. When m is even, the situation is simpler. Then the existence of a real root of $h(t)$ other than 1 is not a problem and θ_1 can be taken to be any number from $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$. Provided that the cone K_0 defined above has the mentioned property (irrespective of the parity of m), the same argument also shows that $\gamma(A) = (n-1)(m-1)+1$. In view of the above discussion, we pose the following:

Conjecture. The maximum value of $\gamma(K)$ as K runs through all n-dimensional polyhedral cones K with m extreme rays equals $(n-1)(m-1)+1$ when m is even or m and n are both odd, and equals $(n-1)(m-1)$ when m is odd and n is even.

We have obtained the following partial result, which supports the above conjecture.

Observation. When m is even, the maximum value of $\gamma(K)$ as K runs through all 4-dimensional polyhedral cones K with m extreme rays is $3(m-1)+1$; when m is odd, the maximum value is at least $3(m-1)$.

When m is even, consider the polyhedral cone $K_0 := \mathrm{pos}\{y_1, \dots, y_m\}$, where for $j = 1, \dots, m$,

$$y_j = \begin{bmatrix} \cos\frac{(j-1)2\pi}{m} \\ \sin\frac{(j-1)2\pi}{m} \\ (-1)^{j-1} \\ 1 \end{bmatrix}.$$

We contend that $y_{m-2} + y_{m-1} + y_m \in \partial K_0$. It is readily seen that we have

$$y_m = \begin{bmatrix} \cos\frac{2\pi}{m} \\ -\sin\frac{2\pi}{m} \\ -1 \\ 1 \end{bmatrix}, y_{m-1} = \begin{bmatrix} \cos\frac{4\pi}{m} \\ -\sin\frac{4\pi}{m} \\ 1 \\ 1 \end{bmatrix}, y_{m-2} = \begin{bmatrix} \cos\frac{6\pi}{m} \\ -\sin\frac{6\pi}{m} \\ -1 \\ 1 \end{bmatrix}.$$

Let

$$v = \begin{bmatrix} 2\left(\sin\frac{2\pi}{m} - \sin\frac{6\pi}{m}\right) \\ 2\left(\cos\frac{2\pi}{m} - \cos\frac{6\pi}{m}\right) \\ 2\sin\frac{2\pi}{m} - \sin\frac{4\pi}{m} \\ 2\sin\frac{2\pi}{m} + \sin\frac{4\pi}{m} \end{bmatrix}.$$

Using some trigonometric identities, by a little calculation we obtain for every positive integer j

$$\begin{aligned} \langle v, y_j \rangle &= -2\sin\frac{(j+2)2\pi}{m} + 2\sin\frac{2j\pi}{m} + (-1)^{j-1}\left(2\sin\frac{2\pi}{m} - \sin\frac{4\pi}{m}\right) + \left(2\sin\frac{2\pi}{m} + \sin\frac{4\pi}{m}\right) \\ &= \begin{cases} 4\sin\frac{2\pi}{m}\left(1 - \cos\frac{(j+1)2\pi}{m}\right) & \text{when } j \text{ is odd,} \\ 4\sin\frac{j\pi}{m}\left(\cos\frac{j\pi}{m} - \cos\frac{(j+4)\pi}{m}\right) & \text{when } j \text{ is even.} \end{cases} \end{aligned}$$

It follows that $\langle v, y_j \rangle$ equals 0 for $j = m-2, m-1, m$, and is positive for $j = 1, \dots, m-3$. This implies that $y_{m-2} + y_{m-1} + y_m \in \partial K_0$.

Now construct a polyhedral cone K with m extreme rays and a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 in the way as given in the proof of Theorem 7.1. Then the extreme vectors $x_1, \dots, x_m$ of K are respectively close to the extreme vectors $y_1, \dots, y_m$ of K_0, provided that θ, which is chosen from the open interval $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$, is sufficiently close to $\frac{2\pi}{m}$. By what we have done above, the subspace spanned by the extreme vectors y_{m-2}, y_{m-1}, y_m is a supporting hypersubspace for K_0 and the remaining extreme vectors $y_1, \dots, y_{m-3}$ all lie on the same open half-space determined by the hypersubspace. By a continuity argument, it follows that the same can be said for the extreme vectors $x_1, \dots, x_m$ of K. This shows that $x_{m-2}+x_{m-1}+x_m \in \partial K$. Therefore, when m is even, the maximum value of $\gamma(K)$ as K runs through all 4-dimensional polyhedral cones K with m extreme rays is $3(m-1)+1$.

(In more details: Let $B(\theta)$ denote the 4×4 matrix whose 1st, 2nd, 3rd and 4th rows are respectively $x_{m-3}^T, x_{m-2}^T, x_{m-1}^T$ and x_m^T. Let $u(\theta) = (u_1, u_2, u_3, u_4)^T$, where u_j denotes the $(1, j)$-cofactor of $B(\theta)$. By elementary properties of determinants we have $\langle u, x_j \rangle = 0$ for $j = m-2, m-1, m$ and $\langle u, x_{m-3} \rangle = \det B(\theta)$. Note that the matrix $B(\theta)$ is nonsingular (cf. the proof Theorem 7.1 given on p.43). So the vector $u(\theta)$ is nonzero and as θ tends to $\frac{2\pi}{m}$, u tends to a multiple of v.)

Now consider the case when m is odd. We again follow the construction for K and A as given in the proof of Theorem 7.1. But this time we choose θ (again from $(\frac{2\pi}{m}, \frac{2\pi}{m-1})$) sufficiently close to $\frac{2\pi}{m-1}$). Let K_0 denote the polyhedral in $\mathbb{R}^4$ with extreme vectors $y_1, \dots, y_{m-1}$, where

$$y_j = \begin{bmatrix} \cos \frac{(j-1)2\pi}{m-1} \\ \sin \frac{(j-1)2\pi}{m-1} \\ (-1)^{j-1} \\ 1 \end{bmatrix}$$

for $j = 1, \dots m-1$. Since $m-1$ is even, by what we have done above, the subspace spanned by the extreme vectors $y_{m-3}, y_{m-2}, y_{m-1}$ is a supporting hypersubspace for K_0 and the remaining extreme vectors $y_1, \dots, y_{m-4}$ lie on the same open half-space determined by the hypersubspace. So when θ is sufficiently close to $\frac{2\pi}{m-1}$, $\operatorname{span}\{x_{m-3}, x_{m-2}, x_{m-1}\}$ is a supporting hypersubspace for K and the remaining extreme vectors $x_m, x_1, x_2, \dots, x_{m-4}$ all lie on the same open half-space determined by the hypersubspace. This shows that $x_{m-3}+x_{m-2}+x_{m-1} \in \partial K$. As a consequence, we can conclude that when m is odd, the maximum value of $\gamma(K)$ as K runs through all 4-dimensional polyhedral cones K with m extreme rays is at least $3(m-1)$.
By the above argument we cannot tell whether $x_{m-2} + x_{m-1} + x_m \in \partial K$. We are certain that when θ is sufficiently close to $\frac{2\pi}{m-1}$, the extreme vectors $x_2, \dots, x_{m-3}$ lie

on the same open half-space determined by span$\{x_{m-2}, x_{m-1}, x_m\}$. However, since x_1 (as well as x_m) tends to y_1 as θ tends to $\frac{2\pi}{m-1}$, we are unable to tell where x_1 lies when θ is very close to $\frac{2\pi}{m-1}$. We suspect that it lies on the other open half-space, and in view of our knowledge on the minimal cone case we tend to believe that when m is odd the maximum value of $\gamma(K)$ as K runs through all 4-dimensional polyhedral cones K with m extreme rays is precisely $3(m-1)$.

8. Examples, Remarks and Open Questions

Here is an example of a proper cone which does not have finite exponent.

Example 8.1. Let $\|\cdot\|$ denote the norm of $\mathbb{R}^2$ given in [B–L, p.67]. Let K be the proper cone in $\mathbb{R}^3$ given by: $K = \{\alpha\binom{x}{1} : \alpha \geq 0 \text{ and } \|x\| \leq 1\}$. As shown in [B–L], for each positive integer k, we can find some 2-by-2 real matrix B_k such that $\|B_k\| = \|B_k^k\| = 1$ but $\|B_k^{k+1}\| < 1$. Let $A_k = B_k \oplus (1)$. Then it is easy to show that A_k is K-primitive and $\gamma(A_k) = k$. Since k can be arbitrarily large, this shows that for this K we have $\gamma(K) = \infty$. It is also of interest to note that the K-primitive matrices A_k obtained in this example are, in fact, all extreme matrices of the cone $\pi(K)$. The point is, each of them map infinitely many extreme rays of K onto extreme rays.

Let E_n denote the set of values attained by the exponents of primitive matrices of order n. Dulmage and Mendelsohn [D–M] have found intervals in the set $\{1, 2, \ldots, (n-1)^2 + 1\}$ containing no integer which is the exponent of a primitive matrix of order n. These intervals have been called *gaps* in E_n. The problem of determining E_n or the gaps is an intricate problem, but it has been completely resolved. (See, for instance, [B–R].)

For a polyhedral cone (or a proper cone) K, we can consider a similar problem — to determine the set of values attained by the exponents of K-primitive matrices. We expect that for every polyhedral cone K of dimension greater than 2 there are gaps in the set (but at present we *do not* have a proof for this claim). As an example, consider an n-dimensional indecomposable minimal cone K, where n is an odd integer greater than or equal to 5, such that the linear relation on its extreme vectors has the same number of terms on its two sides. Let A be a K-primitive matrix. Then $\gamma(A)$ equals $n^2 - n + 1$ if the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1 (see Theorem 5.3(II)) and equals $n^2 - n$ if the digraph is given by Figure 2 (see Lemmas 5.1(ii) and 5.2(ii)). On the other hand, if the digraph is not given by Figure 1 or Figure 2, then by Lemma 4.1 the length of the shortest circuit in $(\mathcal{E}, \mathcal{P})$ is at most $n-1(= m-2)$ and by Remark 4.6 it follows that $\gamma(A) \leq (n-1)^2+2$. So in this case any integer lying in the closed interval $[n^2 - 2n + 4, n^2 - n - 1]$ cannot be attained as the exponent of some K-primitive matrix.

Question. Question. Let $m \geq 4$ be a positive integer. Determine the set of integers that can be attained as the exponent of a K-primitive matrix, where K is a polyhedral cone with m extreme rays (but arbitrary dimension, within the natural boundaries).

In [H–N] Hartwig and Neumann posed and proved partially the following conjecture:

If A is a primitive matrix, then

$$\gamma(A) \le (m_A - 1)^2 + 1,$$

where m_A is the degree of the minimal polynomial for A.

Their work was based on the observation that for any primitive matrix A, we have

$$d \le m_A - 1,$$

where d is the diameter of the usual digraph associated with A. (The observation itself is a consequence of the inequality $(I + A)^{m_A - 1} \gg O$ for an irreducible nonnegative matrix A. The observation also implies the inequalaity $m_A \ge s$, where s is the length of the shortest circuit in D, because we have $d \ge s - 1$.) When K is a general polyhedral cone and A is a K-primitive matrix, one may wonder whether there is a lower bound for m_A in terms of the diameter of the digraph $(\mathcal{E}, \mathcal{P}(A, K))$. One can readily dispense with the rather naive inequality $m_A - 1 \ge \operatorname{diam}(\mathcal{E}, \mathcal{P}(A, K))$. This is because by Remark 6.3, for any positive integer m, however large it is, there exist a polyhedral cone K in $\mathbb{R}^3$ with m extreme rays and a K-primitive matrix A such the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1. Then the diameter of the digraph is $m - 1$, but $m_A - 1 \le 3 - 1 = 2$. Since we have $\gamma(A) = 2m - 1$ (according to Lemma 6.1(ii)), this example also illustrates that the inequality $\gamma(A) \le (m_A - 1)^2 + 1$ does not hold when A is K-primitive and K is non-simplicial; that is, the above-mentioned conjecture of Hartwig and Neumann is not valid in the non-simplicial cone case.

The next inequality one may try to get at is

$$m_A - 1 + k \ge \operatorname{diam}(\mathcal{E}, \mathcal{P}(A, K)).$$

Here K is an n-dimensional polyhedral cone with m extreme rays and $k = m - n$. In the special case when the pair K and A are such that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 1, the above inequality becomes $m_A = n$. At present we do not even know whether in the special case the latter equality holds. (We know it holds when $n = 3$, in view of Lemma 3.6.)

The above-mentioned conjecture of Hartwig and Neumann has already been resolved. It was established independently by Shen [She] and Neufeld [Neu]. Actually, they obtained a stronger inequality, namely, $\gamma(A) \le d^2 + 1$, where d denotes the diameter of the usual digraph of A. We suspect that, in general, for a K-primitive matrix A, we have

$$\gamma(A) \leq (\mathrm{diam}(\mathcal{E}, \mathcal{P}(A, K)))^2 + 1.$$

The idea of using the concept of minimal generating matrix for a polyhedral cone to obtain results on K-primitive matrices does not look promising. Here is the reason. Consider a non-simplicial polyhedral cone K in $\mathbb{R}^n$ with m extreme extreme rays. Let P be the minimal generating matrix for K (i.e., its column vectors form a set of distinct representatives of the extreme rays of K). Let $A \in \pi(K)$. Choose a nonnegative matrix B with the maximum number of positive entries that satisfies $AP = PB$. As shown in [Tam 4] the usual digraph of B^T is equal to the digraph $(\mathcal{E}, \mathcal{P}(A, K))$. Furthermore, A and the restriction of B^T to $\mathcal{R}(P^T)$, the range space of P^T are similar. However, as shown in the work of [B–T] (see also [Tam 4]), the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ need not be strongly connected. So the nonnegative matrix B need not be irreducible and it is difficult to apply known results on primitive matrices.

In spite of this work, our knowledge on the exponents of primitive matrices over polyhedral cones is scanty. Even for the minimal cone case, there remain many problems to be explored. Here is one:

Question: If K is an n-dimensional minimal cone such that the linear relation on its extreme vectors has p vectors on one side and q vectors on the other side, where $p, q \geq 2, p + q \leq n + 1$, what is $\gamma(K)$?

In the minimal cone case, we are able to determine not only the maximum value of $\gamma(K)$ but also solve the realization problem completely. In contrast, our knowledge on the case $n = 3$ is meagre. For instance, we have not yet characterized all those 3-dimensional minimal cones K with m extreme rays for which $\gamma(A) = 2m - 1$ (or, i.e., for which there exists a K-primitive matrix A such that the digraph $(\mathcal{E}, \mathcal{P})$ is given by Figure 1). It appears that the larger is the difference $m - n$, the harder is the more problem. Besides the simplicial cone case and the minimal cone case, the next simplest case is the almost minimal cone case. We call an n dimensional polyhedral cone *almost minimal* if it has exactly $n + 2$ extreme rays.

Question: Find $\max_K \gamma(K)$, where the maximum is taken over all n-dimensional almost minimal cones K. Determine also the cone K (and the K-primitive matrix A) such that the maximum is attained.

In view of Theorem 7.1, one may ask whether it is true that for every pair of positive integers m, n with $3 \leq n \leq m$ there exist an n-dimensional polyhedral cone K with m extreme rays and a K-primitive matrix A such that the digraph

$(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2. The following example provides an affirmative answer to this question for the special case when $m = 5$ and $n = 3$. It also shows that this is a harder problem (than the one for Figure 1).

Example 8.2. We want to construct a 3-dimensional polyhedral cone K with distinct extreme vectors $x_1, \ldots, x_5$ and a K-primitive matrix A for which $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2 with $m = 5$. Choose $x_1 = (c, -b, 1)^T, x_2 = e_1, x_3 = e_2, x_4 = e_3$ and $x_5 = (1, -a, b)^T$, where e_j is the jth standard unit vector of $\mathbb{R}^3$ and a, b, c are positive numbers to be determined. Let $K = \text{pos}\{x_1, \ldots, x_5\}$ and take

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -a \\ 0 & 1 & b \end{bmatrix}.$$

Then $Ax_2 = x_3, Ax_3 = x_4$ and $Ax_4 = x_5$. Also, Ax_1 equals $(1, c - a, 0)^T$ and is a positive linear combination of x_2 and x_3 provided that $c > a$. By direct calculation $Ax_5 = (b, 1 - ab, b^2 - a)^T$. The condition that Ax_5 is a positive linear combination of x_2 and x_3. leads to the equation

$$(b, 1 - ab, b^2 - a) = \alpha(c, -b, 1) + \beta(1, 0, 0),$$

with positive unknowns α and β. It is readily seen that the equation has a solution if and only if $b \neq 0$ and $a = (1 + b^3)/2b$; then the solution is unique and is given by $\alpha = b^2 - a$ and $\beta = b - (b^2 - a)c$. With $a = (1 + b^3)/2b$, α becomes $(b^3 - 1)/2b$, which is positive, whenever b is greater than 1. (Then certainly $a > 0$.) Note also that for $b > 1$, sufficiently close to 1, we have

$$b - \alpha a = b - \frac{b^6 - 1}{4b^2} > 0.$$

So for $c > a$, sufficiently close to a, we have $\beta = b - \alpha c > 0$.

We also require that $x_1, \ldots, x_5$ are indeed the distinct extreme vectors of K. It is readily checked that x_2 cannot be written as a nonnegative linear combination of x_1, x_3, x_4 and x_5. Similarly, $x_3 \notin \text{pos}\{x_1, x_2, x_4, x_5\}$ and $x_4 \notin \text{pos}\{x_1, x_2, x_3, x_5\}$. A little calculation shows that the inner product between $(0, b, a)^T$ and each of the vectors x_2, x_3, x_4, x_5 is nonnegative, whereas the inner product between $(0, b, a)^T$ and x_1, which is $a - b^2 = (1 - b^3)/2b$, is negative whenever $b > 1$. So $x_1 \notin \text{pos}\{x_2, x_3, x_4, x_5\}$. Note also that $\langle (b, c, 0)^T, x_i \rangle \geq 0$ for $i = 1, 2, 3, 4$ and $\langle (b, c, 0)^T, x_5 \rangle = b - ac$. Now

$b - a^2$ equals $\frac{2b^3-1-b^6}{4b^2}$ and is negative for $b > 1$, sufficiently close to 1 (as the derivative of $2b^3 - 1 - b^6$ is negative for $b > 1$). So for $c > a$, sufficiently close to a, we have $b - ac < 0$; then $x_5 \notin \mathrm{pos}\{x_1, x_2, x_3, x_4\}$.

Now choose $b > 1$ sufficiently close to $1, a = (1 + b^3)/2b$ and $c > a$ sufficicently close to a. Then $x_1, \ldots, x_5$ are indeed the distinct extreme vectors of K and the matrix A defined above (in terms of a and b) belongs to $\pi(K)$. The fact that Ax_5 is a positive linear combination of x_1 and x_2 guarantees the presence of the arcs $(\Phi(x_5), \Phi(x_1))$ and $(\Phi(x_5), \Phi(x_2))$ in $(\mathcal{E}, \mathcal{P}(A, K))$. But we still need to show that $x_1 + x_2$ is a boundary vector of K, in order to make sure that the preceding two arcs are the only arcs with initial vertex $\Phi(x_5)$. Let $z = (0, 1, b)^T$. Then we have $\langle z, x_j \rangle$ equals 0 for $j = 1, 2$ and is positive for $j = 3, 4, 5$. So $z \in K^*$ (the dual cone of K) and $x_1 + x_2 \in \partial K$. Similarly, we can also show that $x_2 + x_3$ is a boundary vector of K (by considering the vector e_3 of K^*); thus, $(\Phi(x_1), \Phi(x_2))$ and $(\Phi(x_1), \Phi(x_3))$ are the only arcs with initial vertex $\Phi(x_1)$. Now it is clear that the digraph $(\mathcal{E}, \mathcal{P}(A, K))$ is given by Figure 2.

By direct calculation, $A^7 x_2$ is a positive linear combination of $x_4 + x_5$, which belongs to ∂K, as x_4, x_5 are neighborly extreme vectors, in view of Lemma 6.1. On the other hand, $A^8 x_2$, being a positive linear combination of x_1, x_2 and x_5, belongs to $\mathrm{int}\, K$. So $\gamma(A, x_2) = 8$, and by Lemma4.2 we have $\gamma(A) = 8$.

Before we end the example, we would like to point out that if we let b tends to 1 and c tends to a, then in the limit we have $a = b = c = 1$ and the almost minimal cone K becomes the minimal cone generated by e_1, e_2, e_3 and $(1, -1, 1)^T$. (The vectors x_1 and x_5 have collapsed to $(1, -1, 1)^T$.)

Finally, one may also consider the following problem dual to the problem at issue:

Question: Given positive integers m, n with $m \geq n$, determine

$$\min\{\gamma(K) : K \text{ is an } n\text{-dimensional polyhedral cone with } m \text{ extreme rays }\}.$$

Acknowledgement A large part of this work was carried out during the period from 2004, November to 2005, March when the second author visited the Department of Mathematics of Technion as a Lady Davis visiting professor. He would like to take this opportunity to thank the university for its hospitality.

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The Omega Limit Set of a Point under a Cone-Preserving Linear Map

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Theorem. [Denjoy-Wolff, 1926]

Given $f : \Delta \to \Delta$ (*open unit disc*), *analytic,*

no fixed point

Then

$\exists b \in \partial\Delta$ *s.t.* $(f^k x)_{k=1}^{\infty} \to b$

$\forall x \in \Delta$, $f^k = f \circ f \circ \cdots \circ f$ (*k times*).

Theorem. [Beardon, 1997)]

$D \subset \mathbb{R}^n$, *bdd. open convex, strictly convex*

$f : D \to D$, *nonexpansive* $(d(f(x), f(y)) \le d(x, y))$

no fixed point

d, Hilbert's metric

Then

$\exists z \in \partial D$ *s.t.* $\lim_{k\to\infty} \|f^k(x) - z\| = 0 \, \forall x \in D$.

Without strict convexity, no longer holds.

Given $f : D \to D$, $x \in D \subseteq \mathbb{R}^n$,

$\omega(x; f) := \{$Lts of converg. subseq. of $(f^k x)_{k=1}^{\infty}\}$.

(*omega limit set of x under f*)

Theorem. [Karlsson and Noskov, 2002]

Same hypotheses but D need not strictly convex

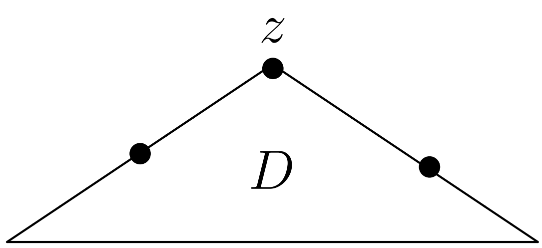
Then

$\forall x \in D$, $\exists z \in \omega(x; f)$ *s.t.* $\forall y \in \omega(x; f)$, $\overline{zy} \subseteq \partial D$.

Conjecture [Nussbaum]: *Same hypotheses*

$\forall x \in D$, $\operatorname{conv} \omega(x; f) \subset \partial D$.

That is,



cannot happen.

K, proper cone: closed pointed full convex cone

$\pi(K) := \{A, \text{ linear} : AK \subseteq K\}$.

Theorem. [Lins and Nussbaum, Theorems 2 and 3]

Given:

$A \in \pi(K)$ *s.t.* $A(\operatorname{int} K) \subseteq \operatorname{int} K$;

$K^* := \{q : q(x) \geq 0 \ \forall x \in K\}$, *dual cone*;

$q \in \operatorname{int} K^*$; $K_q := \{x \in K : q(x) = 1\}$;

$T : \operatorname{ri} K_q \to \operatorname{ri} K_q$, $T(x) = Ax/q(Ax)$.

No fixed point

Then $\forall x \in \operatorname{ri} K_q$

(i) $\operatorname{conv}(\omega(x;T)) \subseteq \operatorname{rbd} K_q$.

(ii) *For polyhedral* K,

$\omega(x;T)$ *finite*; $\omega(x;T) \subset \operatorname{ri} F, K_q \neq F(\text{face})$,

indep. of x.

Remarks. Lins and Nussbaum based work on:

1. Earlier papers, mainly on infinite-dim. spaces:

   P. Bushell, Hilbert's metric and positive contraction mappings in Banach space, *Arch. Rat. Mech. Anal.* **52** (1973), 330-338.

   and

   R.D. Nussbaum, Iterated nonlinear maps and Hilbert's projective metric, *Memoirs Amer. Math. Soc.*, **391** (1988).

2. A number of auxiliary results:

   Lemma 1, Theorem 1, Proposition 2, Lemma 2 and Lemma 3,

   hold for a contin. order-preserving map, homog. of deg. 1.

3. Part(a) of their Theorem 1 invokes a result of Sine and Roehrig on a topological group.

4. The only place using finite-dimensional tool:

   Lemma 2. If $A \geq 0$, non-nilpotent, then $\lim_{k\to\infty} \frac{A^{kp}}{(\rho(A))^{kp}k^{\nu-1}} \geq 0$ for some $p, \nu \in \mathbb{Z}_+$.

5. Seem unaware of Birkhoff's similar proof for

   Theorem. [Birkhoff, Schaefer, Vandergraft] *Given $A \in \mathcal{M}_n(\mathbb{R}), \exists K$ such that $A \in \pi(K)$ iff A satisfies the Perron-Schaefer condition*:

   $$\rho(A) \in \sigma(A) \textit{ and}$$
   $$\nu_A(\lambda) \leq \nu_A(\rho(A)) \ \forall \lambda \in \sigma(A), |\lambda| = \rho(A).$$

   The lemma still holds if $A \in \pi(K)$ for a polyhedral cone K.

Theorem. [T.] *Given non-nilpotent* $A \in \pi(K)$,

$$Tx = Ax/\|Ax\|, x \in S^{n-1} \cap \operatorname{int} K$$

$$\nu := \nu_A(\rho(A))$$

$$\Lambda = \{\lambda \in \sigma(A) : |\lambda| = \rho(A), \nu_A(\lambda) = \nu_A(\rho(A))\}.$$

($\Lambda \neq \emptyset$ *by Perron-Schaefer condition*)

$orbit_T(x) := \{x, Tx, T^2x, \ldots\}$.
$M := \mathbb{R}^n \cap \oplus_{\lambda\in\Lambda}[(\lambda I_n - A)^{\nu-1}\mathcal{N}((\lambda I_n - A)^{\nu})]$,
Then
$\forall x \in K, \|x\| = 1$ *s.t.* $\Phi((A+I)^{n-1}x = K$, (A-*invar. face of* K *generated by* x), *we have*

(i) $\Phi(\omega(x;T)) = \Phi(M \cap K)$, *and*

$$\omega(x;T) \subseteq \begin{cases} \mathrm{ri}(M \cap K) & \text{if } \mathrm{orbit}_T(x) \cap \mathrm{int}\, K \neq \emptyset \\ \mathrm{rbd}(M \cap K) & \text{otherwise} \end{cases}$$

(ii) *If A has no eigenvector in* $\mathrm{int}\, K$, *then*

$$\Phi(M \cap K) \subseteq \partial K.$$

(iii) *If $\frac{1}{\rho(A)}\Lambda$ consists of roots of unity* (*in particular, if K is polyhedral*), *then*

$$\omega(x;T) \text{ is a finite set } \forall x$$

.

Proof of (ii): Suppose that $\Phi(M \cap K) \nsubseteq \partial K$. Then $\exists y \in \mathrm{ri}(M \cap K) \cap \mathrm{int}\, K$. We have

$$y = y_1 + y_2 + \cdots + y_h,$$

where y_j is an eigenvector corresp. to λ_j, $\lambda_j \in \Lambda$ for $j = 1, \ldots, h$ (and one of λ_js equals $\rho(A)$). Choose a nonzero polynomial $v(t)$ with nonnegative coefficients such that $v(0) = 1$ and $v(\lambda_j) = 0$ for $\lambda_j \neq \rho(A), j = 1, \ldots, h$. Then $v(A)y$ is an eigenvector of A (corresp. to $\rho(A)$) that belongs to $\mathrm{int}\, K$. ■

The existence of the polynomial $v(t)$ follows from:

Lemma. *For every complex number α off the non-negative real axis there exist positive numbers $w_0, \ldots, w_q$ such that $\sum_{p=0}^{q} w_p \alpha^p = 0$.*

Given $A \in \mathcal{M}_n, 0 \neq x \in \mathbb{C}^n$.

$x = x_1 + \cdots + x_m$, where $x_1, \ldots, x_m$ general. eigenvectors corresp. to $\lambda_1, \ldots, \lambda_m$.

$\rho_x(A) := \max_{1 \leq j \leq m} |\lambda_j|$ (local spec. radius)

$\mathrm{ord}_A(x) := \max\{\mathrm{ord}_A(x_j) : |\lambda_j| = \rho_x(A)\}$.

$\Lambda_x := \{\lambda_j : |\lambda_j| = \rho_x(A), \mathrm{ord}_A(x_j) = \mathrm{ord}_A(x)\}$.

$x_{1j} := (A - \lambda_j)^{\mathrm{ord}_A(x_j)-1} x_j$, eigenvector corresp. to λ_j.

A satisfies the *local Perron-Schaefer condition* at x if $\rho_x(A) \in \Lambda_x$.

Fact. If $A \in \pi(K)$, then A satisfies the local PS condition at $x, \forall 0 \neq x \in K$.

Theorem. [Tam] *Let $A \in \mathcal{M}_n(\mathbb{R})$, $x \in \mathbb{R}^n$ such that $\rho_x(A) > 0$.*

Let $x = x_1 + \cdots + x_m$ be the representation as a sum of general. eigenvectors.

$\Lambda_x = \{\lambda_1, \ldots, \lambda_h\}$, $\lambda_j = \rho_x(A)e^{i\theta_j}, 1 \leq j \leq h$.

Then $y \in \omega(x;T)$ iff y is of the form

$$\frac{\alpha_1 x_{10} + \cdots + \alpha_h x_{1h}}{\|\alpha_1 x_{10} + \cdots + \alpha_h x_{1h}\|},$$

where $\alpha_1, \ldots, \alpha_h \in \mathbb{C}$ s.t. $\exists$ subseq. $(r_k)_{k=1}^{\infty}$ of the seq. of natural numbers satisfying

$$\lim_{k\to\infty} e^{i(r_k\theta_j - \mathrm{ord}_A(x)+1)} = \alpha_j$$

for $j = 1, \ldots, h$. Consequently, $\omega(x;T)$ is finite iff every elt. of Λ_x equals $\rho_x(A)$ times a root of unity.

Theorem. *Let A be an $n \times n$ real matrix, and let x be a given nonzero vector of $\mathbb{R}^n$. The following conditions are equivalent:*

(a) *A satisfies the local Perron-Schaefer condition at x.*

(b) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*

(c) *The convex cone* $\mathrm{cl}(\mathrm{pos}\,\{A^i x : \ i = 0, 1, \ldots\})$ *is pointed.*

(d) *There is a closed, pointed convex cone C containing x such that $AC \subseteq C$.*

[Q] Can we ask a similar question for a non-expansive map ?

Theorem. *Let $A \in \pi(K)$ be non-nilpotent. For each nonzero distinguished eigenvalue λ of A, let W_λ be the direct sum of all (real) eigenspaces of A corresponding to eigenvalues with modulus equal to λ. Then*

$$\cup_x \omega(x;T) = \oplus_\lambda (W_\lambda \cap K) \subseteq \mathrm{core}_K(A),$$

where the union is taken over all $x \in S^{n-1} \cap K$ such that $\rho_x(A) > 0$ and the direct sum is taken over all nonzero distinguished eigenvalues λ of A.

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http://www.math.nthu. edu.tw/tjm/

$$\omega(x;f) = \begin{cases} \mathrm{orbit}_f(x) & \text{if } x \text{ is a periodic point of} f \\ \mathrm{cl}\,\mathrm{orbit}_f(x) & \text{otherwise} \end{cases}$$

出席國際會議報告

會議名稱：第十三屆國際線性代數學會會議

會議地點：荷蘭 阿姆斯特丹 Vrije 大學

會議時間：95 年 7 月 18 日至 21 日

報 告 人：淡江大學數學系　譚必信

撰寫日期：95 年 7 月 26 日

《國際線性代數學會會議》今年輪到在荷蘭阿姆斯特丹 Vrije 大學舉行，參加者來自世界各地，共約 250 人。

在四個全天的會議中總共安排了 6 個一小時、8 個 45 分鐘的大會演講，約 125 個 25 分鐘的分組報告及 7 個迷你會議（含 93 場各 25 分鐘的演講），其主題分別為：Structured Matrices、Positive Matrices、Numerical Linear Algebra、Matrix Canonical Forms、Matrix and Operator Inequalities、Matrices in Indefinite Inner Product Spaces 及 Linear Algebra in Statistics。分組報告也有分主題，包括：

Graphs、Core Linear Algebra、Canonical Forms、Eigenvalue Problems、Nonnnegative Matrices、Applications、Operator Theory、System Theory、Numerics、Matrix Factorizations、Algebra、Perturbations、Numerical Ranges、Geometry/Graphs、Polynomials 及 Matrix Patterns 等等。

這次會議的主題與最近幾屆的相差不遠，但我發現 Structured Matrices 及 Indefinite Inner Product 這兩個題目特別受到重視—它們各佔了一個迷你會議及兩場大會演講。

這屆的 Hans Schneider Prize 是由美國的 Richard Varga 及 Richard Brualdi 分別獲得，而這次會議的 Hans Schneider Prize 演講則是由 Brualdi 主講，講題為 “Diagonals and Cycles in Matrices”。另外，Robert Plemmons 及 Stephen Kirkland 分別為會議的 LAA Speaker 及 LAMA Lecturer，他們的講題分別為 “Nonnegative Matrix Factorizations and Applications” 及 “Perron Values and the Evolution of Dispersal”。

除了大會報告以外，會議的演講都是（分六或七組）平行進行。本人主要選擇聆聽主題跟自己興趣相關的幾個分組報告。其中包括：Positive Matrices、Matrix and Operator Inequalities 及 Core Linear Algebra 等等。

本人的演講是安排在會議第一天下午的分組 “Operator Theory”，講題為 “The Omega Limit Set of a Point under a Cone-Preserving Linear Map”，我的演講引起了一些注意。

開會前一天下午我也有參加在 Vrije 大學舉辦的一個 Pre-conference Workshop，主題為 Positive Systems，總共有三場 1 小時的演講，我特別感到興趣的是 Bas Lemmens 所講的 “Nonlinear Perron-Frobenius Theory and Non-expansive Maps Dynamics”，因為這個講題與我的新研究方向相關。有幸能認識 Lemmens 並與他交換一些研究心得。

攜回資料：會議議程及摘要一本。