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ON THEORY MULTIPLE CONTRACTION

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To my grandmother Maria Adelaide,
for everything.

Abstract

The one which is considered the standard model of theory change was presented in [AGM85] and is known as the AGM model. In particular, that paper introduced the class of partial meet contractions. In subsequent works several alternative constructive models for that same class of functions were presented, e.g.: safe/kernel contractions ([AM85, Han94]), system of spheres-based contractions ([Gro88]) and epistemic entrenchment-based contractions ([Gär88, GM88]).

Besides, several generalizations of such model were investigated. In that regard we emphasise the presentation of models which accounted for contractions by sets of sentences rather than only by a single sentence, i.e. multiple contractions. However, until now, only two of the above mentioned models have been generalized in the sense of addressing the case of contractions by sets of sentences: The partial meet multiple contractions were presented in [Han89, FH94], while the kernel multiple contractions were introduced in [FSS03].

In this thesis we propose two new constructive models of multiple contraction functions, namely the system of spheres-based and the epistemic entrenchment-based multiple contractions which generalize the models of system of spheres-based and of epistemic entrenchment-based contractions, respectively, to the case of contractions (of theories) by sets of sentences. Furthermore, analogously to what is the case in what concerns the corresponding classes of contraction functions by one single sentence, those two classes are identical and constitute a subclass of the class of partial meet multiple contractions.

Additionally, and as the first step of the procedure that is here followed to obtain an adequate definition for the system of spheres-based multiple contractions, we present a possible worlds semantics for the partial meet multiple contractions analogous to the one proposed in [Gro88] for the partial meet contractions (by one single sentence).

Finally, we present yet an axiomatic characterization for the new class(es) of multiple contraction functions that are here introduced.

Keywords:

Belief Change; Theory Contraction; Multiple Contraction; Possible Worlds Semantics; System of Spheres; Epistemic Entrenchment.

Resumo

Aquele que é considerado o modelo padrão de mudança de teorias foi introduzido em [AGM85] e é conhecido por modelo AGM. Em particular, esse artigo apresenta a classe das contrações *partial meet*. Em trabalhos posteriores foram apresentados vários modelos construtivos alternativos para essa mesma classe de funções, e.g.: contrações *safe/kernel* ([AM85, Han94]), contrações baseadas em sistemas de esferas ([Gro88]) e contrações baseadas em relações de entrincheiramento epistémico ([Gär88, GM88]).

Além disso, foram investigadas várias generalizações desse modelo. Nesse sentido destacamos a apresentação de modelos que tinham em consideração as contrações por conjuntos de sentenças − expressões suscetíveis de serem verdadeiras ou falsas − em vez de apenas por uma única sentença, i.e. contrações múltiplas. No entanto, até o momento, apenas dois dos modelos acima mencionados foram generalizados no sentido de considerar o caso de contrações por conjuntos de sentenças: As contrações múltiplas *partial meet* foram apresentadas em [Han89, FH94], enquanto que as contrações múltiplas *kernel* foram introduzidas em [FSS03].

Nesta tese propomos dois novos modelos construtivos de funções de contração múltipla, nomeadamente as contrações múltiplas baseadas em sistemas de esferas e as contrações múltiplas baseadas em relações de entrincheiramento epistémico. Tais modelos generalizam os modelos de contrações baseadas em sistemas de esferas e de contrações baseadas em relações de entrincheiramento epistémico, respectivamente, ao caso de contrações (de teorias) por conjuntos de sentenças. Além disso, tal como no caso das classes correspondentes de funções de contração por uma única sentença, verifica-se que as duas classes de contrações múltiplas aqui introduzidas são idênticas e constituem uma subclasse da classe das contrações múltiplas *partial meet*.

Adicionalmente, e como primeiro passo do procedimento que aqui seguimos para obter uma definição adequada para as contrações múltiplas baseadas em sistemas de esferas, apresentamos uma semântica de mundos-possíveis para as contrações múltiplas *partial meet* análoga à que foi proposta em [Gro88] para as contrações *partial meet* (por uma única sentença).

Finalmente, apresentamos ainda uma caracterização axiomática da(s) nova(s) classe(s) de funções de contração múltipla que são aqui introduzidas.

Palavras-chave:

Mudança de Crenças; Contração de Teorias; Contração Múltipla; Semântica de Mundos Possíveis; Sistemas de Esferas; Entrincheiramento Epistémico.

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Chapter 1

Introduction

The central goal underlying the research area commonly known as *logic of theory change* is the study of the changes which can occur in the belief state of a rational agent[1] when he receives new information. More precisely, when an agent is faced with evidence of some new fact (in the sense that such information is not included in his current set of beliefs) he is forced to change his set of beliefs,[2] and the main purpose of most of the works in the area of logic of theory change is to investigate and model how such modifications on the belief state of an agent occur. Two main distinct kinds of such changes can be intuitively identified, namely *incorporation* and *contraction*, where, loosely speaking, by incorporation we mean the addition of one or more new beliefs to the set of beliefs of the agent, and by contraction we mean removal of one or several beliefs from the belief state of the agent.

Hence, one of the main concerns of the logic of theory change consists in modeling how information is removed from the set of beliefs of an agent, in other words, one of its fundamental purposes is to find appropriate ways of defining contraction functions. Roughly speaking, under the assumption that the beliefs of an agent are represented by sentences, given a set of beliefs (i.e. a set of belief-representing sentences), such a function receives a (or a set of) sentence(s) that is (are) intended to be removed from it and returns a new set of beliefs that no longer contains the given sentence(s).

For the investigation of such processes essentially two approaches are usually followed: On the one hand contraction functions can be characterized by identifying conditions − commonly referred to as *postulates* − which are natural to be demanded from this kind of operations, having in mind the fulfilment of the rationality criteria under consideration at the moment. On the other hand, explicit (constructive) definitions of contraction functions can be presented. Nonetheless, those two approaches can be seen as being complementary to each other and, in fact, in general, the studies of belief contraction functions attempt to conciliate the two approaches namely by providing, simultaneously, a constructive definition of a certain kind of contraction functions and a set of postulates which exactly charac-

[1]Here, by a *rational agent* we mean an abstract entity (which may represent a computational system, a robot, a person, etc.) from which rational reactions are expected.

[2]Notice that when an agent receives information which was already present in his set of beliefs then no change at all should occur in such set.

terizes the class of functions so obtained. The results which consist of (univocally) identifying a certain class of contraction functions with a certain set of postulates are called *representation theorems* or *axiomatic characterizations*.

The first and most widely known constructive model of a contraction operation is the *partial meet contraction* which was introduced by Carlos Alchourrón, Peter Gärdenfors and David Makinson in [AGM85]. The model of contraction functions presented in that paper has acquired the status of standard model in the area of logic of theory change. Nevertheless, several other well known constructions have been proposed after that one, namely, *safe contraction* [AM85], *system of spheres-based contraction* [Gro88] and *epistemic entrenchment-based contraction* [Gär88, GM88].

However, each of the above mentioned kinds of contraction functions only models processes consisting of the removal of a single sentence from a *belief set* or *theory* − a set of sentences which is closed under logical consequence − thus originating a new belief set (or theory). This fact yields, as it was promptly pointed out by several researchers (e.g. in [AM82, Mak87, Fuh88, Han89, Fuh91, Nie91, FH94]), two major limitations of all those approaches:

1. Only accounting for the case when the belief state of an agent is represented by a set of sentences that is closed under logical consequence (i.e. a theory).

2. Considering only the removal of a single sentence.

Hence, it became necessary to *improve* those constructions in two ways:

> "(...) there is a need for operators for theory change that can account for changes in sets not closed under logical consequence and also for changes consisting of the addition or retraction of sets with more than one element."
> (Sven Ove Hansson, in [Han89, p. 119])

This led, in particular, to the introduction of several constructions which extended and generalized the above mentioned models in one (or both) of the following ways:

1. They could also be applied even if the set of sentences representing the beliefs of an agent is not (required to be) closed under logical consequence. Such sets are called *belief bases*.

2. They accounted for contractions by (possibly non-singleton) sets of sentences instead of (only) by a single sentence.

The functions which address the latter concern, i.e. which model the removal of (possibly non-singleton) sets of sentences from a given set of beliefs are commonly known as *multiple contractions*.[3] One of the first constructive models of multiple contractions to be presented was the class of *partial meet multiple contractions*

[3] The term *multiple contraction* was introduced by Fuhrmann in [Fuh88].

([Han89, Han91a, FH94]).[4] The definition of such functions consists of a generalization of the definition of the partial meet contractions (by one single sentence) of [AGM85] which accounts for the case of contractions by sets of sentences.

After the publication of Fuhrmann and Hansson's paper [FH94], several years passed until another explicit definition of a multiple contraction function was presented. In fact, the next kind of multiple contraction function to be presented were the *kernel multiple contractions*. Those functions, introduced by Fermé, Saez and Sanz in [FSS03], result of the generalization to the case of multiple contraction of the *kernel contraction functions*, which were originally presented in [Han94] and that can, on their turn, be seen as a generalization of the safe contractions of [AM85].

The multiple contraction functions referred in the above paragraphs are the only generalizations (in order to account for contractions by sets of sentences) so far presented in the literature of the constructions of contractions (of belief sets by one single sentence) which we have mentioned further above. Thus, some of the primordial models of contraction functions (of belief sets by one single sentence) have not yet been generalized in that way.

Indeed, the number of explicit constructions of multiple contraction functions so far presented is still quite smaller than the number of existing models for *singleton contraction*.[5] Having such fact in mind, the main goal of this text is to contribute for the reduction of that gap by introducing some new explicit definitions of multiple contraction functions. Such goal is more detailedly explained in the following section.

1.1 Aims

As we have remarked above, only a couple of the (above mentioned) seminal models for singleton contraction have been, until now, generalized to the case of multiple contractions (rather than only by a single sentence), namely the models of *partial meet contraction* and of *kernel contraction* (which, on its turn, is a generalization of the model of *safe contraction*). Having that fact in mind, the main goal of the present thesis is precisely to present new explicit constructions of multiple contraction functions which generalize – in the sense of accounting (also) for the contraction by sets of sentences – other two of those seminal models of contraction (of a belief set by a single sentence), namely the *system of spheres-based contractions* of [Gro88] and the *epistemic entrenchment-based contractions* of [Gär88, GM88].

It is well known that the classes of system of spheres-based (singleton) contractions and of epistemic entrenchment-based (singleton) contractions are identical and constitute a subclass of the class of partial meet (singleton) contractions.[6] Having

[4] The operations that we are designating here by *partial meet multiple contractions* were originally called *simple partial meet contractions*, in [Han89, Han91a] and, latter, $\perp$*-based partial meet* $(\perp -pm)$ *contractions*, in [FH94].

[5] Throughout this text we will use the expression "singleton contraction" to designate a contraction by a single sentence. This designation is supported by the possibility of identifying the contraction of a set of beliefs by a single sentence α with the (multiple) contraction of that same set of beliefs by the singleton set $\{\alpha\}$.

[6] To be more precise, it is known that both the classes of system of spheres-based (singleton) contractions and of epistemic entrenchment-based (singleton) contractions coincide with the class

that in mind, it is our intention that those interrelations between different constructions also holds among the classes of multiple contraction functions which generalize each of those models of singleton contraction.

Furthermore, it is also our goal to obtain an axiomatic charaterization for the new class(es) of multiple contraction functions that we plan to propose.

Hence, the central aims of the present thesis can be summarized as follows:

1. To propose a definition for the *system of spheres-based multiple contractions*, assuring that such functions:
   - Can be seen as a generalization of the construction of the system of spheres-based contractions proposed in [Gro88] which accounts for the case of contractions by (possibly non-singleton) sets of sentences.
   - Are partial meet multiple contractions.
2. To propose a definition for the *epistemic entrenchment-based multiple contractions* which is such that:
   - It generalizes the definition of the epistemic entrenchment-based contractions proposed in [Gär88, GM88] in order to account for the case of contractions by sets of sentences.
   - The class of epistemic entrenchment-based multiple contractions coincides with the class of system of spheres-based multiple contractions (mentioned in Point 1. above).
3. To present an axiomatic characterization for the system of spheres-based multiple contractions and the epistemic entrenchment-based multiple contractions. At this point it is worth to remark that such an axiomatic characterization can also be thought of as a generalization, to the case of multiple contractions, of the well know representation theorems which state that the classes of system of spheres-based (singleton) contractions and of epistemic entrenchment-based (singleton) contractions coincide with the class of (contraction) functions that satisfy the basic and supplementary AGM postulates for belief set (singleton) contraction.

1.2 Main Contributions

As it is natural to expect, having in mind those which were the aims of this work (that we have exposed in the previous section), among the main contributions of the present thesis we can identify the following ones: the presentation of explicit definitions for system of spheres-based multiple contractions and for epistemic-entrenchment multiple contractions and the obtention of an axiomatic characterization of such functions. We notice here that those two classes of multiple contraction functions are the first generalizations (to the multiple contraction level), of

of *transitively relational partial meet contractions* (introduced in [AGM85]), which is a (proper) subclass of the class of partial meet contractions.

constructive models of (singleton) contraction functions which satisfy the basic and the supplementary AGM postulates for belief set contractions, to be defined and axiomatically characterized.

The mentioned explicit construction of system of spheres-based multiple contractions is presented in Chapter 7, where it is also shown that such are a generalization of the system of spheres-based (singleton) contractions (of [Gro88]) and that any system of spheres-based multiple contraction is a partial meet multiple contraction. The main results of that chapter have been accepted for publication and shall appear in [FR].

As for the explicit definition of the epistemic entrenchment-based multiple contraction, it is presented in Chapter 8. In that chapter it is shown that the class of epistemic entrenchment-based multiple contractions is such that: (a) it is a generalization of the class of epistemic entrenchment-based (singleton) contractions (of [Gär88, GM88]) and (b) it coincides with the class of system of spheres-based multiple contractions.

In what concerns the above mentioned axiomatic characterization of the system of spheres-based and the epistemic entrenchment-based multiple contractions, such representation theorem is presented in Chapter 9.

The definitions and results mentioned in the above paragraphs fulfil those which were the aims of this work and are, therefore, among the main contributions of the present thesis. However, there are some other results in this work which also deserve to be highlighted here.

Indeed, the process of obtaining the above mentioned new definitions of theory multiple contraction functions, led to the obtention of some other related results whose motivation and content we briefly explain in the following paragraphs.

In order to generalize the construction of system of spheres-based contractions proposed by Grove ([Gro88]) in a way that covers the case of contractions by sets of sentences, we felt the need to start by obtaining a characterization of the partial meet multiple contraction in terms of possible worlds. Such possible worlds semantics of the partial meet multiple contractions is presented in Chapter 6 and consists essentially of the generalization (to the case of contractions by sets of sentences) of the method for constructing partial meet (singleton) contractions through a propositional approach[7] which was exposed in [Gro88, Han99b]. The main results of that chapter will appear in the paper [RF].

On the other hand, in our quest for an appropriate definition for the epistemic entrenchment-based multiple contractions which assured, in particular, that the class of such functions coincided with the class of system of spheres-based multiple contractions, we realized that it was convenient to start by detailedly clarifying the strong interrelation between systems of spheres and epistemic entrenchment relations. Therefore, we decided to start the investigation of an adequate definition for epistemic entrenchment-based multiple contractions by showing in a direct way the well known fact that (in the context of contractions by one single sentence) the class of system of spheres-based contractions coincides with the class of epistemic entrenchment-based contractions. More precisely, we proved that such identity holds

[7]Here, by "propositional approach" we mean a method which is based in selections among propositions (i.e. sets of possible worlds).

by means of a two-steps procedure: first we clarified that a certain condition (namely condition ($\leq -\mathbb{S}$)) relating an epistemic entrenchment relation $\leq$ and a system of spheres $\mathbb{S}$ is necessary and sufficient to assure that the $\leq$-based contraction and the $\mathbb{S}$-based contraction are identical, and after that we exposed that given an arbitrary system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ such that the mentioned condition holds and, vice versa, given an arbitrary epistemic entrenchment relation $\leq$ it is possible to construct a system of spheres $\mathbb{S}$ such that that same condition is satisfied.

However, we must clarify here that a (different) direct proof for the fact that those two classes of (singleton) contraction functions are identical can be found in [RP99] and, on the other hand such a proof can also be seen as following immediately from the combination of certain results of [Gro88, Gär88, PW95]. Therefore, the presentation of a direct proof for that fact cannot be considered an original contribution. Nevertheless, the explicit proof for that fact that is presented in the present thesis, more precisely in Section 4.2, as well as some of the intermediate results which are part of such proof (namely, Theorem 4.2.6 and Observation 4.2.11) are original and, furthermore, the proofs that are presented for each of them in this thesis are completely independent from the analogous results which can be found in some of the above mentioned papers.

1.3 Organization of the Thesis

In order to make this thesis as self-contained as possible in Chapters 2 − 5 we include all the background needed to ensure the readability of this work.

In **Chapter 2** we briefly introduce the main objects of study of the research area designated by *Logic of Theory Change* and we present the main concepts underlying such subject.

In **Chapter 3**, we present the primordial constructive models of contraction functions of belief sets by one single sentence (namely, *partial meet contraction*, *safe/kernel contraction*, *system of spheres-based contraction* and *epistemic entrenchment-based contraction*), and we expose axiomatic characterizations for each of those kinds of contraction functions. Then, in **Chapter 4** we present the interrelations among those models which have, so far, been proven to hold in a direct way. Of that chapter we highlight the direct proof of the equality of the classes of *system of spheres-based contraction* and of *epistemic entrenchment-based contraction* that is presented in Section 4.2 since it differs from any of the proofs for that fact that have so far been presented in the literature.

Chapter 5 is dedicated to the introduction of some of the main constructive models of multiple contraction functions which have, so far, been presented in the literature. More precisely, in that chapter we present the models of *partial meet multiple contraction* and of *kernel multiple contraction* which consist of generalizations to the case of contractions by (possibly non-singleton) sets of sentences of the models of *partial meet (singleton) contraction* and *kernel (singleton) contraction*. We close that chapter with a section which presents some works which are closely related to the topic of multiple contraction.

The readers of this work who are familiarized with the basic concepts of the area of logic of theory change and, more specifically, with the known basic results concerning the study of (singleton) contraction and multiple contraction of theories, may skip all the above described introductory chapters and consult them only to get acquainted with the notation that shall be used throughout the remainder of the text as well as whenever some of the definitions or results there included is mentioned throughout the remaining chapters.

The main contributions of the present thesis are exposed in Chapters 6 − 9.

In **Chapter 6** we present a characterization of the partial meet multiple contractions in terms of possible worlds. More precisely, we present a way of obtaining the partial meet multiple contractions by means of a propositional approach. Such possible worlds semantics for the partial meet multiple contractions is, roughly speaking, a generalization to the multiple contraction case of the possible worlds semantics for the partial meet (singleton) contraction presented in [Gro88, Han99b].

Then, based on the above mentioned possible worlds semantics for partial meet multiple contraction, in **Chapter 7**, we expose a method for using systems of spheres as a tool for constructing multiple contraction functions. More precisely we present a definition for the *system of spheres-based multiple contractions* which constitute a new kind of multiple contractions that (a) are a generalization to the multiple contraction level of the system of spheres-based (singleton) contractions proposed by Grove ([Gro88]) and (b) are partial meet multiple contractions (analogously to what is the case in what concerns the corresponding classes of singleton contraction functions).

Afterwards, making use of the strong interrelation between systems of spheres and epistemic entrenchment orderings (which is detailedly exposed in Section 4.2), in **Chapter 8** we obtain a definition for the *epistemic entrenchment-based multiple contractions* which is such that (a) it generalizes Gärdenfors and Makinson's definition of epistemic entrenchment-based (singleton) contractions ([Gär88, GM88]) to the case of contraction by sets of sentences, and (b) the class of such functions coincides with the class of the above mentioned *system of spheres-based multiple contraction* (analogously to what is the case regarding the corresponding classes of singleton contractions).

The following chapter, i.e. **Chapter 9**, is devoted to the presentation of an axiomatic characterization of the newly defined classes of multiple contraction functions, i.e. a representation theorem for the system of spheres-based multiple contractions and the epistemic entrenchment-based multiple contractions.

Finally, **Chapter 10** contains a brief overview of the main achievements of this thesis as well as an exposition of some topics of future research which arise naturally from the investigation reported in the present text.

Regarding the structure of this dissertation it remains only to remark here that, in order to improve the readability of the thesis all the proofs provided for original results presented in the thesis are deferred to the **Appendix**.

1.4 Formal Preliminaries

We will assume a language $\mathcal{L}$ that is closed under all Boolean connectives. By abuse of notation we shall use the same letter $\mathcal{L}$ to denote also the set of the well formed formulae of that language. The elements of $\mathcal{L}$ will be referred to as *sentences* (of $\mathcal{L}$). Lower-case Greek letters (with or without subscripts) – e.g. $\alpha, \alpha_i, \beta, \ldots$ – other than γ and σ, will be used to denote sentences (i.e. elements of $\mathcal{L}$).[8] $\top$ stands for an arbitrary tautology and $\bot$ for an arbitrary contradiction. Upper-case Roman letters (with or without subscripts) – e.g. $A, A_i, B, \ldots$ – shall denote sets of sentences (i.e. subsets of $\mathcal{L}$). The conjunction of all elements of a finite and non-empty set of sentences A shall be denoted by $\bigwedge A$.

We shall make use of a consequence operation Cn that takes each set of sentences to another set of sentences and which satisfies the standard Tarskian properties ([Tar30]):

- *inclusion*: $A \subseteq Cn(A)$,
- *monotony*: if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$,
- *idempotence*: $Cn(A) = Cn(Cn(A))$,

Furthermore we will assume that Cn also satisfies:

- *supraclassicality*: if α can be derived from A by classical truth-functional logic, then $\alpha \in Cn(A)$,
- *deduction*: if $\beta \in Cn(A \cup \{\alpha\})$, then $(\alpha \to \beta) \in Cn(A)$,
- *compactness*: if $\alpha \in Cn(A)$, then $\alpha \in Cn(A')$ for some finite subset A' of A.

The following alternative notations will sometimes be used: $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\alpha \vdash \beta$ for $\beta \in Cn(\alpha)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\alpha \nvdash \beta$ for $\beta \notin Cn(\alpha)$, and $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. A set of sentences H shall be said to be *closed under logical consequence* or *logically closed* whenever it is such that $H = Cn(H)$ – such a set is called a *belief set* or *theory*. We shall use $\mathbf{K}$ to represent a belief set (hence, throughout this text $\mathbf{K}$ is always a set of sentences such that $\mathbf{K} = Cn(\mathbf{K})$). Sometimes, although not very often, other capital upper-case Roman letters in bold face shall be used to represent belief sets or theories. We shall

[8] The Greek letters γ and σ will be used to denote *selection functions* (cf. Definition 3.1.5) and *incision functions* (cf. Definition 3.2.2), respectively.

use $\mathbf{K}_\perp$ to denote the inconsistent belief set (containing all $\mathcal{L}$-sentences). A belief set is a consistent complete theory (of $\mathcal{L}$) if and only if it is a maximal consistent subset (of $\mathcal{L}$). We shall denote the set of all theories of $\mathcal{L}$ by $\mathcal{T}_\mathcal{L}$ and the set of all consistent complete theories of $\mathcal{L}$ by $\mathcal{M}_\mathcal{L}$. Consistent complete theories shall play an essential role further ahed, and we will often use the terms *possible world* and *world* to refer to one such theory.

Upper-case Roman calligraphic letters (with or without subscripts) – e.g. $\mathcal{M}, \mathcal{N}_i, \mathcal{W}, \ldots$ – other than $\mathcal{L}$, $\mathcal{T}$ and $\mathcal{P}$, shall be used to denote subsets of $\mathcal{M}_\mathcal{L}$. Such sets are called *propositions*. The letter $\mathcal{P}$ is reserved to represent power sets, more precisely, given an arbitrary set S, we shall denote the power set of S by $\mathcal{P}(S)$ and the set of finite subsets of S by $\mathcal{P}_\mathcal{F}(S)$.

Given a set of sentences R, the set consisting of all the possible worlds that contain R will be denoted by $\|R\|$ and referred to as the *set of possible worlds for* R. The elements of $\|R\|$ are the R-worlds. For a sentence $\varphi \in \mathcal{L}$, $\|\varphi\|$ is an abbreviation of $\|\{\varphi\}\|$ and the elements of $\|\varphi\|$ are the φ-worlds. To any set of possible worlds $\mathcal{V}$ we associate a belief set $Th(\mathcal{V})$ given by $Th(\mathcal{V}) = \bigcap \mathcal{V}$ (under the assumption that $\bigcap \emptyset = \mathcal{L}(= \mathbf{K}_\perp)$)[9].

Upper-case Roman letters in *mathbb* font (with or without subscripts) – e.g. $\mathbb{M}, \mathbb{N}_i, \mathbb{W}, \ldots$ – shall be used to denote subsets of $\mathcal{P}(\mathcal{M}_\mathcal{L})$. In particular $\mathbb{S}$ is reserved to represent a system of spheres (cf Definition 3.3.5).

[9]Notice that this is a natural assumption in the present context. Indeed, for any set of possible worlds $\mathcal{U} \subseteq \mathcal{M}_\mathcal{L}$ it holds that $\bigcap \mathcal{U} = \{\alpha : \forall M \in \mathcal{U}, \alpha \in M\} = \{\alpha \in \mathcal{L} : \forall M \in \mathcal{U}, \alpha \in M\}$ (where this last equality holds due to the fact that the possible worlds are, by definition, subsets of $\mathcal{L}$), therefore, in the particular case of the empty subset of $\mathcal{M}_\mathcal{L}$ we have that $\bigcap \emptyset = \{\alpha \in \mathcal{L} : \forall M \in \emptyset, \alpha \in M\} = \mathcal{L}$ (since for any $\alpha \in \mathcal{L}$ the condition "$\forall M \in \emptyset, \alpha \in M$" is vacuously true).

Chapter 2

The Logic of Theory Change

In this chapter we introduce the main concepts concerning the logic of theory change. The main purpose for this is to provide the reader with the preliminary material necessary for the understanding of the remaining contents of this text and also to leave it clear where the main contributions of this thesis stand within the general picture of the current state of the art of the area of belief revision. Nevertheless, we notice that completer texts on the basics and the state of the art of the research area known (among other alternative designations) as logic of theory change can be found in [Gär88, Han99b, Pep08, Han09] (references on which the introductory chapters of this thesis are mostly based).

2.1 Introduction

The *Logic of Theory Change* (also known e.g. as *Belief Revision*, *Belief Change*, *Belief Dynamics*) is a research area which studies the dynamics of knowledge and beliefs. More precisely, its main target is to model the way a rational agent's[1] set of beliefs/knowledge is updated when he receives new information.

We start by clarifying at this stage that although, from a philosophical perspective, *knowledge* and *belief* are different entities, such distinction is beyond the scope of the present thesis. For our purposes it is simpler and enough to see them as different designations for the same concept. Hence, throughout this text the expressions *knowledge (set)* and *set of beliefs* of an agent will be used with the same meaning.

The two basic and more intuitive kinds of change that can occur to the belief state of some agent are the following:

- *incorporation* or *addition*: when the belief state is updated in order that a new belief (which was previously not included in the agent's set of beliefs) is added to it.

- *contraction* or *derogation*: when the belief state is updated in order that a certain (previous) belief of the agent (i.e., an element of its (previous) set of beliefs) is contracted (removed) from it (in the sense that the new belief state of the agent no longer contains such (previous) belief).

Each one of these kinds of change will be more thoroughly and formally analysed and explained below, after introducing the basic elements of any theoretical framework for modelling the dynamics of belief. However, at this point, it is worth (and natural) to anticipate that, if it is assumed that the belief states are, as long as that is possible, consistent (i.e., they do not contain contradictory beliefs)[10] then the process of *incorporation* takes a different form in each of the two following alternative possible cases:

1. the new belief (which is intended to be included in the agent's set of beliefs) is consistent with the (prior) belief state.

2. the belief intended to be added to the belief state is inconsistent with the beliefs (previously) included in such set of beliefs.

The above mentioned (possible) assumption of *consistency* of the belief states is an example of a *rationality criteria*. Rules of that kind play a central role in any process of change of belief state. Therefore, as we shall clarify further ahead, the logic of theory change aims at presenting a *theoretical framework* for modelling the dynamics of belief which includes rationality principles that govern each of the different kinds of changes of belief states.

In the following section we introduce the basic elements of any formal framework to model belief change. Throughout this chapter, following [Gär88], we shall sometimes refer to such a framework as an Epistemological Theory.

2.2 The Elements of an Epistemological Theory

In this section, which is essentially based on [Gär88], we clarify the meaning of *epistemological theory* and introduce the main epistemic factors that form the core of such theories.

Generally speaking, an *epistemological theory* is the theoretical framework which embeds the different models of epistemic states and their dynamics. Such a theory aims at investigating the changes in beliefs, therefore it must provide, among other things, a representation of the *epistemic elements* − namely, the *epistemic states*, the *epistemic attitudes*[11], *epistemic inputs* and the *epistemic changes* or *changes of beliefs* − and the *criteria of rationality* that govern the epistemic dynamics.

2.2.1 Epistemic States

The *epistemic states* or *belief states* are the most fundamental factor in an epistemological theory. Such entities are representations of actual or possible cognitive states of some agent at a given moment.

[10] Notice that it is natural to make such assumption since, in general, a rational agent is not expected to believe simultaneously in contradictory facts.

[11] The epistemic attitudes are also known as *doxastic attitudes*. In fact, to be more precise, in philosophical terminology "epistemic" means related to knowledge and "doxastic" means related to beliefs (e.g. [Han99b, p. 6]). However, as we have already mentioned above, along this text we shall not distinguish between knowledge and belief.

There are several models of epistemic states. Among them we mention the following ones:

1. **Sentential Models**: An epistemic state is represented as a *set of sentences* from some given object language. It is intended that each of the sentences included in that set represents one of the agent's beliefs in the modelled state.

   In general such sets are required to be consistent, since a rational agent is not expected to have contradictory beliefs. Sometimes it is also assumed that such belief state-representing sets are logically closed. Such assumption is made whenever the agent is considered to be *logically omniscient*, in the sense that all the logical consequences of his beliefs are also in his set of beliefs.

   Depending on whether this latter assumption is or is not made the epistemic states of (the sentential model) are represented, respectively, by the following kinds of sets of sentences:

   - **Belief Sets or Theories** - sets of sentences that are closed under logical consequence.
   - **Belief Bases** - sets of sentences that are not (necessarily) closed under logical consequence.

2. **Bayesian Models**: A state of belief is represented by a *probability measure* defined over some object language or over some space of events.

3. **Possible Worlds Models**: In these models an epistemic state is represented by a set of possible worlds (i.e. a *proposition*). The interpretation of such a set is that the agent associated with the modelled state knows that the "actual world" is a member of that set, but it can be **any** of its members.

2.2.2 Epistemic Attitudes

The *epistemic attitudes* or *doxastic attitudes*[11] describe the status of various elements of belief that are contained in an epistemic state. For example, a person can accept or not accept a particular statement as true or may judge it to be certain, probable (with a certain degree of probability), or possible. These are some of the different possible attitudes towards a given statement. The expressions of the natural language which are used to describe epistemic attitudes are called *epistemic judgements*. Some of the most common examples are that someone "believes that α", "is convinced that α", "finds it extremely likely that α", "does not know whether α", "finds it possible that α" and "finds α more likely than β".

Obviously the set of epistemic attitudes which are expressible in a certain epistemological theory depends on what kind of models are used to represent the epistemic states. In formal systems it is usually preferable to consider a small number of epistemic attitudes. This is due to the fact that, although the existence of a lot of epistemic attitudes increases the expressivity of the system, an excessive number of those also makes the theoretical study of the system harder or even impossible.

For example, in a sentential model, where the epistemic states are represented by belief sets three epistemic attitudes that can be considered are: *acceptance*, *rejection* and *indetermination*. More precisely, assuming that the epistemic state of an agent is represented by a belief set, given an arbitrary (belief-representing) sentence α there are only the following three possibilities for the epistemic attitude concerning α:

1. α is *accepted*, which means that α belongs to the set representing the epistemic state;

2. α is *rejected*, which is to say that the negation of α is included in the *state*;

3. α may be *indetermined*, which means that neither α nor its negation are elements of the relevant set of sentences.

Furthermore, if such epistemic state is consistent (i.e. it is represented by a belief set $\mathbf{K}$ such that $\mathbf{K} \neq \mathbf{K}_{\perp}$) then one and only one of the mentioned epistemic attitudes concerning α is withheld.

To finish this summary description of the meaning of the concept of *epistemic attitudes* we remark that, in general, the epistemic attitudes can be described in terms of a *valuation* of the items in a model of an epistemic state. For example, in the case of a sentential model the valuation is given by the membership relation, and in probabilistic models the valuation is given by the probability measure.

2.2.3 Epistemic Inputs

The *epistemic inputs* are pieces of (external) information that may provoke changes in an epistemic state. Moreover it is assumed that a change of a state of belief can only be motivated by such *external forces*. In other words, only an epistemic input can alter the belief state of an agent (to another belief state). These inputs can be seen as a result of the observation of certain phenomena or of the exchange of information between agents.

However, since the content of the epistemic states is at focus, the *form* of the epistemic input is not relevant. What matters is the *effect* the input has on a given epistemic state. Thus, rather than providing a general account of the different forms of epistemic inputs, for our purposes it is enough to define them abstractly by describing their effects on epistemic states.

Furthermore, in this brief introduction of the concept of epistemic inputs and their role in the context of an epistemological theory we shall only describe in more detail a very particular type of epistemic inputs, namely the kind of epistemic inputs which can be identified with a change (caused by them) in the epistemic attitude towards a single sentence.

The typology of the epistemic inputs depends on the kind of model of epistemic states considered. To clarify this fact let us analyse the case when belief states are modelled by logically closed sets of sentences (which express exactly the beliefs that are accepted in the modelled state). Having in mind the three epistemic attitudes that we have mentioned above that can be considered in such sentential models and assuming, additionally, that the epistemic states are consistent (and, consequently, only one epistemic attitude is withheld with respect to any given sentence) it follows that there are the following six possible changes of epistemic attitudes concerning a certain sentence:

(i) From *indetermined* to *accepted*.

(ii) From *indetermined* to *rejected*.

(iii) From *accepted* to *rejected*.

(iv) From *rejected* to *accepted*.

(v) From *accepted* to *indetermined*.

(vi) From *rejected* to *indetermined*.

Having in mind the above list of the possible changes of epistemic attitudes, we can identify the following two basic kinds of changes that may occur on a consistent epistemic state (as a consequence of a change of epistemic attitude towards a single sentence):

1. A sentence α, which was not originally accepted, may be *incorporated* in (i.e. *added* to) the set of beliefs.[12] (Each of the changes of epistemic attitudes (i)$-$(iv) above corresponds to a change of epistemic state of this kind)

2. A sentence α that was accepted may be *contracted* (or *derogated* or *removed*) from the belief state.[13] (Each of the changes of epistemic attitudes (v)$-$(vi) above corresponds to a change of epistemic state of this kind)

Therefore, recalling that we are restricting our attention to epistemic inputs which can be identified with a change of epistemic attitude towards one single sentence, it follows that, when the epistemic states are represented by belief sets, essentially two different classes of epistemic inputs can be considered: (a) one consisting of the epistemic inputs which lead to the *incorporation* (or *addition*) of a (previously not accepted) sentence to the (original consistent) belief state and (b) (another) one containing the epistemic inputs which provoke the *contraction* (or *derogation*) of a (previously accepted) sentence from a (original consistent) belief state.

However, it is convenient to remark here that, in general, a change of epistemic attitude towards a certain sentence implies a change in the epistemic attitude towards other sentences also (for instance if consistency is required to be preserved). To clarify this statement, let us consider, for example, a change of epistemic attitude of the type (iii) above. Let α be a sentence which is neither a tautology nor a contradiction and consider an epistemic input that causes the epistemic attitude towards α to change from accepted (in the previous belief state) to rejected (in the new belief state). Then on the one hand the original belief set contained α but not $\neg\alpha$ while, on the other hand, the new belief set contains $\neg\alpha$ but, under the imposition that the

[12] Notice that if α is an inconsistent sentence then it can not be *incorporated* in the set of beliefs unless the requirement of consistency of the (resulting) belief state is dropped.

[13] Notice that all tautologies are accepted in any epistemic state (modelled by a logically closed set of sentences). Therefore if α is a tautology it can not be *contracted* from any belief state.

belief states are consistent (as long as possible), it does not contain α. Therefore, we can observe that, while the epistemic attitude towards α changed from accepted to rejected, the opposite change occurred in what concerns the epistemic attitude towards $\neg\alpha$.

Having the above in mind, we are led to notice that the class of epistemic inputs which we have identified above as corresponding to the addition of a certain sentence to the belief state can be subdivided into two quite different subclasses: on the one hand the class of epistemic inputs identified with changes of the kinds (i)−(ii) (when the sentence that is added to the belief state (as a consequence of the epistemic input) is consistent with the previous beliefs) and, on the other hand, the class of epistemic inputs that correspond to changes of the kinds (iii)−(iv) (when the sentence that is added to the belief state contradicts the previous beliefs). Hence, in what follows, we shall often consider the subdivision of the class of epistemic inputs in three (rather than two) subclasses.

Having seen this it is convenient to clarify here that throughout this text by identifying the epistemic inputs with a change of epistemic attitude towards one single sentence (namely, by referring to an epistemic input as the *addition* or as the *contraction* of a sentence to/from a belief state) we do not mean that such change is the only one that such an input causes in the belief state. By that we mean, however, that the change with which the epistemic input is identified is the only one that is indeed mandatory (under that input) and, therefore, such an input[14]

(a) leads to the change with which it is identified, and

(b) causes the least possible amount of changes concerning the epistemic attitudes (towards the remaining sentences), in the sense that (as a consequence of that input) only do take place those changes which are strictly necessary in order to assure the change mentioned in (a) indeed occurs.

In summary, given a belief state, the epistemic inputs can be identified with the causes which lead to either the addition (which takes two different forms depending on whether the sentence that is added contradicts or not the previous beliefs) or the derogation of a single sentence from a given belief state. Nevertheless, in general, when it is assumed that the belief states are modelled by sets of sentences which are logically closed and (as long as possible) consistent, an epistemic input causes more changes to a given belief state than the simple addition[15] or contraction of a single

[14] At this point we remark that the requirement made in (a) that the change that is *suggested* by a certain epistemic input does indeed occur is related to the underlying assumption that the new information is always accepted − such assumption is known as (the principle of) *primacy (of the new information)*. On the other hand, the rationale underlying the constraint imposed by (b) is known as *(the principle of) minimal change/loss of the previous beliefs* and, as it is reasonable to expect, such property is a basic requirement of most epistemological theories. We shall describe (in more detail) in Subsection 2.2.5 the *rationality criteria* which are more usually assumed to govern epistemological theories.

[15] Notice that even in the case when the epistemic input under consideration is identified with the addition of a sentence which is consistent with the previous beliefs, in general, some more changes occur in the belief state (for example, in order to maintain the logical closure of the corresponding set of beliefs).

sentence from it, however, it is natural to assume that such additional *collateral* changes are as few as possible.

In what follows we will analyse in more detail what are the different effects that each of the above mentioned different kinds of epistemic inputs have in any given belief state and how such changes of epistemic states can be modelled. However, from the above remarks it is already obvious at this stage that, generally speaking, any epistemic input causes the addition of several sentences to the belief state and, simultaneously, the derogation of several (other) sentences from the belief set.

A general way of describing epistemic inputs, which seems to be applicable to most kinds of models of epistemic states, is to formulate the inputs as *constraints* on the resulting state of belief. On this approach the addition of a sentence α corresponds to the constraint that α be accepted in the new epistemic state, and the derogation of α corresponds to the constraint that α be indetermined in the new state of belief.

Another, more abstract, way of representing epistemic inputs is to describe them as *functions* taking states of belief into new states of belief. This approach consists, then, in identifying the epistemic inputs with the *effects* they have on different epistemic states.

2.2.4 Epistemic Changes

The *epistemic changes* or *changes of belief* are the modifications that occur in the epistemic states. Therefore, recalling that an epistemic input is assumed to be the only possible cause for a change of an epistemic state, the types of changes of epistemic states *admitted* in a given epistemological theory are exclusively determined by the kinds of epistemic inputs that are *available* in such framework.

Now let us see what the above remark means (and implies) in the case of sentential models on which the epistemic states are modelled by belief sets. As we have seen in the previous subsection, when such a model is considered there are essentially two basic kinds of epistemic inputs, namely the ones that lead to the addition of a sentence to the (original) belief state and the ones that lead to the derogation of a sentence from the (original) belief set.[16] Therefore, according to the previous paragraph, we can conclude that in sentential models there are basically two kinds of epistemic changes, namely ***incorporations*** (or *additions*) and ***contractions*** (or *derogations*) (with the obvious meanings).

In this regard Sven Ove Hansson [Han99b, pp. 7 − 8] pointed out that the following principle is a fundamental assumption in belief dynamics introduced by Isaac Levi [Lev77]:

Decomposition principle [Fuh89] (first version) (as cited in [Han99b, pp. 8]):

[16]Recall, however, that, in the previous subsection, to justify the possibility of subdividing the epistemic inputs in (only) such two subclasses we have made use of the fact that we had decided to take into consideration only epistemic inputs which can be identified with a change of epistemic attitude towards one single sentence.

> Every legitimate belief change is decomposable into a sequence of incorporations and contractions.

Next we characterize more formally each of those two kinds of epistemic changes. In order to do that, we identify an epistemic state with the belief set that represents it and describe the epistemic changes in terms of belief sets. More precisely, in what follows an incorporation (respectively, a contraction) shall be seen as an operator (or a function) which receives a belief set (that represents the original epistemic state) and a sentence to be *added* to (resp. *removed* from) it and returns a belief set (which represents the new epistemic state).

So, let $\mathbf{K}$ be a belief set and α be an arbitrary (belief-representing) sentence. In what follows we investigate which properties should be satisfied by any belief set which can be the result of either the *incorporation* or the *contraction* of α (in/from $\mathbf{K}$).

Let us start by considering the case of contraction. In the above terms, the first remark we have to make with respect to contraction is that, for any non-tautological sentence α (i.e., for any $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$)[13], if $\mathbf{K}-\alpha$ is a belief set that results from the contraction of α from $\mathbf{K}$ then it must be such that $\alpha \notin \mathbf{K}-\alpha$. Furthermore, $\mathbf{K}-\alpha$ must also satisfy the following condition $\mathbf{K}-\alpha \subseteq \mathbf{K}$. Indeed, recalling that, as we have remarked in the previous subsection we are assuming that an epistemic input causes the least possible amount of changes in the (previous) belief state, we notice that, although some other sentences $\mathbf{K}$ apart from α may need to be left out from $\mathbf{K}-\alpha$ in order to assure that this latter set is logically closed and does not contain α, there is no need to include in $\mathbf{K}-\alpha$ any sentences which were not originally in $\mathbf{K}$ in order to assure that those two constraints are satisfied.

Notice yet that, since, in the present context, belief sets are being used as representations of epistemic states of a rational agent, we ought to demand also that the resulting belief set $\mathbf{K}-\alpha$ does not contain contradictory beliefs. However, that requirement does not need to be made explicitly because it is an immediate consequence of $\mathbf{K}-\alpha \subseteq \mathbf{K}$ that $\mathbf{K}-\alpha$ is consistent (i.e. $\mathbf{K}-\alpha \neq \mathbf{K}_\perp$) as long as the original belief set $\mathbf{K}$ is so. It is worth noticing that, in most cases, given a belief set $\mathbf{K}$ and a sentence $\alpha \in \mathbf{K} \setminus Cn(\emptyset)$, to obtain a belief set $\mathbf{K}-\alpha$ satisfying all the above mentioned required conditions (i.e., such that $\alpha \notin \mathbf{K}-\alpha \subseteq \mathbf{K}$) many more sentences of $\mathbf{K}$ apart from α itself must be left out from such new belief set. However, on the other hand, $\mathbf{K}-\alpha$ shall contain as many sentences as possible from the original belief set $\mathbf{K}$.[17] This kind of change is called ***contraction*** of $\mathbf{K}$ with respect to α.

We now turn to the case of incorporations. The only obvious constraint that a belief set $\mathbf{K} * \alpha$ which results of the incorporation of α in $\mathbf{K}$ must satisfy is that $\alpha \in \mathbf{K} * \alpha$. Thus the simplest and most intuitive way of constructing (explicitly) a belief set satisfying that condition is to add α set-theoretically to $\mathbf{K}$ and afterwards obtaining the logical closure of the resulting set. This kind of change is called ***expansion*** of $\mathbf{K}$ by α and is more formally introduced in the following definition:

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[17] Notice that if this additional constraint is not required then $Cn(\emptyset)$ would be an acceptable result for any contraction operation.

Definition 2.2.1 **([Lev77])** *Let* $\mathbf{K}$ *be a belief set and* α *be an arbitrary sentence. The* expansion *of* $\mathbf{K}$ *by* α*, is the belief set* $\mathbf{K}{+}\alpha$ *defined by:*

$$\mathbf{K}{+}\alpha = Cn(\mathbf{K} \cup \{\alpha\}).$$

At this point it is convenient to introduce some notation that shall be used throughout this thesis.

Notation 2.2.2 *Throughout this text, given a belief set* $\mathbf{K}$ *and a sentence* α *the notation "*$\mathbf{K}{+}\alpha$*" shall always be used as a representation of the set* $Cn(\mathbf{K} \cup \{\alpha\})$*.*

Since $\mathbf{K}{+}\alpha$ is a belief set and it is obvious that $\alpha \in \mathbf{K}{+}\alpha$ we might be tempted to simply define the result of adding a sentence α to a belief set $\mathbf{K}$ as the expansion of $\mathbf{K}$ by α. However, although in some cases that is indeed the most adequate option, such is not always the case, essentially because in some situations $\mathbf{K}{+}\alpha$ is the inconsistent belief set $\mathbf{K}_{\perp}$.

Indeed, since we are using the belief sets as representations of epistemic states of a rational agent, it is natural to require that (as long as possible) such belief sets are consistent. And, if such imposition is made, then we must distinguish between two types of incorporations. More precisely, when considering the incorporation of a sentence α in a (consistent) belief set $\mathbf{K}$ the following two possibilities have to be taken into account:

1. α is *consistent* with the beliefs in $\mathbf{K}$ (i.e. $\neg\alpha \notin \mathbf{K}$).

2. α *contradicts* the beliefs in $\mathbf{K}$ (i.e. $\neg\alpha \in \mathbf{K}$).

In the first case, the result of the incorporation of α in $\mathbf{K}$ can be simply modelled by the (above defined) expansion of $\mathbf{K}$ by α.

In what concerns the second case we must start by distinguishing the two following possibilities: (i) α is a contradiction and (ii) α is a consistent sentence.

If (i) is the case then the impositions that $\alpha \in \mathbf{K} * \alpha$ and that $\mathbf{K} * \alpha$ is closed under logical consequence lead to one only possible result for the incorporation of α in $\mathbf{K}$, namely the inconsistent belief set $\mathbf{K}_{\perp}$ (i.e. $\mathbf{K} * \alpha = \mathbf{K}_{\perp}$)[18].

Hence it only remains to consider the case when α is not a contradiction but $\neg\alpha \in \mathbf{K}$. In this case, the expansion of $\mathbf{K}$ by α is the inconsistent belief set (i.e. $\mathbf{K}{+}\alpha = \mathbf{K}_{\perp}$). However, it is possible to fulfil the goal of obtaining a belief set which contains α and is consistent. Therefore, if it is required that the result of the incorporation of α in $\mathbf{K}$ is, as long as possible, consistent, in this case $\mathbf{K} * \alpha$ should not be taken to be simply the expansion of $\mathbf{K}$ by α.

Indeed, in this situation, in order to obtain a set $\mathbf{K} * \alpha$ in the desired conditions (i.e., such that $\alpha \in \mathbf{K} * \alpha$ and $\mathbf{K} * \alpha$ is consistent), some other modifications need to be made to the original belief set apart from the set-theoretical addition of α to $\mathbf{K}$ and subsequent logical closure of the resulting set. More precisely, enough of the sentences in the original belief set need to be left out from $\mathbf{K} * \alpha$ in order to assure that $\neg\alpha$ is not a logical consequence of the remaining ones. However, such

[18] Cf. Footnote 12.

additional modifications to the original belief set should be only the ones which are strictly necessary in order to assure that the desired properties hold. This is because, as we have remarked in the previous subsection, it is assumed that an epistemic input causes (only) the least possible amount of changes to the (previous) belief state necessary to assure that the (single) change with which it is identified does indeed take place (the so-called *the principle of minimal change of the previous beliefs*).

The kind of change, consisting of the *incorporation* of a sentence α in a belief set $\mathbf{K}$ such that $\neg\alpha \in \mathbf{K}$ (whether or not such sentence α is contradictory), is called ***revision*** of $\mathbf{K}$ by α and the resulting belief set is denoted by $\mathbf{K}{*}\alpha$.

The interrelation between the three, above presented, basic kinds of epistemic changes in epistemological theories where the belief states are modelled by belief sets, namely *expansion*, *contraction* and *revision*, shall be exposed in Section 2.3.

2.2.5 Criteria of Rationality

The *criteria of rationality* (or *rationality criteria*) are basic principles that stand on the metalevel of an epistemological theory and which are responsible for *governing* the other factors of the theory. Indeed, however we did not make an explicit reference to that fact, several rationality criteria have underlain the above presented descriptions of the epistemic elements of an epistemological theory.

To clarify the above remark, next we indicate explicitly some of the rationality criteria that have been considered in such descriptions and what has been their influence in some of the choices and distinctions there made. More precisely, we will exemplify, in particular, how the choice of a specific way of modelling the epistemic states and their changes depends on the rationality criteria underlying the epistemological theory under consideration.

Thus, in what concerns the influence of rationality criteria in the choice of the representation for the epistemic states we start by recalling that we have several times mentioned *consistency* as an intuitive characteristic (requirable) of an epistemic state - notice that, although we have not previously used such designation for it, *consistency* is a rationality criterion - and we have clarified that in the context of sentential models such requirement corresponds to the imposition that the sets of sentences (used as representations of epistemic states) are consistent (whether they are belief sets or belief bases). Regarding this same aspect, and once again with respect to sentential models, we have also observed that the use of belief sets or belief bases to represent the epistemic states is determined by whether the rationality criterion of *logical omniscience* (which is modelled by the *logical closure* of the sets of sentences used to represent the epistemic states) is or is not required. On the other hand, to see how a rationality criterion can be responsible for the determination of the different possibilities for admissible kinds of epistemic changes it is enough to recall that, when the epistemic states are modeled by belief sets, we have seen that there are two basic kinds of changes - incorporations and contractions - and that two subcases of incorporations must be considered separately - expansions and revisions - if and only if the rationality criterion of *consistency* (of the epistemic states) is considered.

In the above paragraph we have already mentioned two rationality criteria, however there are many more such criteria which are worth mentioning. Hence, below we present a list of some of the most common criteria of rationality (see e.g. [Dal88, Gär88]):

1. *Principle of Categorical Matching* or *Adequacy of Representation* - After a change the resulting epistemic state should have the same kind of representation as the initial epistemic state.

2. *Irrelevance of Syntax* - The result of a change should not depend on the syntax (or representation) of neither the previous beliefs nor the new information (acquired through some epistemic input).

3. *Primacy (of the new information)* - The new information should always be accepted.

4. *Consistency* - The new epistemic state should be consistent, if possible.

5. *Logical Omniscience* - The logical consequences of the beliefs that are accepted in a certain epistemic state should also be accepted in that epistemic state.

6. *Minimal Change/Loss* (of the previous beliefs) - The new epistemic state should retain as much as possible of the old beliefs.

7. *Fairness* - If there are several epistemic states that are (valid) candidates to represent the result of a certain change operation then none of them should be arbitrarily chosen.

The above listed principles constitute some of the most intuitive properties that can be used to characterize any given belief change process.

Notice, however, that this does not mean that any process of belief change must satisfy all the above principles. Not all the above criteria are desirable at all times. In fact, some of these principles contradict each other in certain situations. In this respect another important fact must be remarked here: any given epistemological theory besides from a (simple) list of rationality criteria that are used to govern the dynamics of epistemic change must provide an explicit (and unambiguous) indication of the order of priority among those principles.

We have already exposed above where the rationality criteria of *consistency* and *logical omniscience* are present throughout the preceding part of this section. In what follows we expose/recall that, and how, some other of the above listed rationality criteria have also been considered (however, in some cases, only implicitly) throughout the descriptions of the epistemic elements previously introduced. Furthermore, below, we also highlight some of the stages of the previous descriptions where a certain order of priority among different rationality criteria has been clearly taken into account.

Thus, we start by observing that the *principle of categorical matching* underlies all the above mentioned descriptions since we have tacitly assumed that, although there are several possible representations of epistemic states, in each epistemological theory only one of such possible alternative representations is used to model the epistemic states. Other rationality criterion that has been implicitly assumed in all the foregoing part of this section is the *primacy (of the new information)*. Some above presented explicit manifestations of the assumption of such rationality criterion have been (a) the imposition that α is included in the result of the incorporation of α in $\mathbf{K}$, for any sentence α and any belief set $\mathbf{K}$, and (b) the requirement that α is not included in the belief set resulting of the contraction of α from $\mathbf{K}$, for any belief set $\mathbf{K}$ and any sentence $\alpha \in \mathbf{K} \setminus Cn(\emptyset)$. Finally it remains only to remark that the *principle of minimal change/loss of the previous beliefs* has also been mentioned explicitly when we described the epistemic inputs. More precisely, in Subsection 2.2.3, when assuming that the epistemic inputs are identified with a change of epistemic attitude towards a (single) sentence, the satisfaction of the principle of minimal change has been assured by the imposition, there made, that the changes caused by an epistemic input on a belief state are only the strictly necessary to assure that the change (of epistemic attitude) with which that epistemic input is identified effectively occurs (and, that throughout such process, no belief is removed or added to the belief state unnecessarily).

We close this subsection by bringing to light some of the stages, of the above presented descriptions of the elements of an epistemological theory, in which we have (although only implicitly) assumed some priority among two different rationality criteria.

We start by noticing that the situations that we shall recall below and interpret from the perspective of a certain order of priority among rationality criteria have all appeared (in the previous subsection) in contexts when the belief states had been assumed to be represented by belief sets. Before moving on to present such examples it is convenient to remark that, in those circumstances, the principle of categorical matching (or the adequacy of representation) and the rationality criteria known as logical omniscience can be seen as being equivalent, in the sense that to require that all the epistemic states are represented by the same kind of entity (in this case by belief sets), i.e. that the principle of categorical matching is satisfied, is the same as requiring that all the belief states are (represented by entities) such that they include all the logical consequences of the beliefs they contain, i.e. that logical omniscience holds.

Having said this, the first obvious example that we wish to highlight of an (implicit) assumption of a priority among rationality criteria has occurred when we have clarified that it is not possible to remove a tautology from a belief state (modelled by a belief set). Notice that imposing that restriction corresponds to assuming that the principle of categorical matching is considered more important than the rationale of the primacy of the new information (or, in other words, the principle of categorical matching is given priority over the assurance of primacy of the new information). This is the case because, in that situation, under which those two rationality criteria are incompatible (in the sense that, if one of them is fulfilled then the other one is unsatisfied[19]) and, therefore, only one of them can hold, we have opted by the fulfilment of first one.

Another assumption of a certain order of priority among rationality principles

[19]Notice that this is indeed the case when a certain epistemic input leads to the removal of a tautology from the belief state

has been (implicitly) made when we have stated that the result of the revision of a belief set by a contradictory sentence is the inconsistent belief set. Indeed, such statement corresponds to the assumption that the rationale of primacy of the new information is given higher priority than the one of consistency.

We have also assumed in our previous descriptions that the property of consistency (of the belief states) is more important than the minimal loss of previous information. Indeed it is that assumption that explains the fact that the contraction of the inconsistent belief set (which contains all sentences) with respect to any non-tautological sentence is contracted gives rise to a (consistent) belief set which does not contain that sentence (nor, eventually, many others) and, therefore, contains much less information than the original belief set.

We must, however remark that, the above described situation can also be seen, more adequately, as a result of a much more general and relevant assumption regarding orders of priority among postulates, namely the consideration that the primacy of the new information is more fundamental than the minimal change of previous beliefs. Notice that, in fact, this latter mentioned assumption is one of the most central ones underlying most epistemological theories since, if such priority was reversed then no changes at all would occur in any epistemic state (regardless of the existence or not of some epistemic input).[20]

With the above remark, we finish our description of the main concepts underlying an epistemological theory.

2.3 The Three Basic Kinds of Epistemic Changes in Belief Sets-based Models and the Interrelations Among Them

As we have exposed in Subsection 2.2.4, in sentential models in which the epistemic states are represented by belief sets there are essentially three basic kinds of changes of belief state, namely *expansions*, *contractions* and *revisions*.

In the present section we aim to present some interconnections among those three kinds of belief change functions as well as to introduce the terminology and notation (concerning such operators) which we shall use throughout this text.

We start by remarking that so far we have only presented an explicit definition to the first one of those three change operations. More precisely, we have provided an explicit meaning for the expansion of a belief set by a sentence in Definition 2.2.1. This is because, in what concerns contractions and revisions there are several acceptable ways of defining operations which satisfy the properties required from such operations. Further ahed we will present a few of the constructions, which have been proposed in the literature, of functions satisfying those properties. However

[20] It is convenient, however, to notice that, although the rationality criterion of *primacy (of the new information)* is required in most of the existing formal models in belief revision and, in particular, is assumed in all the models that shall be presented throughout this thesis, there are also some models of non-prioritised belief revision (where the new information is not always accepted). A survey of some of those models is presented in [Han99a].

in this section we will only present some interrelations between those two kinds of operations which are independent of the precise definition that is chosen for each of them.

Now we recall that, in what concerns revision, if α is a contradiction there is only one belief set which can be the result of the revision of $\mathbf{K}$ by α, namely the inconsistent belief set $\mathbf{K}_\perp$ (i.e., if $\vdash \neg\alpha$ then $\mathbf{K}{*}\alpha = \mathbf{K}_\perp$). Now we consider the problem of revising a belief set $\mathbf{K}$ by a non-contradictory sentence α, such that $\neg\alpha \in \mathbf{K}$.

The most intuitive procedure to obtain a (consistent) belief set which satisfies the conditions that we have stated in Subsection 2.2.4 that a belief set must satisfy in order for it to be acceptable as the result of the revision of (a belief set) $\mathbf{K}$ by (a non-contradictory sentence) α has been proposed by Isaac Levi [Lev77] and consists of the following two steps:

1. finding a belief set which can be the result of the contraction of $\mathbf{K}$ with respect to $\neg\alpha$ (using the terminology that we have introduced in the above mentioned subsection).

2. taking the expansion of this latter belief set by α as the result of the revision of $\mathbf{K}$ by α.

Such process for obtaining the result of the revision of a belief set $\mathbf{K}$ by a sentence α, by means of a sequence of adequate contractions and expansions can be written in the form of an equation as follows:

$$\mathbf{K}{*}\alpha = (\mathbf{K}{-}\neg\alpha){+}\alpha \tag{2.1}$$

Since it is suggested by some of the remarks of Isaac Levi in [Lev77], the above identity is know as the *Levi identity* [Gär81, AM82]. Such identity provides an explicit way of defining a revision function $*$ on $\mathbf{K}$ from a (given) contraction function $-$ on $\mathbf{K}$.

It is worth noticing that the Levi identity can be seen as an *equational version* of the *Decomposition principle* presented on page 17. More precisely, if we accept that any desirable revision function can be defined from a contraction function by means of the Levi identity then the mentioned principle can be more precisely stated as follows:

> *Decomposition principle (second version)* [Han99b, pp. 17]:
>
> Every legitimate belief change is decomposable into a sequence of expansions and contractions.

We have, thus, seen how revisions can be defined from contractions. Now we present a widely known way of proceeding conversely, i.e. of defining a contraction by means of a revision. Such process can be described as follows: Let $\mathbf{K}$ be a belief set and α be any sentence, if $\mathbf{K}{*}\neg\alpha$ results of the revision of $\mathbf{K}$ by $\neg\alpha$, then the set $\mathbf{K}{-}\alpha$ given by:

$$\mathbf{K}{-}\alpha = \mathbf{K} \cap (\mathbf{K}{*}\neg\alpha) \tag{2.2}$$

is a possible result of a contraction of $\mathbf{K}$ by α, in the sense that it satisfies all the properties that we have mentioned in Subsection 2.2.4 that should be satisfied by a set in order for it to be acceptable as the result of such an operation.

Equation (2.2) is known as the *Harper identity* (e.g. [Gär88]) since the idea underlying that equation was first presented by William Harper in [Har77]. We notice, however, that in the literature such equation is also sometimes referred to as the *Gärdenfors identity* (e.g. [AM82, pp.27], [Mak85, pp.352]) since it has also been proposed by Gärdenfors in [Gär81].[21]

We notice that as an immediate consequence of the Levi and Harper identities we can conclude that contractions and revisions are interdefinable.

Now we clarify some of the terminology and notation which we shall use throughout the present text when referring to operations of expansion, contraction and revision.

Hence, throughout this thesis we shall use the expression "contraction function" (or simply "contraction") to refer to a function[22]

$$\begin{array}{rcl} - : \mathcal{T}_{\mathcal{L}} \times \mathcal{L} & \to & \mathcal{T}_{\mathcal{L}} \\ (\mathbf{K}, \alpha) & \mapsto & \mathbf{K}{-}\alpha \end{array},$$

where the image of a pair $(\mathbf{K}, \alpha)$ by $-$ is represented by $\mathbf{K}{-}\alpha$. Often throughout this text we will consider a fixed belief set $\mathbf{K}$ and in such situation we shall use the expression "contraction function on $\mathbf{K}$" (or simply "contraction on $\mathbf{K}$") to refer to a function[23]

$$\begin{array}{rcl} - : \mathcal{L} & \to & \mathcal{T}_{\mathcal{L}} \\ \alpha & \mapsto & \mathbf{K}{-}\alpha \end{array},$$

where the image of a sentence α by $-$ is represented by $\mathbf{K}{-}\alpha$. In this regard it remains only to remark that either it is seen as the image of a pair $(\mathbf{K}, \alpha)$ (by a contraction function $-$) or as the image of a sentence α (by a contraction function on $\mathbf{K}$) the set (denoted by) $\mathbf{K}{-}\alpha$ consists of a belief set which is acceptable (in the context under consideration at the moment when such a function is mentioned) as a possible result of the contraction of the belief set $\mathbf{K}$ with respect to the sentence α. Such set shall, sometimes, be referred to as "the result of the contraction of $\mathbf{K}$ by α" or as "the image of α by the contraction $-$ on $\mathbf{K}$".

Analogous terminology and notation shall be used for the case of revision. More precisely, throughout this thesis, when dealing with revision we shall use the notation and terminology which results of replacing in the above paragraph all the occurrences of the word "contraction" by the word "revision" and all the occurrences of the symbol "$-$" by the symbol "$*$".

Now that we have finished the introduction of the main components of any epistemological theory, we shall, conveniently, expose a concrete belief change model.

[21] In this respect see also Footnote 30.

[22] In general we shall use the symbol $-$ to designate such a function, however some variants of this symbol (most of which shall be obtained simply by adding superscripts and/or subscripts to it) shall also be sometimes used for that effect.

[23] We shall use the same symbols, $-$ (or some variant of it), to denote both "contraction functions" and "contractions on (a fixed belief set) $\mathbf{K}$".

More precisely, as a first example of an epistemological theory, in the following section we shall present the most widely known among all the sentential models so far proposed in the literature, in which belief states are represented by belief sets.

However, before moving on to the presentation of that model, we must first clarify some issues concerning how such an epistemological theory (i.e. a sentential model in which belief states are represented by belief sets) can be defined. In that respect, first of all we recall that, in that kind of models, there are essentially three types of basic changes of belief state, namely *expansion*, *contractions* and *revisions*.

Hence, any given sentential model is essentially characterized by the definition provided within such model for each of the belief change operations, namely expansions, contractions and revisions.

Moreover, recalling that in what concerns expansion there is a quite straightforward, consensual and unambiguous definition for such operation (see Definition 2.2.1), we can conclude that an epistemological theory where belief states are modelled by belief sets is essentially determined by the way it models the other two kinds of change operations (which are *admissible* in such class of theories), namely contractions and revisions.[24]

In that respect we should notice at this point that there are mainly two ways of characterising the operations of contraction/revision in any given epistemological theory: one way is *postulation* and consists in presenting a list of properties which every such operation must satisfy (in order to be *accepted* as an operation of contraction/revision within that model) − such properties are designated by "contraction/revision postulates" (or simply by "postulates") − and studying their consequences and interrelations, and the other one is to provide explicit definitions for such operations, i.e. to expose a way of constructing a belief set which can be the results of the contraction/revision of any given belief set $\mathbf{K}$ by any sentence α in such model and then investigating the main properties satisfied by the operations thus defined.

The contraction/revision operations underlying most known belief change (sentential) models have been defined by means of both of those approaches, i.e. the class of contraction/revision functions of most such models has been defined both by presenting a list of the properties that are required to be satisfied by each of its members, and also by providing an explicit constructive way of obtaining each of the functions in such class.[25]

In such cases, we use the designation "representation theorem" to refer to the result that establishes that the class of functions that satisfy the given list of postulates coincides with the class of functions which can be obtained by the mentioned explicitly presented procedure. Such result is also sometimes designated by "axiomatic characterization", since they characterize the explicitly defined functions in terms of postulates (or axioms). Representation theorems are, thus, very use-

[24] In fact, some of such theories contain an explicit description of only one of those two classes of operations and the definition of the other one is obtained by making use of the above exposed interdefinability between those two types of operations.

[25] In fact some change function (of certain epistemic models) have even been characterized by different sets of postulates (instead of only one such set). Also, for some classes of change operations, more that one method for constructing its members have been proposed.

ful results because, on the one hand, they explicit the intuitions which motivate the constructions presented namely by exposing a set of postulates which precisely characterize the functions thus defined, and, on the other hand, they confirm that a certain explicit mechanism for building operations leads to the construction of exactly those functions which satisfy precisely those properties (postulates) that were required to be fulfilled by such an operation.

2.4 The AGM Paradigm

At this moment we are finally in a position to expose a concrete belief change model. More precisely, in this section we will present the one which has acquired the status of standard model in the belief change literature. Such model is commonly denominated by *AGM model* – after the initials of its three creators: Carlos Alchourrón, Peter Gärdenfors and David Makinson – and has been originally presented in the paper [AGM85] (which is indeed an extension of several other previous works of its authors (e.g. [Gär78, AM81, AM82, Gär82])).

The AGM model is a sentential model within which the belief states are represented by belief sets and three types of belief change operations are considered in that framework: expansion, contractions and revisions. In what concerns expansion, that operation is defined, in such model, just as presented in Definition 2.2.1 above. Regarding the other two kinds of operations, we start by remarking that, in their model, the AGM trio, in compliance with Levi's arguments in [Lev77], takes contractions as primitive and revisions as the functions which are obtained from a contraction through the Levi identity (equation (2.1)). Anyway, there is not a straightforward definition for the classes of contraction and of revision functions (as it is the case with expansion) in that framework and several different methods have been made use of in the literature to describe such classes of belief change functions.

As we have already mentioned above, there are two main distinct ways of defining a certain class of belief change functions: one way consists in presenting a list of postulates which must be satisfied by any function in such class and the other way consists in presenting an explicit procedure to construct all such functions. In the following subsections we shall present those two classes (i.e. the class of contractions and the class of revisions of the AGM model) by means of the postulation approach. More precisely in Subsection 2.4.1 (resp. Subsection 2.4.2) we shall expose the postulates considered in the AGM-model as the essential properties that a function must satisfy in order for it to be accepted as a contraction (resp. revision) function. Afterwards, in Subsection 2.4.3 we show that the classes of contractions and of revisions determined by those sets of postulates are indeed interconnected by means of the Levi and Harper identities. More precisely, we shall see there that the class of revision functions coincides with the class of functions which can be obtained from one of the contraction functions by means of the Levi identity (equation (2.1)),[26] and,

[26] We must notice, however, that this fact is not a consequence of the lists of postulates presented to describe each class of operations. Rather, this is the case because, as we have clarified above, it is a foundational assumption of the AGM model that revision functions are precisely those functions which can be obtained from a contraction function by means of the Levi identity (eq. (2.1)).

conversely, the class of contraction functions coincides with the class of functions which can be obtained from one of the revision functions by means of the Harper identity (equation (2.2)).

2.4.1 The AGM (Postulates for) Contraction Functions

In this subsection we shall introduce the AGM model for contraction functions by presenting the list of postulates which characterizes them.

In the AGM model (introduced in [AGM85][27]) a function $-$ from $\mathcal{T}_{\mathcal{L}} \times \mathcal{L}$ to $\mathcal{T}_{\mathcal{L}}$ (resp. from $\mathcal{L}$ to $\mathcal{T}_{\mathcal{L}}$) is a contraction function (resp. a contraction function on $\mathbf{K}$) if and only if, for any belief set $\mathbf{K}$ (resp. for a certain belief set $\mathbf{K}$) and any sentences α and β, the following properties[28] are satisfied:[29]

- **Closure:** $\mathbf{K}-\alpha$ is a belief set (i.e. $\mathbf{K}-\alpha = Cn(\mathbf{K}-\alpha)$).
- **Inclusion:** $\mathbf{K}-\alpha \subseteq \mathbf{K}$.
- **Vacuity:** If $\alpha \notin \mathbf{K}$, then $\mathbf{K}-\alpha = \mathbf{K}$.
- **Success:** If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin \mathbf{K}-\alpha$.
- **Extensionality** (or **Preservation**)**:** If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.
- **Recovery:** $\mathbf{K} \subseteq Cn((\mathbf{K}-\alpha) \cup \{\alpha\})$.

The above postulates are commonly known as ***basic Gärdenfors postulates for contraction*** or ***basic AGM postulates for contraction*** and their designations and formulations are quite *self-explanatory* of the meaning and the motivation behind each of them: *closure* assures that the result of contracting a belief set by any given sentence is still a belief set; *inclusion* tells us that the contraction of a belief set must give rise to a subset of it; *vacuity* states that the outcome of the operation of contracting a belief set by a sentence which is not in that set is precisely the original belief set; *success* imposes that the result of the contraction of a certain belief set by a sentence does not contain the contracted sentence (provided that such sentence is not a tautology); *extensionality* says that the result of contracting a belief set by a certain sentence coincides with the result of contracting that same belief set by any other sentence which is logically equivalent to that (original) one; finally *recovery* means that if a belief set is contracted by a certain sentence and, subsequently, the result of such contraction is expanded (see Definition 2.2.1) by that same sentence then the outcome of this second operation contains the original belief set, in other

[27] It must be noticed, however, that such paper is an extension of some previous works by its authors (e.g. [Gär78, AM81, AM82, Gär82]).

[28] These properties have been originally stated in [Gär78, Gär82]. However, here we present them with formulations that are more similar to the ones found in [AGM85] and, moreover, we follow precisely the same order of presentation used in the latter mentioned paper.

[29] Here we are using the terminology and notation which we have introduced in page 25. Hence, in particular, in the formulations of the postulates, for any sentence α, $\mathbf{K}-\alpha$ denotes the image of the pair $(\mathbf{K}, \alpha)$ (resp. the image of α) by $-$, when that function is regarded as a contraction function (resp. as a contraction function on $\mathbf{K}$).

words, this postulate states that all the sentences which are discarded in the process of contracting a belief set by a certain sentence (as well as the remaining ones) are present in the outcome of the expansion, by that same sentence, of the belief set which results of mentioned contraction.

Further explanation of the motivations which underlay each of the above listed postulates can be found, for example, in [Mak85, Gär88, GR95, Han99b]. Throughout this thesis the class of functions which satisfy basic AGM postulates for contraction will be of particular interest.

Definition 2.4.1 *A contraction function* $- : \mathcal{T}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{T}_{\mathcal{L}}$ *is a* basic AGM contraction *(or, simply, an* AGM contraction*) if it satisfies the basic AGM postulates for contraction, namely closure, inclusion, vacuity, success, extensionality and recovery (for any belief set* $\mathbf{K}$ *and any sentences* α *and* β*).*[29]

Given a (fixed) belief set $\mathbf{K}$*, a contraction function on* $\mathbf{K}$*,* $- : \mathcal{L} \to \mathcal{T}_{\mathcal{L}}$*, is a* basic AGM contraction on $\mathbf{K}$ *(or, simply, an* AGM contraction on $\mathbf{K}$*) if it satisfies the basic AGM postulates for contraction, namely closure, inclusion, vacuity, success, extensionality and recovery (for any sentences* α *and* β*).*[29]

Before closing this subsection it is convenient to remark here, additionally, that, speaking in terms of the epistemological concepts introduced in Section 2.2, the above listed postulates reflect, in an *equational form*, those which are the rationality criteria underlying the AGM framework (regarding the operations of contraction). Moreover, the order by which the postulates have been introduced indicates the order of priority among those rationality criteria which is assumed in the AGM framework.

In particular we emphasise the following correspondences between some the above postulates and some of the rationality criteria mentioned in Subsection 2.2.5: *closure* is, in this context, an equational version of the *principle of categorical matching* (or *adequacy of representation*); *success* is connected to the criterion of *primacy (of the new information)*; *extensionality* corresponds to the principle of *irrelevance of syntax* and, finally, the postulates of *inclusion*, *vacuity* and *recovery* are all essentially related to the criterion of *minimal change/loss* (of the previous beliefs).

In what concerns the priorities among the rationality criteria which are assumed in the AGM framework (concerning the operation of contraction) we remark only the following (most evident) one: Since the postulate of closure comes, in the above presented list, before the postulate of success and the formulation of this latter postulate is conditional on the sentence to be removed not being a tautology, we can conclude that, in the AGM framework the *principle of categorical matching* is given priority over the *primacy (of the new information)*.

Besides from the above introduced, so called, basic AGM postulates for contraction, the following two postulates (presented in [AGM85]), referring to the contraction by conjunctions, also play an important role in the AGM model:

- **Conjunctive overlap** (or **Intersection**)**:** $(\mathbf{K}-\alpha) \cap (\mathbf{K}-\beta) \subseteq \mathbf{K}-(\alpha \wedge \beta)$.
- **Conjunctive inclusion** (or **Conjunction**)**:** $\mathbf{K}-(\alpha \wedge \beta) \subseteq \mathbf{K}-\alpha$ whenever $\alpha \notin \mathbf{K}-(\alpha \wedge \beta)$.

Throughout this text we shall often use the expression "***supplementary AGM postulates for contraction***" to refer to the two postulates above.

These postulates are defended and the motivation behind each of them is detailedly exposed, for example, in [Mak85, Gär88, GR95, Han99b]. In particular, we anticipate here that these postulates are essential in some of the representation theorems that we shall present further ahead.

Definition 2.4.2 *A contraction function (resp., a contraction function on a belief set* $\mathbf{K}$*) – is a* supplementary AGM contraction *(resp., a* supplementary AGM contraction on $\mathbf{K}$*) if it satisfies the basic and the supplementary AGM postulates for contraction, namely closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap and conjunctive inclusion, where* $\mathbf{K}$ *is an arbitrary (resp., fixed) belief set and* α *and* β *are arbitrary sentences.*[29]

2.4.2 The AGM (Postulates for) Revision Functions

In this subsection we present the class of revision functions of the AGM model of belief change by means of the postulation approach, analogously to what we have done in the previous subsection with respect to contraction.

The following postulates, which have been presented in [AGM85] (following [Gär78, Gär82]),[30] constitute the essential properties required from a function $*$ from $\mathcal{T_L} \times \mathcal{L}$ to $\mathcal{T_L}$ (resp. from $\mathcal{L}$ to $\mathcal{T_L}$) in order for it to be considered a revision function (resp. a revision function on $\mathbf{K}$) within the AGM model:[31]

- **Closure:** $\mathbf{K}{*}\alpha$ is a belief set (i.e. $\mathbf{K}{*}\alpha = Cn(\mathbf{K}{*}\alpha)$).
- **Success:** $\alpha \in \mathbf{K}{*}\alpha$.
- **Inclusion:** $\mathbf{K}{*}\alpha \subseteq \mathbf{K}{+}\alpha$.
- **Vacuity:** If $\neg\alpha \notin \mathbf{K}$ then $\mathbf{K}{+}\alpha \subseteq \mathbf{K}{*}\alpha$.
- **Consistency:** $\mathbf{K}{*}\alpha = \mathbf{K}_\perp$ iff $\vdash \neg\alpha$.

[30] We must notice that, in fact, the list of postulates that we expose here coincides, rather, with the one presented in [Gär88].

In the original list of postulates for revision presented in [AGM85] (following [Gär78, Gär82]) there was a combined version of *inclusion* and *vacuity* instead of having them separately. Besides, that original list of postulates also included the following postulate:

- **($*$ 6)** $(\mathbf{K}{*}\alpha) \cap \mathbf{K} = \mathbf{K}{-}\neg\alpha$;

which was later excluded from the axioms.

By considering, instead, Gärdenfors' [Gär88] slightly modified list of postulates for revision (in the AGM model) that we present here, it is possible to treat revision and contraction as totally independent functions.

Notice yet that the postulate **($*$ 6)** is equivalent to the Harper identity (equation (2.2)) that we have presented in the previous section.

[31] Here we are using the terminology and notation which we have introduced in page 25. Hence, in particular, in the formulations of the postulates, for any sentence α, $\mathbf{K}{*}\alpha$ denotes the image of the pair $(\mathbf{K}, \alpha)$ (resp. the image of α) by $*$, when that function is regarded as a revision function (resp. as a revision function on $\mathbf{K}$).

- **Extensionality** (or **Preservation**): If $\vdash \alpha \leftrightarrow \beta$, then $\mathbf{K}{*}\alpha = \mathbf{K}{*}\beta$.

Notice that, when the above postulates are regarded as properties of a revision function $* : \mathcal{T}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{T}_{\mathcal{L}}$, each of them must be read as having an implicit universal quantification of the sentences occurring in its formulation over $\mathcal{L}$ as well as a universal quantification of $\mathbf{K}$ over the set of all theories $\mathcal{T}_{\mathcal{L}}$, i.e., as if their statements started with the expression: "For any belief set $\mathbf{K}$ and any sentence α" (or by "For any belief set $\mathbf{K}$ and any sentences α and β" in the case of the postulate of extensionality).

On the other hand, when the above postulates are considered as properties of a revision function on (the fixed belief set) $\mathbf{K}$, $* : \mathcal{L} \to \mathcal{T}_{\mathcal{L}}$, then each of them must be read as having an universal quantification (only) of the sentences occurring in its formulation over $\mathcal{L}$, i.e., as if their statements started with the expression: "For any sentence α" (or by "For any sentences α and β" in the case of the postulate of extensionality).

We must also remark here that, although the designations of some the above postulates are identical to the designations of some of the contraction postulates (that we have presented in the previous subsection), throughout this text, whenever some of those postulate designations (which are common to both of the mentioned sets of postulates) occurs, it will be clear by the context wether it refers to the contraction or to the revision postulate with that name.

The above postulates are commonly known as ***basic Gärdenfors postulates for revision*** or ***basic AGM postulates for revision*** and the meaning and motivation underlaying each of them can be straightforwardly understood from their formulations and designations: *closure* imposes that the result of the revision of a belief set by a sentence is a belief set; *success* assures that the outcome of the revision of a belief set by a certain sentence contains that sentence; *inclusion* tells us that the belief set which results of the expansion of a belief set by a sentence is a superset of the outcome of the revision of that same belief set by the same sentence; *vacuity* means that the result of the revision of a belief set by a sentence which is consistent with it contains the outcome of the expansion of that belief set by such sentence;[32] *consistency* assures that the result of the revision of a belief set by a sentence is consistent unless such sentence is itself inconsistent; finally *extensionality* tells us that revising a belief set by a certain sentence is the same as revising such belief set by another sentence which is equivalent to that one.

For more detailed explanations and motivations for these postulates see e.g. [Mak85, Gär88, GR95, Han99b]. In the following definition we formally introduce the class of revision functions which satisfy the above mentioned basic AGM postulates for revision.

[32] At this point it is interesting to notice that it follows from the combination of *inclusion* and *vacuity* that the result of the revision of a belief set by a sentence which is consistent with it is identical to the outcome of the expansion of that belief set by that sentence. This fact is expressed by the following postulate

- **($*$ 3)** If $\neg\alpha \notin \mathbf{K}$ then $\mathbf{K}{*}\alpha = \mathbf{K}{+}\alpha$;

which was present (instead of *inclusion* and *vacuity*) in the original list of postulates for revision presented in [AGM85] (following [Gär78, Gär82]).

Definition 2.4.3 *A revision function* $* : \mathcal{T}_\mathcal{L} \times \mathcal{L} \to \mathcal{T}_\mathcal{L}$ *is a* basic AGM revision *(or, simply, an* AGM revision*) if it satisfies the basic AGM postulates for revision, namely closure, success, inclusion, vacuity, consistency and extensionality (for any belief set* $\mathbf{K}$ *and any sentences* α *and* β*).*[29]

Given a (fixed) belief set $\mathbf{K}$*, a revision function on* $\mathbf{K}$*,* $* : \mathcal{L} \to \mathcal{T}_\mathcal{L}$*, is a* basic AGM revision on $\mathbf{K}$ *(or, simply, an* AGM revision on $\mathbf{K}$*) if it satisfies the basic AGM postulates for revision, namely closure, success, inclusion, vacuity, consistency and extensionality (for any sentences* α *and* β*).*[29]

Analogously to what we have done in the previous subsection (concerning contraction), by analysing the above set of postulates (as well as the order by which those postulates have been presented and the conditionals present in some of them), we can easily identify the main rationality criteria which are assumed to govern the operation of revision in the AGM model (and the order of priority among those principles).

In this regard we start by observing that, the imposition that a revision function must satisfy the postulate of *consistency* is obviously connected to the assumption of the rationality criterion which we have designated by that same name in Subsection 2.2.5. On the other hand, in what concerns the remaining of the above introduced revision postulates, just as in the case of the homonymous contraction postulates: *closure* is directly related to the *principle of categorical matching* (or *adequacy of representation*), *success* is an equational version the criterion of *primacy (of the new information)*, *inclusion* and *vacuity* are essentially related to the criterion of *minimal change/loss* (of the previous beliefs) and *extensionality* corresponds to the principle of *irrelevance of syntax*.

Now we emphasise a couple of noteworthy priorities among rationality criteria which are assumed in the AGM model (in what concerns revision): On the one hand *primacy (of the new information)* is given priority over *consistency*, in the sense that, according to the AGM model when a consistent belief set is revised by an inconsistent sentence, such sentence is accepted in the resulting belief set which must, therefore, in such case, be the inconsistent belief set. On the other hand, in the AGM framework, *consistency* has priority over *minimal change/loss* (of the previous beliefs) since, when the inconsistent belief set $\mathbf{K}_\perp$ is revised by a non-contradictory sentence, the resulting belief set is assumed to be consistent (and, therefore, a lot of beliefs in the original inconsistent belief set are abandoned).

Just as in the case of contraction, apart from the set of basic postulates, in the AGM framework, two additional postulates for revision, originally proposed by Gärdenfors [Gär78, Gär82], are considered, namely:

- **Superexpansion:** $\mathbf{K}*(\alpha \wedge \beta) \subseteq (\mathbf{K}*\alpha)+\beta$.
- **Subexpansion:** If $\neg\beta \notin \mathbf{K}*\alpha$ then $(\mathbf{K}*\alpha)+\beta \subseteq \mathbf{K}*(\alpha \wedge \beta)$.

In what follows the two postulates above shall be sometimes referred to as the ***supplementary Gärdenfors postulates for revision*** or the ***supplementary AGM postulates for revision***. Further explications and motivations for the consideration of such couple of postulates can be found, for example, in [Mak85, Gär88, GR95, Han99b].

2.4.3 Interrelation Between AGM Revision and AGM Contraction (Postulates)

In Section 2.3 we have presented the Levi and Harper identities (equations (2.1) and (2.2), respectively) which can be used in order to define a revision function on a belief set $\mathbf{K}$ from a (given) contraction function on $\mathbf{K}$ and vice versa, respectively. In what follows we shall analyse those two identities in terms of the postulates for contraction and for revision postulates that we have introduced in the two previous subsections.

More precisely, in this subsection, we will present some results which expose that: (a) the revision function obtained from an AGM contraction (respectively, from an AGM contraction which satisfies the supplementary AGM postulates for contraction) by means of the Levi identity (equation (2.1)) is an AGM revision (resp., an AGM revision which satisfies the supplementary AGM postulates for revision), and (b) the contraction function obtained from an AGM revision (respectively, from an AGM revision which satisfies the supplementary AGM postulates for revision) through the Harper identity (equation (2.2)) is an AGM contraction (resp., an AGM contraction which satisfies the supplementary AGM postulates for contraction). Furthermore we shall also expose a result which states that, if we regard the Levi and Harper identities as functions (from the class of all AGM contraction to the class of all AGM revisions and vice versa, respectively) then they are the inverse (functions) of each other. We state this facts precisely and more detailedly in the following set of observations.

Observation 2.4.4 ([Gär78, Gär82]) [33]

Let $-$ be a contraction function and $*$ be the revision function defined from $-$ by means of the Levi identity (i.e. equation (2.1)). If $-$ satisfies (the contraction postulates) inclusion*,* vacuity*,* success *and* extensionality *then $*$ satisfies all the basic AGM postulates for revision, namely,* closure*,* success*,* inclusion*,* vacuity*,* consistency *and* extensionality*.*[34]

Observation 2.4.5 ([AGM85, Observations 3.1 and 3.2]) [35]

Let $-$ be a contraction function that satisfies (the contraction postulates) closure*,* inclusion*,* vacuity*,* success *and* extensionality *and let $*$ be the revision function defined from $-$ by means of the Levi identity (i.e. equation (2.1)). Then the following statements hold:*

[33] The references [Gär78, Gär82] presented here for this observation are the ones provided for this result within the proof of [AGM85, Observation 2.3] as well as in [Mak87, pp. 389]. A proof for this observation can also be found in [Gär88, pp. 215] and in [GM88, pp. 91].

[34] It is worth noticing that, according to this result, it is not necessary that the contraction function $-$ satisfies the postulates *closure* and *recovery* in order to assure that the revision function $*$ satisfies all the basic AGM postulates for revision. For that to hold it is enough that $-$ satisfies all the remaining AGM postulates for contraction.

[35] Part (a) of this observation coincides with the right-to-left part of [AGM85, Observation 3.1] and part (b) coincides with the right-to-left part of [AGM85, Observation 3.2]. A proof for this result can also be found in [Gär88, pp. 215].

(a) If $-$ also satisfies recovery *and* conjunctive overlap*, then $*$ satisfies* superexpansion.[36]

(b) If $-$ also satisfies conjunctive inclusion*, then $*$ satisfies* subexpansion.

The two observations above assert that the revision function that is obtained from an AGM contraction function by means of the Levi identity (equation (2.1)) is an AGM revision function. Furthermore, from the above results, we can also conclude that an AGM contraction function which also satisfies the two supplementary AGM postulates for contraction originates, through the Levi identity (equation (2.1)), an AGM revision function which, additionally, satisfies the two supplementary postulates for revision.

The following observations show that the converse procedure is possible by means of the Harper identity (equation (2.2)).

Observation 2.4.6 ([Gär78, Gär82]) [37]

Let $*$ be a revision function and $-$ be the contraction function defined from $*$ by means of the Harper identity (i.e equation (2.2)). If $*$ satisfies (the revision postulates) closure*,* success*,* vacuity*,* consistency *and* extensionality*, then $-$ satisfies all the basic AGM postulates for contraction, namely* closure*,* inclusion*,* vacuity*,* success*,* extensionality *and* recovery.[38]

Observation 2.4.7 ([AGM85, Observations 3.1 and 3.2]) [39]

Let $*$ be a revision function that satisfies all the basic AGM postulates for revision and $-$ be the contraction function defined from $*$ by means of the Harper identity (i.e. equation (2.2)). Then the following statements hold:

(a) If $*$ also satisfies superexpansion*, then $-$ satisfies* conjunctive overlap.

(b) If $*$ also satisfies subexpansion*, then $-$ satisfies* conjunctive inclusion.

From the results so far presented in this subsection we can conclude that the Levi and Harper identities can be used in order to define maps from the class of all AGM contraction functions to the class of all AGM revision functions and vice versa, respectively. In what follows we shall formally introduce such maps and their strong interconnection.

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[36]Notice that, according to [Fer01, Observation 20], if $-$ satisfies the contraction postulates *closure*, *inclusion*, *vacuity*, *success*, *extensionality* and *conjunctive overlap*, but does not satisfy *recovery*, then $*$ does not, in general, satisfy *superexpansion*.

[37]The references [Gär78, Gär82] presented here for this observation are the ones provided for this result in [Mak87, pp. 389]. A proof for this observation can be found in [Gär88, pp. 215] and in [GM88, pp. 91 – 92].

[38]Notice that, according to this result, it is not necessary that the revision function $*$ satisfies the postulate *inclusion* in order to assure that the contraction function $-$ is an AGM contraction. For that to hold it is enough that $*$ satisfies all the remaining basic AGM postulates for revision.

[39]Part (a) of this observation coincides with the left-to-right part of [AGM85, Observation 3.1] and also with part (a) of [Gär88, Theorem 3.5]. In what concerns part (b), it is identical to the left-to-right part of [AGM85, Observation 3.2] as well as to part (b) of [Gär88, Theorem 3.5].

Let $\mathbf{K}$ be a belief set. We shall denote the class of all AGM contractions on $\mathbf{K}$ by $-_{\mathbf{AGM}}\mathbf{K}$ and the class of all AGM revisions on $\mathbf{K}$ by $*_{\mathbf{AGM}}\mathbf{K}$.

The Levi identity (equation (2.1)) naturally leads to the definition of a function $r : -_{\text{AGM}}\mathbf{K} \rightarrow *_{\text{AGM}}\mathbf{K}$ such that, for any AGM contraction $- \in -_{\text{AGM}}\mathbf{K}$, its image $r(-) = *$ is the AGM revision (cf. Observation 2.4.4) defined by: $\mathbf{K}*\alpha = (\mathbf{K}-\neg\alpha)+\alpha$, for any $\alpha \in \mathcal{L}$.

Conversely, having in mind the Harper identity (equation (2.2)), we are led to the following definition of a function $c : *_{\text{AGM}}\mathbf{K} \rightarrow -_{\text{AGM}}\mathbf{K}$, such that, given an arbitrary AGM revision $* \in *_{\text{AGM}}\mathbf{K}$, it holds that $c(*) = -$, where $-$ is the AGM contraction (cf. Observation 2.4.6) defined by: $\mathbf{K}-\alpha = \mathbf{K} \cap (\mathbf{K}*\neg\alpha)$, for any $\alpha \in \mathcal{L}$.

Now that we have introduced the maps r and c between the classes of AGM revisions and of AGM contractions we are in a position to present, as the final result of the present subsection, the following observation, which essentially asserts that those two functions (r and c) are such that each of them is the inverse (function) of the other one.

Observation 2.4.8 ([Gär78, Gär82]) [40]

Let $\mathbf{K}$ *be a belief set and* r *and* c *be the functions introduced in the above paragraphs. Then, the following statements hold:*

(a) If $-$ *is an AGM contraction on* $\mathbf{K}$*, then* $c(r(-)) = -$.[41]

(b) If $*$ *is an AGM revision on* $\mathbf{K}$*, then* $r(c(*)) = *$.[42]

The above result tell us that given an AGM contraction $-$ (respectively an AGM revision $*$) on $\mathbf{K}$, if $*$ is the AGM revision (resp. $-$ is the AGM contraction) on $\mathbf{K}$ obtained from $-$ (resp. from $*$) by means of the Levi identity (resp. Harper identity), then the contraction (resp. revision) function that is obtained from $*$ (resp. from $-$) through the Harper identity (resp. Levi identity) coincides with the (original) contraction function $-$ (resp. revision function $*$).

In summary, the set of results presented in this subsection provides a definite evidence of the adequacy of the Levi and Harper identities as procedures for obtaining (AGM) revisions from (AGM) contractions and vice versa, respectively.

On the other hand, these results also allow us to conclude that revision and contraction functions are very strongly interconnected. Indeed, as we have seen, those two classes of functions are interdefinable by means of the Levi and Harper identities.

As a consequence of this fact, presenting methods for explicitly constructing AGM contractions is equivalent to providing explicit constructive definitions of AGM

[40] The references [Gär78, Gär82] presented here for this observation are the ones provided for this result in [Mak87, pp. 389]. A proof that the statements in this observation indeed hold can be found in [Gär88, pp. 70 – 71].

[41] More generally, as it is stated (and proven) in [Han99b, Observation 3.56], it holds that in order for the identity $c(r(-)) = -$ to be satisfied it is enough that $-$ satisfies the contraction postulates of *closure*, *inclusion*, *vacuity*, *extensionality* and *recovery*.

[42] In fact, according to the statement of [Han99b, Observation 3.57] (and its proof) the identity $r(c(*)) = *$ is satisfied as long as $*$ satisfies the revision postulates of *closure*, *success*, *inclusion* and *extensionality*.

revisions (since the Levi and Harper identities can be used to "translate" an explicit construction of a contraction function into an explicit definition of a revision function and vice versa, respectively).

Hence, when creating/exposing a constructive model of belief change functions it is enough to focus on explicitly defining either revision or contraction functions (since ones can be defined from the others). As we have already remarked above Levi [Lev77] considers contractions as more primitive than revisions (and, therefore, takes revisions as the functions which are obtained from a contraction through the Levi identity (equation (2.1))). Furthermore, as we have also already mentioned, in [AGM85] the AGM trio, in accordance with Levi's arguments, takes contractions as primitive and revisions as derived from them by means of the Levi identity (equation (2.1)).[43]

Having all that in mind, and also because that is convenient for what remains of this thesis, in what follows we shall focus almost exclusively on contraction functions (and give only very little attention to their associated revision functions). In particular we dedicate the following chapter to the presentation of several constructive models (only) for AGM contractions (and to the analysis of the interrelations among them) and, throughout such chapter, we shall only mention revision functions whenever that is necessary in order to obtain or explain some results related to the constructions of contractions that we shall present.

[43] Notice also that the fact, already mentioned in Footnote 30, that the Harper identity was included as postulate **($*$ 6)** in the original list of postulates for revision presented in [AGM85] (following [Gär78, Gär82]), further enforces the idea that contraction is a more primitive operation than revision.

Chapter 3

Some Constructive Models of AGM Contractions

In Section 2.4 we have introduced the one which is currently seen as the standard model of theory change, the so called AGM model.

It is now time to present some explicit constructions for functions of theory change. Having all the above in mind, as well as what is the background needed for the remainder of this work[44], throughout the present chapter we shall focus our attention essentially in such explicit definitions for contraction functions (rather than revision functions).[45]

Hence, in this chapter we will expose some explicit definitions of contraction functions as well as axiomatic characterizations for each of them. As it will be clear from the respective representation theorems, all the contraction functions that we shall present are AGM contractions.

Our presentation of such models in the present chapter will follow the chronological order by which each of those classes of contraction functions have been originally presented in the literature.

Afterwards, in Chapter 4, we will present the interconnection among the different explicit constructions of AGM contractions introduced here.

3.1 Partial Meet Contractions

In this section we present the most widely known constructive model of contraction functions. Such contraction functions are called *partial meet contractions* and were originally presented in [AGM85].

Given a belief set $\mathbf{K}$, a partial meet contraction on $\mathbf{K}$ is such that the result of the contraction of $\mathbf{K}$ by any given sentence is built upon a selection from the maximal subsets of $\mathbf{K}$ that do not imply that sentence. In what follows we present the formal definition of such functions.

[44]Recall that the topic of the present thesis is essentially related to contractions (rather than revisions) of belief sets.

[45]Notice, however, that, as we have seen above, from any explicit definition of a contraction function it is possible to obtain, through the Levi identity (equation (2.1)), an explicit definition of a revision function.

We start by presenting the concepts of *remainder set* and of *remainders*, originally introduced in [AM81, p. 128].

Let $\mathbf{K}$ be a belief set and B be any set of sentences, the *remainder set* of $\mathbf{K}$ by B (or the *remainder set* of $\mathbf{K}$ *modulo* B) is the set of maximal subsets of $\mathbf{K}$ that do not imply any element of B and is denoted by $\mathbf{K}\perp B$. The elements of $\mathbf{K}\perp B$ are the *remainders* (of $\mathbf{K}$, by B). This concepts are more formally introduced in the following definition:

Definition 3.1.1 ([AM81, p. 128]) *Let* $\mathbf{K}$ *be a belief set and* B *be any set of sentences. The* remainder set *of* $\mathbf{K}$ *by* B*, denoted* $\mathbf{K}\perp B$ *is the set such that* $X \in \mathbf{K}\perp B$ *if and only if:*

1. $X \subseteq \mathbf{K}$.

2. $Cn(X) \cap B = \emptyset$.

3. *There is no set* X' *such that* $X \subset X' \subseteq \mathbf{K}$ *and* $Cn(X') \cap B = \emptyset$.

The elements of $\mathbf{K}\perp B$ *are the* remainders *(of* $\mathbf{K}$*, by* B*).*

For any sentence α*,* $\mathbf{K}\perp\alpha$ *is an abbreviation of* $\mathbf{K}\perp\{\alpha\}$ *and is called the* remainder set *of* $\mathbf{K}$ *by* α*. In other words* $\mathbf{K}\perp\alpha$ *is the set of maximal subsets of* $\mathbf{K}$ *that do not imply* α *(and such sets are called* remainders *(of* $\mathbf{K}$*, by* α*)).*

The following observation exposes that, since we are assuming that the consequence operation Cn is compact, given a belief set $\mathbf{K}$ and any set of sentences B the remainder set $\mathbf{K}\perp B$ is non-empty as long as B does not contain any tautology (i.e. $B \cap Cn(\emptyset) = \emptyset$).

Observation 3.1.2 ([AM81, Observation 2.2]) [46] *Let* $\mathbf{K}$ *be a belief set and* B *be any set of sentences. Then,* $\mathbf{K}\perp B \neq \emptyset$ *if and only if* $B \cap Cn(\emptyset) = \emptyset$ *(provided that the consequence operation* Cn *is compact).*

An immediate consequence of the above observation is the fact (which we shall need to refer to further ahed) that given a belief set $\mathbf{K}$ and two sets of sentences B and C, a necessary and sufficient condition for $\mathbf{K}\perp B = \mathbf{K}\perp C$ is that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C. This result is more formally stated in the following observation.

Observation 3.1.3 ([Han99b, Observation 1.39]) *Let* $\mathbf{K}$ *be a belief set and* B *and* C *be any set of sentences. Then the two following conditions are equivalent:*

1. $\mathbf{K}\perp B = \mathbf{K}\perp C$

2. *For all subsets* X *of* $\mathbf{K}$*:* $Cn(X) \cap B \neq \emptyset$ *if and only if* $Cn(X) \cap C \neq \emptyset$.

[46] We notice that the left-to-right part of this result is an immediate consequence of the monotony of Cn. On the other hand, the right-to-left implication follows from the compactness of Cn and Zorn's lemma. (Cf. [AM81, Proof of Observation 2.2])

It is also convenient to highlight here that the remainders of a belief set are also belief sets as stated in the following observation.

Observation 3.1.4 ([AGM85]) *Let* $\mathbf{K}$ *be a belief set and* B *be a set of sentences. If* $X \in \mathbf{K}\perp B$ *then* X *is logically closed, i.e.* $X = Cn(X)$.

Now we introduce the notion of *selection function* which is the basic construct underlying the definition of the partial meet contractions.

Definition 3.1.5 ([AGM85]) *Let* $\mathbf{K}$ *be a belief set. A* selection function *for* $\mathbf{K}$ *is a function* γ *such that for all sentences* α*:*

1. *If* $\mathbf{K}\perp\alpha$ *is non-empty, then* $\gamma(\mathbf{K}\perp\alpha)$ *is a non-empty subset of* $\mathbf{K}\perp\alpha$*, and*
2. *If* $\mathbf{K}\perp\alpha$ *is empty, then* $\gamma(\mathbf{K}\perp\alpha) = \{\mathbf{K}\}$.

We are now in a position to present, in the following definition, the class of partial meet contractions.

Definition 3.1.6 (Partial meet contractions [AGM85]) *Let* $\mathbf{K}$ *be a belief set and* γ *a selection function for* $\mathbf{K}$*. The* partial meet contraction *on* $\mathbf{K}$ *that is generated by* γ *is the operation* $-_\gamma$ *such that for all sentences* α*:*

$$\mathbf{K} -_\gamma \alpha = \bigcap \gamma(\mathbf{K}\perp\alpha).$$

An operation $-$ *on* $\mathbf{K}$ *is a* partial meet contraction *if and only if there is a selection function* γ *for* $\mathbf{K}$ *such that for all sentences* α *it holds that* $\mathbf{K} - \alpha = \mathbf{K} -_\gamma \alpha$.

Now we present the definition of two limiting particular cases of partial meet contractions:

Definition 3.1.7 ([AM82, AGM85]) [47]
Let $\mathbf{K}$ *be a belief set.*

1. *An operation* $-$ *on* $\mathbf{K}$ *is a* maxichoice contraction *if and only if it is a partial meet contraction generated by a selection function* γ *such that for all sentences* α*, the set* $\gamma(\mathbf{K}\perp\alpha)$ *has exactly one element.*[48]
2. *The* full meet contraction *on* $\mathbf{K}$ *is the partial meet contraction* $\sim$ *that is generated by the selection function* γ *such that for all sentences* α*, if* $\mathbf{K}\perp\alpha$ *is non-empty, then* $\gamma(\mathbf{K}\perp\alpha) = \mathbf{K}\perp\alpha$*, i.e., the full meet contraction* $\sim$ *is the contraction operation on* $\mathbf{K}$ *defined by*

$$\mathbf{K} \sim \alpha = \begin{cases} \bigcap \mathbf{K}\perp\alpha & \text{, if } \nvdash \alpha \\ \mathbf{K} & \text{, if } \vdash \alpha \end{cases},$$

[47] It is convenient to remark that, although we present maxichoice and full meet contractions here as particular cases of partial meet contractions, they have been introduced before these latter (more general) contraction functions, in [AM82]. In that paper maxichoice contractions were called *choice contractions*, and full meet contraction was designated simply by *meet contraction*.

[48] In [AM82] a function γ defined on the collection of all non-empty remainder sets, such that $\gamma(\mathbf{K}\perp\alpha) \in \mathbf{K}\perp\alpha$ (for every non-empty remainder set $\mathbf{K}\perp\alpha$) is called a *choice function*.

for all sentences α.[49]

Finally we present another special class of Partial Meet Contractions.

Definition 3.1.8 ([AGM85]) *A selection function* γ *for a belief set* $\mathbf{K}$ *is* (transitively) relational *over* $\mathbf{K}$ *if and only if there is a (transitive) relation* $\sqsubseteq$ *over* $\mathbf{K}\Delta\mathcal{L} = \bigcup_{\varepsilon\in\mathcal{L}} \mathbf{K}\bot\varepsilon$[50] *such that for all* $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$*:*

$$\gamma(\mathbf{K}\bot\alpha) = \{B \in \mathbf{K}\bot\alpha : B' \sqsubseteq B \textit{ for all } B' \in \mathbf{K}\bot\alpha\}.$$

The above formula is called the marking-off identity *and* $\sqsubseteq$ *is the* marking-off relation.

A partial meet contraction function $-$ *is* (transitively) relational *over* $\mathbf{K}$ *if and only if it is determined by some selection function that is so.*

Now we present one first representation theorem which clarifies that the class of partial meet contractions coincides exactly with the class of all contraction functions that satisfy the basic AGM postulates for contractions.

Observation 3.1.9 ([AGM85, Observation 2.5]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then* $-$ *is a partial meet contraction on* $\mathbf{K}$ *if and only if it satisfies the postulates of closure, inclusion, vacuity, success, extensionality and recovery.*

The following result consists of an axiomatic characterization of the class of transitively relational partial meet contractions. It exposes that such class coincides with the class of all contraction functions which satisfy the basic and the supplementary AGM postulates for contraction.

Observation 3.1.10 ([AGM85, Corollary 4.5]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a partial meet contraction function on* $\mathbf{K}$*. Then* $-$ *is transitively relational over* $\mathbf{K}$ *if and only if* $-$ *satisfies both conjunctive overlap and conjunctive inclusion.*

Before finishing this section we present an alternative axiomatic characterization for the partial meet contractions which has been obtained by Sven Ove Hansson as a consequence of his investigation regarding the generalization of the partial meet contractions to the case of contraction of belief bases (i.e., sets of sentences which are not necessarily closed under logical consequence).

Observation 3.1.11 ([Han92, Han93, Han91a]) [51]

Let $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then* $-$ *is a partial meet contraction on* $\mathbf{K}$ *if and only if it satisfies, for any sentences* α *and* β*, the following postulates:*

---

[49] Notice that in the explicit definition of the full meet contraction that we have presented here we have implicitly made use of the fact that, according to Observation 3.1.2, $\mathbf{K}\bot\alpha \neq \emptyset$ if and only if $\nvdash \alpha$.

[50] The notation $\mathbf{K}\Delta\mathcal{L}$ that we use here to represent the set $\bigcup_{\varepsilon\in\mathcal{L}} \mathbf{K}\bot\varepsilon$ was introduced in [FH94, Definition 11]. In [Han99b, Definition 1.63] such a set is designated by *subremainder set* and its elements are called the *subremainders* of $\mathbf{K}$.

[51] We must notice that, in fact, the axiomatic characterization for partial meet contractions on a belief set $\mathbf{K}$ that is presented in this observation has been proven (in the cited references) to hold for the case of contractions on any (not necessarily logically closed) set of sentences A.

- Success*: If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(\mathbf{K}-\alpha)$;*
- Inclusion*:* $\mathbf{K}-\alpha \subseteq \mathbf{K}$*;*
- Relevance*: If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}-\alpha$ then there is a set K' such that $\mathbf{K}-\alpha \subseteq K' \subseteq \mathbf{K}$ and $\alpha \notin Cn(K')$ but $\alpha \in Cn(K' \cup \{\beta\})$;*
- Uniformity*: If it holds for all subsets K' of $\mathbf{K}$ that $\alpha \in Cn(K')$ if and only if $\beta \in Cn(K')$, then $\mathbf{K}-\alpha = \mathbf{K}-\beta$.*

The postulates of *relevance* and *uniformity* for contraction used in the above axiomatic characterization have been originally proposed in [Han91a, Han91b] and [Han92], respectively. A detailed explanation of the motivation for those two postulates can be found, for example, in [Han99b].

3.2 Safe/Kernel Contractions

In this section we present other two constructive models of AGM contraction functions for belief sets, namely, *smooth (or saturated) kernel contraction* and *safe contraction*.

Safe contraction functions have been originally presented in [AM85] and are such that every safe contraction is a partial meet contraction and it is possible to identify a subclass of the class of safe contractions which coincides with the class of transitively relational partial meet contractions.

Latter, in [Han94], a new kind of contraction functions was introduced, the kernel contractions, which can be seen as a generalization of the safe contractions. Kernel contractions are such that the class of (all) partial meet contractions is a proper subclass of the class of kernel contractions. However, in that same paper, a subclass of the class of kernel contractions has been identified, namely the class of *smooth (or saturated) kernel contractions*, which is such that a contraction function on a belief set is a smooth (or saturated) kernel contraction if and only if it is a partial meet contraction (on that same belief set).

In this section, instead of following the chronological order by which kernel contractions and safe contractions have been originally presented in the literature, we start by presenting the most general construction, i.e. the kernel contractions, in Subsection 3.2.1 and only afterwards, in Subsection 3.2.2, we introduce the class of safe contractions as a special particular kind of kernel contractions.

3.2.1 Kernel Contraction

In this subsection we present the kernel contraction functions which were introduced by Sven Ove Hansson in [Han94]. Such contraction functions are defined for any set of sentences (not necessarily closed under logical consequences) and kernel contractions for belief sets appear as a special particular case of the general theory.

The definition of kernel contraction is based on the concepts of kernel sets and of incision function which we present in the following two definitions.

Definition 3.2.1 ([Han94]) *Let A be a set of sentences and α be a sentence in $\mathcal{L}$. Then $X \in A \perp\!\!\!\perp \alpha$ if and only if:*

1. $X \subseteq A$.

2. $X \vdash \alpha$.

3. *If* $Y \subset X$ *then* $Y \nvdash \alpha$.

The set $A \perp\!\!\!\perp \alpha$ is called the kernel set of A with respect to α *and its elements are the* α-kernels *of A.*

Definition 3.2.2 ([Han94]) *An* incision function σ *for a set (of sentences) A is a function such that for all sentences α:*

(i) $\sigma(A \perp\!\!\!\perp \alpha) \subseteq \bigcup(A \perp\!\!\!\perp \alpha)$.

(ii) if $\emptyset \neq X \in A \perp\!\!\!\perp \alpha$, *then* $X \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.

We are now in a position to present the definition of kernel contraction.

Definition 3.2.3 ([Han94]) *Let σ be an incision function for A. The* kernel contraction $-_\sigma$ *for A based on σ is defined as follows:*

$$A -_\sigma \alpha = A \setminus \sigma(A \perp\!\!\!\perp \alpha).$$

An operator $-$ for A is a kernel contraction if and only if there is some incision function σ for A such that $A - \alpha = A -_\sigma \alpha$ for all sentences α.

The following observation presents an axiomatic characterization for kernel contraction.

Observation 3.2.4 ([Han94]) *The operator $-$ for (a given set of sentences) A is a kernel contraction if and only if it satisfies, for any sentences α and β, the following postulates:*

- Success*: If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(A - \alpha)$;*

- Inclusion*: $A - \alpha \subseteq A$;*

- Uniformity*: If it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $A - \alpha = A - \beta$;*

- Core-retainment*: If $\beta \in A$ and $\beta \notin A - \alpha$, then there is some set A' such that $A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.*

The postulate of *core-retainment* for contraction used in the above axiomatic characterization has been originally proposed in [Han91b]. An explanation of the motivation for such postulate can be found, for example, in [Han99b].

As it has been noted in [Han94, Han99b], since *core-retainment* is implied by *relevance*, it follows immediately from the axiomatic characterizations (of partial meet and kernel contractions) presented in Observations 3.1.11 and 3.2.4 that all partial meet contractions are kernel contractions. Nevertheless, the converse inclusion does not hold, i.e., not all kernel contractions are partial meet contractions (for a counterexample see, e.g., [Han94, Sec. 5]). Having this in mind, in what follows we present a special kind of kernel contractions, the so-called *smooth kernel contractions*, which are such that, in what concerns the particular case of contractions of belief sets, the class of partial meet contractions coincides with the class of smooth kernel contractions.

Definition 3.2.5 ([Han94, Definition 11]) *An incision function σ for a set A is* smooth *if and only if it holds for all subsets B of A that if $B \vdash \beta$ and $\beta \in \sigma(A \perp\!\!\!\perp \alpha)$, then $B \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset$.*

A kernel contraction is smooth *if and only if it is based on a smooth incision function.*

In [Han94] Hansson presented an alternative way of constructing smooth kernel contractions. More precisely, in that paper another particular kind of kernel contraction, the so-called *saturated kernel contraction*, has been introduced which is such that the class of saturated kernel contractions coincides with the class of smooth kernel contractions. In what follows we present the definition of saturated kernel contraction and a formal result stating the above mentioned interrelation between smooth kernel contractions and saturated kernel contractions.

Definition 3.2.6 ([Han94, Definition 14]) *Let σ be an incision function for A. The* saturated kernel contraction *$-^s_\sigma$ for A that is associated with σ is defined, for any sentence α, as follows:*

$$A -^s_\sigma \alpha = A \cap Cn(A -_\sigma \alpha),$$

where $-_\sigma$ is the kernel contraction $-_\sigma$ for A based on σ.[52]

An operator $-$ for A is a saturated kernel contraction if and only if there is some incision function σ for A such that $A - \alpha = A -^s_\sigma \alpha$ for all sentences α.

Observation 3.2.7 ([Han94, Observation 15]) *An operator $-$ for a set A is a saturated kernel contraction if and only if it is a smooth kernel contraction.*

The following observation asserts that when only contraction of belief sets is under consideration there is no distinction between smooth (or saturated) kernel contractions and partial meet contractions.

[52]Notice that, it follows from Definition 3.2.3 that the saturated kernel contraction $-^s_\sigma$ for A based on σ is such that

$$A -^s_\sigma \alpha = A \cap Cn(A \setminus \sigma(A \perp\!\!\!\perp \alpha)),$$

for all sentences α.

Observation 3.2.8 ([Han94, Theorem 16]) *Let* $\mathbf{K}$ *be a belief set. Then* $-$ *is a smooth kernel contraction on* $\mathbf{K}$ *if and only if it is a partial meet contraction on* $\mathbf{K}$.

It is worth remarking here that in [Han94] the left-to-right part of the above result is proven in a direct way. More precisely, such proof is essentially based on an explicit definition of a selection function from an incision function. On the other hand, the proof of the right-to-left part of the above result is based on certain axiomatic characterizations of each of those two contraction operation and on interrelations between some different sets of postulates.

In this regard it is worth mentioning here that, in [FFKI06], Falappa, Fermé and Kern-Isberner have presented ways of defining an incision function σ_γ from a selection function γ and vice versa. Furthermore, in that same paper, they have identified which properties are necessary and sufficient to require from selection and incision functions in order to assure that $\gamma_{\sigma_\gamma} = \gamma$ and $\sigma_{\gamma_\sigma} = \sigma$.

3.2.2 Safe Contraction

Safe contraction will be initially presented here as a special particular case of kernel contraction as it is done in [Han99b]. However, we notice once again that, historically, the operation of safe contraction was introduced by Alchourrón and Makinsson [AM85] much earlier than the more general construction of kernel contraction was proposed in [Han94] and, for that reason, we shall also present further below the original definition of safe contraction which was formulated without the need of any of the concepts related to kernel contractions that we have introduced in the previous subsection.

In order to be able to define safe contraction in terms of kernel contractions we need to start by introducing some additional concepts, namely the ones presented in the following four definitions.

Definition 3.2.9 ([Han99b, Definition 2.38]) *Let* A *be a set of sentences. A kernel selection function for* A *is a function* s *such that for all* $X \in \{X | X \in A \perp\!\!\!\perp \alpha$ *for some* $\alpha\}$*:*

1. $s(X) \subseteq X$
2. $s(X) \neq \emptyset$ *if* $X \neq \emptyset$.

Definition 3.2.10 ([Han99b, Definition 2.39]) *Let* s *be a kernel selection function for a set* A*. Then an incision function* σ *is the cumulation of* s *if and only if for all sentences* α*:*

$$\sigma(A \perp\!\!\!\perp \alpha) = \bigcup \{s(X) | X \in A \perp\!\!\!\perp \alpha\}$$

Definition 3.2.11 ([Han99b, Definition 2.40]) *A kernel selection function* s *for a set* A *is based on a relation* $\prec$ *if and only if for all* $X \in A \perp\!\!\!\perp \alpha$*:*

$\beta \in s(X)$ *if and only if* $\beta \in X$ *and there is no* $\delta \in X$ *such that* $\delta \prec \beta$.

An incision function is based on a relation $\prec$ *if and only if it is the cumulation of some kernel selection function for that is based on* $\prec$.

Definition 3.2.12 **([AM85])** *Let A be a set of sentences and $\prec$ a relation over A. We say that $\prec$ is a hierarchy over A if and only if it satisfies the non-circularity (or acyclicity) condition: for all positive integers n, if $\{\alpha_1, \ldots, \alpha_n\} \subseteq A$, then it is* not *the case that $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_n \prec \alpha_1\}$.*

Having the above definitions in mind we can now present the definition of safe contraction.

Definition 3.2.13 **([AM85])** *Let $\prec$ be a hierarchy over a set of sentences A. Let σ be the incision function that is based on $\prec$ and $-_\sigma$ the kernel contraction based on σ. The operation of safe contraction $-_s$ based on $\prec$ is defined as follows:*

$$A{-_s}\alpha = A \cap Cn(A{-_\sigma}\alpha)$$

Remark 3.2.14 *Notice that, according to Definitions 3.2.6 and 3.2.13, it holds that every safe contraction function is a saturated kernel contraction.*

The original presentation of safe contraction, in [AM85], followed slightly different steps and, although the approach presented in that paper is equivalent to the one exposed above, some of the concepts introduced there, by Alchourrón and Makinson, deserve mentioning. Therefore we will now expose their construction of safe contraction.

Definition 3.2.15 **([AM85])** *Let A be a set of sentences and $\prec$ be a hierarchy over A. An element β of A is safe with respect to α (modulo $\prec$) if and only if β is not a minimal element (under $\prec$) of any minimal subset (under inclusion) B of A such that $\alpha \in Cn(B)$. Equivalently, if and only if every inclusion-minimal subset $B \subseteq A$ satisfying $B \vdash \alpha$ either does not contain β or else contains some sentence δ such that $\delta \prec \beta$.*[53]

We write A/α for the set of all elements of A that are safe with respect to α.

Now we present the original definition of safe contraction which was based on the concept of safe elements introduced above.

Definition 3.2.16 **([AM85])** *The operation of safe contraction over a set A (modulo a hierarchy $\prec$ and given a consequence operation Cn) is defined by the equation*

$$A{-_s}\alpha = A \cap Cn(A/\alpha)$$

In the case that $\mathbf{K}$ is a theory, since $\mathbf{K}/\alpha \subseteq \mathbf{K}$ we have $Cn(\mathbf{K}/\alpha) \subseteq Cn(\mathbf{K}) = \mathbf{K}$ so that $\mathbf{K}{-_s}\alpha = Cn(\mathbf{K}/\alpha)$.

[53]Notice that, making use of the concept (and notation) of *kernel set of A with respect to α*, the definition of a safe element can be formulated in the following alternative way:

An element β of A is *safe* with respect to α (modulo $\prec$) if and only if

$$\text{For all } B \in A \perp\!\!\!\perp \alpha, \text{ if } \beta \in B, \text{ then there is some } \delta \in B \text{ such that } \delta \prec \beta.$$

The following remark clarifies the equivalence between both the approaches presented for the construction of safe contractions (see Definitions 3.2.13 and 3.2.16).

Remark 3.2.17 *Let A be a set of propositions and $\prec$ a hierarchy over A. If σ is the incision function based on $\prec$, then*

$$\sigma(A \perp\!\!\!\perp \alpha) = A \backslash A/\alpha$$

Hence

$$A -_\sigma \alpha = A/\alpha$$

In [AM85] Alchourrón and Makinsson proved that every safe contraction is an AGM contraction. This fact is more formally stated in the following observation.

Observation 3.2.18 ([AM85, Observation 3.2]) *If $-$ is a safe contraction function then $-$ satisfies the basic AGM postulates for contraction (i.e. closure, inclusion, vacuity, success, extensionality and recovery).*

Combining the above observation with the axiomatic characterization for partial meet contractions presented in Observation 3.1.9 we can immediately conclude that, when considering contraction functions on belief sets, every safe contraction function is a partial meet contraction function ([AM85, Corollary 3.3]).[54]

In what follows we introduce some properties which can be required of a relation over a set of sentences A as well as some interrelations among such properties. Furthermore we expose some direct relation between the fact that a given safe contraction function satisfies certain contraction postulates and the fact that the hierarchy on which such safe contraction is based satisfies some of the mentioned properties. More precisely, given a safe contraction $-_s$ on a belief set $\mathbf{K}$ based on a hierarchy $\prec$ over $\mathbf{K}$, we state some interconnections between the properties (among the ones that we will introduce below) which are satisfied by $\prec$ and the postulates which are satisfied by $-_s$.

Definition 3.2.19 ([AM86, pp. 192]) *A relation $\prec$ over a set A is regular if and only if it satisfies the two following properties:*
Continuing-up *([AM85]): For all $\alpha, \beta, \delta \in A$, if $\alpha \prec \beta$ and $\beta \vdash \delta$, then $\alpha \prec \delta$.*
Continuing-down *([AM85]): For all $\alpha, \beta, \delta \in A$, if $\alpha \vdash \beta$ and $\beta \prec \delta$, then $\alpha \prec \delta$.*

In [AM85], Alchourrón and Makinsson proved that:

1. ([AM85, Observations 4.3 and 5.3]) If a safe contraction $-$ on a belief set $\mathbf{K}$ is based on a hierarchy that satisfies continuing-up or continuing-down, then $-$ satisfies the postulate of conjunctive overlap;

[54]Notice that this result is also an immediate consequence of Remark 3.2.14 and Observations 3.2.7 and 3.2.8.

2. ([AM85, Observation 6.2]) For a safe contraction $-$ to satisfy the postulate of conjunctive inclusion it is enough that such safe contraction is based on a hierarchy $\prec$ which satisfies continuing-up or continuing-down as well as the following property:

   Virtual connectivity ([AM85]): For all $\alpha, \beta, \delta \in A$, if $\alpha \prec \beta$ then either $\alpha \prec \delta$ or $\delta \prec \beta$.[55]

The most advanced formal result that has been obtained by Alchourrón and Makinsson for safe contraction in [AM85] is the following corollary of the above mentioned facts: if a safe contraction function on a belief set is determined by a regular and virtually connected hierarchy, then it satisfies the basic and the supplementary AGM postulates and, therefore, it is a transitively relational partial meet contraction (cf. [AM85, Corollary 6.3]).

Afterwards, in [AM86] the same authors showed that, for the case of contractions of finite (modulo Cn) theories, any operation is a safe contraction function determined by a regular and virtually connected hierarchy if and only if it is a transitively relational partial meet contraction function (cf. [AM86, Theorem 2]). We notice however that this result was obtained in a direct way, i.e. its proof does not make use of any representation theorems for the contraction functions under consideration.

Finally, Rott [Rot92b] proved that, every contraction function, on a (possibly infinite) theory, that satisfies both the basic and the supplementary AGM postulates can be represented as a safe contraction function generated by a regular and virtually connected hierarchy (cf. [Rot92b, Corollary of Theorem 4]).

All the above exposed facts can be summed up in the following representation theorem:

Observation 3.2.20 ([AM85, Rot92b]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then* $-$ *is a safe contraction, based on a regular and virtually connected hierarchy, if and only if it satisfies both the basic and the supplementary AGM postulates (i.e. closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap, and conjunctive inclusion).*

It follows from Observations 3.1.9, 3.1.10 and 3.2.20 that a contraction function on a belief set is a safe contraction, based on a regular and virtually connected hierarchy, if and only if it is a transitively relational partial meet contraction.

3.3 System of Spheres-based Contractions

In this section we expose how AGM contraction functions satisfying the basic and the supplementary AGM postulates for contraction can be constructed using a propositional approach. Such strategy was pioneered by Adam Grove [Gro88].

For convenience, due to the central role they play in the present section, we recall here some concepts already introduced in Section 1.4.

[55] At this point it is worth noticing that, according to [AM85, Remark 6.1 $-$ (b)], if a hierarchy is virtually connected, then it satisfies continuing-up if and only if it satisfies continuing-down.

We must start by reminding that a *consistent complete theory* or a *possible world* (or, simply a *world*) is a maximal consistent subset of $\mathcal{L}$, and that, in the present text, we will denote the set of all possible worlds by $\mathcal{M}_{\mathcal{L}}$.[56]

Next we introduce some basic definitions and notations which clarify how propositions (i.e., sets of possible worlds) can be used to represent sets of sentences.

Definition 3.3.1 ([Gro88]) *Let R be a set of sentences of $\mathcal{L}$. The set of possible worlds that contain R is denoted by $\|R\|$, i.e.,*

$$\|R\| = \{M \in \mathcal{M}_{\mathcal{L}} : R \subseteq M\}.$$

If R is inconsistent this will be the empty set, $\emptyset$. The elements of $\|R\|$ are the R-worlds.

For a sentence $\varphi \in \mathcal{L}$, $\|\varphi\|$ is an abbreviation of $\|\{\varphi\}\|$. The elements of $\|\varphi\|$ are the φ-worlds.

Throughout the remainder of this work, given a set of sentences R, the set $\|R\|$ will be sometimes referred to as the *set of possible worlds for R*.

In particular, in Grove's model, a belief set $\mathbf{K}$ is identified with the set $\|\mathbf{K}\|$ of all consistent complete theories (i.e., maximal consistent subsets) of $\mathcal{L}$ that contain $\mathbf{K}$.

Conversely, any set of possible worlds $\mathcal{V} \subseteq \mathcal{M}_{\mathcal{L}}$ has an associated belief set:

Definition 3.3.2 ([Gro88]) *Let $\mathcal{V} \subseteq \mathcal{M}_{\mathcal{L}}$. The theory associated to $\mathcal{V}$ is $Th(\mathcal{V}) = \bigcap \mathcal{V}$. If $\mathcal{V} = \emptyset$, then $Th(\mathcal{V})$ will be taken to be the inconsistent theory $\mathcal{L}(= \mathbf{K}_{\perp})$.*[57]

The following observation presents some properties relating belief sets and sets of possible worlds which will be necessary further ahead.

Observation 3.3.3 ([Gro88, pp. 158]) *Let $\mathbf{H}, \mathbf{K}$ be belief sets and $\mathcal{U}, \mathcal{V}$ be sets of possible worlds. Then:*

(a) $Th(\|\mathbf{K}\|) = \mathbf{K}$ (if the underlying logic is compact).

(b) $Th(\mathcal{V})$ is consistent if and only if $\mathcal{V}$ is nonempty.

(c) For any sentence $\varphi \in \mathcal{L}$, $Th(\mathcal{V} \cap \|\varphi\|) = Cn(Th(\mathcal{V}) \cup \{\varphi\})$.

(d) If $\mathcal{U} \subseteq \mathcal{V}$, then $Th(\mathcal{V}) \subseteq Th(\mathcal{U})$.

(e) If $\mathbf{H} \subseteq \mathbf{K}$, then $\|\mathbf{K}\| \subseteq \|\mathbf{H}\|$.

[56] Notice that, having in mind the concept of remainder set introduced in Definition 3.1.1, it holds that $\mathcal{M}_{\mathcal{L}} = \mathcal{L} \perp \{\perp\}$.

[57] Notice that this is a natural assumption in the present context. Indeed, for any set of possible worlds $\mathcal{U} \subseteq \mathcal{M}_{\mathcal{L}}$ it holds that $\bigcap \mathcal{U} = \{\alpha : \forall M \in \mathcal{U}, \alpha \in M\} = \{\alpha \in \mathcal{L} : \forall M \in \mathcal{U}, \alpha \in M\}$ (where this last equality holds due to the fact that the possible worlds are, by definition, subsets of $\mathcal{L}$), therefore, in the particular case of the empty subset of $\mathcal{M}_{\mathcal{L}}$ we have that $\bigcap \emptyset = \{\alpha \in \mathcal{L} : \forall M \in \emptyset, \alpha \in M\} = \mathcal{L}$ (since for any $\alpha \in \mathcal{L}$ the condition "$\forall M \in \emptyset, \alpha \in M$" is vacuously true).

Remark 3.3.4 *It follows immediately from Observation 3.3.3 - (a),(d) and (e) that, if* $\mathbf{H}, \mathbf{K}$ *are belief sets then* $\mathbf{H} \subseteq \mathbf{K}$ *if and only if* $\|\mathbf{K}\| \subseteq \|\mathbf{H}\|$*. That is, it holds that to a greater number of beliefs it corresponds a smaller set of possible worlds and vice versa.*

Now, before proceeding to present the model of belief set contraction proposed by Grove [Gro88], it is convenient to highlight the intuition behind the construction of contraction functions using the possible worlds approach: If we wish to contract a belief set $\mathbf{K}$ by a sentence φ (such that $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$) the result must be a new belief set which is a (proper) subset of $\mathbf{K}$ that does not contain φ. In terms of possible worlds, having Remark 3.3.4 in mind, this means that the set of possible worlds for the new belief set must, on the one hand, contain all $\mathbf{K}$-worlds, and on the other hand contain (in addition) some $\neg\varphi$-worlds.

The central contribution of Adam Grove's paper of 1988 consists precisely of the presentation of a way, which is based on the above described *possible worlds-based view* of a contraction function, of obtaining contraction functions that satisfy the basic and supplementary AGM postulates for contraction. In other words, the main result of [Gro88] has been the obtention of an alternative way of (explicitly) constructing transitively relational partial meet contractions, which is based upon intersections of (adequately chosen) families of possible worlds. In what follows we present such method.

The alternative modelling proposed in the mentioned paper is in form similar to the "sphere" semantics for counterfactuals proposed by Lewis [Lew73]. In particular, Grove's construction is based on the generalization, presented in the following definition, of the concept of a system of spheres introduced in [Lew73, pp. 14].

Definition 3.3.5 (System of Spheres [Gro88]) *Let* $\mathcal{X}$ *be a subset of* $\mathcal{M}_{\mathcal{L}}$*. A system of spheres (abrev. S.S.), or spheres' system, centred on* $\mathcal{X}$ *is a collection* $\mathbb{S}$ *of subsets of* $\mathcal{M}_{\mathcal{L}}$*, i.e.* $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$*, that satisfies the following conditions:*

($\mathbb{S}1$) $\mathbb{S}$ *is totally ordered with respect to set inclusion; that is, if* $\mathcal{U}, \mathcal{V} \in \mathbb{S}$*, then* $\mathcal{U} \subseteq \mathcal{V}$ *or* $\mathcal{V} \subseteq \mathcal{U}$*.*

($\mathbb{S}2$) $\mathcal{X} \in \mathbb{S}$*, and if* $\mathcal{U} \in \mathbb{S}$ *then* $\mathcal{X} \subseteq \mathcal{U}$*.*

($\mathbb{S}3$) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ *(and so it is the largest element of* $\mathbb{S}$*).*

($\mathbb{S}4$) For every $\varphi \in \mathcal{L}$*, if there is any element in* $\mathbb{S}$ *intersecting* $\|\varphi\|$ *then there is also a smallest element in* $\mathbb{S}$ *intersecting* $\|\varphi\|$*.*

The elements of $\mathbb{S}$ *are called spheres.*

Condition ($\mathbb{S}4$) above is called the limit assumption ([Lew73, pp. $19-20$*]).*

For any consistent sentence $\varphi \in \mathcal{L}$*, the smallest sphere in* $\mathbb{S}$ *intersecting* $\|\varphi\|$ *is denoted by* $\mathbb{S}_\varphi$ *and* $f_{\mathbb{S}}(\varphi)$ *denotes the set consisting of the* φ*-worlds* closest *to* $\mathcal{X}$*, i.e.,*

$$f_{\mathbb{S}}(\varphi) = \|\varphi\| \cap \mathbb{S}_\varphi.$$

Remark 3.3.6 *Let* $\varphi \in \mathcal{L}$ *be a sentence such that* $\|\varphi\| \neq \emptyset$*. Then* $\mathbb{S}_\varphi = \bigcap\{\mathcal{G} \in \mathbb{S} : \mathcal{G} \cap \|\varphi\| \neq \emptyset\}$*.*

A graphical representation of a system of spheres centred on $\mathcal{X}$ highlighting the sets $\mathbb{S}_\varphi$, $\|\varphi\|$ and $f_\mathbb{S}(\varphi)$, for a given sentence φ, is depicted in Figure 3.1.[58]

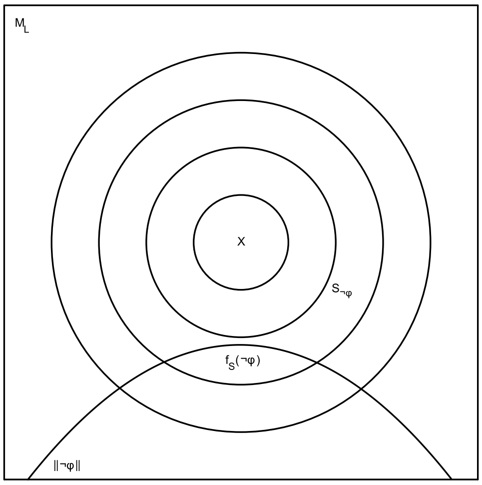

Figure 3.1: Schematic representation of a system of spheres centred on $\mathcal{X}$ displaying the sets $\mathbb{S}_\varphi$, $\|\varphi\|$ and $f_\mathbb{S}(\varphi)$, for some $\varphi \in \mathcal{L} \setminus Cn(\emptyset)$.

Based on the above presented notion of a system of spheres centred on a set of possible worlds Grove observed that it is natural to define a contraction function on a belief set as it is described in the following definition.

Definition 3.3.7 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. The* $\mathbb{S}$-based contraction *on* $\mathbf{K}$ *is the contraction operation* $-_\mathbb{S}$ *defined, for any* $\varphi \in \mathcal{L}$*, by:*

$$\mathbf{K}-_\mathbb{S}\varphi = \begin{cases} Th(\|\mathbf{K}\| \cup f_\mathbb{S}(\neg\varphi)) & \text{, if } \not\vdash \varphi \\ \mathbf{K} & \text{, if } \vdash \varphi \end{cases} .$$

An operation $-$ *on* $\mathbf{K}$ *is a* system of spheres-based contraction on $\mathbf{K}$ *if and only if there is a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$*, such that, for all sentences* $\varphi \in \mathcal{L}$*,* $\mathbf{K}-\varphi = \mathbf{K}-_\mathbb{S}\varphi$*.*

The following remark presents an alternative equivalent definition of a *system of spheres-based contraction on* $\mathbf{K}$.

Remark 3.3.8 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $-_\mathbb{S}$ *be the* $\mathbb{S}$*-based contraction on* $\mathbf{K}$*. Then, taking Observation 3.3.3-(a) into account, since* $f_\mathbb{S}(\neg\varphi) \subseteq \|\mathbf{K}\|$ *for every* φ *such that* $\varphi \notin \mathbf{K}$*, it holds that*

$$\mathbf{K}-_\mathbb{S}\varphi = \begin{cases} Th(\|\mathbf{K}\| \cup f_\mathbb{S}(\neg\varphi)) & \text{, if } \not\vdash \varphi \text{ and } \varphi \in \mathbf{K} \\ \mathbf{K} & \text{, if } \vdash \varphi \text{ or } \varphi \notin \mathbf{K}, \end{cases}$$

for any $\varphi \in \mathcal{L}$*.*

[58]Figure 3.1 has been adapted from [Gär88, Figure 4.2].

Figure 3.2 contains a possible graphical representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, for a belief set $\mathbf{K}$. The shaded region in the mentioned figure represents the set $\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)$, where φ is a sentence such that $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. Notice that in these conditions, if $-_{\mathbb{S}}$ is the $\mathbb{S}$-based contraction on $\mathbf{K}$ then $\mathbf{K}-_{\mathbb{S}}\varphi$ consists of the intersection of all the worlds included in the shaded region, i.e., $\mathbf{K}-_{\mathbb{S}}\varphi = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi))$.

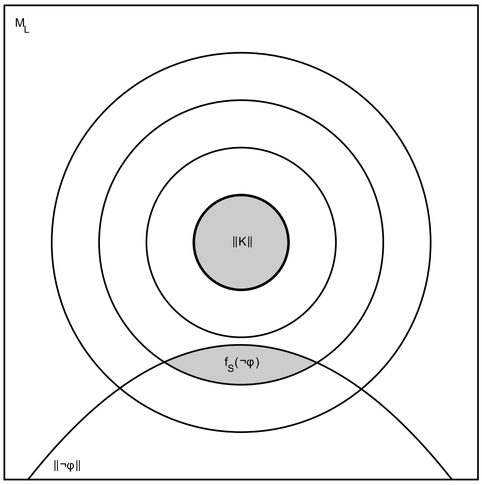

Figure 3.2: Schematic representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ (where $\mathbf{K}$ is a belief set) highlighting the worlds in $\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)$, for a given $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. The intersection of all the worlds in the shaded region originates the set $\mathbf{K}-_{\mathbb{S}}\varphi$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based contraction on $\mathbf{K}$.

Intuitively, given a belief set $\mathbf{K}$, a system of spheres centred on $\|\mathbf{K}\|$ can be regarded as an ordering on the set of possible worlds, which can be interpreted as a measure of plausibility of alternative worlds given the current belief state (represented by $\mathbf{K}$). The closer a possible world is to $\|\mathbf{K}\|$, the more plausible it is. From this point of view, the contraction of $\mathbf{K}$ by φ, as defined above, consists of the theory determined by the set of possible worlds which is the union of the most plausible $\neg\varphi$-worlds with the set of possible worlds for $\mathbf{K}$ (i.e. $\|\mathbf{K}\|$).

Next we present an axiomatic characterization for the system of spheres-based contractions. In that regard we start by remarking that, according to [Gro88, Theorems 1 and 2], given a belief set $\mathbf{K}$, a revision function on $\mathbf{K}$ is a system of spheres-based revision[59] if and only if it satisfies the basic and the supplementary AGM postulates for revision (i.e. closure, success, inclusion, vacuity, consistency, extensionality, superexpansion and subexpansion).[60]

Now, since the system of spheres-based contractions are precisely the contraction functions which can be obtained from a system of spheres-based revision by

[59] Here by *system of spheres-based revision* we mean a revision function that is obtained from a system of spheres–based contraction (see Def. 3.3.7) by means of the Levi identity (eq. (2.1)).

[60] It must be noted here that, in [PST96, PT97], Priest, Surendonk and Tanaka have pointed out an error in [Gro88, Proof of Theorem 1] (more precisely, in the demonstration presented by Grove that a system of spheres-based revision[59] satisfies the supplementary AGM postulates for revision, i.e. *superexpansion* and *subexpansion*). However, in the mentioned papers, two possible ways to correct that proof have been presented, assuring that the statement of Grove's theorem is valid.

means of the Harper identity (equation (2.2)) (cf. Footnote 59 and Observation 2.4.8), combining the above mentioned axiomatic characterization for the system of spheres-based revisions[59] with Observations 2.4.6 and 2.4.7, the following axiomatic characterization for the $\mathbb{S}$-based contraction on $\mathbf{K}$ is immediately obtained:

Observation 3.3.9 ([Gro88]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then* $-$ *is a system of spheres-based contraction if and only if it satisfies the basic and the supplementary AGM postulates (i.e. closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap, and conjunctive inclusion).*

Before closing the present section it is worth to remark that combining the above observation with Observations 3.1.9 and 3.1.10 we can immediately conclude that the class of system of spheres-based contractions coincides with the class of transitively relational partial meet contractions. Nevertheless, in Section 4.1 we shall show in a direct way that a contraction function is a transitively relational partial meet contraction if and only if it is a system of spheres-based contraction.

3.4 Epistemic Entrenchment-based Contractions

This section is devoted to the presentation of the class of *epistemic entrenchment-based contractions* originally introduced in [Gär88, GM88]. The main virtue of such class is the fact that it coincides with the class of contraction functions which satisfy the basic and supplementary AGM postulates for contraction.

We start by introducing in the following definition the concept of *epistemic entrenchment relation*[61] which is the fundamental concept underlying the definition of the above mentioned class of contraction functions.

Definition 3.4.1 (Epistemic Entrenchment relation [Gär88, GM88]) *An ordering of* epistemic entrenchment *(abrev. E.E.) with respect to a belief set* $\mathbf{K}$ *is a binary relation* $\leq$ *on* $\mathcal{L}$ *which satisfies the following postulates:*

(EE1) For all $\alpha, \beta, \delta \in \mathcal{L}$*, if* $\alpha \leq \beta$ *and* $\beta \leq \delta$ *then* $\alpha \leq \delta$*. (Transitivity)*

(EE2) For all $\alpha, \beta \in \mathcal{L}$*, if* $\alpha \vdash \beta$ *then* $\alpha \leq \beta$*. (Dominance)*

(EE3) For all $\alpha, \beta \in \mathcal{L}$*,* $\alpha \leq \alpha \wedge \beta$ *or* $\beta \leq \alpha \wedge \beta$*. (Conjunctiveness)*

(EE4) When $\mathbf{K} \neq \mathbf{K}_{\perp}$*,* $\alpha \notin \mathbf{K}$ *iff* $\alpha \leq \beta$ *for all* $\beta \in \mathcal{L}$*. (Minimality)*

(EE5) If $\beta \leq \alpha$ *for all* $\beta \in \mathcal{L}$*, then* $\vdash \alpha$*. (Maximality)*

As explained in [Gär88, Sec. 4.6] and [GM88], given an epistemic entrenchment relation $\leq$, the notation $\alpha \leq \beta$ (where α and β are two sentences) is seen as meaning that "β is at least as epistemically entrenched as α". A detailed discussion of rationale behind each of the postulates (EE1)$-$(EE5), chosen as characteristic properties

[61] The epistemic entrenchment relations can be seen as dual to the binary relations on $\mathcal{L}$ proposed by Grove [Gro88, Sec. 3], which we present in Section *A.2* (of the Appendix) under the designation of *Grovean relations*. In that same section this remark is explained in further detail.

of an epistemic entrenchment relation, can be found in, e.g., [Gär88, Sec. 4.6] and [Han99b, Sec. 2.10].

In [Gär88, Sec. 4.6] and [GM88] it is also clarified that the main motivation underlying the introduction of concept of *epistemic entrenchment* is that it is used to determine priorities among sentences in processes of contractions and revisions. More precisely, the guiding idea regarding the use of an epistemic entrenchment relation in order to explicitly define revision or contraction functions is simply that, when a belief set $\mathbf{K}$ is revised or contracted, the sentences in $\mathbf{K}$ that are given up are the ones which have the *lowest* degrees of epistemic entrenchment.

By convenience we now introduce some notation:

Notation 3.4.2 *In what follows, given an E.E. relation $\leq$ as defined above, for any $\alpha, \beta \in \mathcal{L}$ we will denote the negation of $\alpha \leq \beta$ by $\alpha \not\leq \beta$.*

The strict part of the binary relation $\leq$ is denoted by $<$, i.e.

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha < \beta \text{ if and only if } \alpha \leq \beta \text{ and } \beta \not\leq \alpha,$$

and $\alpha \not< \beta$ shall denote the negation of $\alpha < \beta$. It follows from the connectivity of $\leq$ (Observation 3.4.3-(i)) that

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha < \beta \text{ if and only if } \beta \not\leq \alpha. \tag{3.1}$$

The notation $\alpha =_{E.E.} \beta$ (for any two sentences α and β) will be used with the following meaning:

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha =_{E.E.} \beta \text{ if and only if } \alpha \leq \beta \text{ and } \beta \leq \alpha.$$

The negation of $\alpha =_{E.E.} \beta$ will be denoted by $\alpha \neq_{E.E.} \beta$.

The following observation presents some further properties that are satisfied by any E.E. relation:

Observation 3.4.3 ([GM88, Lemma 3]) *If a binary relation $\leq$ satisfies (EE1), (EE2) and (EE3) then the following properties also hold:*

(i) For all $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \beta$ or $\beta \leq \alpha$ (Connectivity).

(ii) For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\beta \wedge \delta \leq \alpha$, then $\beta \leq \alpha$ or $\delta \leq \alpha$.

(iii) For all $\alpha, \beta \in \mathcal{L}$, $\alpha < \beta$ iff $\alpha \wedge \beta < \beta$.

(iv) For all $\alpha, \beta, \delta \in \mathcal{L}$,if $\delta \leq \alpha$ and $\delta \leq \beta$, then $\delta \leq \alpha \wedge \beta$.

(v) For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \leq \beta$, then $\alpha \leq \alpha \wedge \beta$.

Now we proceed to what is the main goal of the present section, i.e. the presentation of the definition and axiomatic characterization of the *epistemic entrenchment-based contractions* which have been introduced in [Gär88, GM88].

Definition 3.4.4 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$.

The $\leq$-based contraction *on* $\mathbf{K}$ *is the contraction operation* $-_{\leq}$ *defined, for any* $\alpha \in \mathcal{L}$, *by:*[62]

$$\mathbf{K}-_{\leq}\alpha = \begin{cases} \{\beta \in \mathbf{K} : \alpha < \alpha \vee \beta\} & \text{, if } \nvdash \alpha \\ \mathbf{K} & \text{, if } \vdash \alpha. \end{cases} \qquad (C_{-_{\leq}})$$

An operation $-$ *on* $\mathbf{K}$ *is an* epistemic entrenchment-based contraction *on* $\mathbf{K}$ *if and only if there is an epistemic entrenchment relation with respect to* $\mathbf{K}$ *such that, for all sentences* $\alpha \in \mathcal{L}$, $\mathbf{K}-\alpha = \mathbf{K}-_{\leq}\alpha$.

Apart from presenting a way of defining a contraction operation based on an epistemic entrenchment relation (by means of condition ($C_{-_{\leq}}$)), Gärdenfors and Makinsson [Gär88, GM88] have also exposed a way of proceeding to the converse construction. More precisely, they have suggested the following condition to define and epistemic entrenchment relation ($\leq$) with respect to a belief set $\mathbf{K}$ by means of a (given) contraction function ($-$) on $\mathbf{K}$:

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K}-\alpha \wedge \beta \text{ or } \vdash \alpha \wedge \beta, \qquad (C_{\leq})$$

The following two observations, obtained in [GM88],[63] present more formally the interrelation between conditions ($C_{-_{\leq}}$) and ($C_{\leq}$).

Observation 3.4.5 ([GM88, Theorem 4]) *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$. *Then the* $\leq$*-based contraction function, which is uniquely determined by* ($C_{-_{\leq}}$)*, satisfies the basic and the supplementary AGM postulates for belief set contraction (i.e. closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap, and conjunctive inclusion). Furthermore, if* $-$ *is the* $\leq$*-based contraction then condition* ($C_{\leq}$) *is satisfied.*

Observation 3.4.6 ([GM88, Theorem 5]) *Let* $\mathbf{K}$ *be a belief set. If* $-$ *is a contraction function on* $\mathbf{K}$ *that satisfies the basic and the supplementary AGM postulates for belief set contraction (i.e. closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap, and conjunctive inclusion), then the ordering* $\leq$ *that is uniquely determined by* ($C_{\leq}$) *satisfies (EE1)-(EE5). Furthermore,* $\mathbf{K}-\alpha = \mathbf{K}-_{\leq}\alpha$ *for any sentence* α*, where* $-_{\leq}$ *is the* $\leq$*-based contraction defined by condition* ($C_{-_{\leq}}$)*.*

---

[62] In [GM88], condition ($C_{-_{\leq}}$) is formulated in the following alternative (equivalent) way:

$$\beta \in \mathbf{K}-_{\leq}\alpha \text{ if and only if } \beta \in \mathbf{K} \text{ and either } \alpha < \alpha \vee \beta \text{ or } \vdash \alpha.$$

[63] To be more accurate we must note that Observations 3.4.5 and 3.4.6 are indeed, partially, a consequence of [Gro88, Theorems 3 and 4] (see Observation A.2.3 (of the present thesis)). More precisely, the axiomatic characterization of the E.E.-based contractions can be derived from the combination of [Gro88, Theorems 3 and 4] and [Gär88, Theorems 3.2 – 3.5] (see Observations 2.4.4 – 2.4.7 (of the present thesis)). Gärdenfors [Gär88] proved, in this way, a weaker version of these two results which is stated in [Gär88, Theorem 4.30]. A more detailed explanation of this remark is presented in Section A.2 (of the Appendix).

It follows immediately from the two observations above that the class of epistemic entrenchment-based contractions coincides with the class of AGM contractions which satisfy (also) the supplementary AGM postulates for contraction. Furthermore, from those two results we can yet conclude that there is a one-to-one correspondence between the class of epistemic entrenchment relations and the class of the contraction functions which satisfy the basic and the supplementary postulates for contraction. This fact is more formally stated in the following corollary.

Corollary 3.4.7 ([GM88, Corollary 6]) *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $-$ *be a contraction function on* $\mathbf{K}$ *that satisfies all the basic and supplementary AGM postulates for belief set contraction. Then:*

(i) If $\leq$ *is the ordering determined from* $-$ *by* $(C_{\leq})$ *then the* $\leq$*-based contraction function* $-_{\leq}$*, determined by* $(C_{-_{\leq}})$*, is identical to* $-$*.*

(ii) If $-$ *coincides with the* $\leq$*-based contraction function* $-_{\leq}$*, determined by* $(C_{-_{\leq}})$*, then the epistemic entrenchment relation that is obtained from* $-$ *by means of* $(C_{\leq})$ *is identical to* $\leq$*.*

We close this section with the following list of results concerning some properties that are satisfied by any epistemic entrenchment relation and which will be useful further ahead.

Observation 3.4.8 (see e.g. [Han99b, Observation 2.75]) *Let* $\leq$ *be a relation that satisfies transitivity (i.e., if* $\alpha \leq \beta$ *and* $\beta \leq \delta$*, then* $\alpha \leq \delta$*) and let* $<$ *be defined by* $\alpha < \beta$ *if and only if* $\alpha \leq \beta$ *and not* $\beta \leq \alpha$*. Then* $\leq$ *also satisfies:*

1. *If* $\alpha \leq \beta$ *and* $\beta < \delta$*, then* $\alpha < \delta$
2. *If* $\alpha < \beta$ *and* $\beta \leq \delta$*, then* $\alpha < \delta$
3. *If* $\alpha < \beta$ *and* $\beta < \delta$*, then* $\alpha < \delta$ *(quasi-transitivity)*

Observation 3.4.9 ([GR95, Lemma 4.2.1-(i)]) *Let* $\leq$ *be a relation that satisfies (EE1) and (EE2). Then it also satisfies:*

If $\vdash \alpha \leftrightarrow \alpha'$ *and* $\vdash \beta \leftrightarrow \beta'$*, then* $\alpha \leq \beta$ *if and only if* $\alpha' \leq \beta'$*. (Intersubstitutivity)*[64]

Observation 3.4.10 ([Foo90]) [65]

Let $\leq$ *be a relation that satisfies (EE1), (EE2) and (EE3). Then it also satisfies: If* $\delta < \alpha$ *and* $\delta < \beta$*, then* $\delta < \alpha \wedge \beta$*. (Conjunction up)*[66]

[64] The denomination *intersubstitutivity* has been introduced in [Han99b, Observation 2.92]. In [GR95, Lemma 4.2.1-(i)] this property has been presented with a slightly different formulation and has been there called *extensionality*.

[65] This result is an immediate consequence of [GM88, Lemma 3 – (i) and (ii)] (which appear in the present thesis as Observation 3.4.3 – (i) and (ii)).

[66] The designation *conjunction up* has been introduced in [Rot92c], where this property is also thoroughly motivated and explained.

Observation 3.4.11 ([Foo90]) *Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE5). If $\alpha \in Cn(\emptyset)$ then for all $\delta \in \mathcal{L} \setminus Cn(\emptyset)$, $\delta < \alpha$, where $<$ denotes the strict part of $\leq$.*

Proof. A proof for this observation, which differs from the one presented in [Foo90], can be found in page 165. ∎

Observation 3.4.12 [67] *Let $\mathbf{K}$ be a belief set and $\leq$ be a relation that satisfies (EE1) and (EE4). Then $\alpha \notin \mathbf{K}$ if and only if for all $\beta \in \mathbf{K}$, $\alpha < \beta$, where $<$ denotes the strict part of $\leq$.*

Proof. A proof for this observation can be found in page 165. ∎

Lemma 3.4.13 *Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE3) and let $<$ denote the strict part of $\leq$. Then the following statements hold:*

(i) If $\alpha < \beta \vee \delta$ and $\alpha < \varepsilon \vee \neg\beta$, then $\alpha < \varepsilon \vee \delta$.

(ii) If $\alpha < \alpha \vee \neg\beta$ then $\beta \leq \alpha$.

(iii) If $\delta \leq \alpha$, $\alpha < \alpha \vee \beta$ and $\delta < \delta \vee \neg\alpha$, then $\delta < \delta \vee \beta$.

Proof. A proof for this lemma can be found in page 166. ∎

[67] We notice that the left-to-right part of this statement is a slightly stronger version of [Han99b, Observation 2.49 − 1.].

Chapter 4

Interrelations Among Different Models of AGM Contractions

In the previous chapter we have presented several constructive models for AGM contractions as well as for AGM contractions satisfying the basic and the supplementary AGM postulates for contraction (of belief sets). The present chapter is dedicated to the exposition of some of the results so far presented in the literature concerning the obtention of direct proofs of the equivalence between some of such explicit definitions of contraction functions. In that respect we present more thoroughly (namely in Sections 4.1 and 4.2) the interrelations (between constructions) which involve those classes of contraction functions that we will generalize in the latter chapters of this thesis.

We start by clarifying, by means of the following two observations, that the different ways of explicitly obtaining AGM contraction functions and AGM contractions which satisfy the supplementary AGM postulates for contraction, presented in the previous chapter, can essentially be divided in two classes, each of them containing a set of equivalent definitions.

Observation 4.0.1 *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then the following statements are equivalent:*

1. $-$ *satisfies the basic AGM postulates for contraction, i.e closure, inclusion, vacuity, success, extensionality and recovery.*
2. $-$ *is a partial meet contraction.*
3. $-$ *is a smooth (or saturated) kernel contraction.*

Observation 4.0.2 *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then the following statements are equivalent:*

1. $-$ *satisfies the basic and the supplementary AGM postulates for contraction, i.e closure, inclusion, vacuity, success, extensionality, recovery, conjunctive overlap and conjunctive inclusion.*
2. $-$ *is a transitively relational partial meet contraction.*

3. $-$ *is a safe contraction based on a regular and virtually connected hierarchy.*

4. $-$ *is a system of spheres-based contraction.*

5. $-$ *is an epistemic entrenchment-based contraction.*

Notice that the above observations are immediate consequences of the results presented in the previous chapter. More precisely, on the one hand Observation 4.0.1 follows from Observations 3.1.9, 3.2.7 and 3.2.8 while, on the other hand, Observation 4.0.2 is a consequence of Observations 3.1.9, 3.1.10, 3.2.20, 3.3.9, 3.4.5 and 3.4.6.

As remarked above, all the equivalences stated in Observations 4.0.1 and 4.0.2 follow trivially from the axiomatic characterizations that we have introduced in the previous chapter, for each of the contraction functions under consideration. However in the literature, some equivalences between some of those different definitions of contraction functions have been proven to hold in a direct way. In what follows we indicate the references where some of those proofs have been originally presented.

In [AM86] Alchourrón and Makinson have presented an explicit one-to-one correspondence between the class of transitively relational partial meet contractions and the class of safe contractions based on a regular and virtually connected hierarchy, when considering only contraction functions on theories that are finite modulo logical equivalence.

The relationship between the transitively relational partial meet contractions and the system of spheres-based contractions has been thoroughly investigated in [Gro88]. In Section 4.1 we present the main tools and arguments used by Grove, in that paper, to show in a direct way (rather than only by means of axiomatic characterizations of such functions) that the transitively relational partial meet contractions and the system of spheres-based contractions define precisely the same class of contraction functions.

A direct proof of the fact that any transitively relational partial meet contractions is an epistemic entrenchment-based contraction and vice versa has been presented by Rott in [Rot91].

Latter, in [Rot92b], Rott has also shown in a direct way that the class of safe contractions coincides with the class of epistemic entrenchment-based contractions.

Finally, by combining some results of [Gro88], [Gär88] and [PW95] it is possible to establish a direct way of showing that the class of epistemic entrenchment-based contractions coincides with the class of system of spheres-based contractions. In Section 4.2 we present such a direct proof for that fact.

All the above mentioned interconnections relate two different kinds of contraction functions that satisfy the basic and the supplementary AGM postulates for contraction. Now we make some brief remarks concerning some analogous results concerning some different definitions of AGM contraction functions which do not necessarily satisfy the supplementary AGM postulates for contraction. In this respect firstly we recall that, as mentioned in Remark 3.2.14, it follows immediately from the definitions of saturated kernel contraction and of safe contraction presented in [Han94, Han99b] (see Definitions 3.2.6 and 3.2.13) that every safe contraction function is a smooth (or saturated) kernel contraction. Secondly we remind that, according to Observation 3.2.8, a contraction function on a belief set is a smooth (or saturated) kernel contraction on **K** if and only if it is a partial meet contraction. Furthermore, we note that in [Han94] the proof of the left-to-right part of the latter mentioned equivalence is made in a direct way, however, the right-to-left part is there proven by means of some known axiomatic characterizations for those functions.

In Figures 4.1 and 4.2 we illustrate, by means of a diagram, those of the, above mentioned, equivalences between different models for belief set contraction which have so far been proven to hold, in a direct way. Furthermore, in such diagrams we also indicate the references where each of those equivalences have been (directly) shown to hold. Figure 4.1 presents the mentioned interconnections between different constructions/definitions for AGM contractions and Figure 4.2 plays the same role but concerning contraction functions which satisfy the basic and the supplementary AGM postulates for contraction.

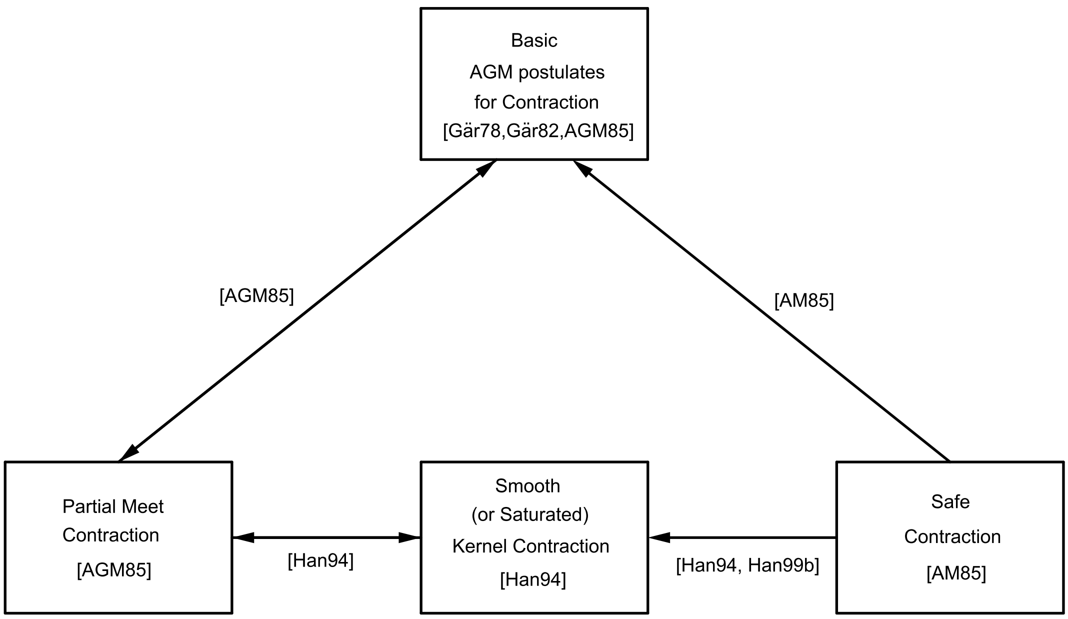

Figure 4.1: Diagram of some of the (so far) explicitly proven equivalences between different modellings of AGM contraction functions.

Having in mind what are the goals of this work, we devote the following two sections to the exposition of direct ways of showing that the class of transitively relational partial meet contractions coincides with the class of system of spheres-based contractions and that the classes of epistemic entrenchment-based contractions and of system of spheres-based contractions are identical. Hence, in Section 4.1 we shall show in a direct way that every transitively relational partial meet contraction is a system of spheres-based contraction and vice versa, and in Section 4.2 we will prove explicitly that every epistemic entrenchment-based contraction is a system of spheres-based contraction and vice versa.

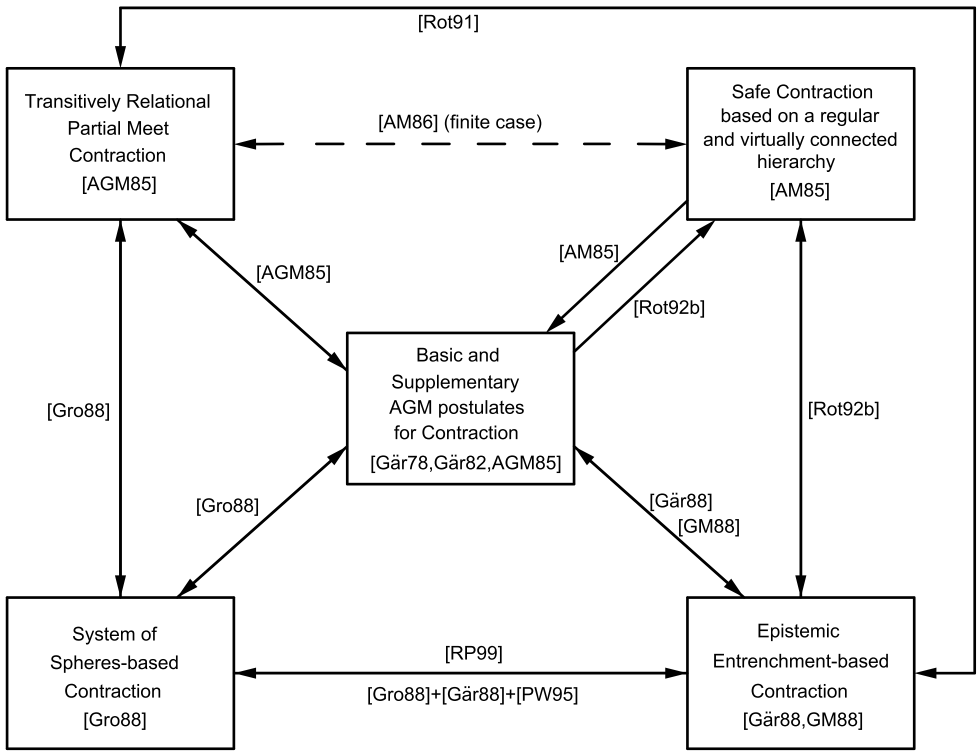

Figure 4.2: Diagram of some of the (so far) explicitly proven equivalences between different modellings of AGM contraction functions which satisfy (additionally) the supplementary AGM postulates for contractions.

4.1 Possible Worlds Semantics for Partial Meet Contraction and Interrelation Between Partial Meet Contractions and System of Spheres-based Contractions

As we have mentioned in Section 3.3, Grove [Gro88] has introduced the system of spheres-based contractions (cf. Definition 3.3.7) with the intention of providing an (alternative) explicit way of obtaining contraction functions that satisfy the basic and supplementary AGM postulates for contraction.

Observation 3.3.9 assures that Grove's goal has been achieved. Furthermore, as we have stated in the end of the above mentioned section, an immediate consequence of that observation is the fact that the class of system of spheres-based contractions coincides with the class of transitively relational partial meet contractions.

In the present section, which is essentially based in [Gro88, Sec. 4] and [Han99b, Sec. 3.23], we will start by presenting a noteworthy result consisting of a characterization of the partial meet contractions in general (rather than only the ones that are transitively relational) in terms of possible worlds. Such result is often regarded as providing a (possible worlds) semantics for partial meet contraction. Afterwards, based on such *possible worlds-based view* of a partial meet contraction function, we shall expose in a direct way that a contraction function is a transitively relational partial meet contraction if and only if it is a system of spheres-based contraction. That is, we will expose, without making use of any axiomatic characterizations for those contraction functions, that those two constructions are simply two alternative methods for constructing the same (class of) contraction functions.

First of all we recall the main ideas underlying the concept of partial meet contraction and the intuition behind the construction of contraction functions using the possible worlds approach. On the one hand, the partial meet contraction of a belief set $\mathbf{K}$ by a non-tautological sentence α is the result of the intersection of a family of remainders chosen, by means of a selection function (see Definition 3.1.5), from among all the elements of the remainder set $\mathbf{K}\perp\alpha$ (see Definition 3.1.1). On the other hand, in a possible worlds-based approach for belief set contraction, the contraction of $\mathbf{K}$ by α, consists of the intersection of a family of possible worlds which is the union of $\|\mathbf{K}\|$ with a subset of $\|\neg\alpha\|$ (i.e. with a set composed of some selected $\neg\alpha$-worlds).

Having the above paragraph in mind, in order to show how a partial meet contraction can be obtained by means of a possible worlds-based method for constructing contractions, we start by exposing, in the following observation, the very tight interconnection between the sets $\mathbf{K}\perp\alpha$ and $\|\neg\alpha\|$.

Observation 4.1.1 (Grove's bijection ([Gro88, Sec. 4])) *Let $\mathbf{K}$ be a belief set and α be a sentence such that $\alpha \in \mathbf{K}$. Then:*

1. *If $X \in \mathbf{K}\perp\alpha$, then there is some $W \in \|\neg\alpha\|$ such that $\|X\| = \|\mathbf{K}\| \cup \{W\}$ (and, therefore, it follows from Observations 3.1.4 and 3.3.3-(a) that $X = Th(\|\mathbf{K}\| \cup \{W\})$).*

2. *If $W \in \|\neg\alpha\|$, then $X = Th(\|\mathbf{K}\| \cup \{W\}) \in \mathbf{K}\perp\alpha$ and $\|X\| = \|\mathbf{K}\| \cup \{W\}$.*

3. *The function*

$$\begin{array}{rcl} g : \|\neg\alpha\| & \to & \mathbf{K}\perp\alpha \\ W & \mapsto & Th(\|\mathbf{K}\| \cup \{W\}) \end{array}$$

   is a bijection.

Throughout this text we will refer to the bijection introduced in the above observation (the function g) as *Grove's bijection*. Such one-to-one correspondence between the elements of $\mathbf{K}\perp\alpha$ and the possible worlds in $\|\neg\alpha\|$ leads to a possible alternative definition for the remainder set $\mathbf{K}\perp\alpha$ (see Definition 3.1.1) which we present in the following remark.

Remark 4.1.2 *Let $\mathbf{K}$ be a belief set, α be a sentence such that $\alpha \in \mathbf{K}$ and X be set of sentences. Then, it follows from the Observation 4.1.1 that $X \in \mathbf{K}\perp\alpha$ if and only if there is some $W \in \|\neg\alpha\|$ such that $X = Th(\|\mathbf{K}\| \cup \{W\})$ (and $\|X\| = \|\mathbf{K}\| \cup \{W\}$).*

Having this in mind we observe that the remainder set $\mathbf{K}\bot\alpha$ *(see Definition 3.1.1), can be defined in the following alternative way:*[68]

$$\mathbf{K}\bot\alpha = \begin{cases} \{Th(\|\mathbf{K}\| \cup \{W\}) : W \in \|\neg\alpha\|\} & \text{, if } \alpha \in \mathbf{K} \\ \{\mathbf{K}\} & \text{, if } \alpha \notin \mathbf{K}. \end{cases}$$

From another perspective we can say that, given a subset of $\mathcal{M}_{\mathcal{L}}$ *of the form*

$$\|\mathbf{K}\| \cup \mathcal{W},$$

where $\mathcal{W}$ *is a singleton set such that* $\mathcal{W} \cap \|\mathbf{K}\| = \emptyset$*, there is some* $\alpha \in \mathbf{K}$ *such that* $X = Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\bot\alpha$ *and* $\|X\| = \|\mathbf{K}\| \cup \mathcal{W}$ *(in fact this holds for any* $\alpha \in \mathbf{K}$ *such that* $\mathcal{W} \subseteq \|\neg\alpha\|$*).*

Given a belief set $\mathbf{K}$ *and a sentence* $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$*, in Figure 4.3 we present a possible graphical representation of the set of possible worlds for a remainder* $H \in \mathbf{K}\bot\varphi$*. Moreover, we notice that, according to Grove's bijection, in Figure 4.3, each point (i.e. world) of the region* $\|\neg\varphi\|$ *corresponds to one and only one set (i.e. remainder) in* $\mathbf{K}\bot\varphi$*.*

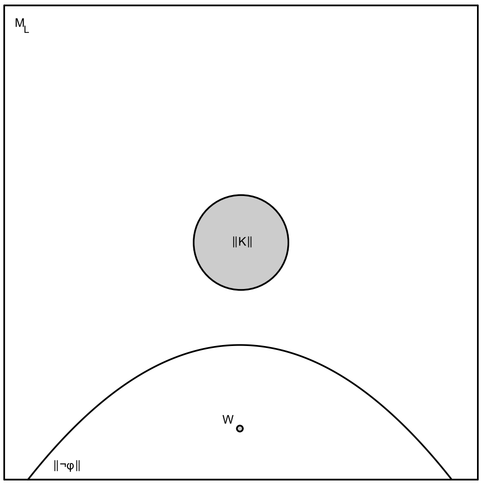

Figure 4.3: Possible graphical representation of the set $\|\mathbf{K}\| \cup \{W\}$, where $\mathbf{K}$ is a belief set, $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$ and $W \in \|\neg\varphi\|$. The set of sentences H consisting of the intersection of all the worlds in the shaded region belongs to $\mathbf{K}\bot\varphi$ and $\|H\| = \|\mathbf{K}\| \cup \{W\}$.

The above remark evidences that, given a sentence $\alpha \in \mathbf{K}$, a selection of some of the remainders in $\mathbf{K}\bot\alpha$ corresponds to a selection of some of the worlds in $\|\neg\alpha\|$ and vice versa. Having this in mind and recalling that our first goal in this section is to show how the partial meet contractions can be constructed by means of a possible worlds-based approach, we introduce in the following definition the concept of *propositional selection function*. Further below such functions shall play in the construction of partial meet contractions as intersections of families of possible worlds the same role that is played by the selection functions in the definition of partial meet contractions (Definition 3.1.6).

[68] Notice that if $\alpha \in Cn(\emptyset)$ then $\{Th(\|\mathbf{K}\| \cup \{W\}) : W \in \|\neg\alpha\|\} = \emptyset$.

Definition 4.1.3 ([Han99b, Definition 3.65]) *Let $\mathcal{M}$ be a proposition. A* propositional selection function *for $\mathcal{M}$ is a function f such that for any sentence α:*

(i) $f(\|\alpha\|) \subseteq \|\alpha\|$.

(ii) *If $\|\alpha\| \neq \emptyset$ then $f(\|\alpha\|) \neq \emptyset$.*

(iii) *If $\mathcal{M} \cap \|\alpha\| \neq \emptyset$, then $f(\|\alpha\|) = \mathcal{M} \cap \|\alpha\|$.*

Finally we are in a position to present the following observation, which illustrates how the partial meet contractions can be constructed following a possible worlds-based approach.

Observation 4.1.4 ([Gro88]) *Let $\mathbf{K}$ be a belief set. An operation $-$ on $\mathbf{K}$ is a* partial meet contraction *if and only if there is a propositional selection function f for $\|\mathbf{K}\|$ such that for any sentence α:*

$$\mathbf{K} - \alpha = Th(\|\mathbf{K}\| \cup f(\|\neg\alpha\|)).$$

It is worth noticing that, from a different perspective we can say that the above observation provides a possible worlds semantics to the partial meet contraction operations, in the sense that it illustrates how the partial meet contractions can be described in terms of possible worlds.

A straightforward corollary of Observation 4.1.4 is the following characterization of maxichoice contractions and of the full meet contraction (Definition 3.1.7) in terms of possible worlds:

Corollary 4.1.5 ([Gro88]) *Let $\mathbf{K}$ be a belief set and f be a propositional selection function for $\|\mathbf{K}\|$. If $-$ is the contraction function on $\mathbf{K}$ such that for any sentence α:*

$$\mathbf{K} - \alpha = Th(\|\mathbf{K}\| \cup f(\|\neg\alpha\|))$$

then the following statements hold:

1. *the function $-$ is a maxichioce contraction if and only if f is such that $f(\|\alpha\|)$ contains at most one element, i.e., $f(\|\alpha\|) \subseteq \|\alpha\|$ is either the empty set (whenever $\|\alpha\| = \emptyset$) or a singleton set (whenever $\|\alpha\| \neq \emptyset$), for all sentences α such that $\|\mathbf{K}\| \cap \|\alpha\| = \emptyset$.*

2. *If f is the propositional selection function such that $f(\|\alpha\|) = \|\alpha\|$, for all sentences α such that $\|\mathbf{K}\| \cap \|\alpha\| = \emptyset$, then $-$ is the full meet contraction.*

Next we make some remarks, which complement the above result, concerning the possible worlds semantics for maxichoice contraction and for the full meet contraction, more precisely, in what follows we shall describe the composition of the sets $\|\mathbf{K} - \varphi\|$, when $-$ is either a maxichoice contraction or the full meet contraction on the belief set $\mathbf{K}$ and φ is an arbitrary sentence.

Regarding maxichoice contractions we notice that, given a belief set $\mathbf{K}$ and a contraction function $-$ on $\mathbf{K}$, combining the above corollary with Observations 3.3.3 $-(a)$ and 4.1.1 $-$ 2., we can conclude that $-$ is a maxichoice contraction if and only if the following conditions are satisfied:

(a) $\mathbf{K} - \varphi = Th(\|\mathbf{K}\|) = \mathbf{K}$ and, therefore, $\|\mathbf{K} - \varphi\| = \|\mathbf{K}\|$, for all sentences φ such that $\vdash \varphi$ or $\varphi \notin \mathbf{K}$.

(b) $\mathbf{K} - \varphi = Th(\|\mathbf{K}\| \cup f(\|\neg\varphi\|))$ and $\|\mathbf{K} - \varphi\| = \|\mathbf{K}\| \cup f(\|\neg\varphi\|)$, for all sentences φ such that $\nvdash \varphi$ and $\varphi \in \mathbf{K}$, where f is a propositional selection function for $\|\mathbf{K}\|$ such that $f(\|\neg\varphi\|) = \{W\}$, with $W \in \|\neg\varphi\|$.

Figure 4.3 can be seen as a possible graphical representation of the set $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup f(\|\neg\varphi\|)$ (which corresponds to the shaded region in the mentioned figure), where $\mathbf{K}$ is a belief set, $-$ is a maxichoice contraction function on $\mathbf{K}$, $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$, $W \in \|\neg\varphi\|$ and f is a propositional selection function for $\|\mathbf{K}\|$ such that $f(\|\neg\varphi\|) = \{W\}$.

In what concerns the *full meet contraction*, we present in the following remark the composition of the set $\|\mathbf{K} \sim \varphi\|$, where $\mathbf{K}$ is a belief set, $\sim$ is the full meet contraction on $\mathbf{K}$ and φ is an arbitrary sentence.

Remark 4.1.6 *A contraction function $\sim$ is the full meet contraction on a belief set $\mathbf{K}$ if and only if, for any $\varphi \in \mathcal{L}$, $\mathbf{K} \sim \varphi = Cn(\mathbf{K} \sim \varphi)$ and:*

$$\|\mathbf{K} \sim \varphi\| = \begin{cases} \|\mathbf{K}\| \cup \|\neg\varphi\| & \text{, if } \varphi \in \mathbf{K} \setminus Cn(\emptyset) \\ \|\mathbf{K}\| & \text{, otherwise} \end{cases}.$$

Proof. A proof for this remark can be found in page 171. ∎

Figure 4.4 contains a graphical representation of the set $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$, where $\mathbf{K}$ is a belief set, $\sim$ is the full meet contraction function on $\mathbf{K}$ and $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. We also notice that, in these conditions, since $\mathbf{K} \sim \varphi$ is a belief set, it follows from Observation 3.3.3-(a) that, $\mathbf{K} \sim \varphi$ consists of the intersection of all the worlds included in the shaded region, i.e., $\mathbf{K} \sim \varphi = Th(\|\mathbf{K}\| \cup \|\neg\varphi\|)$.

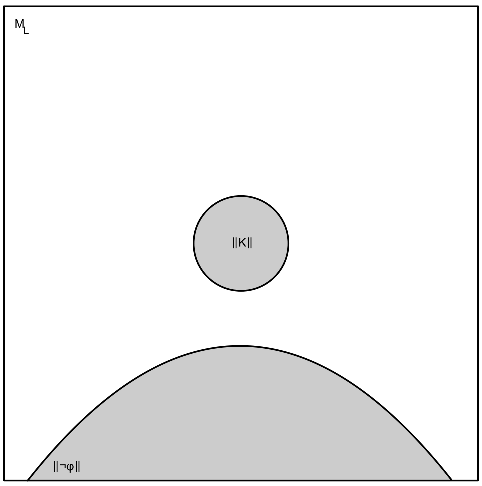

Figure 4.4: Graphical representation of the set $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$, where $\mathbf{K}$ is a belief set, $\sim$ is the full meet contraction function on $\mathbf{K}$ and $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. The set $\mathbf{K} \sim \varphi$ consists of the intersection of all the worlds included in the shaded region.

At this point it is interesting to notice that the following observation, originally presented in [AM82], emerges as an immediate consequence of the above remark.

Observation 4.1.7 ([AM82, Observation 2.1]) *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *be the operator of full meet contraction on* $\mathbf{K}$*. Then for all sentences* $\varphi \in \mathcal{L}$*:*

$$\mathbf{K} \sim \varphi = \begin{cases} \mathbf{K} \cap Cn(\neg\varphi) & ,\text{if } \varphi \in \mathbf{K} \\ \mathbf{K} & ,\text{if } \varphi \notin \mathbf{K} \end{cases}.$$

Proof. A proof for this observation, which differs from the one presented in [AM82], can be found in page 172. ■

Above we have presented explicitly the composition of the set of possible worlds for $\mathbf{K} - \varphi$ when $-$ is either a maxichoice contraction or the full meet contraction on the belief set $\mathbf{K}$ and φ is an arbitrary sentence. Next we consider the general case of (any) *partial meet contractions*. That is, in what follows we clarify how the set $\|\mathbf{K} - \varphi\|$ is formed when $-$ is an arbitrary partial meet contraction function on $\mathbf{K}$.

Observation 4.1.8 (Based on [Gro88]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a partial meet contraction on* $\mathbf{K}$*. Then, for any* $\varphi \in \mathcal{L}$*:*

1. *If* $\vdash \varphi$ *or* $\varphi \notin \mathbf{K}$*, then* $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\|$
2. *If* $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$*, then* $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$*, where* $\mathcal{M}$ *is a non-empty subset of* $\|\neg\varphi\|$*.*

Proof. A proof for this observation can be found in page 173. ■

Figure 4.5 contains a possible graphical representation of the set $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$, where $-$ is a partial meet contraction on the belief set $\mathbf{K}$, $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$ and $\mathcal{M}$ is a non-empty subset of $\|\neg\varphi\|$.

We have, thus, detailedly described partial meet contractions in terms of possible worlds. Now we move to our second and last goal of the present section, namely the clarification of the interrelation between partial meet contractions and the system of spheres-based contractions.

In fact, in the end of Section 3.3 we have already anticipated that it follows immediately from Observations 3.3.9, 3.1.9 and 3.1.10 that the class of system of spheres-based contractions coincides with the class of transitively relational partial meet contractions. However, we wish to show here how it is possible to conclude in a direct way (rather than by means of representation theorems for each of those two kinds of contractions) that a contraction function is a transitively relational partial meet contraction if and only if it is a system of spheres-based contraction.

In this regard we start by noticing that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ it can be easily shown that the function f from $\{\|\alpha\| : \alpha \in \mathcal{L}\}$ to $\mathcal{P}(\mathcal{M}_{\mathcal{L}})$ defined by:[69]

$$f(\|\alpha\|) = \begin{cases} f_{\mathbb{S}}(\alpha) & ,\text{if } \|\alpha\| \neq \emptyset \\ \emptyset & ,\text{if } \|\alpha\| = \emptyset \end{cases},$$

[69] Such a function f is called a *sphere-based propositional selection function* in [Han99b, Definition 3.73].

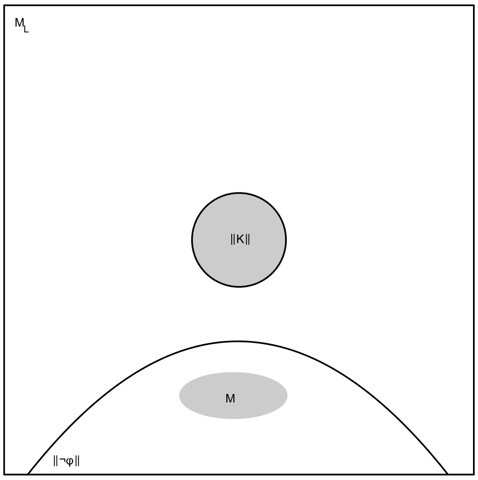

Figure 4.5: Possible graphical representation of the set $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$, where $-$ is a partial meet contraction on the belief set $\mathbf{K}$, $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$ and $\mathcal{M}$ is a non-empty subset of $\|\neg\varphi\|$.

for any sentence α, is a propositional selection function (cf. Definition 4.1.3) such that for any sentence φ:

$$\mathbf{K}-_{\mathbb{S}}\varphi = Th(\|\mathbf{K}\| \cup f(\|\neg\varphi\|)).$$

Therefore we can conclude immediately from Observation 4.1.4 that the $\mathbb{S}$-based contraction $-_{\mathbb{S}}$ is a partial meet contraction.

The above reasoning proves, in a direct way, that every system of spheres-based contraction is a partial meet contraction. However, the converse implication does not hold, i.e., not every partial meet contraction is a system of spheres-based contraction. Nevertheless, it is in that fact that rests one of the main virtues of Grove's system of spheres-based contractions. Indeed, the major achievement of Grove's construction rests in the fact that a contraction function on a belief set $\mathbf{K}$ is a system of spheres-based contraction if and only if it is a transitively relational partial meet contraction ([Gro88, Sec. 4]).

In what remains of the present section, based in [Gro88, Sec. 4], we will show that this indeed holds. More precisely, in what follows, making use of Grove's bijection (Observation 4.1.1-3), we will explicitly show how the systems of spheres can be used to construct transitively relational partial meet contractions and vice versa.

Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. We start by defining a relation $\sqsubseteq$ on $\mathbf{K}\Delta\mathcal{L} = \bigcup_{\varepsilon\in\mathcal{L}} \mathbf{K}\perp\varepsilon$[50]. Notice that if $\mathbf{K} = \mathcal{L}$ is the inconsistent belief set then $\mathbf{K} \notin \mathbf{K}\Delta\mathcal{L} = \mathbf{K}\Delta\mathbf{K} = \bigcup_{\varphi\in\mathbf{K}} \mathbf{K}\perp\varphi$. On the other hand, if $\mathbf{K}$ is a consistent belief set then $\mathbf{K}\perp\psi = \{\mathbf{K}\}$ for any $\psi \in \mathcal{L} \setminus \mathbf{K}$ and it follows that $\mathbf{K}\Delta\mathcal{L} = (\mathbf{K}\Delta\mathbf{K}) \cup \{\mathbf{K}\}$. Having the above in mind we first define $\sqsubseteq$ on the set $\mathbf{K}\Delta\mathbf{K} = \bigcup_{\varphi\in\mathbf{K}} \mathbf{K}\perp\varphi$. It follows from Grove's bijection (Observation 4.1.1-3) that $\mathbf{K}\Delta\mathbf{K} = \bigcup\{Th(\|\mathbf{K}\| \cup \{W\}) : W \in \mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|\}$. Now let G and H be two arbitrary elements of $\mathbf{K}\Delta\mathbf{K}$, then $G = Th(\|\mathbf{K}\| \cup \{W_1\})$ and $H = Th(\|\mathbf{K}\| \cup \{W_2\})$ for some $W_1, W_2 \in \mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$. In these conditions, the relation $\sqsubseteq$ is, roughly speaking, defined by stating that $G \sqsubseteq H$ if and only if W_2 is at least as *close* to $\|\mathbf{K}\|$ as W_1.

Formally this can be written in the following way:

$$G \sqsubseteq H \text{ iff } \bigcap\{\mathcal{M} \in \mathbb{S} : W_2 \in \mathcal{M}\} \subseteq \bigcap\{\mathcal{W} \in \mathbb{S} : W_1 \in \mathcal{W}\}^{70}.$$

As we have seen above, if $\mathbf{K}$ is the inconsistent belief set then $\mathbf{K}\Delta\mathcal{L} = \mathbf{K}\Delta\mathbf{K}$ and therefore the relation $\sqsubseteq$ is already defined on the whole set $\mathbf{K}\Delta\mathcal{L}$. On the other hand, if $\mathbf{K}$ is a consistent belief set then $\mathbf{K}\Delta\mathcal{L} = (\mathbf{K}\Delta\mathbf{K}) \cup \{\mathbf{K}\}$ and, in that case, (so that it is defined on all of $\mathbf{K}\Delta\mathcal{L}$) we add the following condition to the definition of the relation $\sqsubseteq$:

$$J \sqsubseteq \mathbf{K}, \text{ for all } J \in \mathbf{K}\Delta\mathcal{L}$$

Note that it follows from the conditions satisfied by $\mathbb{S}$ (see Definition 3.3.5) and the transitivity of the set inclusion relation ($\subseteq$) that the above defined relation $\sqsubseteq$ is a transitive (and connected) relation on $\mathbf{K}\Delta\mathcal{L}$. Hence, the selection function γ for $\mathbf{K}$ defined by $\gamma(\mathbf{K}\perp\alpha) = \{B \in \mathbf{K}\perp\alpha : B' \sqsubseteq B \text{ for all } B' \in \mathbf{K}\perp\alpha\}$ if $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$, and $\gamma(\mathbf{K}\perp\alpha) = \mathbf{K}$, if $\alpha \in Cn(\emptyset)$, is transitively relational over $\mathbf{K}$ and consequently the partial meet contraction generated by γ is a transitively relational partial meet contraction (see Definition 3.1.8). Moreover, it is easily verified that the transitively relational partial meet contraction thus obtained coincides with the $\mathbb{S}$-based contraction.

It is obvious that this process can be reversed, i.e., that starting with a transitive relation associated to a given transitively relational partial meet contraction we can obtain a system of spheres which determines the same contraction function as the original one. For a given belief set $\mathbf{K}$, such construction of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ from a marking-off relation $\sqsubseteq$ on $\mathbf{K}\Delta\mathcal{L}$, such that the $\mathbb{S}$-based contraction coincides with the transitively relational partial meet contraction based on the relation $\sqsubseteq$ (according to Definition 3.1.8), is described in more detail in [Gro88, Sec. 4][71].

From all the above we can conclude that the following observation holds:

Observation 4.1.9 ([Gro88, Sec. 4]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a contraction function on* $\mathbf{K}$*. Then* $-$ *is a* transitively relational partial meet contraction *if and only if it is a* system of spheres-based contraction*.*

Therefore, as remarked by Grove [Gro88, Sec. 4], the spheres' based model and the representations in terms of (transitively relational partial) meet contraction functions are *dual* to each other, in the sense that their quite distinct forms are essentially due to the fact that they use different ways of representing possible new theories. In the spheres modelling such representation is based on subsets of $\mathcal{M}_{\mathcal{L}}$ rather than actual theories as it is the case in [AGM85].

[70] Note that, for $i \in \{1,2\}$, $\bigcap\{\mathcal{M} \in \mathbb{S} : W_i \in \mathcal{M}\} \in \mathbb{S}$. Moreover, $\bigcap\{\mathcal{M} \in \mathbb{S} : W_i \in \mathcal{M}\}$ is, indeed, the smallest sphere in $\mathbb{S}$ containing the world W_i.

[71] In fact Grove's [Gro88] construction of a system of spheres from a marking-off relation consists of two steps: first ([Gro88, p. 160]) it is highlighted "that a system of spheres is really an ordering on the set of worlds" that satisfies certain properties or, more precisely, that a system of spheres induces an ordering (satisfying certain properties) on the set of worlds, and conversely. And afterwards ([Gro88, Sec. 4]) Grove's bijection is used to *translate* the marking-off relation $\sqsubseteq$ into an ordering on the set of worlds which satisfies the required conditions for it to *correspond* to a system of spheres.

Having all the above in mind we notice that Figure 3.2 can be seen as a representation of a transitively relational partial meet contraction. Indeed, given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a sentence $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$, according to Observation 4.1.9, there is a transitively relational partial meet contraction $-$ on $\mathbf{K}$ such that $\mathbf{K} - \varphi = \mathbf{K} -_{\mathbb{S}} \varphi = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi))$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based contraction on $\mathbf{K}$, and the indicated figure contains precisely a graphical representation of the set $\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)$, with $\mathbf{K}$, $\mathbb{S}$ and φ in the mentioned conditions.

4.2 Interrelation Between System of Spheres--based Contractions and Epistemic Entrenchment-based Contractions

In this section we will essentially present some results revealing the very close connection between epistemic entrenchment-relations and systems of spheres. In particular, we will show that such interconnection is strong to the point that, given a belief set $\mathbf{K}$, the class of the epistemic entrenchment-based contractions on $\mathbf{K}$ coincides with the class of the system of spheres-based contractions on $\mathbf{K}$.

Several results concerning such deep interconnection between systems of spheres and epistemic entrenchment relations have been presented in the literature devoted to the study and development of the AGM framework. In what follows we will briefly mention some of the main of such results. In [PW95, Theorem 6.1] Peppas and Williams presented an explicit condition, relating an epistemic entrenchment and a system of spheres, under which two such structures yield the same revision function. Having that result in mind it is also worth to remark here that combining some results of [Gro88] and [Gär88] it is possible to conclude that for any system of spheres there is an epistemic entrenchment relation such that the above mentioned condition holds and vice versa. Therefore, making use of all those results it can be shown that the class of system of spheres-based revisions coincides with the class of epistemic entrenchment-based revisions.[72]

Latter, in [RP99, Sec. 12], Rott and Pagnucco have shown that, given a belief set $\mathbf{K}$, for any system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ there is an epistemic entrenchment relation such that the $\mathbb{S}$-based contraction coincides with the $\leq$-based contraction on $\mathbf{K}$, and vice versa. However, in the mentioned paper, the authors have not presented explicitly a condition relating a system of spheres and an epistemic entrenchment relation which is necessary and sufficient for the contraction functions based on each of those two structures to be identical.

Our exposition in the present section results of a combination of the approaches of [PW95] and [RP99]. Nevertheless, however it is inspired by the mentioned references, the exposition that follows departs quite significantly from the presentations of each of those two papers and all the proofs presented below are original (as well as the

[72] The revision functions here named *system of spheres-based revisions* (resp. *epistemic entrenchment-based revisions*) are the ones that can be obtained from a system of spheres-based contraction (see Definition 3.3.7) (resp. an epistemic entrenchment-based contraction (see Definition 3.4.4)), by means of the Levi identity (equation (2.1)).

results to which such proofs correspond) and made explicitly (in the sense that they do not make use of any of the results presented in the above mentioned works). As mentioned above our main goal in the present section is to prove in a direct way that the class of system of spheres-based contractions coincides with the class of epistemic entrenchment-based contractions. We anticipate here that in what follows such proof will be, loosely speaking, composed of two parts: On the one hand a condition relating a system of spheres $\mathbb{S}$ and an epistemic entrenchment relation $\leq$ is presented which is necessary and sufficient to assure that the $\mathbb{S}$-based contraction and the $\leq$-based contraction are identical. On the other hand it is shown that, given an arbitrary system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ such that $\mathbb{S}$ and $\leq$ satisfy the above mentioned condition and vice versa.

Finally, before moving on to the presentation of such proof, it is convenient to emphasise here also that most of the results that we shall introduce in the present section shall play a fundamental role further ahead (namely, in Chapter 8).

Now, as the first step towards the obtention of the proof pretended in the present section we present, in the following observation, a way of defining an epistemic entrenchment relation by means of a system of spheres:

Observation 4.2.1 ([Gro88, Gär88]) *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. If* $\leq$ *is the binary relation on* $\mathcal{L}$ *defined in the following way:*

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \textit{ iff either } \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta} \textit{ or } \vdash \beta, \qquad (\leq -\mathbb{S})$$

then $\leq$ *is an epistemic entrenchment relation with respect to* $\mathbf{K}$*, i.e.,* $\leq$ *satisfies conditions (EE1)-(EE5).*

Given a belief set $\mathbf{K}$, Figure 4.6 contains a representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and illustrates a situation in which $\alpha, \beta \in \mathbf{K} \setminus Cn(\emptyset)$ are two sentences such that $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ (moreover, $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$) and, therefore, it holds that $\alpha \leq \beta$ (moreover, $\alpha < \beta$), if $\leq$ is the E.E. relation defined from $\mathbb{S}$ by condition ($\leq -\mathbb{S}$).

In addition to being, as we have just seen, suitable to construct an E.E. relation by means of a given system of spheres, we will see further ahead that condition ($\leq -\mathbb{S}$) is a necessary and sufficient condition for the $\mathbb{S}$-based contraction to coincide with the $\leq$-based contraction.

In order to show that, we start by presenting some results which introduce some conditions equivalent to ($\leq -\mathbb{S}$).

Observation 4.2.2 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then condition (*$\leq -\mathbb{S}$*) is satisfied if and only if*

$$\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset),\ \alpha \leq \beta \textit{ iff } \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}. \qquad (\leq -\mathbb{S}')$$

Proof. A proof for this observation can be found in page 173. ∎

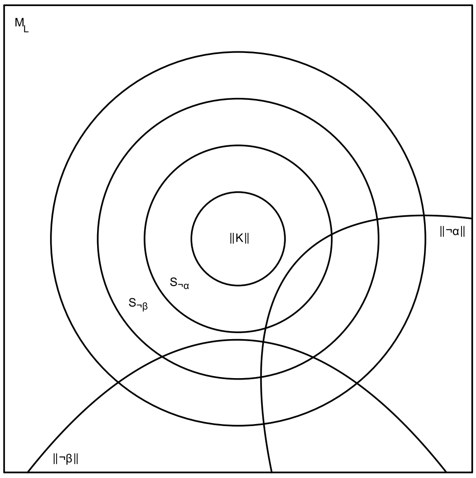

Figure 4.6: Schematic representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ (where $\mathbf{K}$ is a belief set) illustrating a possible representation for the sets $\|\neg\alpha\|$ and $\|\neg\beta\|$ in a situation when $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$ are such that $\alpha \leq \beta$ (more precisely $\alpha < \beta$), where $\leq$ is the E.E. relation defined from the S.S. $\mathbb{S}$ by means of condition ($\leq -\mathbb{S}$).

Remark 4.2.3 *According to Observation 4.2.2, condition ($\leq -\mathbb{S}'$) is equivalent to condition ($\leq -\mathbb{S}$) when $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$ and $\leq$ is an epistemic entrenchment relation with respect to $\mathbf{K}$, for a given belief set $\mathbf{K}$.*

Nevertheless, it is worth noticing that condition ($\leq -\mathbb{S}'$) can not be used (instead of condition ($\leq -\mathbb{S}$)) for the purpose of defining (as it was done in Observation 4.2.1) an epistemic entrenchment relation by means of a system of spheres. This is because if we used that conditions (rather than ($\leq -\mathbb{S}$)), we would only obtain a binary relation in $\mathcal{L} \setminus Cn(\emptyset)$. In other words, the binary relation thus defined would not comprise any information concerning tautologies (i.e., tautologies would not be in relation with any sentence).

The following observation introduces two more conditions which are equivalent to condition ($\leq -\mathbb{S}'$) and therefore, according to Observation 4.2.2, are also equivalent to condition ($\leq -\mathbb{S}$).

Lemma 4.2.4 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Then the following three conditions are equivalent:*

1. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha \leq \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ *(notice that this is condition ($\leq -\mathbb{S}'$)).*
2. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha < \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ *and* $\alpha =_{E.E.} \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.
3. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha < \beta \Rightarrow \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ *and* $\alpha =_{E.E.} \beta \Rightarrow \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.

Proof. A proof for this lemma can be found in page 174. ■

Now we introduce one more lemma concerning the interrelation between epistemic entrenchment orderings and systems of spheres which we will make use of, right afterwards, in the proof of a theorem that states the above mentioned connection between condition ($\leq -\mathbb{S}$) and the interrelation between E.E.-based contractions and S.S.-based contractions.

Lemma 4.2.5 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If* $\mathbb{S}$ *and* $\leq$ *satisfy condition* ($\leq -\mathbb{S}$)*, then for any* $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$ *and any* $\beta \in \mathcal{L}$ *it holds that*

$$\alpha < \alpha \vee \beta \text{ iff } f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|.$$

Proof. A proof for this lemma can be found in page 175. ■

We are now in a position to prove the above mentioned fact that condition ($\leq -\mathbb{S}$) is a necessary and sufficient condition for the $\mathbb{S}$-based contraction to coincide with the $\leq$-based contraction, as it is formally stated in the following theorem:

Theorem 4.2.6 [73] *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based contraction on* $\mathbf{K}$*,* $-_{\leq}$ *(cf. condition* ($C_{-_{\leq}}$) *in Definition 3.4.4), and the* $\mathbb{S}$*-based contraction on* $\mathbf{K}$*,* $-_{\mathbb{S}}$ *(cf. Definition 3.3.7), coincide, i.e.,*

$$\forall \varphi \in \mathcal{L},\ \mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi,$$

if and only if condition ($\leq -\mathbb{S}$) *is satisfied.*

Proof. A proof for this theorem can be found in page 175. ■

At this point, combining Observation 4.2.1 and Theorem 4.2.6, we can conclude that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, we can define an E.E. relation $\leq$ with respect to $\mathbf{K}$, such that the $\leq$-based contraction on $\mathbf{K}$ and the $\mathbb{S}$-based contraction on $\mathbf{K}$ coincide (and, consequently, at this stage we can already conclude that every system of spheres-based contraction is an epistemic entrenchment-based contraction).

In the sequence we will show that the converse construction is also possible. That is, given a belief set $\mathbf{K}$ and an E.E. relation $\leq$ with respect to $\mathbf{K}$, we will show how to define a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ such that the $\mathbb{S}$-based contraction on $\mathbf{K}$ is identical to the $\leq$-based contraction on $\mathbf{K}$.[74]

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[73]Notice that this theorem is equivalent to Theorem 6.1 of [PW95] - such equivalence is proven in Section *B.2* (of the Appendix). However, we must remark here that such proof makes use of the representation theorems (more precisely of the construction-to-postulates part of those theorems) for the system of spheres-based contractions and for epistemic entrenchment-based contractions. Therefore, since it is our goal in the present section to prove that the classes of system of spheres-based contractions and of epistemic entrenchment-based contractions coincide without making use of axiomatic characterizations for such functions, here we present a direct proof for this theorem (rather than using the mentioned equivalence between this result and Theorem 6.1 of [PW95]).

Notice also that the right-to-left part of this theorem is similar to Observation 3.76 of [Han99b].

[74]Notice that, alternatively, we could just show, based on the proof of [Gro88, Theorem 4], that a system of spheres $\mathbb{S}$ in these conditions indeed exists, without however presenting its explicit

We start by noticing that it follows from Theorem 4.2.6 that, to achieve our goal, it is enough to assure that the given E.E. relation $\leq$ and the system of spheres $\mathbb{S}$ that we shall define (based on it) satisfy condition ($\leq -\mathbb{S}$).

Now we proceed to the construction of such a system of spheres (from an E.E. relation).[75]

Let $\mathbf{K}$ be a belief set and $\leq$ be an E.E. relation with respect to $\mathbf{K}$.

To each sentence $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$ we associate a subset $\mathcal{W}_{\alpha_i}$ of $\mathcal{M}_{\mathcal{L}}$ (which, as we shall see shortly, will be the smallest sphere - in the system of spheres that we are about to present - intersecting $\neg\alpha_i$) defined by:

$$\mathcal{W}_{\alpha_i} = \|\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}\|. \tag{4.1}$$

In the following remark we highlight some facts concerning the composition of the set $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$[76] on the right-hand-side of the above equation.

Remark 4.2.7 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an E.E. relation with respect to* $\mathbf{K}$*. Then:*

(i) If $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$ *then* $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K} \setminus \{\alpha \in \mathcal{L} : \alpha \leq \alpha_i\}$*.*

(ii) If $\alpha_i \in \mathcal{L} \setminus \mathbf{K}$ *then* $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K}$*.*

Proof. A proof for this remark can be found in page 177. ■

Now, before using the sets $\mathcal{W}_{\alpha_i}$ to construct a system of spheres centred on $\|\mathbf{K}\|$, we introduce some results that characterize them and show how such sets are related among each other. Such results will play an important role in the rest of the present section.

Lemma 4.2.8 *Let* $\leq$ *be an E.E. relation with respect to a belief set* $\mathbf{K}$ *and* $\alpha_i, \alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$*. Then the following statements hold:*

(i) If $\alpha_i < \alpha_j$ *then* $\alpha_j \in Th(\mathcal{W}_{\alpha_i})$ *or, which is the same,* $\mathcal{W}_{\alpha_i} \subseteq \|\alpha_j\|$*.*

(ii) $\mathcal{W}_{\alpha_j} \cap \|\neg\alpha_j\| \neq \emptyset$*.*

Proof. A proof for this lemma can be found in page 178. ■

Lemma 4.2.9 *Let* $\leq$ *be an E.E. relation with respect to a belief set* $\mathbf{K}$ *and* $\alpha_i, \alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$*. Then the following statements hold:*

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construction by means of $\leq$. However, if we followed this alternative way for proving that fact, then we would not fully accomplish our main goal in the present section of proving that the classes of S.S.-based contractions and of E.E.-based contractions coincide, without making use of axiomatic characterization for such functions. This remark is explained and justified in further detail in Section B.2 (of the Appendix).

[75] A similar construction has been presented by Rott and Pagnucco in [RP99, Sec. 12]. However, there are some slight but noteworthy differences between their exposition and our presentation below.

[76] A set of sentences of the form $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$ has been called an *E.E.-cut* (or simply *cut*) in [Rot91, p. 159]. Compare also with the (alternative) notion of *cut* introduced in [Gro88, p. 165], which is *dual* to this one – cf. Observation A.2.4 of Section A.2 (of the Appendix).

(i) $\alpha_i =_{E.E.} \alpha_j$ *if and only if* $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$.

(ii) $\alpha_i < \alpha_j$ *if and only if* $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$.

Proof. A proof for this lemma can be found in page 178. ∎

Remark 4.2.10 *Let* $\leq$ *be an E.E. relation with respect to a belief set* $\mathbf{K}$ *and for any* $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$ *let* $\mathcal{W}_{\alpha_i}$ *be the set defined in* (4.1).

It follows from Lemma 4.2.8-(ii) that, for any $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$*, it holds that* $\mathcal{W}_{\alpha_i} \neq \emptyset$ *and, equivalently, we have that* $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$ *is a consistent set of sentences, for any* $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$.

Now consider the class $\mathbb{S}'$ *of subsets of* $\mathcal{M}_{\mathcal{L}}$ *defined by:*

$$\mathbb{S}' = \{\mathcal{W}_{\alpha_i} : \alpha_i \in \mathcal{L} \setminus Cn(\emptyset)\}.$$

The smallest (with respect to set inclusion) element in $\mathbb{S}'$ *is* $\mathcal{W}_{\perp}$*, where* $\perp$ *stands for an arbitrary contradiction. Indeed, it follows immediately from (EE2) that* $\perp \leq \delta$ *for any* $\delta \in \mathcal{L}$ *so, from Lemma 4.2.9, we can conclude that* $\mathcal{W}_{\perp} \subseteq \mathcal{W}_{\delta}$ *for any* $\delta \in \mathcal{L} \setminus Cn(\emptyset)$.

Having the above in mind, from (EE4) and Lemma 4.2.9 we can conclude that, if $\mathbf{K}$ *is a consistent belief set, then* $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\perp}$*, for any sentence* $\alpha_i \in \mathcal{L} \setminus \mathbf{K}$.

Moreover, when $\mathbf{K}$ *is a consistent belief set, if* $\alpha_i \in \mathcal{L} \setminus \mathbf{K}$ *then it follows from Remark 4.2.7-(ii) and equation* (4.1) *that* $\mathcal{W}_{\alpha_i} = \|\mathbf{K}\|$.

Finally, it follows from the above stated facts that, if $\mathbf{K}$ *is a consistent belief set, then* $\|\mathbf{K}\| = \mathcal{W}_{\perp}$ *(and is the smallest element in* $\mathbb{S}'$*).*

At last, in the following observation, given a belief set $\mathbf{K}$ and an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$, we show that, making use of the sets $\mathcal{W}_{\alpha_i}$, it is possible to construct a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that $\leq$ and $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$).

Observation 4.2.11 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an E.E. relation with respect to* $\mathbf{K}$*, and* $\mathbb{S}'$ *be the class of subsets of* $\mathcal{M}_{\mathcal{L}}$ *defined by:*

$$\mathbb{S}' = \{\mathcal{W}_{\alpha_i} : \alpha_i \in \mathcal{L} \setminus Cn(\emptyset)\},$$

where, for any $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$*,* $\mathcal{W}_{\alpha_i}$ *is the set defined in* (4.1)*, i.e.,*

$$\mathcal{W}_{\alpha_i} = \|\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}\|,$$

then the following statements hold:

(i) If $\mathbf{K} \neq \mathcal{L}$ *(i.e.,* $\mathbf{K}$ *is a consistent belief set), then the set* $\mathbb{S} = \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$.

(ii) If $\mathbf{K} = \mathcal{L}$*, then the set* $\mathbb{S} = \{\emptyset\} \cup \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$.

Moreover, in both cases $\mathbf{K} \neq \mathcal{L}$ *and* $\mathbf{K} = \mathcal{L}$*, it holds that the (respective) thus constructed system of spheres* $\mathbb{S}$ *and the given E.E. relation* $\leq$ *satisfy condition* ($\leq -\mathbb{S}$).

Proof. A proof for this observation can be found in page 179. ■

The main results of the present section can be summarized in the following way: Given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$, Theorem 4.2.6 states that condition ($\leq -\mathbb{S}$) is a necessary and sufficient condition for the $\leq$-based contraction on $\mathbf{K}$ and the $\mathbb{S}$-based contraction on $\mathbf{K}$ to be identical. On the other hand Observations 4.2.1 and 4.2.11 assert that, given an arbitrary system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ there is an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ such that condition ($\leq -\mathbb{S}$) is satisfied and vice versa, respectively. Thus, from all these results, we can immediately conclude that every system of spheres-based contraction is an epistemic entrenchment-based contraction and vice versa, as we formally state in the following corollary:

Corollary 4.2.12 *Let* $\mathbf{K}$ *be a belief set. An operation* $-$ *on* $\mathbf{K}$ *is an epistemic entrenchment-based contraction on* $\mathbf{K}$ *if and only if it is a system of spheres-based contraction on* $\mathbf{K}$*.*

We must note here that the equivalence that is established in the above corollary has already been stated as part of Obsevation 4.0.2, since, as we have clarified immediately after the mentioned observation, such result is an obvious consequence of the representation theorems for the epistemic entrenchment-based contraction (see Observations 3.4.5 and 3.4.6) and for the system of spheres-based contraction (see Observation 3.3.9). Nevertheless, it is worth emphasising that the results presented in this section allow us to conclude in a direct way (rather than by means of the above mentioned axiomatic characterizations) that the class of epistemic entrenchment-based contractions on a belief set $\mathbf{K}$ coincides with the class of system of spheres-based contractions on that same belief set.

Chapter 5

Multiple Contraction

This chapter is devoted to the presentation of some definitions and results concerning an extension of the AGM framework which has been motivated by the awareness of the need to develop models of theory contractions that covered the case of (simultaneous) contraction by several sentences (rather than just by one single sentence). More precisely, in what follows we will expose some models which are essentially generalizations of some of the explicit definitions of AGM contraction functions, that were presented in the previous chapter, in order to account for the case of contractions of a set by (not necessarily singleton) sets of sentences (rather than only by a single sentence).

As it is clear from the two previous chapters as well as from the presentation of the AGM model in Section 2.4, such model only accounted for changes (of a belief set) by a single sentence. Since such model has acquired, immediately after its publication, the status of standard model of belief change, most of the initial studies of belief change considered also only change operations involving only one (single) sentence. Nevertheless, not much time has passed until the need for extending the AGM model to cover changes by sets of sentences was noticed by some researchers.

The idea of contraction by a set of sentences rather than by a single sentence was introduced by Fuhrmann in [Fuh88], where the term *multiple contraction* was originally proposed to designate such an operation.[77] Hansson [Han89], Niederée [Nie91], and Rott [Rot92a] also studied the theory of multiple change operations. A survey of all these works was presented in [FH94], where Fuhrmann and Hansson also introduce three ways of generalizing the belief sets partial meet (singleton) contraction functions of [AGM85] to the contraction of belief sets by (non-singleton) sets of sentences: the $\perp$-*based partial meet* ($\perp -pm$) *contraction*, the Δ-*based partial meet contraction* and the $\angle$-*based partial meet* ($\angle - pm$) *contraction*.

Previously, Hansson had already investigated generalizations of that kind, namely in [Han89, Han91a], where two constructive methods for multiple contraction functions of belief sets by finite (not necessarily singleton) sets were presented: the *simple partial meet contraction* and the *composite partial meet contraction*. Then, in [Han92, Han93, Han91a], the former of these models was extended to cover the

[77] Originally the term *multiple contraction* was introduced to designate contractions by non-singleton sets. However, in this text we shall use such expression to refer to contractions by any (not necessarily non-singleton) set of sentences.

case of multiple contractions of sets of beliefs that are not (necessarily) closed under logical consequence (i.e. of belief bases) by possibly infinite sets. In [Fuh97] Fuhrmann presents a survey which compiles some of the above mentioned constructions of multiple contraction functions.

To close this very brief summary of the initial studies on multiple contraction we mention just one more reference, namely the paper [FSS03], presented several years after the above mentioned ones, where Fermé, Saez and Sanz presented two ways of generalizing the (singleton) kernel contraction functions of [Han94] (cf. Subsection 3.2.1) to the case of the contraction of belief bases by any set of sentences.

In what follows we will summarize the main concepts and results that were presented in the above mentioned works and analyse the interrelations among such different approaches for multiple contraction.

5.1 Meaning and Motivation for Multiple Contraction

By *multiple contraction* of a belief set $\mathbf{K}$ by a set of sentences B we mean the removal of the set B from $\mathbf{K}$. However, this can be interpreted either as:

- the removal of all elements of B from $\mathbf{K}$ (i.e. meaning that the result, say $\mathbf{K}\div[B]$, of the multiple contraction of $\mathbf{K}$ by B must be such that $B\cap\mathbf{K}\div[B]=\emptyset$).

or as

- the removal of at least one of the elements of B from $\mathbf{K}$ (i.e. meaning that the result, say $\mathbf{K}\div\langle B\rangle$, of the multiple contraction of $\mathbf{K}$ by B must be such that $B\not\subseteq\mathbf{K}\div\langle B\rangle$).

In [FH94], a (multiple) contraction function of the first kind above described is designated by *package contraction* whereas a contraction of the second type is called *choice contraction* and in the present text we adopt those designations. In that same paper, Fuhrmann and Hansson presented, for belief sets, two operations of the first kind and one of the second kind. Concerning package contraction they suggested the following operations: *partial meet package contraction* and *subremainder contraction*.[78] Regarding choice contraction, they introduced the *partial meet choice contraction*.[79]

Several arguments supporting the usefulness and necessity of multiple contraction functions can be found in the literature, e.g. [Han89, Han91a, Nie91, FH94, Fuh97]. In this respect, in what follows we summarize some of the arguments providing motivations for the consideration of package contractions which have been presented in [FH94, pp. 43−44]. Thus, as it was remarked in the mentioned reference, a simple

[78] The designations used here were in fact introduced in [Han99b]. Originally, in [FH94], these operations had been called $\perp$-*based partial meet* ($\perp -pm$) *contraction* and Δ-*based partial meet contraction*, respectively.

[79] This designation was presented in [Han99b]. In [FH94] this operation had been called $\angle$-*based partial meet* ($\angle - pm$) *contraction*.

evidence of the usefulness and the necessity of the study of package contractions is the fact that the sets which are intuitively acceptable as possible results of the package contraction of a theory **K** by a set of sentences, say $\{\alpha, \beta\}$, are, in general, different from every set which can be the result of either of the following operations:

1. contracting **K** by $\alpha \wedge \beta$,
2. contracting **K** by $\alpha \vee \beta$,
3. first contracting by α and then (contracting the result of such contraction) by β, or vice versa,
4. intersecting the results of contracting **K** by α and of contracting **K** by β.

Now we briefly expose the informal justifications presented in [FH94] for the inadequateness of identifying the multiple contraction of **K** by $\{\alpha, \beta\}$ with any of the above listed operations. Hence, in what concerns the first operation, to conclude that its use is inadequate in this context it is enough to observe that to remove a conjunction it suffices to remove one of the conjuncts. Regarding the inappropriateness of the second operation we just remark that, however the removal of a disjunction from a theory implies the removal of both disjuncts, the converse does not hold, i.e. in order to remove the set $\{\alpha, \beta\}$ from **K** it is not necessary to remove the sentence $\alpha \vee \beta$ from **K** (to see that this is so it is enough to consider the case when $\beta = \neg\alpha$).[80] Now, in order to clarify that the third operation is unsuitable to be used as the definition of the multiple contraction of **K** by $\{\alpha, \beta\}$, it is enough to remark that, on the one hand, the result of first contracting by α and then by β is not, in general, identical to that of first contracting by β and then by α, and, on the other hand it is implicit in the notion of multiple contraction that, in such a process, all the sentences to be contracted are treated equally.

Finally, to justify that the fourth above mentioned operation is also inadequate to be adopted as the standard way of obtaining a set which is a suitable result for the result of the (package) removal of the set $\{\alpha, \beta\}$ from **K**, Fuhrmann and Hansson informally suggested that the set resulting of such operation may be too small to represent the multiple contraction. However, in [FH94, pp. 62], the same authors have presented a formal proof of such inadequateness by showing that one of the constructive models of multiple contraction functions that were introduced in that paper, namely the partial meet package contraction functions[78] (which we shall present in Section 5.3), are such that, in general, the result of the partial meet package contraction of **K** by a set B is not identical to the set which results of the intersection of the results of contracting **K** by each of the sentences in B.

Still regarding the exposition of some motivation for package contractions it is convenient to remark here that, in [FH94, Sec. 12], it has also been stated that, generally speaking, apart from the four above mentioned operations, any other attempt of reducing the package contraction of a theory to a certain combination of contractions by a single sentence shall fail.

[80] At this point it is worth to remark that in [Han91a, pp. 17] Hansson presents an example illustrating that the result of the multiple contraction of **K** by $\{\alpha, \beta\}$ does not necessarily coincide with the result of neither of the operations 1. and 2. above.

The above exposition supports that it is pertinent and adequate to study the operation of package contraction on its own, since such operation is independent of the operation of contraction by a single sentence.

In what concerns the motivations exposed in the literature for the study of choice contraction, we highlight the example provided in [FH94, pp. 44−45] to justify their usefulness. Nevertheless, as remarked by Hansson [Han99b, pp. 134], the operation of package contraction seems to be more interesting than that of choice contraction, both from an intuitive and from a formal point of view. Such remark is more formally supported by the fact that, contrary to what is the case with package contraction, it holds that the operation of choice contraction can, at least in some cases, be reduced to the operation of contraction by one single sentence. More precisely, it can be shown that the operation of choice contraction that was introduced in [FH94], namely the partial meet choice contraction[79], is such that the operation of (partial meet choice) contraction of a theory by any finite set of sentences B is similar to the operation of partial meet contraction of such theory by the single sentence consisting of the conjunction of all the elements of B ([FH94, Observation 17]).

At this point we must remark that, on the one hand the present thesis essentially concerns the study of (the above mentioned) partial meet multiple contractions and, on the other hand, all the results in that respect that we shall present only account for the case of (multiple) contractions by finite sets. Hence, having the remarks made in the above paragraph in mind, from this point onwards we shall only consider multiple contractions of the *package kind* since that is the only one of the two kinds that is interesting in the context of the partial meet modelling and considering only contractions by finite sets.

Thus, throughout the rest of this text, the expression *multiple contraction* shall be assumed to have the meaning which was above attributed to the expression *package contraction*, i.e., in all that follows, by multiple contraction of a theory $\mathbf{K}$ by a set of sentences B we mean the removal of all the elements of B from $\mathbf{K}$. More formally, the expression "multiple contraction function" (or simply "multiple contraction") will be used to designate a function[81]

$$\begin{array}{rcl} \div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) & \to & \mathcal{T}_{\mathcal{L}} \\ (\mathbf{K}, \mathbf{B}) & \mapsto & \mathbf{K}\div B \end{array},$$

where the image of a pair $(\mathbf{K}, \mathbf{B})$ by $\div$ is represented by $\mathbf{K}\div B$ and is a belief set which is acceptable (in the context under consideration at the moment when such a function is mentioned) as a possible result of the (multiple) contraction of the set B from the belief set $\mathbf{K}$. Such set shall, sometimes, be referred to as "the result of the multiple contraction of $\mathbf{K}$ by B". It is also convenient to remark here that, throughout this text, almost every time a multiple contraction function is considered that is done in a context where a certain belief set $\mathbf{K}$ was previously fixed and, in such cases, we shall designate the multiple contraction function under consideration by "multiple contraction function on $\mathbf{K}$" (or simply "multiple contraction on $\mathbf{K}$").

[81] In general we shall use the symbol $\div$ to denote such a function, however some variants of this symbol (most of which shall be obtained simply by adding superscripts and/or subscripts to it) shall also be sometimes used for that effect.

Furthermore, in such situations, the multiple contraction under consideration will be seen as a function

$$\begin{array}{rcl} \div : \mathcal{P}(\mathcal{L}) & \to & \mathcal{T}_{\mathcal{L}} \\ B & \mapsto & \mathbf{K}\div B \end{array},$$

where the image of a set of sentences B by $\div$, is a set which results of the (multiple) contraction of the set B from $\mathbf{K}$ (according to the definition under consideration at that moment for such operation) and is represented by $\mathbf{K}\div B$.

Let $\mathcal{P}_{\mathcal{F}}(\mathcal{L})$ denote the set of finite subsets of $\mathcal{L}$ and $\mathbf{K}$ denote an arbitrary (but fixed) belief set. Throughout this text, we will use the expression *multiple contraction by finite sets* (respectively *multiple contraction on* $\mathbf{K}$ *by finite sets*) to refer to a function $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}_{\mathcal{F}}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ (respectively $\div : \mathcal{P}_{\mathcal{F}}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$) whose value for an argument $(\mathbf{K}, B)$ (resp. for any argument B) is a set that satisfies the conditions that we have mentioned in the above paragraph that should be satisfied by any set that can be accepted as the result of the multiple contraction of $\mathbf{K}$ by B. The value of a *multiple contraction by finite sets* (respectively *multiple contraction on* $\mathbf{K}$ *by finite sets*) $\div$ for an argument $(\mathbf{K}, B)$ (resp. for an argument B) shall be denoted by $\mathbf{K}\div B$.

We notice that the above introduced notation and terminology are analogous to the ones introduced in Section 2.3 (see page 25) to represent and refer to contractions by a single sentence.

It is also convenient to note here that, given a belief set $\mathbf{K}$ and a sentence α, the basic properties that are naturally expected to be satisfied by a set $\mathbf{K}-\alpha$ which is the result of the contraction of $\mathbf{K}$ by (the single sentence) α are precisely the same requirements that a set $\mathbf{K}\div\{\alpha\}$ should satisfy in order to be acceptable as a possible result of the multiple contraction of $\mathbf{K}$ by the singleton set $\{\alpha\}$. Having this in mind, from this point onwards we shall often use the expression *singleton contraction* to refer to a contraction by one single sentence. Nevertheless, whenever we use only the term *contraction* we shall be referring also to a contraction by one single sentence (unless it is obvious from the context that at that given moment such term is being used as an abbreviation of multiple contraction).

5.2 Postulates for Multiple Contraction

In the present section we introduce a list of postulates for multiple contraction and present some results revealing some of the interconnections among them. Such postulates are essentially properties which are, in general, intuitively requirable from a multiple contraction function. In the sequence those postulates will be used in the axiomatic characterizations that we shall present for each of the multiple contraction functions that we will consider in this thesis.

Hence, the following are some of the postulates which have been proposed in the literature (e.g. [Han89, Han91a, Han92, FH94, FSS03]) as properties which are naturally expectable from a multiple contraction function (or from a multiple contraction function on a certain belief set $\mathbf{K}$) $\div$:[82]

[82]Notice, that the designations and formulations here presented for each of those postulates are

- **Package closure:** $\mathbf{K}\div B$ is a belief set (i.e. $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$).
- **Package inclusion:** $\mathbf{K}\div B \subseteq \mathbf{K}$.
- **Package vacuity:** If $B \cap \mathbf{K} = \emptyset$, then $\mathbf{K}\div B = \mathbf{K}$.
- **Package success:** If $B \cap Cn(\emptyset) = \emptyset$, then $B \cap \mathbf{K}\div B = \emptyset$.
- **Package extensionality:** If for every sentence α in B there is a sentence β in C such that $\vdash \alpha \leftrightarrow \beta$, and vice versa, then $\mathbf{K}\div B = \mathbf{K}\div C$.
- **Package recovery:** $\mathbf{K} \subseteq Cn((\mathbf{K}\div B) \cup B)$.
- **Finite Package recovery:** If B is finite, then $\mathbf{K} \subseteq Cn((\mathbf{K}\div B) \cup B)$.
- **Package uniformity:** If every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C, then $\mathbf{K}\div B = \mathbf{K}\div C$.
- **Package relevance:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}\div B$, then there is a set K' such that $\mathbf{K}\div B \subseteq K' \subseteq \mathbf{K}$ and $B \cap Cn(K') = \emptyset$ but $B \cap Cn(K' \cup \{\beta\}) \neq \emptyset$.
- **Package core-retainment:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}\div B$, then there is a set K', such that $K' \subseteq \mathbf{K}$ and $B \cap Cn(K') = \emptyset$ but $B \cap Cn(K' \cup \{\beta\}) \neq \emptyset$.

When these postulates are seen as properties of a multiple contraction function $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ (resp. a multiple contraction function on $\mathbf{K}$, i.e. $\div : \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$), in each of the above formulations B and C are arbitrary sets of sentences and $\mathbf{K}$ is an arbitrary (resp. fixed) belief set. On the other hand, in a situation when the above postulates are considered as properties of a multiple contraction function *by finite sets* $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}_{\mathcal{F}}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ (resp. a multiple contraction function on $\mathbf{K}$ *by finite sets* $\div : \mathcal{P}_{\mathcal{F}}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$) then in the above formulations B and C are arbitrary *finite* sets of sentences and $\mathbf{K}$ is an arbitrary (resp. fixed) belief set.

The postulates of *P-closure*, *P-inclusion*, *P-vacuity*, *P-success*, *P-extensionality*, *P-recovery* and *finite P-recovery* are generalizations of the basic AGM postulates (for contraction) that cover the case of multiple contraction and were originally presented in [Han89]. Analogously, the postulates of *P-uniformity* and *P-relevance* are generalizations of the postulates of *uniformity* and *relevance* for contraction (presented in Observation 3.1.11) and have been introduced in [Han91a, Han92].[83] The postulate of *P-core-retainment* generalizes the singleton contraction postulate of *core-retainment* (presented in Observation 3.2.4) to the case of multiple contraction and was originally introduced in [FSS03].

We close this section with some results presenting several interrelations among postulates which shall be used further ahead (more precisely, in Chapter 9).

similar to the ones that can be found in [FH94, Han99b] which do not always coincide exactly with the ones that were used in the reference where such postulate was originally introduced.

[83] A detailed exposition of the motivations that lead to the formulations above presented for each of those postulates can be found in [FH94, Sections 5 – 6].

Lemma 5.2.1 (**[FH94, Lemma** 3 - 1.]) *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. If* $\div$ *satisfies* P-inclusion *and* P-relevance*, then it satisfies* P-closure*.*

Observation 5.2.2 (**[FH94, Observation** 10]) *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. If* $\div$ *satisfies* P-relevance*, then it satisfies* finite P-recovery*.*

Finally we observe that, it follows from the two above results that, if $\div$ is a multiple contraction function on a belief set $\mathbf{K}$ that satisfies *P-inclusion*, *P-uniformity* and *P-relevance* then it also satisfies *P-closure*, *P-vacuity*, *P-extensionality* and *finite P-recovery*.

To see that this indeed holds let $\mathbf{K}$ be a belief set and $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *P-inclusion*, *P-uniformity* and *P-relevance*. Then, it follows immediately from Lemma 5.2.1 and Observation 5.2.2 that $\div$ satisfies *P-closure* and *finite P-recovery*. On the other hand, it is easy to check that *P-vacuity* follows from *P-relevance* and that *P-uniformity* entails *P-extensionality*. Hence, we can conclude that $\div$ satisfies also *P-vacuity* and *P-extensionality*.

The following observation formally states the interrelation among postulates for multiple contraction that we have just shown to hold.

Observation 5.2.3 *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. If* $\div$ *satisfies* P-inclusion*,* P-uniformity *and* P-relevance *then it also satisfies* P-closure*,* P-vacuity*,* P-extensionality *and* finite P-recovery*.*

5.3 Partial Meet Multiple Contraction

Partial meet multiple contractions[84] are essentially the result of the generalization of the partial meet contraction functions of [AGM85], presented in Section 3.1, to the case of contractions by (possibly non-singleton) sets. This class of operations has been the first explicit construction of multiple contractions and was introduced in [Han89].

We start by recalling the concepts of *remainders* and of *remainder set* which were already presented in Definition 3.1.1, since they play a central role in the definition of partial meet multiple contraction. Thus, given a belief set $\mathbf{K}$ and a set of sentences B, according to the above mentioned definition, the *remainders* of $\mathbf{K}$, by B are the maximal subsets of $\mathbf{K}$ that do not imply any element of B. Furthermore, the set formed by all the *remainders* (of $\mathbf{K}$, by B) is the *remainder set* of $\mathbf{K}$ by B (or the *remainder set* of $\mathbf{K}$ *modulo* B) and is denoted by $\mathbf{K}\perp B$.

[84]In [Han99b] these operations have been (more appropriately) designated by *partial meet package contractions* in order to be distinguished from the *partial meet choice contractions* but, since in the present text we will only consider the first of this two kinds of operations, we will use the more general designation *partial meet multiple contractions*. In [Han89] these operations had been called *simple partial meet contractions* and in [FH94] they had been designated by $\perp$*-based partial meet* $(\perp -pm)$ *contractions*.

Now we introduce the definition of *package selection function*, which is a straightforward generalization of the notion of *selection function* (presented in Definition 3.1.5).

Definition 5.3.1 ([Han89, FH94]) *Let* $\mathbf{K}$ *be a belief set. A* package selection function *for* $\mathbf{K}$ *is a function* γ *such that for all sets of sentences* B*:*

1. *If* $\mathbf{K}\perp B$ *is non-empty, then* $\gamma(\mathbf{K}\perp B)$ *is a non-empty subset of* $\mathbf{K}\perp B$*, and*

2. *If* $\mathbf{K}\perp B$ *is empty, then* $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$*.*

Finally we are in a position to introduce the definition of *partial meet multiple contraction* which is a generalization of Definition 3.1.6 to the case of contractions by sets of sentences.

Definition 5.3.2 (Partial meet multiple contraction [Han89, FH94]) *Let* $\mathbf{K}$ *be a set of sentences and* γ *be a package selection function for* $\mathbf{K}$*. The partial meet multiple contraction on* $\mathbf{K}$ *that is generated by* γ *is the operation* $\div_\gamma$ *such that for all sets of sentences* B*:*

$$\mathbf{K} \div_\gamma B = \bigcap \gamma(\mathbf{K}\perp B).$$

A multiple contraction function $\div$ *on* $\mathbf{K}$ *is a partial meet multiple contraction if and only if there is some package selection function* γ *such that* $\mathbf{K}\div B = \mathbf{K}\div_\gamma B$ *for all sets of sentences* B*.*

Finally we present, just as we did in the singleton case (see Definition 3.1.7), the definition of two limiting particular cases of partial meet multiple contractions:

Definition 5.3.3 *Let* $\mathbf{K}$ *be a belief set.*

1. *A multiple contraction function* $\div$ *on* $\mathbf{K}$ *is a* maxichoice multiple contraction *if and only if it is a partial meet multiple contraction generated by a package selection function* γ *such that for all sets* B*, the set* $\gamma(\mathbf{K}\perp B)$ *has exactly one element.*

2. *The* full meet multiple contraction *on* $\mathbf{K}$ *is the partial meet multiple contraction* $\dot{\sim}$ *that is generated by the package selection function* γ *such that for all sets* B*, if* $\mathbf{K}\perp B$ *is non-empty, then* $\gamma(\mathbf{K}\perp B) = \mathbf{K}\perp B$*, i.e., the multiple full meet contraction* $\dot{\sim}$ *is the contraction operation on* $\mathbf{K}$ *defined by:*

$$\mathbf{K}\dot{\sim}B = \begin{cases} \bigcap \mathbf{K}\perp B & , \text{ if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & , \text{ if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

for all sets B*.*[85]

In Section 5.5 we shall expose an axiomatic characterization for the partial meet multiple contraction of belief sets which was presented in [FH94].

[85]Notice that in the explicit definition of the full meet contraction that we have presented here we have implicitly made use of the fact that, according to Observation 3.1.2, $\mathbf{K}\perp B \neq \emptyset$ if and only if $B \cap Cn(\emptyset) = \emptyset$.

5.4 Kernel Multiple Contraction

In this section we introduce another method for constructing multiple contractions which is based on the operation of kernel (singleton) contraction presented in Subsection 3.2.1. More precisely, in what follows we present the definition of *kernel multiple contraction* which is a generalization of the operation of kernel (singleton) contraction (presented in Definition 3.2.3) that accounts for contractions by sets of sentences. The class of kernel multiple contractions was originally introduced in [FSS03].[86]

Analogously to what was done in the mentioned subsection, we start by introducing the definition of *package kernel set of A with respect to B* for any sets of sentences A and B. Such set is formed by all the minimal subsets of A that imply some element of B as it is more formally exposed in the following definition.

Definition 5.4.1 (Package Kernel Set [FSS03, Definition 8]) *Let A and B be two sets of sentences. The package kernel set of A with respect to B, denoted $A \perp\!\!\!\perp_P B$ is the set such that $X \in A \perp\!\!\!\perp_P B$ if and only if:*

1. *$X \subseteq A$.*
2. *$B \cap Cn(X) \neq \emptyset$.*
3. *If $Y \subset X$ then $B \cap Cn(Y) = \emptyset$.*

Next we present the definition of *package incision function for (a set) A*, which, roughly speaking, is a function that selects at least one element from each one of the sets in $A \perp\!\!\!\perp_P B$, for any set B. Such concept is more detailedly and formally introduced in the following definition:

Definition 5.4.2 ([FSS03, Definition 10]) *A function σ is an* incision function for *A if and only if, for all sets B:*

1. *$\sigma(A \perp\!\!\!\perp_P B) \subseteq \bigcup A \perp\!\!\!\perp_P B$.*
2. *If $\emptyset \neq X \in A \perp\!\!\!\perp_P B$, then $X \cap \sigma(A \perp\!\!\!\perp_P B) \neq \emptyset$.*

We can now, finally, present the definition of kernel multiple contraction.[87]

Definition 5.4.3 (Kernel Multiple Contraction [FSS03, Definition 11]) *Let σ be an incision function for A. The* kernel multiple contraction $\approx_\sigma$ *for A based on σ is defined as follows:*

$$A \approx_\sigma B = A \setminus \sigma(A \perp\!\!\!\perp_P B).$$

A multiple contraction function $\div$ for A is a kernel multiple contraction *if and only if there is some package incision function σ for A such that $A \div B = A \approx_\sigma B$ for all sets B.*

In the following section we present the axiomatic characterization for the kernel multiple contraction which was obtained in [FSS03].

[86] It is convenient to remark here that, just as it was the case for their singleton counterparts, the original definition and corresponding representation theorems for the kernel multiple contraction functions accounted for the (multiple) contraction of any set of sentences (not necessarily closed under logical consequences).

[87] In [FSS03] such functions were denominated by *kernel package contractions*.

5.5 Representation Theorems and Interrelation Between Partial Meet and Kernel Multiple Contraction of Belief Sets

In this section we present representation theorems for each one of the two kinds of multiple contraction functions that were introduced in the two previous sections and, afterwards, based on such characterizations and on the interrelations among postulates, we clarify how those two classes of multiple contractions are related.

We start by exposing, in the following observation, the axiomatic characterization for partial meet multiple contractions on belief sets which was presented in [FH94]:

Observation 5.5.1 ([FH94, Theorem 9]**)** [88]
Let $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. Then* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *if and only if it satisfies the postulates of* P-inclusion*,* P-success*,* P-uniformity *and* P-relevance*.*

Since some of the postulates introduced in Section 5.2 were not mentioned in the statement of the above observation, it is convenient to remark here that, according to Lemma 5.2.1 and Observations 5.2.2 and 5.2.3, it follows from the above axiomatic characterization, that a partial meet multiple contraction function on a belief set satisfies (also) the postulates of *P-closure*, *finite P-recovery*, *P-vacuity* and *P-extensionality*.

At this point it worth to mention yet that, in [Li98], Jun Li has proven that partial meet multiple contraction functions in general do not satisfy the postulate of *P-recovery* (in the infinite case). More precisely, according to [Li98, Observations 2.1 and 2.2], *P-recovery* does not hold neither for the full meet multiple contraction nor for maxichoice multiple contraction.

It is now time to expose, in the following observation, the axiomatic characterization for kernel multiple contraction that Fermé, Saez and Sanz presented in [FSS03].

Observation 5.5.2 ([FSS03, Theorem 6]**)** *A multiple contraction function* $\div$ *for a set of sentences* A *is a kernel multiple contraction if and only if it satisfies the postulates of* P-inclusion*,* P-uniformity*,* P-core-retainment *and*

- P-success'*: If* $B \cap Cn(\emptyset) = \emptyset$*, then* $B \cap Cn(A \div B) = \emptyset$*.*[89]

[88] Similar results to the one stated in this observation can be found in [Han92, Theorem 1] and [Han93, Theorem 1] (see also [Han91a], where those two papers were included and summarized before they were published). In fact those two theorems are more general than the one that is presented here since, they provide an axiomatic charatezization for the class of partial multiple contraction functions on *belief bases* rather than (only) for (its subclass of) partial meet multiple contractions on belief sets.

[89] It is convenient to notice here that in this observation by *P-inclusion*, *P-uniformity* and *P-core-retainment* we mean the postulate whose formulation is identical to the formulation presented in Section 5.2 for the postulate with the same name, with $\mathbf{K}$ replaced by A. On the other hand, the formulation of the postulate of *P-success'* present in this axiomatic characterization, is the result of a slight modification of the statement of the postulate of *P-success* presented in Section

Next we briefly highlight some interrelations among the postulates used in the axiomatic characterizations presented in Observations 5.5.1 and 5.5.2, in order to be able to obtain some result regarding the relationship between the classes of partial meet multiple contractions and of kernel multiple contractions (on belief sets).

In all that follows we assume that $\div$ is a multiple contraction function on a belief set $\mathbf{K}$.

We start by remarking that if $\div$ satisfies *P-success'* then it (also) satisfies *P-success*, but the converse does not hold in general. However, making use of Lemma 5.2.1, it can be easily shown that if $\div$ satisfies P-inclusion, *P-relevance* and *P-success* then it (also) satisfies *P-success'*. Finally, we note that it is obvious that if $\div$ satisfies *P-relevance* then it (also) satisfies *P-core-retainment*.

Taking into account the above mentioned interrelations among postulates, we can immediately conclude from Observations 5.5.1 and 5.5.2 that if $\div$ is a partial meet multiple contraction on a belief set $\mathbf{K}$ then it is (also) a kernel multiple contraction on $\mathbf{K}$. Therefore the following corollary holds:

Corollary 5.5.3 *Let* $\mathbf{K}$ *be a belief set. Then every partial meet multiple contraction function on* $\mathbf{K}$ *is a kernel multiple contraction function on* $\mathbf{K}$.

Furthermore, from the above mentioned interrelations among postulates it follows, additionally, that *P-success* can be replaced by *P-success'* in the axiomatic characterization for partial meet multiple contraction presented above. That is, from the mentioned interrelations among postulates we can conclude that Observation 5.5.1 remains valid if we replace in its statement the postulate of *P-success* by the postulate of *P-success'*.[90]

In Figure 5.1 we present a diagram that summarizes all the results included in the present section, namely the axiomatic characterizations for the partial meet multiple contractions on belief sets and for the kernel multiple contractions on belief sets and the fact that the class of partial meet multiple contractions for belief sets is contained in the class of kernel multiple contractions for belief sets. In this regard it is convenient to clarify that the set of postulates associated, in that figure, to each of the two mentioned operations constitutes indeed a possible axiomatic characterization for such operation, however, in the case of partial meet multiple contraction the postulate of *P-core-retainment* is redundant in the axiomatic characterization that is associated to that class of functions in such figure.

5.6 Related Works

In this section we shall make a very brief exposition of some works (other than the ones already mentioned throughout the present chapter) where multiple change

5.2 which must be made in order to get a formulation that is adequate for a property of multiple contractions of belief bases (notice that if $\div$ is one such operation on a not logically closed set of sentences A then it may not hold that $A \div B = Cn(A \div B)$ for all sets B).

[90]Notice that, in this context, *P-success'* designates the postulate whose formulation coincides with the formulation presented in Observation 5.5.2 for the postulate with the same name, but with A replaced by $\mathbf{K}$.

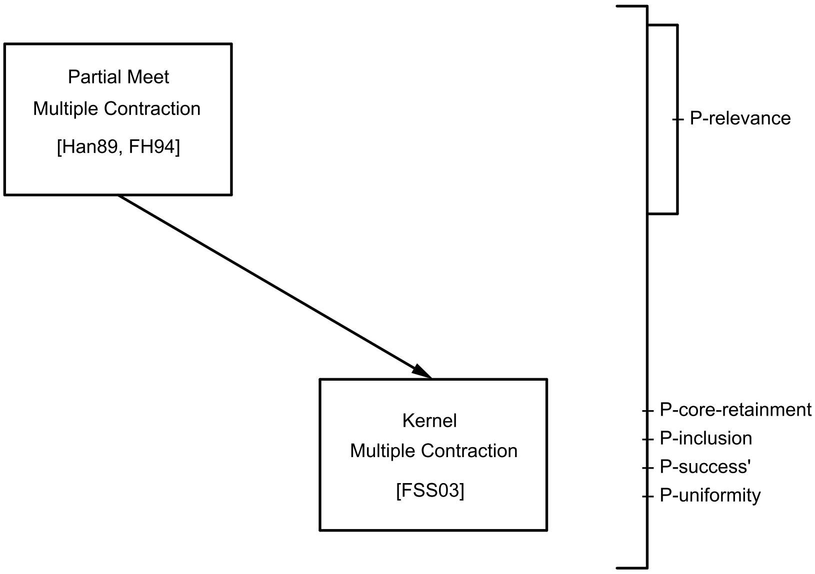

Figure 5.1: Diagram exposing axiomatic characterizations for the *partial meet* and the *kernel* multiple contractions of belief sets and the interrelation between those two classes of multiple contraction functions. In this diagram *P-inclusion*, *P-uniformity*, *P-relevance*, *P-core-retainment* and *P-success'* represent the postulates for multiple contraction *of belief sets* so designated in the present text.

operations are studied. We must, however, notice that it is not our intention to include here an exhaustive list of all the studies and results so far reported in the literature which can be considered to be in some sense related to the topic of multiple contraction. Rather, in the present section we shall mention only those works which are more closely related to the investigation reported in Chapters 6 − 9 below. Furthermore, at this point, we remark also that some of the natural topics of future research (cf. Section 10.2) which arise from the study presented in this text consist in the investigation of the interconnections as well as of the possible combinations of the works that we shall refer in this section with the new contributions exposed in the above mentioned chapters of the present thesis.

All the works that we shall mention throughout this section are concerned with two kinds of multiple change operations which differ from multiple contraction. To be more precise, those works are dedicated to the study of *multiple revision* − i.e., the generalization to the case of changes by sets of the operation of (singleton) revision (mentioned in Section 2.3) − and of *set contraction* − a kind of multiple change operation which differs from both *multiple contraction* and *multiple revision*.

Thus, we start by introducing some notation and terminology concerning such kind of operations: Analogously to what is the case in what concerns revisions of belief sets by a single sentence, given a belief set $\mathbf{K}$, generally speaking, by a *multiple revision on* $\mathbf{K}$ we mean a function, say $\oplus$, from $\mathcal{P}(\mathcal{L})$ to $\mathcal{T}_{\mathcal{L}}$ whose value at an arbitrary set of sentences B, denoted $\mathbf{K}\oplus B$ and which we refer to as *the (result) of the (multiple) revision* $\oplus$ *of* $\mathbf{K}$ *by* B is a belief set such that, if possible, it (i) contains the set B, (ii) is consistent, and (iii) contains as many sentences of $\mathbf{K}$ as it is possible.

At this point it is also convenient to remark that in what follows by the (result of the) *expansion* of a belief set $\mathbf{K}$ by a set of sentences B we mean the belief set $Cn(\mathbf{K} \cup B)$ which we shall represent by $\mathbf{K}+B$.[91]

This section is organized as follows: In Subsection 5.6.1 we present a generalization of the Levi identity (i.e. equation (2.1)) to the case of changes of belief sets by sets of sentences proposed by Sven Ove Hansson ([Han91a, Han92]). Then in Subsection 5.6.2 we introduce another kind of multiple change operations, namely the *set contraction*, which was presented by Dongmo Zhang [Zha96] and we mention the constructive models for those functions that were exposed in [Zha96, ZCZC97, ZF01, Pep]. Afterwards, in Subsection 5.6.3 we present a way of defining multiple revision functions which is based on a system of spheres, that was originally introduced by Pavlos Peppas [Pep04] and generalizes Grove's [Gro88] construction of system of spheres-based (singleton) revisions in order to account for the revision of belief sets by sets of sentences.

5.6.1 Generalized Levi Identity

As we have mentioned in Section 2.3, Isaac Levi, in [Lev77], sustained that revision should not be seen as a basic change operation but, rather, as an operation derived from (the basic operations of) contraction and expansion, in the sense that the only acceptable revision operations are those that can be obtained from some contraction operations by means of the procedure described by equation (2.1) (latter designated by *Levi identity*). Based on that idea Sven Ove Hansson, in [Han91a, Han92] (see also [Han99b, Section 3.17]) proposed a way of generalizing the Levi identity to the case of multiple change operations, i.e., in the mentioned references Hansson proposed a way of defining a multiple revision function from any given multiple contraction. In what follows we present such construction.

We start by recalling that, the idea underlaying the Levi identity is that in order to revise a belief set $\mathbf{K}$ by a sentence α we should first obtain a subset of $\mathbf{K}$ which is consistent with α and afterwards expand such set by α. The generalization of this idea to the case of multiple change operations is quite straightforward: The (result of the) multiple revision of a belief set $\mathbf{K}$ by a set of sentences B should be the result of the expansion by B of a subset of $\mathbf{K}$ which is consistent with B.

However, the generalization of equation (2.1) to the multiple level is not that immediate. In fact, the Levi identity consists of an equational description of the above described two step process which is based on the fact that the result of the removal of $\neg\alpha$ from $\mathbf{K}$ (i.e. any set $\mathbf{K}-\neg\alpha$, where $-$ is a contraction on $\mathbf{K}$) is a subset of $\mathbf{K}$ which is consistent with α. Nevertheless, there is not an equally obvious (analogous) way of obtaining a subset of $\mathbf{K}$ which is consistent with a given set of sentences B, i.e., it is not a trivial task to obtain, by means of a given multiple

[91] We must notice that this is a straightforward generalization to the case of changes (of belief sets) by sets of sentences of the concept of expansion of a belief set by a single sentence presented in Definition 2.2.1. For that reason and because there is no risk of ambiguity we shall use the same terminology (*expansion*) and notation (+) to represent both the operation of expansion of a belief set by a single sentence and the operation of expansion of a belief set by a set of sentences.

contraction $\div$ on $\mathbf{K}$, a subset of $\mathbf{K}$ which is consistent with B.

Indeed, by analogy with what is the case in the singleton contraction level, given a belief set $\mathbf{K}$, an arbitrary set of sentences B and a multiple contraction $\div$ on $\mathbf{K}$, in order to obtain, by means of $\div$, a subset of $\mathbf{K}$ which is consistent with B, it is necessary to start by generalizing the concept of *negation* (of a sentence) to the case of sets of sentences. Or, to be more precise, it is necessary to find a way of obtaining, for any given set of sentences B, another set of sentences, say $\neg B$, which is such that the result of the multiple contraction of $\mathbf{K}$ by $\neg B$ (i.e. $\mathbf{K}\div\neg B$) is (a subset of $\mathbf{K}$ which is) consistent with B.

Having this in mind, in [Han91a, Han92, Han99b], given any set of sentences B, Hansson showed that a set $\neg B$ satisfying the above mentioned conditions, can be obtained as described in the following definition:

Definition 5.6.1 ([Han91a],[Han92],[Han99b, Definition 3.35]) *Let B be any set of sentences. The* negation of B*, denoted $\neg B$, is the set (of sentences) such that $\beta \in \neg B$ if and only if β is either*

1. $\bot$
2. *a negation of some sentence in B, or*
3. *a (finite) disjunction of sentences which are negations of elements of B.*

Afterwards, making use of the above introduced concept of *negation of a set (of sentences)*, Hansson proposed, in [Han91a, Han92], the following straightforward generalization of the Levi identity to the case of changes by sets:

$$\mathbf{K}\oplus B = Cn((\mathbf{K}\div\neg B) \cup B) \tag{5.1}$$

In [Han91a] two alternative designations were used to refer to the above equation, namely *Levi identity for sets* and *generalized Levi identity*. Throughout the present text we shall use only the latter of those designations to refer to equation (5.1).

At this point it is also worth to remark that, for finite sets of sentences B, an alternative simpler way of obtaining a set whose contraction from a belief set $\mathbf{K}$ is (a subset of $\mathbf{K}$) consistent with B was presented in [Han91a, Han92, Han99b]. Such a set was designated by *sentential negation of B* in [Han99b] and is defined as follows:

Definition 5.6.2 ([Han91a],[Han92, Def. 11],[Han99b, Def. 3.36]) *Let B be any finite set of sentences. The* sentential negation of B*, denoted $n(B)$, is the set (of sentences) such that:*

1. *If $B = \emptyset$, then $n(B) = \{\bot\}$.*
2. *If B is a singleton set, $B = \{\beta\}$, then $n(B) = \{\neg\beta\}$.*
3. *If $B = \{\beta_1, \ldots, \beta_m\}$ for some $m > 1$, then $n(B) = \{\neg\beta_1 \vee \ldots \vee \neg\beta_m\}$.*

We must yet mention that in [Han99b] it was shown that if $\div$ is a partial meet multiple contraction on a belief set $\mathbf{K}$, then the result of removing (by means of that function) the negation of B, $\neg B$, coincides with the result of removing the sentential negation of B, $n(B)$, from $\mathbf{K}$. This fact is more formally stated in the following observation.

Observation 5.6.3 ([Han91a],[Han99b, Observation 3.37]) *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a partial meet multiple contraction on* $\mathbf{K}$*. Then, for any finite set of sentences* B*, it holds that* $\mathbf{K}{\div}\neg B = \mathbf{K}{\div}n(B)$*.*

5.6.2 Set Contraction

In this subsection we briefly present the notion of *set contraction* – a kind of multiple change operation which was introduced by Dongmo Zhang, in [Zha96].[92]

The operation of *set contraction* can not be seen as a generalization to the case of changes of belief sets by sets of sentences of any of the three basic operations of change of belief sets (cf. Section 2.3). However we may say that it was inspired by the Levi identity (i.e. equation (2.1)) which provides a method for defining (singleton) revisions from (singleton) contractions. Indeed, given a belief set $\mathbf{K}$, in general, the expression *set contraction on* $\mathbf{K}$ designates a function, say $\ominus$ from $\mathcal{P}(\mathcal{L})$ to $\mathcal{T}_{\mathcal{L}}$ whose output for an arbitrary argument B, denoted $\mathbf{K}{\ominus}B$, is a logically closed subset of $\mathbf{K}$ which is consistent with the set B (i.e. $(\mathbf{K}{\ominus}B) \cup B$ is a consistent set of sentences).

In [Zha96], the class of *set contractions* was defined as consisting of the family of functions from $\mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L})$ to $\mathcal{T}_{\mathcal{L}}$ that satisfied a certain set of eight postulates, namely the postulates $(\ominus 1) - (\ominus 8)$ listed in [ZF01, Subsection 2.2], which are essentially a generalization for set contraction of the basic and the supplementary AGM postulates for contraction.[93] Later, in [ZCZC97], another postulate for set contraction was proposed, namely the postulate $(\ominus 6_S)$ (see, e.g., [ZF01, pp. 531]) which is equivalent to $(\ominus 6)$ in the presence of $(\ominus 7)$ and $(\ominus 8)$. Following [ZF01], throughout this text we shall refer to the postulates $(\ominus 1) - (\ominus 5)$ and $(\ominus 6_S)$ as the *basic postulates for set contraction* and to postulates $(\ominus 7)$ and $(\ominus 8)$ as the *supplementary postulates for set contraction*.

In that same paper Zhang presented also the definition of *set revision*[94] by indicating a list of eight postulates, namely postulates $(\otimes 1) - (\otimes 8)$ listed in [ZF01, Subsection 2.1], whose satisfaction by a function $\otimes : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ is a necessary and sufficient condition for that function to be a *set revision*. Those postulates are the result of the generalization to the case of revision by sets of sentences of the basic and the supplementary AGM postulates for revision and, as it is remarked in

[92] The term *set contraction* which is currently used for designating the operations that we will present in this subsection was proposed in [ZF01]. In [Zha96] those operations were designated by *general contractions*.

[93] We must notice that the list of eight postulates for set contraction presented in [ZF01] constitutes, in fact, a revised version of the one that was (originally) presented in [Zha96]. Nevertheless, those two sets of postulates are equivalent.

[94] The term *set revision* is used as meaning the same as *multiple revision*. Such term was proposed in [ZF01]. In [Zha96] those operations were designated by *general revisions*.

[ZF01], that set of postulates is the result of the refinement by Lindström ([Lin91]) and Nayak ([Nay94]) of the list of postulates for multiple belief revision that was introduced by Fuhrmann ([Fuh88]).

Still in [Zha96], based on the concept of *set contraction* Zhang proposed a generalization of the Levi identity (i.e. equation (2.1)) which differs from the one proposed by Hansson (that we have presented in the previous subsection). More precisely, in the mentioned paper, it was shown that if $\ominus$ is a set contraction on a belief set $\mathbf{K}$, then the operation $\otimes$ defined, for any set of sentences B, by the following equation:

$$\mathbf{K} \otimes B = (\mathbf{K} \ominus B) + B \tag{Def $\otimes$}$$

is a set revision on $\mathbf{K}$.

Furthermore, given a set revision function $\otimes$ on $\mathbf{K}$, in [Zha96] it was also shown that the operation $\ominus$ obtained from it by means of the following generalization of the Harper identity (i.e. equation (2.2)):

$$\mathbf{K} \ominus B = (\mathbf{K} \otimes B) \cap \mathbf{K} \tag{Def $\ominus$}$$

is a set contraction on $\mathbf{K}$.

In the following observation the above mentioned facts are stated more formally and detailedly:

Observation 5.6.4 ([Zha96]) *The following statements hold:*

(i) If the function $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ satisfies the postulates $(\ominus 1) - (\ominus 8)$, then the function $\otimes : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ obtained from $\ominus$ by means of equation (Def $\otimes$) satisfies the postulates $(\otimes 1) - (\otimes 8)$ as well as the identity (Def $\ominus$).

(ii) If the function $\otimes : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ satisfies the postulates $(\otimes 1) - (\otimes 8)$, then the function $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ obtained from $\otimes$ by means of equation (Def $\ominus$) satisfies the postulates $(\ominus 1) - (\ominus 8)$ as well as the identity (Def $\otimes$).

Other contribution of [Zha96] which is worth to mention here consists of a constructive model for set contraction which is based on the concept of *nicely-ordered partition (NOP)* of a belief set (introduced in [Zha96, Definition 5.2]). Regarding the latter mentioned concept, in that same paper, on the one hand it is shown that a NOP is, in a certain sense, equivalent to an epistemic entrenchment ordering[95] but, on the other hand, it is argued that the notion of NOP is more intuitive than that of epistemic entrenchment.

Latter, in [ZCZC97], regarding the relationship between the operations of multiple contraction and of set contraction it was stated that a set contraction can be defined by means of a partial meet multiple contraction (cf. Definition 5.3.2 and Observation 5.5.1) and, conversely a partial meet multiple contraction can be partially defined in terms of a set contraction.

[95] More precisely, in [Zha96, Section 6] it is shown how a NOP can be obtained from a binary relation $\leq$ on $\mathcal{L}$ satisfying the epistemic entrenchment postulates (EE1)−(EE3) (cf. Definition 3.4.1) and, conversely, it is exposed how a NOP can be used to define a binary relation $\leq$ on $\mathcal{L}$ satisfying the epistemic entrenchment postulates (EE1)−(EE4).

Before explicitly presenting the results on that respect that were exposed in [ZCZC97] it is convenient to observe here that, as we have described in the previous subsection, the generalization of the Levi identity to the case of multiple change operations presented by Hansson is based on the concept of negation of a set of sentences (cf. Definition 5.6.1) which was introduced with the goal of assuring that, given a belief set $\mathbf{K}$, a multiple contraction $\div$ on $\mathbf{K}$ and any set of sentences B, the result of the multiple contraction of $\mathbf{K}$ by $\neg B$ (i.e. $\mathbf{K}\div\neg B$) would be (a set of sentences) consistent with B while, on the other hand, the operation of set contraction proposed by Zhang is such that the result of set contracting $\mathbf{K}$ by the set B itself, i.e. $\mathbf{K}\ominus B$, is consistent with B.

Having remarked this, it is now time to paraphrase here two observations that were presented (without however an associated proof) in the mentioned paper:

Observation 5.6.5 ([ZCZC97]) *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a partial meet multiple contraction on* $\mathbf{K}$*. Then the function* $\ominus : \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *(on* $\mathbf{K}$*) defined, for any set of sentences* B*, by:*

$$\mathbf{K}\ominus B = \mathbf{K}\div\overrightarrow{B},$$

where $\overrightarrow{B} = \{\alpha \in \mathcal{L} : \exists\beta_1, \ldots \beta_n \in B(\alpha = \neg\beta_1 \vee \cdots \vee \neg\beta_n)\}$*, satisfies the basic postulates for set contraction.*[96]

Observation 5.6.6 ([ZCZC97]) *Let* $\mathcal{P}_{\vee}(\mathcal{L}) = \{B \in \mathcal{P}(\mathcal{L}) : \forall\alpha, \beta \in B(\alpha \vee \beta \in B)\}$*,* $\mathbf{K}$ *be a belief set and* $\ominus : \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *be a function on* $\mathbf{K}$ *that satisfies the basic postulates for set contraction. Then the function* $\div : \mathcal{P}_{\vee}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *(on* $\mathbf{K}$*) defined, for any set of sentences* $B \in \mathcal{P}_{\vee}$*, by:*

$$\mathbf{K}\div B = \mathbf{K}\ominus\overrightarrow{B},$$

where $\overrightarrow{B} = \{\neg\alpha : \alpha \in B\}$*, is a partial meet multiple contraction on* $\mathbf{K}$*.*

Afterwards, in [ZCZC97, ZF01], Zhang et al. introduced a partial meet model for set contraction. The functions presented in that context consist of a generalization to the case of set contraction of the partial meet (singleton) contractions. Moreover, in those papers two representation theorems for those functions have been obtained, one involving (only) the basic postulates for set contractions (see, e.g., [ZF01, Theorem 3.13]) and another one involving the basic and the supplementary postulates for set contractions (see, e.g., [ZF01, Theorem 3.26]).

Also in [ZCZC97, ZF01] a definition for *NOP-based contractions* was introduced (see, e.g., [ZF01, Definition 4.4]) which differs from the one that had been originally provided in [Zha96]. Moreover, in those references a representation theorem for such functions has been presented (see, e.g., [ZF01, Theorem 4.19]) which asserts that a function $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ is a NOP-based contraction if and only if it satisfies the (above mentioned) postulates $(\ominus 1) - (\ominus 8)$ (for set contraction) as well as an additional postulate, which, in the mentioned papers, has been designated by *Limit*

[96] At this point is worth to remark that for any **non-empty** set of sentences B it holds that $\overrightarrow{B} = \neg B$, where $\neg B$ is the negation of B introduced in Definition 5.6.1.

Postulate (for set contraction) and denoted by $(\ominus LP)$. Such postulate is essentially a principle concerning the relationship between the result of the set contraction of a belief set by an infinite set and the results of the set contractions (of that same belief set) by the finite subsets of that infinite set and its precise formulation can be found in, e.g., [ZF01, pp. 548].

Recently, in [Pep], Pavlos Peppas has presented a constructive model for set contraction which can be seen as a generalization of the NOP-based contraction in the sense that the class of set contractions there introduced is axiomatically characterized by the postulates $(\ominus 1)-(\ominus 8)$ for set contraction proposed by Dongmo Zhang that we have mentioned in the beginning of the present subsection. It is convenient to emphasise here that, contrary to what is the case in what concerns the representation theorem for the NOP-based contractions presented in [ZCZC97, ZF01], the axiomatic characterization of the functions proposed by Peppas does not include the *limit postulate* and, for that reason, we can conclude that the class of such functions subsumes the one consisting of the NOP-based contractions.

The constructive model for set contraction proposed in [Pep] is based on the concept of *comparative possibility preorder* which is, loosely speaking, a binary relation $\leqslant$ on the set of nonempty sets of sentences, i.e. on $\mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ that satisfies a set of axioms, namely, conditions (CP1)−(CP6) listed in [Pep, Section 4]. In that same paper it is remarked that comparative possibility preorders are a generalization of the epistemic entrenchment relations (in the sense of being a preorder on *sets of sentences* rather than (simply) a preorder on sentences). Furthermore it is there shown that, if $\leqslant$ is a comparative possibility preorder then the binary relation $\preceq$ on $\mathcal{L}$ defined by the following condition:

$$\alpha \preceq \beta \text{ iff } \{\neg\beta\} \leqslant \{\neg\alpha\} \tag{PE}$$

is an epistemic entrenchment (cf. [Pep, Theorem 2]).

Given a belief set $\mathbf{K}$, making use of an arbitrary comparative possibility preorder $\leqslant$ related to $\mathbf{K}$, in [Pep] a function $\dot{-} : \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\} \to \mathcal{T}_{\mathcal{L}}$ is defined by means of the following condition:

$$\alpha \in \mathbf{K}\dot{-}B \text{ iff } \alpha \in \mathbf{K} \text{ and } B \cup \{\neg\alpha\} < B \cup \{\alpha\}. \tag{PC}$$

Moreover, in the mentioned paper it was proven that the class of functions which are defined from a comparative possibility preorder by means of the above condition coincides with the class formed by all the functions $\ominus$ which satisfy postulates $(\ominus 1)-(\ominus 8)$ for set contraction. Part of the proof for such fact that is presented in [Pep] makes use of the binary relation $\leqslant$ on $\mathcal{P}(\mathcal{L})\setminus\{\emptyset\}$ that is defined from a set contraction $\dot{-}$ on a belief set $\mathbf{K}$ by means of the following condition:

$$B \leqslant C \text{ iff } C \text{ is consistent with } \mathbf{K}\dot{-}(B \curlyvee C) \text{ or } B \vdash \bot, \tag{SP}$$

where $B \curlyvee C = \{\alpha \vee \beta : B \vdash \alpha \text{ and } C \vdash \beta\}$.

The above described facts are more detailedly and formally stated in the following observation:

Observation 5.6.7 ([Pep, Theorems $3-5$]) *Let* $\mathbf{K}$ *be a belief set. The following statements hold:*

1. *If $\leqslant$ is a comparative possibility preorder related to* $\mathbf{K}$ *then the function* $\dot{-}$ *constructed from* $\leqslant$ *via condition (PC) is a set contraction function satisfying postulates* $(\ominus 1) - (\ominus 8)$ *as well as condition (SP).*

2. *If* $\dot{-}$ *is a set contraction function satisfying* $(\ominus 1) - (\ominus 8)$ *then the binary relation* $\leqslant$ *constructed from* $\dot{-}$ *by means of condition (SP) is a comparative possibility preorder related to* $\mathbf{K}$ *which satisfies condition (PC).*[97]

In what follows we shall designate the set contraction function $\dot{-}$ that is generated from a comparative possibility preorder $\leqslant$ by means of condition (PC) by *$\leqslant$-based set contraction* or, more generally, by *comparative possibility preorder-based set contraction*.

According to the above observation the comparative possibility preorder-based set contractions are axiomatically characterized by the postulates $(\ominus 1) - (\ominus 8)$. In particular this means that such functions are independent from the *limit postulate*. Nevertheless, in [Pep], having in mind the relevance of such postulate (which is included, in addition to postulates $(\ominus 1) - (\ominus 8)$, in the axiomatic characterization of the NOP-based contractions that was presented in [ZCZC97, ZF01]), Peppas identified three properties that a comparative possibility preorder $\leqslant$ must satisfy in order that the $\leqslant$-based set contraction satisfies the limit postulate (cf. [Pep, Theorem 7]).

5.6.3 System of Spheres-based Multiple Revision

In this subsection we will briefly present the constructive model for multiple revision that was introduced by Pavlos Peppas in [Pep04]. Such model consists of a generalization of the system of spheres-based method for constructing revisions (by one single sentence) proposed by Grove (in [Gro88]) to account for revisions by any nonempty set of sentences.[98]

We must start by remarking that, in the mentioned paper, a *multiple revision* is defined as a function $\oplus : \mathcal{T}_{\mathcal{L}} \times (\mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}) \to \mathcal{T}_{\mathcal{L}}$ which satisfies a certain set of eight postulates – namely the postulates $(K \oplus 1) - (K \oplus 8)$ listed in [Pep04, pp. 365] – which are essentially the generalization to the case of multiple revision of the basic and the supplementary AGM postulates for revision (by one single sentence) presented in Subsection 2.4.2.[99]

In what follows we present the explicit construction of *system of spheres-based multiple revisions* which was proposed by Peppas as well as the main results obtained in [Pep04] regarding such class of functions, among which we emphasise a representation theorem asserting that the mentioned class of functions coincides with

[97] This fact is shown to hold in [Pep, Proof of Theorem 4].

[98] In the mentioned paper it is remarked that if one extends the domain of such functions by identifying the (result of the) revision by the empty set with the (result of the) revision by a singleton containing a tautology then all the results that have been there presented remain valid.

[99] At this point it is worth to observe that the list of postulates for multiple revision $(K \oplus 1) - (K \oplus 8)$ that is presented in [Pep04] coincides with the list of postulates for set revision $(\otimes 1) - (\otimes 8)$ which we have mentioned in the previous subsection.

the class consisting of the functions which satisfy the above mentioned postulates $(K \oplus 1) - (K \oplus 8)$ (cf. Observation 5.6.8).[100]

Having in mind the definition of system of spheres-based revisions (by one single sentence) proposed by Grove in [Gro88], given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, we are naturally led to define the $\mathbb{S}$-based multiple revision on $\mathbf{K}$ as the function $\oplus_{\mathbb{S}} : \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\} \to \mathcal{T}_{\mathcal{L}}$ defined by:

$$\mathbf{K} \oplus_{\mathbb{S}} B = \begin{cases} Th(f_{\mathbb{S}}(B)) & \text{, if } \|B\| \neq \emptyset \\ \mathcal{L} & \text{, otherwise} \end{cases} , \qquad (\oplus_{\mathbb{S}})$$

with $f_{\mathbb{S}}(B) = \|B\| \cap \mathbb{S}_B$, where $\mathbb{S}_B$ is the smallest sphere in $\mathbb{S}$ intersecting $\|B\|$.

Nevertheless, the generalization of Grove's construction is not as straightforward as that. Indeed, in [Pep04] it is observed that, in order for the function $\oplus_{\mathbb{S}}$ defined by $(\oplus_{\mathbb{S}})$ to satisfy the postulates $(K \oplus 1) - (K \oplus 8)$ it is necessary that the system of spheres $\mathbb{S}$ on which such definition is based satisfies the two following conditions:

For every nonempty consistent set of sentences B, there exists a smallest sphere in $\mathbb{S}$ intersecting $\|B\|$. (SM)

For every nonempty $B \subseteq \mathcal{L}$, if there exists a smallest sphere $\mathbb{S}_B$ in $\mathbb{S}$ intersecting $\|B\|$, then $\|B\| \cap \mathbb{S}_B$ is *elementary*. (SD)

The notion of *elementary* set of possible worlds that is mentioned in the formulation of the latter condition above can be introduced as follows: A set $\mathcal{U}$ of possible worlds is *elementary* if and only if it holds that $\|Th(\mathcal{U})\| = \mathcal{U}$.

In [Pep04] the expression *well ranked system of spheres* was introduced to designate a system of spheres that satisfies conditions (SM) and (SD).

We are now in a position to present the following representation theorem, obtained in [Pep04], which asserts that the class of functions defined from well ranked systems of spheres by means of $(\oplus_{\mathbb{S}})$ coincides with the class consisting of all functions that satisfy the postulates $(K \oplus 1) - (K \oplus 8)$.

Observation 5.6.8 ([Pep04, Theorems 5.1 and 5.2]) *Let $\mathbf{K}$ be a belief set and $\oplus$ be a function (from $\mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}$ to $\mathcal{T}_{\mathcal{L}}$). Then $\oplus$ satisfies postulates $(K\oplus 1) - (K\oplus 8)$ if and only if there exists a well ranked system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ such that, for any nonempty set of sentences B, it holds that $\mathbf{K} \oplus B = \mathbf{K} \oplus_{\mathbb{S}} B$, where $\mathbf{K} \oplus_{\mathbb{S}} B$ is the belief set defined by equation $(\oplus_{\mathbb{S}})$.*

In what follows, given a belief set $\mathbf{K}$, we shall use the expression *system of spheres-based multiple revision on* $\mathbf{K}$ to designate any function $\oplus : \mathcal{P}(\mathcal{L}) \setminus \{\emptyset\} \to \mathcal{T}_{\mathcal{L}}$ such that, for any nonempty set of sentences B, it holds that $\mathbf{K} \oplus B = \mathbf{K} \oplus_{\mathbb{S}} B$, where $\mathbb{S}$ is a well ranked system of spheres centred on $\|\mathbf{K}\|$ and $\mathbf{K} \oplus_{\mathbb{S}} B$ is defined from $\mathbb{S}$ by means of $(\oplus_{\mathbb{S}})$.

Throughout this text, given a belief set $\mathbf{K}$ and a well ranked system of spheres centred on $\|\mathbf{K}\|$, we shall refer to the function $\oplus_{\mathbb{S}}$ defined by $(\oplus_{\mathbb{S}})$ as the $\mathbb{S}$-*based multiple revision on* $\mathbf{K}$. We shall also use the expression *system of spheres-based*

[100]We notice that, according to [Pep04, Footnote 8], Lindström ([Lin91]) was the first to state, without proving it, that Grove's results can be generalized to the case of multiple revision.

multiple revision on $\mathbf{K}$ to designate any function that can be generated from a well ranked system of spheres centred on $\|\mathbf{K}\|$ by means of ($\oplus_{\mathbb{S}}$). It follows from the above observation that the system of spheres-based multiple revisions on $\mathbf{K}$ are precisely those (multiple revision) functions that satisfy postulates $(K \oplus 1) - (K \oplus 8)$.

We close this subsection with the presentation of another interesting result obtained in [Pep04], which consists of a way of defining multiple revisions by means of revisions by one single sentence.

In this regard, Peppas started by remarking that, if $\oplus : \mathcal{T}_{\mathcal{L}} \times (\mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}) \to \mathcal{T}_{\mathcal{L}}$ is a function that satisfies postulates $(K \oplus 1) - (K \oplus 8)$, then the function $* : \mathcal{T}_{\mathcal{L}} \times (\mathcal{P}(\mathcal{L}) \setminus \{\emptyset\}) \to \mathcal{T}_{\mathcal{L}}$ defined, for any belief set $\mathbf{K}$ and any sentence α by $\mathbf{K}{*}\alpha = \mathbf{K}{\oplus}\{\alpha\}$, satisfies the basic and the supplementary AGM postulates for revision (by one single sentence). In [Pep04] the function $*$ defined as exposed above is designated by the *restriction to sentences* of the multiple revision function $\oplus$.

With the goal of presenting a way of constructing a multiple revision from a revision by single sentences, Peppas started by remarking that it follows trivially from condition ($\oplus_{\mathbb{S}}$) that, if $\mathbf{K}$ is a belief set, $\oplus$ is a system of spheres-based multiple revision on $\mathbf{K}$, and $*$ is its restriction to sentences then, for any nonempty finite set of sentences B, it holds that $\mathbf{K}{\oplus}B = \mathbf{K}{*}\bigwedge B$ (where for a nonempty finite set of sentences B, $\bigwedge B$ denotes the conjunction of all elements of B).

However, the above identity is no longer valid when B is an infinite set of sentences (since, by assumption, $\mathcal{L}$ does not contain infinite conjunctions). Therefore, in [Pep04], a different way of reducing multiple revision (by infinite sets) to revision by a single sentence had to be presented. In order to expose such reduction strategy we need to start by introducing the following notation: Given a nonempty set of sentences B we shall use the notation $\bigwedge_f(B)$ to represent the set $\bigwedge_f(B) = \{\bigwedge F : F$ is a nonempty, *finite* subset of $B\}$.

Given a multiple revision function $\oplus$ on a belief set $\mathbf{K}$ and its restriction to sentences $*$, Peppas proposed the following identity as a reasonable property interconnecting those two functions:

$$\mathbf{K}{\oplus}B = \bigcap_{\alpha \in \bigwedge_f(B)} ((\mathbf{K}{*}\alpha){+}B)\,. \qquad (\oplus - *)$$

Nevertheless, in [Pep04] it was observed that not all multiple revision functions satisfy the above condition. More precisely, in the mentioned paper it was shown that, a system of spheres-based multiple revision $\oplus$ satisfies condition $(\oplus - *)$ if and only if it is generated, by means of ($\oplus_{\mathbb{S}}$), from a well ranked system of spheres $\mathbb{S}$ which (additionally) satisfies the following condition:

$$\text{For every } G \subseteq \mathbb{S}, \bigcup G \text{ is elementary.} \qquad \text{(SF)}$$

The above mentioned fact is more formally stated in the following observation:

Observation 5.6.9 ([Pep04, Theorem 6.1]) *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a well ranked system of spheres centred on $\|\mathbf{K}\|$, $\oplus$ be the $\mathbb{S}$-based multiple revision on $\mathbf{K}$, and $*$ be its restriction to sentences. Then $\mathbb{S}$ satisfies (SF) if and only if $\oplus$ satisfies $(\oplus - *)$.*

Chapter 6

Possible Worlds Semantics for Partial Meet Multiple Contraction Functions

In this chapter our main goal is to obtain the possible worlds semantics for the partial meet multiple contractions. To achieve that, we will start by finding out a relation between the package remainder set of $\mathbf{K}$ by B and a certain set of possible worlds, which is the analogous, in the multiple case, to the relation between a singleton remainder set of $\mathbf{K}$ by α and the set $\|\neg\alpha\|$, for each sentence $\alpha \in \mathbf{K}$, that is expressed by Grove's bijection (Observation 4.1.1-3). Afterwards, by making use of such relation we present the characterization of the partial meet multiple contractions in terms of possible worlds.

It is also worth to anticipate here that such results will play an essential role in the following chapter, namely in what concerns the construction of a suitable generalization of the definition of the S.S.-based (singleton) contractions to the multiple case. Indeed, having in mind that in the singleton case it was by means of the characterization of the remainders (by a single sentence) in terms of possible worlds (cf. Observation 4.1.1) that we have shown that every S.S.-based contraction is a (transitively relational) partial meet contraction (cf. Observation 4.1.9), it seems natural that, a first step when searching for the generalization of the S.S.-based contraction to the case of multiple contractions (in a way that such new operation is also a partial meet multiple contraction) should be to obtain the characterization of the remainders by sets of sentences (rather than by a single sentence) in terms of possible worlds.

Hence, in what follows we first introduce, in Section 6.1, the possible worlds semantics for the remainders by (not necessarily singleton) sets of sentences. Afterwards, in Section 6.2, based on those results we investigate the possible worlds semantics of the full meet multiple contraction. Finally, in Section 6.3 we present the possible world semantics for the partial meet multiple contractions (in general).

The main results of the present chapter appear also in [RF].

6.1 Possible Worlds Semantics for Package Remainders

We start by observing that, in the singleton case, the main virtue and usefulness of Grove's bijection (Observation 4.1.1-3) consists in the fact that it provides us with the possibility of defining and identifying remainders (by a single sentence) by means of a possible worlds-based approach (cf. Remark 4.1.2).

In this section we will expand that possibility to the multiple case. In other words we will investigate how remainders by sets of sentences (rather than by a single sentence) – which are also designated by *package remainders* (e.g. in [FH94, Definition 5]) – can be characterized in terms of possible worlds.

Let $\mathbf{K}$ be a belief set and B be a set of sentences. We start by noticing that it follows immediately from Definition 3.1.1 that:

$$\text{If } B \cap Cn(\emptyset) \neq \emptyset, \text{ then } \mathbf{K} \perp B = \emptyset.$$

On the other hand, in what concerns the case $B \cap Cn(\emptyset) = \emptyset$, a rough analogy with the singleton case (cf. Remark 4.1.2) might lead us to expect that, if $B \cap \mathbf{K} \neq \emptyset$, it should be the case that if $X \in \mathbf{K} \perp B$ then $\|X\| = \|\mathbf{K}\| \cup \mathcal{W}$, where $\mathcal{W} \subseteq \mathcal{M}_{\mathcal{L}}$ is a set composed of one and only one $\neg\alpha_i$-world for each $\alpha_i \in B \cap \mathbf{K}$. Nevertheless a more careful analysis shows that the generalization of Grove's bijection to the multiple case is not that straightforward since, as we will see further ahead, for some $X \in \mathbf{K} \perp B$ and some $\alpha_i \in B \cap \mathbf{K}$, the set $\|X\|$ may contain more than one $\neg\alpha_i$-world.

However, the slightly weaker result that if $X \in \mathbf{K} \perp B$ then $\|X\| = \|\mathbf{K}\| \cup \mathcal{N}$, where $\mathcal{N}$ contains at least one $\neg\alpha_i$-world for each $\alpha_i \in B \cap \mathbf{K}$, can be shown to hold if B is a finite set, as we can see in the following observation:

Observation 6.1.1 *Let $\mathbf{K}$ be a belief set and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. If $X \in \mathbf{K} \perp B$ then*

$$\|X\| = \|\mathbf{K}\| \cup \mathcal{N},$$

where $\mathcal{N} \subseteq \mathcal{M}_{\mathcal{L}}$ is such that:

1. $\mathcal{N} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B\}$.
2. $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, *for all* $\alpha_i \in B \cap \mathbf{K}$.

Proof. A proof for this observation can be found in page 183. ∎

Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. The above result assures that it holds that the set of possible worlds for any given remainder set $X \in \mathbf{K} \perp B$, with $B \cap Cn(\emptyset) = \emptyset$, must contain at lest one $\neg\alpha_i$-world for each $\alpha_i \in B \cap \mathbf{K}$. However, contrary to what is the case in the singleton version (where the set of possible worlds for any given remainder set of $\mathbf{K}$ by φ, for some $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$, must contain one and only one $\neg\varphi$-world (see Observation 4.1.1-1)), in the multiple case it may happen that, even for some $\alpha_i \in B \cap \mathbf{K}$[101], there is more than one $\neg\alpha_i$-world included in $\|X\|$.

Notwithstanding, in what follows we will see that if $X \in \mathbf{K}\bot B$, then $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$, for some $\mathcal{W} \subseteq \mathcal{M}_{\mathcal{L}}$ such that, on the one hand, $\mathcal{W}$ contains at least one $\neg\alpha_i$-world for each $\alpha_i \in B \cap \mathbf{K}$ and, on the other hand, if $W \in \mathcal{W}$ then W is the only $\neg\alpha_j$-world included in $\mathcal{W}$, for some $\alpha_j \in B \cap \mathbf{K}$. More precisely, we will show that, given a belief set $\mathbf{K}$ and a finite set of sentences B, if $X \in \mathbf{K}\bot B$, then $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$, for some $\mathcal{W} \subseteq \bigcup_{\alpha_i \in B} \|\neg\alpha_i\|$ such that:

(i) $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.

(ii) If $W \in \mathcal{W}$ then there is some $\alpha_i \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_i\| = \{W\}$.

To see that this indeed holds, we start by formally introducing, in the following definition, the set $\mathbb{W}_{\mathbf{K}\bot B}$ composed by all the sets of possible worlds that satisfy conditions (i) and (ii) mentioned above.

Definition 6.1.2 *Let $\mathbf{K}$ be a belief set and B be a set of sentences. We denote by $\mathbb{W}_{\mathbf{K}\bot B}$ the subset of $\mathcal{P}\left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}\right)$ such that $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$ if and only if:*

1. *$\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.*
2. *If $M \in \mathcal{W}$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$.*

Figures 6.1, 6.2, 6.3 and 6.4 contain several possible graphical representations of different sets of the form $\|\mathbf{K}\| \cup \mathcal{W}$, with $\mathcal{W} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$, for some belief set $\mathbf{K}$ and some set of sentences $B \subseteq \mathbf{K} \setminus Cn(\emptyset)$. The mentioned figures illustrate four different situations in what concerns the set $\mathcal{W}$. Figure 6.1 exemplifies a situation in which $B = \{\alpha_1, \alpha_2\}$, $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$ and, furthermore, for each $\alpha_i \in B \cap \mathbf{K}$ there is one and only one $\neg\alpha_i$-world in $\mathcal{W}$. Figure 6.2 illustrates a situation where $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$, but there is one $\alpha_i \in B \cap \mathbf{K}$ such that there is more than one $\neg\alpha_i$-world in $\mathcal{W}$. Figure 6.3 represents an example where $B = \{\alpha_1, \alpha_2\}$ and $\mathcal{W} \notin \mathbb{W}_{\mathbf{K}\bot B}$ because there is one world in $\mathcal{W}$ that is not the only $\neg\alpha_i$-world for any $\alpha_i \in B \cap \mathbf{K}$. Finally, Figure 6.4 represents an example where $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathcal{W} \notin \mathbb{W}_{\mathbf{K}\bot B}$ because $\mathcal{W}$ does not contain any $\neg\alpha_j$-world for some $\alpha_j \in B \cap \mathbf{K}$.

Now, given a belief set $\mathbf{K}$ and a set of sentences B, we investigate the composition of the set $\mathbb{W}_{\mathbf{K}\bot B}$ in each of the following possible situations concerning the set B: (a) $B \cap Cn(\emptyset) \neq \emptyset$, (b) $B \cap \mathbf{K} = \emptyset$ and (c) $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$.

We start by introducing the following Lemma:

Lemma 6.1.3 *Let $\mathbf{K}$ be a belief set, B be a finite set of sentences and $\mathcal{N}$ be a subset of $\mathcal{M}_{\mathcal{L}}$. If $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$, then there is some set $\mathcal{N}'$ such that $\mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\bot B}$.*[102]

[101]Note that if $\alpha_i \in B \setminus \mathbf{K}$ it can obviously be the case that there is more than one $\neg\alpha_i$-world included in $\|\mathbf{K}\|$.

[102]Notice that from $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$, it follows that $B \cap Cn(\emptyset) = \emptyset$. Indeed, assume by *reductio* that there is some $\alpha_j \in B \cap Cn(\emptyset)$. Then, on the one hand $\alpha_j \in B \cap \mathbf{K}$ and, on the other hand $\|\neg\alpha_j\| = \emptyset$. Hence, there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{N} \cap \|\neg\alpha_j\| = \emptyset$, which is a contradiction.

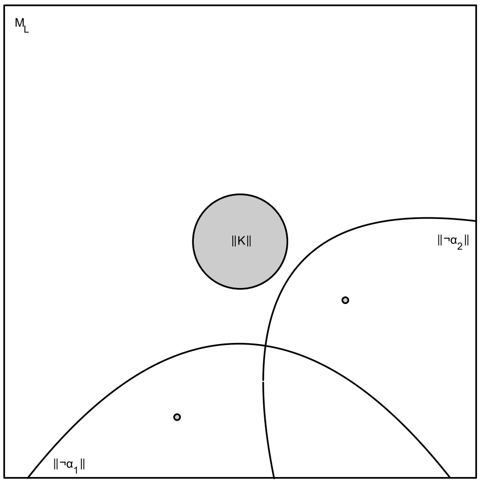

Figure 6.1: Graphical representation of a set of possible worlds of the form $\|\mathbf{K}\| \cup \mathcal{W}$, with $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$, where $\mathbf{K}$ is a belief set and $B = \{\alpha_1, \alpha_2\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

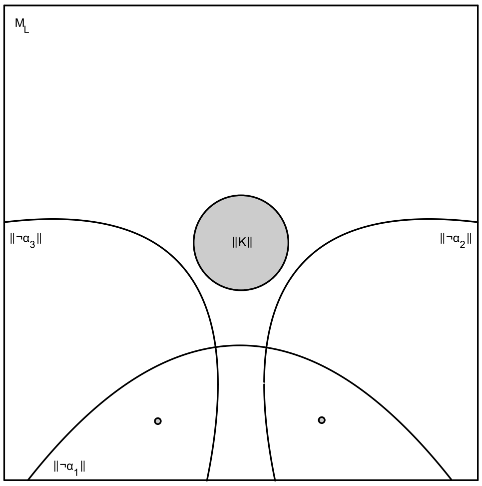

Figure 6.2: Graphical representation of a set of possible worlds of the form $\|\mathbf{K}\| \cup \mathcal{W}$, with $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$, where $\mathbf{K}$ is a belief set and $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

Proof. A proof for this lemma can be found in page 184. ■

Making use of the above lemma we can conclude that the set $\mathbb{W}_{\mathbf{K}\perp B}$ is composed as indicated in the following observation.

Observation 6.1.4 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then the following statements hold:*

1. $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.
2. $B \cap \mathbf{K} = \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$.
3. $B \cap Cn(\emptyset) = \emptyset$ *and* $B \cap \mathbf{K} \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ *and* $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$.

Proof. A proof for this observation can be found in page 185. ■

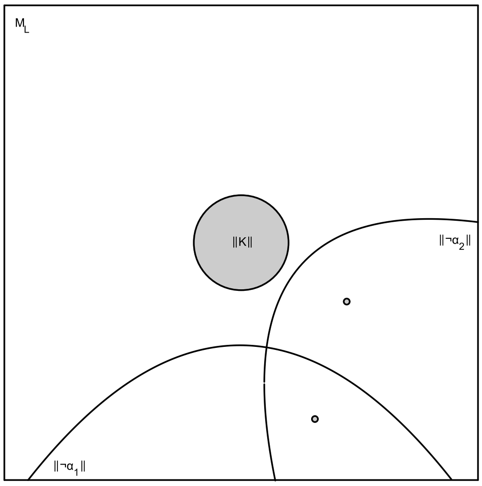

Figure 6.3: Graphical representation of a set of possible worlds of the form $\|\mathbf{K}\| \cup \mathcal{W}$, with $\mathcal{W} \notin \mathbb{W}_{\mathbf{K}\perp B}$, where $\mathbf{K}$ is a belief set and $B = \{\alpha_1, \alpha_2\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

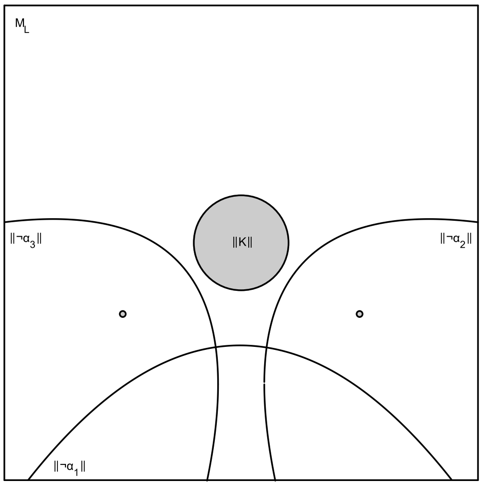

Figure 6.4: Graphical representation of a set of possible worlds of the form $\|\mathbf{K}\| \cup \mathcal{W}$, with $\mathcal{W} \notin \mathbb{W}_{\mathbf{K}\perp B}$, where $\mathbf{K}$ is a belief set and $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

At this point it is convenient to notice that in the special case $\mathbf{K} = Cn(\emptyset)$ it holds, for any finite set of sentences B, that either $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$ or $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$. This fact is explained in more detail in the following remark:

Remark 6.1.5 *Let $\mathbf{K} = Cn(\emptyset)$ and B be a finite set of sentences. Then it follows immediately from Observation 6.1.4 above that:*

1. *$B \cap \mathbf{K} \neq \emptyset$ if and only if $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.*

2. *$B \cap \mathbf{K} = \emptyset$ if and only if $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$.*

Having defined the set $\mathbb{W}_{\mathbf{K}\perp B}$, for a belief set $\mathbf{K}$ and any set of sentences B and, moreover, analysed its composition for different situations concerning the set B, in the following theorem we present the relation between $\mathbb{W}_{\mathbf{K}\perp B}$ and $\mathbf{K}\perp B$.

Theorem 6.1.6 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then:*

1. *If* $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ *then* $X = Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$.
2. *If* $X \in \mathbf{K}\perp B$ *then there is some* $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ *such that* $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.

Proof. A proof for this theorem can be found in page 186. ∎

An immediate consequence of the above observation is the fact that there is a surjective function from $\mathbb{W}_{\mathbf{K}\perp B}$ to $\mathbf{K}\perp B$, for any belief set $\mathbf{K}$ and any finite set of sentences B, as stated in the following corollary. Throughout this chapter such surjection will play an analogous role to the one played by Grove's bijection (see Observation 4.1.1) in the singleton case.

Corollary 6.1.7 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then the function*

$$\begin{array}{rcl} t : \mathbb{W}_{\mathbf{K}\perp B} & \to & \mathbf{K}\perp B \\ \mathcal{W} & \mapsto & Th(\|\mathbf{K}\| \cup \mathcal{W}) \end{array}$$

is surjective.

Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. By means of the above presented surjective function from $\mathbb{W}_{\mathbf{K}\perp B}$ to $\mathbf{K}\perp B$ we can obtain the definition of the latter of these sets in terms of possible worlds that is presented in the following corollary (which, to some extent, generalizes Remark 4.1.2 to the multiple contraction case).

Corollary 6.1.8 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then* $\mathbf{K}\perp B = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}\}$.

The following remark clarifies why and how the the surjective function introduced in Corollary 6.1.7 can be seen as a generalization of Grove's bijection (Observation 4.1.1-3) to the multiple case.

Remark 6.1.9 *Notice that the function* t *introduced in Corollary 6.1.7 above can be seen as a generalization of the function* g *presented in Observation 4.1.1-3 (Grove's bijection) to the case of remainders of* $\mathbf{K}$ *by (not necessarily singleton) sets of sentences* B. *This is true in the sense that, whenever* B *is a singleton set (such that* $B \cap \mathbf{K} \neq \emptyset$*), the function* t *coincides with the function* g. *To see this, let* φ *be an arbitrary sentence such that* $\varphi \in \mathbf{K}$ *and consider on the one hand Grove's bijection* $g : \|\neg\varphi\| \to \mathbf{K}\perp\varphi$ *(as defined in Observation 4.1.1-3) and, on the other hand the function* $t : \mathbb{W}_{\mathbf{K}\perp\{\varphi\}} \to \mathbf{K}\perp\{\varphi\}$ *defined in Corollary 6.1.7. It follows from Definition 6.1.2 that* $\mathbb{W}_{\mathbf{K}\perp\{\varphi\}} = \{\{W\} : W \in \|\neg\varphi\|\}$. *Hence, if we identify each world* $W \in \|\neg\varphi\|$ *with the singleton set* $\{W\}$, *then the set* $\|\neg\varphi\|$ *is identified with the set* $\mathbb{W}_{\mathbf{K}\perp\{\varphi\}}$, *i.e. the functions* g *and* t *can be regarded as having the same domain. On the other hand, since* $\mathbf{K}\perp\varphi = \mathbf{K}\perp\{\varphi\}$, *then* r *and* t *also have the same codomain. And finally, it additionally holds that* $g(W) = Th(\|\mathbf{K}\| \cup \{W\}) = t(\{W\})$, *for every* $W \in \|\neg\varphi\|$ *(or, equivalently, for every* $\{W\} \in \mathbb{W}_{\mathbf{K}\perp\{\varphi\}}$*).*

Observe also that from the above we can conclude that the function t *is a bijective function whenever* B *is a singleton set.*

At this point it is worth noticing that, since any partial meet multiple contraction is, roughly speaking, given by intersections of remainders, the above results will play a very relevant role in the remaining of this work. More precisely, in what remains of the present chapter we will make use of the above presented possible worlds-based characterization of the remainders by sets of sentences to obtain the possible worlds semantics for the partial meet multiple contractions.

6.2 Possible Worlds Semantics for Full Meet Multiple Contraction

Having in mind, our goal of obtaining the characterization of the partial meet multiple contractions in terms of possible worlds, we will start by considering, in this section, the limiting case of the full meet multiple contraction (see Definition 5.3.3).

More precisely, by making use of the surjective function between the sets $\mathbb{W}_{\mathbf{K}\perp B}$ and $\mathbf{K}\perp B$, for a belief set $\mathbf{K}$ and a finite set of sentences B, presented in Corollary 6.1.7, we will show how the full meet multiple contraction by finite sets can be defined when considering a possible worlds-based approach. Afterwards, as a consequence of such characterization of the full meet multiple contraction in terms of possible worlds, we will obtain a syntactic formula for the full meet multiple contraction by finite sets which generalizes the result of Observation 4.1.7 to the multiple contraction case. Additionally, we will identify which worlds compose the set $\|\mathbf{K}\mathrel{\dot\sim}C\|$ for any finite set of sentences C, where $\dot\sim$ is the full meet multiple contraction on $\mathbf{K}$.

We start by recalling that, according to Definition 5.3.3, the full meet multiple contraction $\dot\sim$ on a belief set $\mathbf{K}$ is such that $\mathbf{K}\mathrel{\dot\sim}B = \mathbf{K}$, if $B \cap Cn(\emptyset) \neq \emptyset$, and $\mathbf{K}\mathrel{\dot\sim}B = \bigcap \mathbf{K}\perp B$, if $B \cap Cn(\emptyset) = \emptyset$. Hence, having in mind Corollary 6.1.8, we can easily show that the following observation holds:

Observation 6.2.1 *Let $\mathbf{K}$ be a belief set and $\dot\sim$ be the full meet multiple contraction on $\mathbf{K}$. Then*

$$\mathbf{K}\mathrel{\dot\sim}B = \bigcap\left(\|\mathbf{K}\| \cup \left(\bigcup \mathbb{W}_{\mathbf{K}\perp B}\right)\right) = \mathbf{K} \cap \left(\bigcap\left(\bigcup \mathbb{W}_{\mathbf{K}\perp B}\right)\right),$$ [103]

for all finite sets of sentences B.

Proof. A proof for this observation can be found in page 187. ■

Having seen this, we will now investigate which worlds are included in the set $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$, for any belief set $\mathbf{K}$ and any finite set of sentences B.

Observation 6.2.2 *Let $\mathbf{K}$ be a belief set and B a set of sentences. Then:*

1. *If $B \cap Cn(\emptyset) \neq \emptyset$ or $B \cap \mathbf{K} = \emptyset$ then $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.*

[103] Notice that, in the case $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \emptyset$, the identity $\bigcap(\|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K}\perp B})) = (\bigcap \|\mathbf{K}\|) \cap (\bigcap(\bigcup \mathbb{W}_{\mathbf{K}\perp B}))$ is valid due to the fact that $Th(\emptyset) = \bigcap \emptyset = \mathcal{L}$ (according to Definition 3.3.2). Furthermore note that, in that case, $\mathbf{K} \cap (\bigcap(\bigcup \mathbb{W}_{\mathbf{K}\perp B})) = \mathbf{K} \cap \mathcal{L} = \mathbf{K}$.

2. $\bigcup \mathbb{W}_{\mathbf{K}\perp B} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

3. *If* $B \cap \mathbf{K} = \emptyset$, *then* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

4. *If* $\mathbf{K} = Cn(\emptyset)$ *then* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

Proof. A proof for this observation can be found in page 188. ■

From the above observation we can conclude that there are several conditions on $\mathbf{K}$ and B under which it holds that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$. However this identity does not hold in general. In fact, we will start our study of the composition of the set $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$ for any belief set $\mathbf{K}$ and any finite set B such that $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$[104] by clarifying, by means of (the counterexample provided within the proof of) the following observation that, in general, provided that $\mathbf{K} \neq Cn(\emptyset)$[105], it does not hold that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ (even if $B \cap Cn(\emptyset) = \emptyset$[106]).

Observation 6.2.3 *Let* $\mathbf{K}$ *a belief set and* B *be a set of sentences such that* $\mathbf{K} \neq Cn(\emptyset)$ *and* $B \cap \mathbf{K} \neq \emptyset$. *Then:*

1. *It may not hold that* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ *(whether or not* $B \cap Cn(\emptyset) = \emptyset$[106]*).*

2. *If* M *is a world such that*

   (i) $M \in \|\neg\alpha_j\|$ *for some* $\alpha_j \in B \cap \mathbf{K}$,

   (ii) $M \notin \|\neg\alpha_l\|$ *for every* $\alpha_l \in B \setminus \{\alpha_j\}$, *and*

   (iii) there is some $\alpha_r \in B \cap \mathbf{K}$ *such that* $\|\neg\alpha_r\| \subset \|\neg\alpha_j\|$

   then $M \notin \bigcup \mathbb{W}_{\mathbf{K}\perp B}$.

Proof. A proof for this observation can be found in page 188. ■

Figure 6.5 contains a possible graphical representation of a situation where the world M, there represented, satisfies the conditions stated in Point 2. of the above observation and, therefore, it is such that $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ but $M \notin \bigcup \mathbb{W}_{\mathbf{K}\perp B}$, for a belief set $\mathbf{K}$ and a finite set of sentences B such that $\mathbf{K} \neq Cn(\emptyset)$ and $B \cap Cn(\emptyset) = \emptyset$.

Having seen this, at this point all we can say regarding the explicit description of the worlds that compose the set $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$, for any belief set $\mathbf{K}$ and any finite set

[104] The case $B \cap Cn(\emptyset) \neq \emptyset$ or $B \cap \mathbf{K} = \emptyset$ has already been considered in Observation 6.2.2-1..

[105] See Observation 6.2.2-4..

[106] Notice that if $B \cap Cn(\emptyset) \neq \emptyset$ then it can be trivially shown that, in general, $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} \not\subseteq \bigcup \mathbb{W}_{\mathbf{K}\perp B}$. Indeed, assume $\mathbf{K}$ is a belief set such that $\mathbf{K} \neq Cn(\emptyset)$ and let $\alpha_l \in \mathbf{K} \setminus Cn(\emptyset)$ and B be any finite set of sentences such that $\alpha_l \in B$ and $B \cap Cn(\emptyset) \neq \emptyset$. Then, on the one hand, according to Observation 6.1.4-1., $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$ and on the other hand, from $\|\neg\alpha_l\| \neq \emptyset$ and $\|\neg\alpha_l\| \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ if follows that $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} \neq \emptyset$. Hence $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} \not\subseteq \bigcup \mathbb{W}_{\mathbf{K}\perp B} (= \emptyset)$.

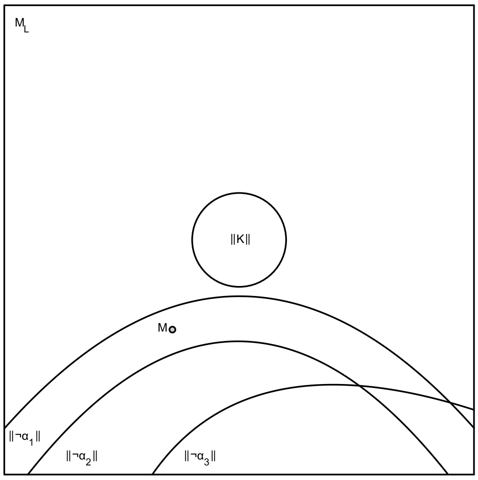

Figure 6.5: Graphical representation of $\mathcal{M}_{\mathcal{L}}$ highlighting a possible world $M \in (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}) \setminus (\bigcup \mathbb{W}_{\mathbf{K}\perp B})$, where $\mathbf{K}$ is a belief set and $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

of sentences B, is that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ and that, in some cases, such inclusion is indeed a proper inclusion. Furthermore we have already identified in Point 2. of the above observation some sufficient conditions for a world M of $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ not to belong to $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$.

Hence, given a belief set $\mathbf{K}$ and a finite set of sentences B, in order to find out which worlds of the set $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ are indeed included in $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$, we shall start by finding a way of defining a subset of $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ that does not contain any world M satisfying (simultaneously) conditions (i)$-$(iii) mentioned in Point 2. of the above observation (and which, therefore, does not belong to $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$).

Having this in mind, we now notice that, if we consider a subset B' of B consisting of those and only those sentences α_i in B such that for any sentence $\alpha_j \in B$, it holds that $\|\neg\alpha_j\| \not\subset \|\neg\alpha_i\|$, then the set $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B' \cap \mathbf{K}\}$ is a subset of $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ in the desired conditions, i.e., if $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B' \cap \mathbf{K}\}$ then M does not simultaneously satisfy the conditions (i)$-$(iii) mentioned above.

Taking this into account, in what follows we will show that the subset of $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ just introduced above coincides indeed with the set $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$.

In order to do that we start by introducing the definition of *normalization of a set of sentences* B, which, for a given set of sentences B, is indeed nothing more than its subset B' that we have already anticipated above.

Definition 6.2.4 (Normalization of a set of sentences) *Let B be a set of sentences. The* normalization *of B is the subset $B_N \subseteq B$, defined in the following way:*[107]

$$B_N = \{\alpha_i \in B : \textit{For all } \alpha_j \in B \ \ \textit{it holds that } \|\neg\alpha_j\| \not\subset \|\neg\alpha_i\|\}.$$

[107] An alternative (but equivalent) definition for the set B_N, which does not make use of the notion of possible world can be found in Definition *F.2.8*

Next, we introduce a couple of lemmas that we will make use of later on when proving that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$, for any belief set $\mathbf{K}$ and any finite set of sentences B.

Lemma 6.2.5 *Let B be a finite set of sentences. If $\alpha_l \in B$ and $\alpha_l \notin B_N$ then there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_l\|$.*

Proof. A proof for this lemma can be found in page 189. ■

Lemma 6.2.6 *Let $\mathbf{K}$ be a belief set, B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ and $f : \{\|\alpha\| : \alpha \in \mathcal{L} \text{ and } \nvdash \neg\alpha\} \to \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ be a function such that if $\alpha \in \mathcal{L}$ and $\nvdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$.*

If $B_f \subseteq B$ is such that:

(i) If $\alpha_j \in B_f$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subseteq \|\neg\alpha_j\|$, for all $\alpha_k \in B_f$;

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_f \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$;

then

1. *For every world $M \in \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ there is some subset $\mathcal{W}_M \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$, such that $M \in \mathcal{W}_M$ and $\mathcal{W}_M \in \mathbb{W}_{\mathbf{K}\perp B}$;*
2. *$\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} \neq \emptyset$ and $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} = \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$.*

Proof. A proof for this lemma can be found in page 190. ■

The following observation is an immediate consequence of the two lemmas above:

Observation 6.2.7 *Let $\mathbf{K}$ be a belief set, B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. Then $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$.*

Proof. A proof for this observation can be found in page 191. ■

We are now in a position to explicitly indicate which worlds compose the set $\bigcup \mathbb{W}_{\mathbf{K}\perp B}$, for any belief set $\mathbf{K}$ and any finite set of sentences B.

Observation 6.2.8 *Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. Then*

$$\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}.$$

Proof. A proof for this observation can be found in page 192. ■

Now we observe that, combining the identity just presented in the above observation with the result of Observation 6.2.1 we can obtain the characterization of the full meet multiple contraction presented in the following corollary (of Observations 6.2.1 and 6.2.8):[108]

[108] Notice that such characterization generalizes the result of [AM82, Observation 2.1] (presented in Observation 4.1.7 of the present thesis) to the multiple contraction case.

Corollary 6.2.9 *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *be the operator of full meet multiple contraction on* $\mathbf{K}$*. Then, under the assumption that* $\bigcap \emptyset = \mathcal{L}$,[109] *it holds that*

$$\begin{aligned} \mathbf{K} \sim B &= \bigcap \left(\|\mathbf{K}\| \cup \left(\bigcup \{ \|\neg \alpha_i\| : \alpha_i \in B_N \cap \mathbf{K} \} \right) \right) \\ &= \mathbf{K} \cap \left(\bigcap \{ Cn(\neg \alpha_i) : \alpha_i \in B_N \cap \mathbf{K} \} \right), \end{aligned}$$

for all finite sets of sentences B.[110]

Proof. A proof for this corollary can be found in page 192. ∎

Next we introduce one last observation concerning the full meet multiple contraction, that explicitly indicates which worlds compose the set $\|\mathbf{K} \sim B\|$, where $\mathbf{K}$ is a belief set, $\sim$ is the operator of full meet multiple contraction on $\mathbf{K}$ and B is a finite set of sentences.

Observation 6.2.10 *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *be the operator of full meet multiple contraction on* $\mathbf{K}$*. Then*

$$\|\mathbf{K} \sim B\| = \|\mathbf{K}\| \cup \left(\bigcup \mathbb{W}_{\mathbf{K} \perp B} \right) = \|\mathbf{K}\| \cup \left(\bigcup \{ \|\neg \alpha_i\| : \alpha_i \in B_N \cap \mathbf{K} \} \right),$$

for all finite sets of sentences B.

Proof. A proof for this observation can be found in page 193. ∎

To close this section we remind the example illustrated in Figure 6.5 and present the set $\|\mathbf{K} \sim B\|$ where $\mathbf{K}$ and B are, respectively, a belief set and a (finite) set of sentences satisfying the conditions proposed in the mentioned example, and $\sim$ is the operator of full meet multiple contraction on $\mathbf{K}$.

Hence, we start by observing that in the situation represented by the mentioned figure it holds that $\mathbf{K}$ is a belief set such that $\mathbf{K} \neq Cn(\emptyset)$ and $B = \{\alpha_1, \alpha_2, \alpha_3\}$, where α_1, α_2 and α_3 are three sentences in $\mathbf{K} \setminus Cn(\emptyset)$ such that $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$, $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$ and $\|\neg\alpha_3\| \not\subset \|\neg\alpha_2\|$. In these conditions we have, according to Definition 6.2.4, that $B_N = \{\alpha_2, \alpha_3\}$ (notice that $\alpha_1 \notin B_N$, because $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$). Then it

[109] Notice that this is a natural assumption and, nonetheless, it is analogous to (and consistent with) the stipulation that $Th(\emptyset) = \bigcap \emptyset = \mathcal{L}$ introduced in Definition 3.3.2. More precisely, the only difference between the two situations is the fact that, while in the mentioned definition the empty set $\emptyset$ is seen as the empty subset of $\mathcal{M}_{\mathcal{L}}$ (cf. Footnote 57), in the present context it is seen as the empty subset of $\mathcal{P}(\mathcal{L})$. Indeed, for any belief set $\mathbf{K}$ and any set of sentences B, we have that $\{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\} \subseteq \mathcal{P}(\mathcal{L})$, hence, in the present context, whenever it is the case that $\{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\} = \emptyset$ we shall regard such empty set $\emptyset$ as a subset of $\mathcal{P}(\mathcal{L})$, i.e. $\emptyset \subseteq \mathcal{P}(\mathcal{L})$. Therefore it follows that $\bigcap \emptyset = \{\alpha : \forall M \in \emptyset, \alpha \in M\} = \{\alpha : \forall M \in (\emptyset \cap \mathcal{P}(\mathcal{L})), \alpha \in M\} = \{\alpha \in \mathcal{L} : \forall M \in \emptyset, \alpha \in M\} = \mathcal{L}$ (where this last equality holds due to the fact that, for any $\alpha \in \mathcal{L}$, the condition "$\forall M \in \emptyset, \alpha \in M$" is vacuously true).

[110] Notice that, on the one hand, if $B \cap \mathbf{K} = \emptyset$ then $B_N \cap \mathbf{K} = \emptyset$ and therefore $\mathbf{K} \cap (\bigcap \{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\}) = \mathbf{K} \cap (\bigcap \emptyset) = \mathbf{K} \cap \mathcal{L} = \mathbf{K}$ (where we have made use of the assumption that $\bigcap \emptyset = \mathcal{L}$). On the other hand, if $B \cap Cn(\emptyset) \neq \emptyset$ then, according to Definition 6.2.4, $B_N = B \cap Cn(\emptyset) \subseteq Cn(\emptyset)$, from which we can conclude that $Cn(\neg\alpha_i) = \mathcal{L}$, for all $\alpha_i \in B_N \cap \mathbf{K}$, and, consequently, $\mathbf{K} \cap \bigcap \{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\} = \mathbf{K} \cap \mathcal{L} = \mathbf{K}$.

follows from Corollary 6.2.9 that $\mathbf{K}\dot{\sim}B = \bigcap(\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})) = \bigcap(\|\mathbf{K}\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$. Furthermore, according to Observation 6.2.10, $\|\mathbf{K}\dot{\sim}B\| = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}) = \|\mathbf{K}\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|$.

Figure 6.6 contains a possible graphical representation of the situation presented above (notice that the sets $\|\neg\alpha_1\|$, $\|\neg\alpha_2\|$, $\|\neg\alpha_3\|$ and $\|\mathbf{K}\|$ are represented in precisely the same positions as in Figure 6.5 in order to allow for a direct comparison between the two illustrations). The shaded region in Figure 6.6 highlights the set $\|\mathbf{K}\dot{\sim}B\| = \|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K}\perp B}) = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$. Hence, as we have already noticed above, the set $\mathbf{K}\dot{\sim}B$ consists of the intersection of all the worlds included in that region.

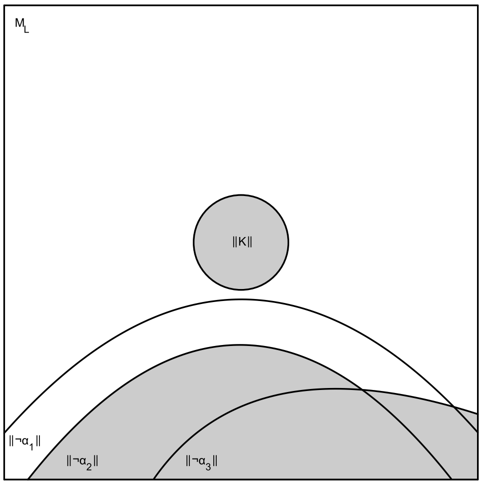

Figure 6.6: Graphical representation of $\mathcal{M}_{\mathcal{L}}$ highlighting the region $\|\mathbf{K}\dot{\sim}B\| = \|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K}\perp B}) = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$, where $\mathbf{K}$ is a belief set, $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

6.3 Possible Worlds Semantics for Partial Meet Multiple Contractions (the General Case)

Now that we have thoroughly exposed the characterization in terms of possible worlds of the full meet multiple contraction, in this last section of the present chapter, we turn our attention to the obtention of the possible worlds semantics of the partial meet multiple contractions in general.

We start by using Observation 6.2.10 to obtain a superset of the set $\|\mathbf{K}\div B\|$, where $\mathbf{K}$ is a belief set, $\div$ is a partial meet multiple contraction on $\mathbf{K}$ and B is a finite set of sentences:

Observation 6.3.1 *Let* $\mathbf{K}$ *be a belief set. If an operation* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *then*

$$\|\mathbf{K}\div B\| \subseteq \|\mathbf{K}\| \cup \left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\right),$$

for all finite sets of sentences B.

Proof. A proof for this observation can be found in page 194. ■

Our next goal is to obtain the characterization of the partial meet multiple contraction functions in terms of possible worlds. In order to do that we will start by proving that, given a belief set $\mathbf{K}$, it holds that $\mathbf{K}\perp B = \mathbf{K}\perp C$ iff $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$, for any finite sets of sentences B and C. However, to that we will need to make use of the following results:

Lemma 6.3.2 *Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. Then $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp(B_N \cap \mathbf{K})}$.*

Proof. A proof for this lemma can be found in page 194. ■

Lemma 6.3.3 *Let $\mathbf{K}$ be a belief set and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for all $\beta_i \in G_N \cap \mathbf{K}$ there is some $\varsigma_i \in H_N \cap \mathbf{K}$ such that $\|\beta_i\| = \|\varsigma_i\|$.*

Proof. A proof for this lemma can be found in page 195. ■

Observation 6.3.4 *Let $\mathbf{K}$ be a belief set and B and C be two finite sets of sentences. Then:*

$$\mathbf{K}\perp B = \mathbf{K}\perp C \textit{ iff } \mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}.$$

Proof. A proof for this observation can be found in page 195. ■

Combining Lemma 6.3.2 with Observation 6.3.4 and having in mind the representation of remainders in terms of possible worlds (i.e. the surjection from $\mathbb{W}_{\mathbf{K}\perp B}$ to $\mathbf{K}\perp B$ presented in Corollary 6.1.7) we can conclude that the remainder set of $\mathbf{K}$ by B coincides with the remainder set of $\mathbf{K}$ by $B_N \cap \mathbf{K}$. Therefore $\mathbf{K}\div B$ is identical to $\mathbf{K}\div(B_N \cap \mathbf{K})$ whenever $\div$ is a partial meet multiple contraction. Thus, we are now in a position to present the characterization of the partial meet multiple contractions by finite sets in terms of possible worlds.

In order to do that we need the following concept of *propositional package selection function* (which has been adapted from Definition 4.1.3).

Definition 6.3.5 (Based on [Han99b, Definition 3.65]) *Let $\mathbf{K}$ be a belief set. A* propositional package selection function *for $\|\mathbf{K}\|$ is a function f such that for all sets of sentences B:*

1. *$f(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$.*
2. *If $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ then $f(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.*

We can now present the last and main result of the present chapter, which provides the characterization of partial meet multiple contractions in terms of possible worlds.

Theorem 6.3.6 *Let* $\mathbf{K}$ *be a belief set. An operation* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *by finite sets if and only if there is a propositional package selection function* f *for* $\|\mathbf{K}\|$ *such that:*

$$\mathbf{K}\div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})\right)\right),$$

for any finite set of sentences B*.*

Proof. A proof for this theorem can be found in page 196. ■

Using the above theorem and Theorem 6.1.6 it can be easily shown that if f is a propositional package selection function for $\|\mathbf{K}\|$ such that, for any set of sentences B, $f(\mathbb{W}_{\mathbf{K}\perp B})$ contains at most one element, then the operation $\div$ defined by $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, for any sets of sentences B, is a maxichoice multiple contraction on $\mathbf{K}$. On the other hand, it is also worth noticing that it follows immediately from Observation 6.2.1 that the operation $\dot\sim$ of full meet multiple contraction on $\mathbf{K}$ is such that $\mathbf{K}\dot\sim B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, where f is the propositional package selection function for $\|\mathbf{K}\|$ defined by $f(\mathbb{W}_{\mathbf{K}\perp B}) = \mathbb{W}_{\mathbf{K}\perp B}$, for all sets of sentences B.

The facts mentioned in the above paragraph are more detailedly and formally stated in the following corollary which provides us with a characterization for maxichoice and full meet multiple contractions, in terms of propositional package selection functions.

Corollary 6.3.7 *Let* $\mathbf{K}$ *be a belief set and* $\div$ *be a multiple contraction function on* $\mathbf{K}$*. Then:*

1. $\div$ *is a maxichoice multiple contraction on* $\mathbf{K}$ *by finite sets if and only if, for all finite sets of sentences* B*,* $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$*, where* f *is a propositional package selection function* f *for* $\|\mathbf{K}\|$ *such that* $f(\mathbb{W}_{\mathbf{K}\perp B})$ *contains at most one element (i.e.,* f *is such that* $f(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$ *whenever* $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$*, and* $f(\mathbb{W}_{\mathbf{K}\perp B})$ *is a singleton subset of* $\mathbb{W}_{\mathbf{K}\perp B}$ *if* $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$*), for any finite set of sentences* B*.*

2. $\div$ *is the full meet multiple contraction on* $\mathbf{K}$ *by finite sets if and only if, for all finite sets of sentences* B*,* $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$*, where* f *is the propositional package selection function for* $\|\mathbf{K}\|$ *defined by* $f(\mathbb{W}_{\mathbf{K}\perp B}) = \mathbb{W}_{\mathbf{K}\perp B}$*, for all finite sets of sentences* B*.*

At this point it is interesting to observe that Theorem 6.3.6 and the above corollary can be regarded also as alternative definitions for partial meet and for maxichoice and full meet multiple contractions, which are based on the concept of propositional package selection function rather than on the concept of package selection function (cf. Definitions 5.3.2 and 5.3.3).

6.4 Summary

In this chapter we have presented the possible worlds semantics for the partial meet multiple contraction that generalizes the possible worlds semantics for partial meet (singleton) contraction presented in [Gro88, Han99b] (which we have exposed in Section 4.1).

To be more precise we have shown (cf. Theorem 6.1.6) that every remainder $X \in \mathbf{K}\perp B$ is such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$ for some set of possible worlds $\mathcal{W}$ such that (i) $\mathcal{W}$ contains at least one $\neg\alpha_i$-world for every $\alpha_i \in B \cap \mathbf{K}$ and (ii) for every world in $\mathcal{W}$ there is a sentence $\alpha \in B \cap \mathbf{K}$ for which that one is the only $\neg\alpha$-world in $\mathcal{W}$. Subsequently (cf. Theorem 6.3.6) we have proven that partial meet multiple contractions can be defined by means of intersections of sets of the form $\|\mathbf{K}\| \cup (\bigcup \mathbb{M})$, where $\mathbb{M}$ consists of a selection of sets of possible worlds that satisfy the above conditions (i) and (ii).

In particular, we have also characterized in terms of possible worlds the upper and lower bounds of that class of functions, i.e., the full meet multiple contraction and the maxichoice multiple contractions (cf. Corollary 6.3.7).

Chapter 7

System of Spheres-based Multiple Contraction Functions

In this chapter we will generalize Grove's construction of a S.S.-based contraction to the multiple case. More precisely, we will propose a way of constructing multiple contraction functions that generalizes the definition of the S.S.-based (singleton) contractions introduced in [Gro88] to the case when we wish to remove from a belief set a (possibly non-singleton) set of sentences rather than only a single sentence. Furthermore, having in mind that in the singleton case the S.S.-based contractions are partial meet contractions (cf. Observation 4.1.9), it is our goal that the new multiple contraction functions that we shall propose constitute a subclass of the class of partial meet multiple contractions.

More formally, in what follows our goal is to define, for any belief set $\mathbf{K}$ and any system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, a multiple contraction operation $\div_{\mathbb{S}}$ on $\mathbf{K}$ – which we will designate by $\mathbb{S}$-*based multiple contraction on* $\mathbf{K}$ – such that:

1. it generalizes (to the multiple contraction level) Grove's $\mathbb{S}$-based (singleton) contraction $-_{\mathbb{S}}$ (cf. Definition 3.3.7), in the sense that, on the one hand $\div_{\mathbb{S}}$ is defined for (possibly non-singleton) sets of sentences (rather than only for single sentences) and, on the other hand, for any sentence α it holds that $\mathbf{K}\div_{\mathbb{S}}\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$.

2. it is a partial meet multiple contraction (cf. Definition 5.3.2).

We will start by defining the operation $\div_{\mathbb{S}}$ on $\mathbf{K}$ for the trivial case when the set B to be removed from $\mathbf{K}$ is such that $B \cap Cn(\emptyset) \neq \emptyset$. Recall that, given a set B in the mention conditions, if $\div$ is a partial meet multiple contraction on $\mathbf{K}$ then $\mathbf{K}\div B = \mathbf{K}$. Hence, since we want $\div_{\mathbb{S}}$ to be a partial meet multiple contraction, we must define $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}$, for all sets B such that $B \cap Cn(\emptyset) \neq \emptyset$.[111]

[111]Notice that such definition is also compatible with the other requirement that we wish this new operation to fulfil, namely that if $B = \{\alpha\}$ then $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}-_{\mathbb{S}}\alpha$. To see this, and noticing that so far we have only defined $\mathbf{K}\div_{\mathbb{S}}B$ for sets B that contain some tautology, let $B = \{\alpha\}$ be a singleton set in these conditions. Then it follows on the one hand that $\vdash \alpha$ and therefore, according to Definition 3.3.7, $\mathbf{K}-_{\mathbb{S}}\alpha = \mathbf{K}$ and, on the other hand, according to our proposal for the (partial) definition of the operation $\div_{\mathbb{S}}$, $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}$. Hence we can conclude that for any singleton set B

Having seen this it is clear that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, the $\mathbb{S}$-*based multiple contraction on* $\mathbf{K}$ $\div_{\mathbb{S}}$ that we wish to construct (i.e., an operation that satisfies conditions 1. and 2. mentioned above) must be defined by $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}$, for all sets B such that $B \cap Cn(\emptyset) \neq \emptyset$. However, as we will see below, it is not that obvious how we shall define $\mathbf{K}\div_{\mathbb{S}}B$ for sets B such that $B \cap Cn(\emptyset) = \emptyset$ in order that $\div_{\mathbb{S}}$ fulfils the desired properties 1. and 2. introduced above.

The remaining of this chapter will be organized in the following way: We consider a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and we start by presenting, in Section 7.1, those that seem to be the most intuitively natural proposals for the definition of the $\mathbb{S}$-*based multiple contraction on* $\mathbf{K}$. However, we accompany those proposals with counterexamples which show that such operations actually do not fulfil the above mentioned requirements.

Afterwards, in Section 7.2 we present a way of constructing a function which satisfies the desired properties, and which, for that reason, we shall adopt as the definition of $\mathbb{S}$-*based multiple contraction on* $\mathbf{K}$.

Most of the results of this chapter have been included also in [FR].

7.1 Unsuccessful Proposals

Keeping in mind that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, our main goal in the present chapter is to construct a multiple contraction function that generalizes the $\mathbb{S}$-based (singleton) contraction on $\mathbf{K}$ and which is a partial meet multiple contraction, in this section we intend to show that the more intuitively appealing ways of constructing such a function are in fact inappropriate regarding the fulfilment of the desired requirements.

Before moving on it is convenient to observe that since, as we have seen above, if $\div_{\mathbb{S}}$ is an operation that satisfies the desired conditions then $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}$, for all sets B such that $B \cap Cn(\emptyset) \neq \emptyset$, in all that follows we only need to investigate how shall $\mathbf{K}\div_{\mathbb{S}}B$ be defined for sets B such that $B \cap Cn(\emptyset) = \emptyset$.

7.1.1 First Unsuccessful Proposal

Taking all the above into consideration, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, it only remains to obtain a suitable definition of $\mathbf{K}\div_{\mathbb{S}}B$ for sets B such that $B \cap Cn(\emptyset) = \emptyset$. Having this in mind, since we want $\div_{\mathbb{S}}$ to satisfy $\mathbf{K}\div_{\mathbb{S}}\{\varphi\} = \mathbf{K}-_{\mathbb{S}}\varphi$, and recalling that if $\nvdash \varphi$ then $\mathbf{K}-_{\mathbb{S}}\varphi = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi))$, one might be tempted to define $\mathbf{K}\div_{\mathbb{S}}B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$, for all sets B such that $B \cap Cn(\emptyset) = \emptyset$. However, as we will see right away, the operation $\div_{\mathbb{S}}$ thus defined is not a partial meet multiple contraction.

So, assume $\mathbf{K}$ is a belief set and $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$ and let

such that $B \cap Cn(\emptyset) \neq \emptyset$ it indeed holds that $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}-_{\mathbb{S}}\alpha$, where $\div_{\mathbb{S}}$ is the operation defined, for such sets, as we proposed above, $\alpha \in Cn(\emptyset)$ is the (only) sentence such that $B = \{\alpha\}$ and $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction on $\mathbf{K}$ (cf. Definition 3.3.7).

$\div^1_{\mathbb{S}}$ be the operation on $\mathbf{K}$ defined by

$$\mathbf{K} \div^1_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset. \end{cases}$$

for any set of sentences B. In what follows we will show that $\div^1_{\mathbb{S}}$ is not a partial meet multiple contraction.

Remark 7.1.1 *Notice that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and having in mind Definition 3.3.7, the above proposed operation $\div^1_{\mathbb{S}}$ on $\mathbf{K}$ can be equivalently defined in the following (alternative) way:*[112]

$$\mathbf{K} \div^1_{\mathbb{S}} B = \begin{cases} \bigcap_{\alpha_i \in B} \mathbf{K} -_{\mathbb{S}} \alpha_i & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset, \end{cases}$$

where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction.[113]

It follows immediately from the definition of $\div^1_{\mathbb{S}}$ that if $B = \{\alpha\}$ then $\mathbf{K} \div^1_{\mathbb{S}} B = \mathbf{K} -_{\mathbb{S}} \alpha$. However, the following simple counterexample, which is based on the example illustrated in Figure 6.5, shows that $\div^1_{\mathbb{S}}$ is not a partial meet multiple contraction on $\mathbf{K}$.[114]

Counterexample 7.1.2 *Let $\mathbf{K}$ be a belief set and $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of sentences such that $B \subseteq \mathbf{K} \setminus Cn(\emptyset)$ and consider the system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ defined by $\mathbb{S} = \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$. Notice that, in these conditions, it follows from Definition 3.3.5 that $f_{\mathbb{S}}(\neg\alpha_1) = \|\neg\alpha_1\|$, $f_{\mathbb{S}}(\neg\alpha_2) = \|\neg\alpha_2\|$ and $f_{\mathbb{S}}(\neg\alpha_3) = \|\neg\alpha_3\|$. Furthermore, assume that the sentences α_1, α_2 and α_3 are such that $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$, $\|\neg\alpha_2\| \not\subseteq \|\neg\alpha_3\|$, $\|\neg\alpha_3\| \not\subseteq \|\neg\alpha_2\|$ and $\|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|) \neq \emptyset$.*

Then, according to the definition of $\div^1_{\mathbb{S}}$ proposed above, we have that $\mathbf{K} \div^1_{\mathbb{S}} B = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_1) \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)) = \bigcap(\|\mathbf{K}\| \cup \|\neg\alpha_1\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$. And from this we can conclude that $\|\neg\alpha_1\| \subseteq \|\mathbf{K} \div^1_{\mathbb{S}} B\|$.

Hence, since $\|\neg\alpha_1\| \cap \|\mathbf{K}\| = \emptyset$ and $\|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|) \neq \emptyset$, it follows that $\|\mathbf{K} \div^1_{\mathbb{S}} B\| \not\subseteq (\|\mathbf{K}\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$.

Now observe that, as we have seen in the end of Section 6.2, in the above conditions, $B_N = \{\alpha_2, \alpha_3\}$. Therefore $\|\mathbf{K} \div^1_{\mathbb{S}} B\| \not\subseteq (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$ and, according to Observation 6.3.1, we can conclude that $\div^1_{\mathbb{S}}$ is not a partial meet multiple contraction.

Figure 7.1 contains a possible graphical representation of the situation presented in this counterexample. The shaded region in that figure represents the set $\|\mathbf{K}\| \cup$

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[112] Notice that this is the case because, whenever $B \cap Cn(\emptyset) = \emptyset$ it holds that $\bigcap_{\alpha_i \in B} \mathbf{K} -_{\mathbb{S}} \alpha_i = \bigcap_{\alpha_i \in B} (Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_i))) = \bigcap_{\alpha_i \in B} (\bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_i))) = \bigcap\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$.

[113] Based on a ranking theoretic approach Spohn [Spo10] suggested a multiple contraction function very similar to operation $\div^1_{\mathbb{S}}$.

[114] An alternative justification for the fact that a multiple contraction function $\div$ on a belief set $\mathbf{K}$ defined by $\mathbf{K} \div B = \bigcap_{\alpha_i \in B} \mathbf{K} - \alpha_i$, for all sets of sentences B, where $-$ is a partial meet (singleton) contraction operation on $\mathbf{K}$, is not, in general, a partial meet multiple contraction on $\mathbf{K}$, can be found in [FH94, p. 62].

$\left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)$. *Then, according to the definition of the operation* $\div^1_{\mathbb{S}}$*, the set* $\mathbf{K} \div^1_{\mathbb{S}} B$ *consists of the intersection of all the worlds highlighted there and so every world in that region belongs to* $\|\mathbf{K} \div^1_{\mathbb{S}} B\|$*. Noticing that in the mentioned figure the sets* $\|\neg\alpha_1\|$*,* $\|\neg\alpha_2\|$*,* $\|\neg\alpha_3\|$ *and* $\|\mathbf{K}\|$ *are represented in precisely the same positions as in Figure 6.6 (in which the shaded region corresponds to the set* $\|\mathbf{K} \dot{\sim} B\|$*), a direct comparison between such two figures allows us to observe that some of the worlds in* $\|\mathbf{K} \div^1_{\mathbb{S}} B\|$ *do not belong to* $\|\mathbf{K} \dot{\sim} B\|$ *(namely the worlds in* $\|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$*) and from this we can immediately conclude that the operation* $\div^1_{\mathbb{S}}$ *is not a partial meet multiple contraction.*

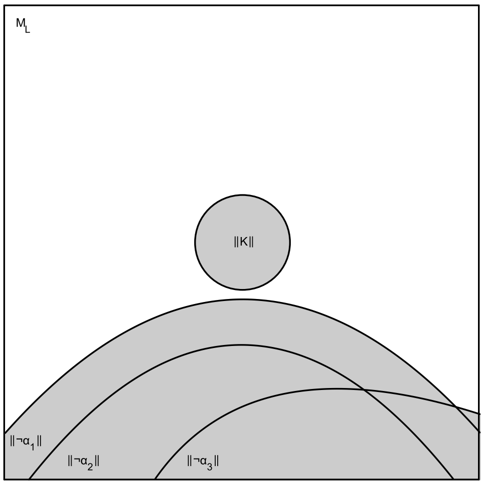

Figure 7.1: Graphical representation of $\mathcal{M}_{\mathcal{L}}$ highlighting the region $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)$, where $\mathbf{K} \neq Cn(\emptyset)$ is a belief set, $\mathbb{S} = \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$ and $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$. (A more detailed explanation of the situation illustrated by this figure can be found in Counterexample 7.1.2.)

Still having in mind the previous counterexample we notice that, in those conditions, according to Observation 6.2.10, if $\dot{\sim}$ is the operator of full meet multiple contraction on $\mathbf{K}$, then $\|\mathbf{K} \dot{\sim} B\| = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}) = \|\mathbf{K}\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|$ (cf. Figure 6.6). Hence, it is not surprising that the function $\div^1_{\mathbb{S}}$ proposed above is not a partial meet multiple contraction.

Indeed, given a finite set B, we already know from Observation 6.3.1 that the worlds in $\mathcal{M}_{\mathcal{L}} \setminus (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$ do not belong to $\|\mathbf{K} \div B\|$, for any partial meet multiple contraction $\div$ on $\mathbf{K}$.

So, given that we want the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$ to be a partial meet multiple contraction, if we define $\mathbf{K} \div_{\mathbb{S}} B$ as an intersection of a set of possible worlds then (at least in the case when B is finite) we must avoid that the worlds in $\mathcal{M}_{\mathcal{L}} \setminus (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$ are included in the set of possible worlds that we choose to intersect to originate the set $\mathbf{K} \div_{\mathbb{S}} B$.

7.1.2 Second Unsuccessful Proposal

Taking into account both the definition of the operation $\div^1_{\mathbb{S}}$ and the last paragraph of the previous subsection, given a belief set $\mathbf{K}$ and a system of spheres

$\mathbb{S}$ centred on $\|\mathbf{K}\|$, we are led to consider an operation similar to operation $\div^1_{\mathbb{S}}$ but with the following slight difference: in case B is a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ the contraction of $\mathbf{K}$ by B is given by the intersection of all the worlds in $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ rather than the intersection of all the worlds in $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)$.

More formally, let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ we will now take into consideration the operation $\div^2_{\mathbb{S}}$ on $\mathbf{K}$ defined in the following way:

$$\mathbf{K} \div^2_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset. \end{cases}$$

Noticing that whenever B is a singleton set $B_N = B$, we can conclude immediately from the above definition that if $B = \{\alpha\}$ then $\mathbf{K} \div^2_{\mathbb{S}} B = \mathbf{K} -_{\mathbb{S}} \alpha$. Nevertheless, by means of the following counterexample we will show that, as it was the case with operation $\div^1_{\mathbb{S}}$, the operation $\div^2_{\mathbb{S}}$ is not a partial meet multiple contraction on $\mathbf{K}$ either.

Counterexample 7.1.3 *Let $\mathbf{K}$ be a belief set, $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of sentences such that $B \subseteq \mathbf{K} \setminus Cn(\emptyset)$ and β be a sentence in $\mathbf{K} \setminus Cn(\emptyset)$. Assume also that the sentences $\alpha_1, \alpha_2, \alpha_3$ and β are such that $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$, $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$, $\|\neg\alpha_1\| \not\subset \|\neg\alpha_3\|$, $\|\neg\alpha_3\| \not\subset \|\neg\alpha_2\|$, $\|\neg\alpha_2\| \cap \|\neg\alpha_3\| \neq \emptyset$, $\|\neg\alpha_3\| \cap \|\neg\beta\| = \emptyset$ and $\|\neg\alpha_2\| \cap \|\neg\beta\| \neq \emptyset$.*[115]

Furthermore, let $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ such that:

(i) $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$ and $f_{\mathbb{S}}(\neg\alpha_2) \cap \|\neg\beta\| \neq \emptyset$ (notice that, since we are assuming that $\|\neg\alpha_3\| \cap \|\neg\beta\| = \emptyset$, from this two conditions, it follows that $f_{\mathbb{S}}(\neg\alpha_2) \setminus f_{\mathbb{S}}(\neg\alpha_3) \neq \emptyset$).

(ii) $\|Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))\| = \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$.[116]

Notice that, in the above conditions we have that $B_N = \{\alpha_2, \alpha_3\}$. Therefore, according to the definition of $\div^2_{\mathbb{S}}$, it follows that $\mathbf{K} \div^2_{\mathbb{S}} B = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)) = \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))$.

Now assume by reductio *that the operation $\div^2_{\mathbb{S}}$ is a partial meet multiple contraction on $\mathbf{K}$. Then, combining Theorem 6.3.6, Definition 6.3.5, Observation 6.1.4-1. and the fact that $B \cap Cn(\emptyset) = \emptyset$, we can conclude that there is some $\mathbb{M}_B \subseteq \mathbb{W}_{\mathbf{K}\perp B}$ such that $\mathbb{M}_B \neq \emptyset$ and $\mathbf{K} \div^2_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup (\bigcup \mathbb{M}_B)\right)$. Consequently, $\|\mathbf{K}\| \cup (\bigcup \mathbb{M}_B) \subseteq \|\mathbf{K} \div^2_{\mathbb{S}} B\|$ and, taking condition (ii) above into account, we obtain that $\|\mathbf{K}\| \cup (\bigcup \mathbb{M}_B) \subseteq \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$. Moreover, noticing that from $\mathbb{M}_B \subseteq \mathbb{W}_{\mathbf{K}\perp B}$ it follows that $(\bigcup \mathbb{M}_B) \cap \|K\| = \emptyset$ (since, according to Definition 6.1.2, $(\bigcup \mathbb{W}_{\mathbf{K}\perp B}) \cap \|K\| = \emptyset$), we can additionally conclude that $(\bigcup \mathbb{M}_B) \subseteq f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$.*

[115] Notice that here the conditions $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$ and $\|\neg\alpha_1\| \not\subset \|\neg\alpha_3\|$ are redundant. Indeed, on the one hand $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$ follows immediately from $\|\neg\alpha_3\| \cap \|\neg\beta\| = \emptyset$ and $\|\neg\alpha_2\| \cap \|\neg\beta\| \neq \emptyset$, and on the other hand, from $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$ and $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$ we can conclude that $\|\neg\alpha_1\| \not\subset \|\neg\alpha_3\|$.

[116] Notice that to see that there is in fact some system of spheres $\mathbb{S}$ that satisfies this condition it is enough to observe that $\|Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))\| \subseteq \|\mathbf{K}\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|$.

Furthermore, it holds that $(\bigcup \mathbb{M}_B) \subseteq f_{\mathbb{S}}(\neg\alpha_3)$. *Indeed, to see this, it is enough to notice that, since* $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$, *if* $M \in f_{\mathbb{S}}(\neg\alpha_2) \setminus f_{\mathbb{S}}(\neg\alpha_3)$[117] *then it follows immediately from Definition 6.1.2 that there is no* $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ *such that* $\mathcal{W} \subseteq f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$ *and* $M \in \mathcal{W}$.

Having seen this we are now in a position to show that the above made assumption that the operation $\div^2_{\mathbb{S}}$ *is a partial meet multiple contraction on* $\mathbf{K}$ *indeed leads to a contradiction.*

In fact, on the one hand, from $\beta \in \mathbf{K}$, $\|\neg\alpha_3\| \cap \|\neg\beta\| = \emptyset$, $(\bigcup \mathbb{M}_B) \subseteq f_{\mathbb{S}}(\neg\alpha_3) \subseteq \|\neg\alpha_3\|$ *and* $\mathbf{K} \div^2_{\mathbb{S}} B = Th\,(\|\mathbf{K}\| \cup (\bigcup \mathbb{M}_B))$ *we can conclude that* $\beta \in \mathbf{K} \div^2_{\mathbb{S}} B$. *However, on the other hand, since* $\mathbf{K} \div^2_{\mathbb{S}} B = \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))$ *and* $f_{\mathbb{S}}(\neg\alpha_2) \cap \|\neg\beta\| \neq \emptyset$ *(see condition (i) above), it follows that* $\beta \notin \mathbf{K} \div^2_{\mathbb{S}} B$. *Therefore, we can conclude that* $\div^2_{\mathbb{S}}$ *is not a partial meet multiple contraction on* $\mathbf{K}$, *as we wished to clarify.*

In Figure 7.2 we present a possible graphical representation of the situation described in the present counterexample. The shaded region represents the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$. *Hence, according to the definition of* $\div^2_{\mathbb{S}}$, *the intersection of all the worlds in that region originates the set* $\mathbf{K} \div^2_{\mathbb{S}} B$. *Furthermore we have above assumed by hypothesis that* $\|\mathbf{K} \div^2_{\mathbb{S}} B\|$ *consists precisely of the worlds highlighted in that figure. An attentive observation of such figure allows us to immediately conclude that the worlds in* $f_{\mathbb{S}}(\neg\alpha_2)$ *(and, in particular, the worlds in* $f_{\mathbb{S}}(\neg\alpha_2) \cap \|\neg\beta\|$*) can not belong to* $(\bigcup \mathbb{M}_B)$ *for any* $\mathbb{M}_B \subseteq \mathbb{W}_{\mathbf{K}\perp B}$ *such that* $(\bigcup \mathbb{M}_B) \subseteq f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$. *Additionally, combining this evidence with, on the one hand Theorem 6.3.6, and on the other hand the above mentioned facts that* $\mathbf{K} \div^2_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$ *and* $\|\mathbf{K} \div^2_{\mathbb{S}} B\| = \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$, *we can conclude that* $\div^2_{\mathbb{S}}$ *is not a partial meet multiple contraction.*

In the referred figure the sets $\|\neg\alpha_1\|$, $\|\neg\alpha_2\|$, $\|\neg\alpha_3\|$ *and* $\|\mathbf{K}\|$ *have been represented in precisely the same positions that in Figure 7.1 so that the differences in the constructions of each of the operations* $\div^1_{\mathbb{S}}$ *and* $\div^2_{\mathbb{S}}$ *are easier to observe just by comparing those two figures. However, we must notice that, when making such comparison, it must be taken into account that the underlying system of spheres is not the same in both of those representations. More precisely, in Figure 7.1 the shaded region represents the set* $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)$, *with* $\mathbb{S} = \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$, *while the highlighted region in Figure 7.2 corresponds to the set* $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ *for some* $\mathbb{S} \neq \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$.

Observing the above counterexample, we notice that, the reason why the proposed operation $\div^2_{\mathbb{S}}$ is not a partial meet multiple contraction is analogous to the reason why the full meet multiple contraction of a belief set $\mathbf{K}$ by a set B is not, in general, given *simply* by the intersection of $\|\mathbf{K}\|$ with all the $\neg\alpha_i$-worlds for all $\alpha_i \in B \cap \mathbf{K}$, i.e., in general, $\mathbf{K} \dot{\div} B \neq \bigcap\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B \cap \mathbf{K}} \|\neg\alpha_i\|\right)\right)$ (cf. Observations 6.2.3 and 6.2.10).

Indeed, and analogously to what was the case exposed in Counterexample 7.1.2, it has been illustrated in Counterexample 7.1.3 that, for some belief sets $\mathbf{K}$ there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a finite set of sentences B such that the set

[117] Notice that in the present conditions if $M \in f_{\mathbb{S}}(\neg\alpha_2) \setminus f_{\mathbb{S}}(\neg\alpha_3)$ then $M \notin \|\neg\alpha_3\|$.

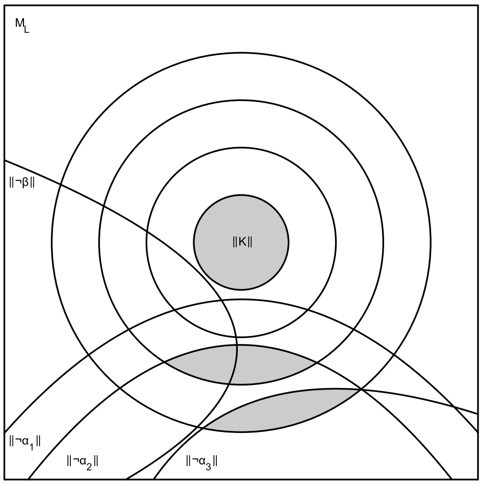

Figure 7.2: Graphical representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ for a belief set $\mathbf{K} \neq Cn(\emptyset)$. The shaded region highlights the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$, where $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$. (A more detailed explanation of the situation illustrated by this figure can be found in Counterexample 7.1.3.)

$\|\mathbf{K} \div^2_{\mathbb{S}} B\|$ includes some worlds that do not belong to any $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} \cap \mathcal{P}(\|\mathbf{K} \div^2_{\mathbb{S}} B\|)$. Such fact together with Theorem 6.3.6 allows us to conclude that the operation $\div^2_{\mathbb{S}}$ is not, in general, a partial meet multiple contraction on $\mathbf{K}$.

Having the previous paragraph in mind, we notice that in order that the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$ is a partial meet multiple contraction, if for a belief set $\mathbf{K}$ and a finite set of sentences B, $\mathbf{K} \div_{\mathbb{S}} B$ is defined as the intersection of a set of possible worlds then it seems that the set of possible worlds that shall be used for that matter must be a subset of the one that was correspondingly used in the definition of the operation $\div^2_{\mathbb{S}}$.

More precisely, given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$, according to the definition of $\div^2_{\mathbb{S}}$, the set $\mathbf{K} \div^2_{\mathbb{S}} B$ is given by the intersection of the worlds in $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$. However, from the above exposed, we conclude that when defining $\mathbf{K} \div_{\mathbb{S}} B$ as the intersection of a set of possible worlds, in a way that the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$ is a partial meet multiple contraction on $\mathbf{K}$, such set of possible worlds must, in general, be a proper subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ at least in the case when B is finite. To be more exact, from all the above we are led to conjecture that (at least when B is finite) the set $\mathbf{K} \div_{\mathbb{S}} B$ should be defined as the intersection of the set of possible worlds that results of removing from $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ all the worlds that do not belong to any $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} \cap \mathcal{P}(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right))$.

In particular we may notice, from Counterexample 7.1.3 that, if $B = \{\alpha_i, \alpha_j\} \subset \mathbf{K}$ and α_i, α_j are such that $\|\neg\alpha_i\| \not\subset \|\neg\alpha_j\|$, $\|\neg\alpha_j\| \not\subset \|\neg\alpha_i\|$, but $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i) \neq \emptyset$ then, when defining the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$ we must prevent that the worlds in $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i)$ are included in the set of possible worlds that we choose to intersect to originate $\mathbf{K} \div_{\mathbb{S}} B$.

Hence, we might expect that it would be enough to define the operation $\div_{\mathbb{S}}$ in the same way that we have defined operation $\div^2_{\mathbb{S}}$ with the only difference that whenever B is a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ then $\mathbf{K} \div_{\mathbb{S}} B$ is given by the intersection of the worlds that remain in the set obtained by discarding from the set of worlds that are intersected to define $\mathbf{K} \div^2_{\mathbb{S}} B$ all the worlds in $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i)$ whenever $\alpha_j \in B \cap \mathbf{K}$ and $\alpha_i \in B$ are such that $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i) \neq \emptyset$. However, in the following subsection we will show that the operation defined in that way is not, in general, a partial meet multiple contraction either.

7.1.3 Third Unsuccessful Proposal

In the sequence of the discussion presented in the end of the previous subsection, in what follows, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, we will introduce an operation $\div^3_{\mathbb{S}}$ on $\mathbf{K}$ such that, for any set of sentences B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $\mathbf{K} \div^3_{\mathbb{S}} B$ is given by the intersection of a set of possible worlds which is a subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ and does not contain any worlds that do not belong to any $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} \cap \mathcal{P}(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right))$.

Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. Having in mind the last paragraph of the previous subsection we start by defining, for any given set of sentences B, the following subset $B_{\mathbb{S},1} \subseteq B$:

$$B_{\mathbb{S},1} = \{\alpha_i \in B : \forall \alpha_k \in B \setminus Cn(\emptyset)\ (f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i) \text{ or } f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|)\}.$$ [118]

Now let $\div^3_{\mathbb{S}}$ be the operation defined on $\mathbf{K}$ by[119]

$$\mathbf{K} \div^3_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset. \end{cases}$$

First we notice that, just as it was the case with the two operations presented in the two previous subsections, the operation $\div^3_{\mathbb{S}}$ also satisfies the following condition: if $B = \{\alpha\}$ then $\mathbf{K} \div^3_{\mathbb{S}} B = \mathbf{K} -_{\mathbb{S}} \alpha$ (observe that whenever B is a singleton set $B_{\mathbb{S},1} = B$).

Nonetheless, this construction also fails to be a partial meet multiple contraction, as we can conclude from the following counterexample:

Counterexample 7.1.4 *Let $\mathbf{K}$ be a belief set and $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be a set of sentences such that $B \subseteq \mathbf{K} \setminus Cn(\emptyset)$. Assume also that the sentences α_1, α_2 and α_3 are such that $\|\neg\alpha_2\| \cap \|\neg\alpha_1\| \neq \emptyset$, $\|\neg\alpha_1\| \setminus \|\neg\alpha_2\| \neq \emptyset$, $\|\neg\alpha_3\| \cap \|\neg\alpha_1\| = \emptyset$ and $\|\neg\alpha_3\| \cap \|\neg\alpha_2\| \neq \emptyset$. Furthermore, let $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ such that $f_{\mathbb{S}}(\neg\alpha_2) \subset f_{\mathbb{S}}(\neg\alpha_1)$, $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$, $f_{\mathbb{S}}(\neg\alpha_3) \neq f_{\mathbb{S}}(\neg\alpha_2)$, $f_{\mathbb{S}}(\neg\alpha_2) \not\subset \|\neg\alpha_3\|$ and $f_{\mathbb{S}}(\neg\alpha_1) \not\subset \|\neg\alpha_3\|$.*[120]

[118] Notice that, if $B \subseteq (\mathbf{K} \setminus Cn(\emptyset))$ and $\mathbb{S} = \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$ then $B_{\mathbb{S},1} = B_N$ (since, in that case, $\forall \alpha_k \in \mathbf{K} \setminus Cn(\emptyset)\ f_{\mathbb{S}}(\neg\alpha_k) = \|\neg\alpha_k\|$). However, in general, neither $B_{\mathbb{S},1} \subseteq B_N$ nor $B_N \subseteq B_{\mathbb{S},1}$.

[119] Notice that whenever B is finite and $B \cap Cn(\emptyset) = \emptyset$ it can be shown that $\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \subseteq \bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)$.

[120] Notice that the three latter conditions are redundant in the present context. Indeed, on the one hand, $f_{\mathbb{S}}(\neg\alpha_1) \not\subset \|\neg\alpha_3\|$ follows immediately from $\|\neg\alpha_3\| \cap \|\neg\alpha_1\| = \emptyset$ and, on the other hand, from $f_{\mathbb{S}}(\neg\alpha_2) \subset f_{\mathbb{S}}(\neg\alpha_1)$ and $\|\neg\alpha_3\| \cap \|\neg\alpha_1\| = \emptyset$ we can conclude that $f_{\mathbb{S}}(\neg\alpha_2) \not\subset \|\neg\alpha_3\|$ and $f_{\mathbb{S}}(\neg\alpha_3) \neq f_{\mathbb{S}}(\neg\alpha_2)$.

Notice that, in the above conditions we obtain that $B_{\mathbb{S},1} = \{\alpha_3\}$*, and consequently, according to the definition of* $\div^3_{\mathbb{S}}$*, it follows that* $\mathbf{K} \div^3_{\mathbb{S}} B = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_3))$.

But then, since $\alpha_1 \in \mathbf{K}$ *and* $f_{\mathbb{S}}(\neg\alpha_3) \subseteq \|\alpha_1\|$ *we can conclude that* $\alpha_1 \in \mathbf{K} \div^3_{\mathbb{S}} B$.

On the other hand, if $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *then, combining Theorem 6.3.6, Definition 6.3.5, Observation 6.1.4-1., we can conclude that (since* B *is such that* $B \cap Cn(\emptyset) = \emptyset$*) there is some* $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ *such that* $\mathcal{W} \subset \|\mathbf{K}\div B\|$*. Then, since according to Definition 6.1.2 the set* $\mathcal{W}$ *is such that* $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$ *for all* $\alpha_i \in B \cap \mathbf{K}$*, we can conclude that* $\alpha_i \notin \mathbf{K}\div B$ *for all* $\alpha_i \in B \cap \mathbf{K}$*.*[121]

Therefore, the fact that $\alpha_1 \in B \cap \mathbf{K}$ *and* $\alpha_1 \in \mathbf{K} \div^3_{\mathbb{S}} B$*, proves that* $\div^3_{\mathbb{S}}$ *is not a partial meet multiple contraction on* $\mathbf{K}$*.*

The situation described in this example is represented in Figure 7.3. There, the shaded region represents the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)$*. Hence, according to the definition of* $\div^3_{\mathbb{S}}$*, the set* $\mathbf{K} \div^3_{\mathbb{S}} B$ *consists of the intersection of all the worlds in that region. Noticing that none of the worlds highlighted in that figure is a* $\neg\alpha_1$*-world we immediately conclude that* $\alpha_1 \in \mathbf{K} \div^3_{\mathbb{S}} B$*. Combining this observation with the fact that it follows from Definition 5.3.2 (and also from Theorem 6.3.6) that, in this circumstances, if* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *then* $\alpha_i \notin \mathbf{K}\div B$ *for all* $\alpha_i \in B \cap \mathbf{K}$*, we obtain that* $\div^3_{\mathbb{S}}$ *is not a partial meet multiple contraction on* $\mathbf{K}$*.*

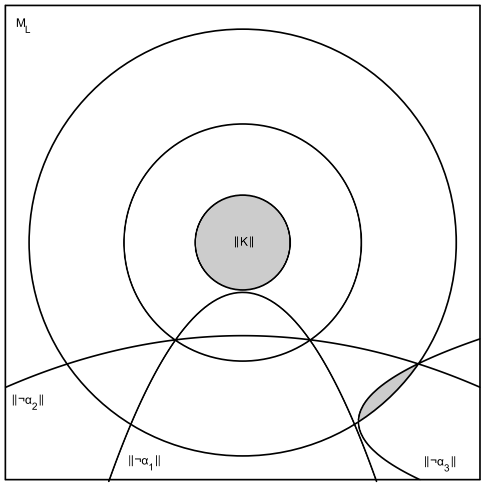

Figure 7.3: Graphical representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ for a belief set $\mathbf{K} \neq Cn(\emptyset)$, where the shaded region highlights the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)$, with $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$. (A more detailed explanation of the situation illustrated by this figure can be found in Counterexample 7.1.4.)

From the above counterexample we notice that, roughly speaking, contrary to what happened in the first two operations that we have proposed, the operation $\div^3_{\mathbb{S}}$

[121] We must notice that this paragraph is nothing more than the proof, made by means of the characterization of the partial meet multiple contractions in terms of possible worlds, that the partial meet multiple contractions satisfy the postulate of *P-Success*, which is a quite well known fact (cf. Observation 5.5.1).

fails to be a partial meet multiple contraction not because the set $\|\mathbf{K} \div^3_{\mathbb{S}} B\|$ contains more worlds than it should, but because it has too few worlds.

Indeed, the problem with the above proposed operation $\div^3_{\mathbb{S}}$ on a belief set $\mathbf{K}$ is that, for some sets B such that $B \cap Cn(\emptyset) = \emptyset$, the set of worlds whose intersection defines the set $\mathbf{K} \div^3_{\mathbb{S}} B$ does not include enough worlds from the set $\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)$.

Recall that, given a belief set $\mathbf{K}$, we wanted the operation $\div^3_{\mathbb{S}}$ on $\mathbf{K}$ to be such that, for all sets B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $\mathbf{K} \div^3_{\mathbb{S}} B$ would be given by the intersection of all the worlds of a subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ (which, according to the definition of the operation $\div^2_{\mathbb{S}}$ in the previous subsection, is the set of worlds that are intersected to define $\mathbf{K} \div^2_{\mathbb{S}} B$). More precisely, we constructed the operation $\div^3_{\mathbb{S}}$ in a way that, for any given set B such that $B \cap Cn(\emptyset) = \emptyset$, the set of worlds whose intersection defines the set $\mathbf{K} \div^3_{\mathbb{S}} B$ is the subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$ obtained by discarding from this latter set all the worlds in $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i)$ for some $\alpha_j \in B \cap \mathbf{K}$ and $\alpha_i \in B$ such that $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i) \neq \emptyset$. However, what we can realize from Counterexample 7.1.4 is that, whenever α_i and α_j are sentences in the just mentioned conditions, the worlds in $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i)$ should indeed not be included in the set of possible worlds whose intersection originates the set $\mathbf{K} \div^3_{\mathbb{S}} B$ but only if, on the other hand, it is assured that after discarding those worlds there is some $\neg\alpha_j$-world left on that set (whose intersection is used to define the set $\mathbf{K} \div^3_{\mathbb{S}} B$). In the next section we will present a way of constructing a set of possible worlds satisfying these conditions.

7.2 System of Spheres-based Multiple Contractions

We start this section by recalling that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, our main goal in the present chapter is to define an operation $\div_{\mathbb{S}}$ which, on the one hand, is a partial meet multiple contraction and, on the other hand, is such that for all sentences $\alpha \in \mathcal{L}$ it holds that $\mathbf{K} \div_{\mathbb{S}} \{\alpha\} = \mathbf{K} -_{\mathbb{S}} \alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction.

Having this in mind, we have clarified in the beginning of this chapter that for sets B such that $B \cap Cn(\emptyset) \neq \emptyset$ the operation $\div_{\mathbb{S}}$ must be defined by $\mathbf{K} \div_{\mathbb{S}} B = \mathbf{K}$. Nevertheless, it is not that clear how shall $\div_{\mathbb{S}}$ be defined for sets B such that $B \cap Cn(\emptyset) = \emptyset$. In the previous section we have proposed several alternative (and somehow intuitively appealing) ways of defining such set $\mathbf{K} \div_{\mathbb{S}} B$ for sets B in those conditions, however, we have also shown there that, although all the proposed operations $\div^1_{\mathbb{S}}$, $\div^2_{\mathbb{S}}$ and $\div^3_{\mathbb{S}}$ are such that if $B = \{\alpha\}$ then $\mathbf{K} \div^1_{\mathbb{S}} B = \mathbf{K} \div^2_{\mathbb{S}} B = \mathbf{K} \div^3_{\mathbb{S}} B = \mathbf{K} -_{\mathbb{S}} \alpha$, none of them is, in general, a partial meet multiple contraction on $\mathbf{K}$.

All the, above mentioned, proposed operations on a belief set $\mathbf{K}$ had the following common characteristic: For any set B such that $B \cap Cn(\emptyset) = \emptyset$, the image of B under each of those functions is defined as the intersection of a certain set of possible worlds which is a subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)$. More precisely, in all those proposals such set of possible worlds (whose intersection defines the image of B under that function) has the form $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B'} f_{\mathbb{S}}(\neg\alpha_i)\right)$ where B' is a subset of B.

Namely, the first operation we have proposed - $\div^1_{\mathbb{S}}$ - consists of the function defined in the above mentioned way, with $B' = B$, then in the definition of operation $\div^2_{\mathbb{S}}$ we have taken B' to be the normalization of B - B_N - and finally in the definition of $\div^3_{\mathbb{S}}$ we have considered $B' = B_{\mathbb{S},1}$, where the set $B_{\mathbb{S},1}$ has been defined with the specific goal of assuring that the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)$ is a subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$. However, as we have observed in the end of the previous section, contrary to what was the case with operation $\div^2_{\mathbb{S}}$ (and also with operation $\div^1_{\mathbb{S}}$), it follows from Counterexample 7.1.4 that operation $\div^3_{\mathbb{S}}$ fails to be a partial meet multiple contraction not because the set of possible worlds whose intersection defines $\mathbf{K} \div^3_{\mathbb{S}} B$, for each set B such that $B \cap Cn(\emptyset) = \emptyset$, has too many worlds, but because (at least for finite sets B) such set has, in general, too few worlds.

Hence, considering a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, and holding on to the idea of defining such operation $\div_{\mathbb{S}}$ in a way that, for any sets B such that $B \cap Cn(\emptyset) = \emptyset$, the set $\mathbf{K} \div_{\mathbb{S}} B$ is given by an intersection of a set of possible worlds of the form $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B'} f_{\mathbb{S}}(\neg\alpha_i)\right)$ with $B' \subseteq B$, from the above we can conclude that such set B' must be such that $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right) \subseteq \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B'} f_{\mathbb{S}}(\neg\alpha_i)\right) \subseteq \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$.

Having this in mind, in what follows, considering a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, we will start by finding out a systematic way of defining, for any given set B such that $B \cap Cn(\emptyset) = \emptyset$, a subset $B_{\mathbb{S}} \subseteq B$ such that:

- $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right) \subseteq \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) \subseteq \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)$.
- The operation $\div_{\mathbb{S}}$ defined by $\mathbf{K} \div_{\mathbb{S}} B = \mathbf{K}$ if $B \cap Cn(\emptyset) \neq \emptyset$, and $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$ if $B \cap Cn(\emptyset) = \emptyset$ is a partial meet multiple contraction on $\mathbf{K}$.

So, let B be a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. In order to find out how to define a set $B_{\mathbb{S}} \subseteq B$ such that the above conditions hold, it is convenient to start by understanding in more detail why the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)$ contains, in general, too few worlds (or, from another point of view, why the set $B_{\mathbb{S},1}$ contains, in general, too few sentences) for the operation $\div^3_{\mathbb{S}}$, defined in the previous section, to be a partial meet multiple contraction on $\mathbf{K}$.

We may observe from its definition, that the set $B_{\mathbb{S},1}$ is, roughly speaking, obtained from B by discarding from it all the sentences α_r for which there is some other sentence α_s in B such that $f_{\mathbb{S}}(\neg\alpha_s) \subset \|\neg\alpha_r\|$ and $f_{\mathbb{S}}(\neg\alpha_s) \neq f_{\mathbb{S}}(\neg\alpha_r)$. However, as we have seen in Counterexample 7.1.4, for some sets B, such definition of $B_{\mathbb{S},1}$ leads to a situation in which for some $\alpha_t \in B \setminus B_{\mathbb{S},1}$ it holds that $\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) \cap \|\neg\alpha_t\| = \emptyset$ (and, therefore, $\alpha_t \in \mathbf{K} \div^3_{\mathbb{S}} B$, which yields that, in that case, $\div^3_{\mathbb{S}}$ is not a partial meet multiple contraction on $\mathbf{K}$). Indeed, as it can be seen in the mentioned counterexample (where we have shown $\alpha_1 \in B$ but $\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) \cap \|\neg\alpha_1\| = \emptyset$), such a situation occurs whenever $\alpha_t \in B \setminus B_{\mathbb{S},1}$ is such that (i) $\alpha_t \in B \cap K$, (ii) there is only one sentence in B, say α_q, which is, roughly speaking, responsible for the fact that $\alpha_t \notin B_{\mathbb{S},1}$, (iii) for all $\alpha_p \in B \setminus \{\alpha_t, \alpha_q\}$ it holds that $f_{\mathbb{S}}(\neg\alpha_p) \cap \|\neg\alpha_t\| = \emptyset$ and (iv) $\alpha_q \notin B_{\mathbb{S},1}$.

Hence, having in mind the way how, for any set B such that $B \cap Cn(\emptyset) = \emptyset$, the set $B_{\mathbb{S},1}$ is obtained as well as the reason (exposed by means of Counterexample 7.1.4) why such set is, in general, too small for our purposes, we are led to assume that, given an arbitrary set B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $B_{\mathbb{S}} \subseteq B$ that we are looking for shall be a superset of $B_{\mathbb{S},1}$ such that, whenever it holds that $f_{\mathbb{S}}(\neg\alpha_s) \subset \|\neg\alpha_r\|$ and $f_{\mathbb{S}}(\neg\alpha_s) \neq f_{\mathbb{S}}(\neg\alpha_r)$, for two sentences $\alpha_r, \alpha_s \in B \cap \mathbf{K}$, then the sentence α_r does not belong to the set $B_{\mathbb{S}}$, but only as long as it is also somehow assured (by the way $B_{\mathbb{S}}$ is defined) that at least one of the sentences $\alpha_q \in B$ such that $f_{\mathbb{S}}(\neg\alpha_q) \subset \|\neg\alpha_r\|$ and $f_{\mathbb{S}}(\neg\alpha_q) \neq f_{\mathbb{S}}(\neg\alpha_r)$ belongs to that set $B_{\mathbb{S}}$. In other words, for any given set B not containing tautologies, it appears that the subset $B_{\mathbb{S}}$ of B that we wish to define should be constructed in a way that, if $\alpha_r, \alpha_s \in B \cap \mathbf{K}$ are two sentences in the above described conditions then they do not simultaneously belong to $B_{\mathbb{S}}$ but, on the other hand, $B_{\mathbb{S}}$ must be such that for all sentences $\alpha_l \in (B \cap \mathbf{K}) \setminus B_{\mathbb{S}}$ there is some $\alpha_k \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_l\|$ and $f_{\mathbb{S}}(\neg\alpha_k) \neq f_{\mathbb{S}}(\neg\alpha_l)$.

More formally, based on the above considerations, we are led to the following conjecture:

Conjecture 7.2.1 *Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. If for any set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$ the set $B_{\mathbb{S}} \subseteq B$ is such that the following conditions hold:*

(i) if $\alpha_r \in B_{\mathbb{S}}$ then $f_{\mathbb{S}}(\neg\alpha_s) \not\subset \|\neg\alpha_r\|$ or $f_{\mathbb{S}}(\neg\alpha_s) = f_{\mathbb{S}}(\neg\alpha_r)$, for all $\alpha_s \in B_{\mathbb{S}}$,

(ii) for all $\alpha_l \in (B \cap \mathbf{K}) \setminus B_{\mathbb{S}}$ there is some $\alpha_k \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_l\|$ and $f_{\mathbb{S}}(\neg\alpha_k) \neq f_{\mathbb{S}}(\neg\alpha_l)$,

then the operation $\div_{\mathbb{S}}$ on $\mathbf{K}$ defined by:

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

for any set of sentences B, is a partial meet multiple contraction on $\mathbf{K}$.

Having the above conjecture in mind, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ in the following definition we present a way of systematically obtaining, for any finite set of sentences B, a subset $B_{\mathbb{S}}$ such that, whenever $B \cap Cn(\emptyset) = \emptyset$, the conditions (i) and (ii) above are satisfied. Afterwards, based on that same conjecture, we will make use of the constructed sets $B_{\mathbb{S}}$ to present a possible definition for the S.S.-based multiple contractions.

At this point we start by remarking that condition (i) above presents in some sense a necessary condition for an element α_r of B to belong to $B_{\mathbb{S}}$ (namely: for all $\alpha_s \in B_{\mathbb{S}}$ it holds that $f_{\mathbb{S}}(\neg\alpha_s) = f_{\mathbb{S}}(\neg\alpha_r)$ or $f_{\mathbb{S}}(\neg\alpha_s) \not\subset \|\neg\alpha_r\|$). We will use that condition as our guideline towards the definition of a set $B_{\mathbb{S}}$ in the desired conditions. Taking this into account and observing that the condition imposed to be satisfied by all elements of $B_{\mathbb{S}}$ is *recurring* (in the sense that it makes reference to the set $B_{\mathbb{S}}$ itself) we are led to believe that the definition of $B_{\mathbb{S}}$ shall follow an iterative procedure.

Having this in mind we suggest the following construction for the set $B_{\mathbb{S}}$:[122]

[122]Further below we provide a more detailed explanation of the intuition behind this definition.

Definition 7.2.2 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$.

Consider a finite set of sentences $B = \{\alpha_1, \ldots, \alpha_n\}$ *such that* $B \setminus Cn(\emptyset) \neq \emptyset$.

Denote by $C_1, \ldots, C_m$ *the (different) equivalence classes in the quotient set of* $(B \setminus Cn(\emptyset))$ *by* $\backsim$, *i.e.* $\{C_1, \ldots, C_m\} = (B \setminus Cn(\emptyset))/_{\backsim}$, *where* $\backsim$ *is the equivalence relation on* $B \setminus Cn(\emptyset)$ *defined by:*

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset),\ \alpha \backsim \beta \ \textit{ iff } \ \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}.$$

Moreover, assume that the equivalence classes $C_1, \ldots, C_m$ *are ordered according to the following condition:*

$$\textit{If } 1 \leq i < j \leq m \textit{ then } \forall \alpha_r \in C_i\ \forall \alpha_s \in C_j\ \ \mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}.$$

Now consider the following list of subsets of B:

$$\begin{array}{rcl} B_0 & = & B \cap Cn(\emptyset) \\ C'_1 & = & C_1 \\ C''_1 & = & \{\alpha_i \in C'_1 : \forall \alpha_j \in C'_1\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\} \\ B_1 & = & C''_1 \end{array}$$

Moreover, if $m > 1$ *for all* $l \in \{2, \ldots, m\}$, *let* C'_l, C''_l *and* B_l *be the sets defined by:*

$$\begin{array}{rcl} C'_l & = & \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\}; \\ C''_l & = & \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}; \\ B_l & = & B_{l-1} \cup C''_l. \end{array}$$

The $\mathbb{S}$-based filtration of B *is the set* $B_{\mathbb{S}} = B_m$.

If D *is a set of sentences such that* $D \subseteq Cn(\emptyset)$ *then the* $\mathbb{S}$-based filtration of D *is the empty set, i.e.* $D_{\mathbb{S}} = \emptyset$.

The following observation presents an alternative (equivalent) way of defining the sets $C'_2, \ldots, C'_m, C''_1, \ldots, C''_m$ introduced in the above definition.

Observation 7.2.3 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \setminus Cn(\emptyset) \neq \emptyset$. *Furthermore, let* $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ *be the subsets of* B *constructed as we described in the above definition. Then:*

(a) For all $l \in \{1, \ldots, m\}$ *it holds that* $C''_l = \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

(b) If $m > 1$ *then for all* $l \in \{2, \ldots, m\}$ *the following identity is satisfied:* $C'_l = \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

Proof. A proof for this observation can be found in page 199. ■

Given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a finite set of sentences B, in what follows we will expose the intuition behind the above definition of the $\mathbb{S}$-*based filtration of* B. In order to do that we must start by remarking that our intention when proposing the above definition was that the set $B_{\mathbb{S}} \subseteq B$ should be such that, whenever B is a set that does not contain any tautology, the following conditions are satisfied:

(i) If $\alpha_j \in B_{\mathbb{S}}$ then $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_{\mathbb{S}}$.

(ii) For all $\alpha_l \in (B \cap \mathbf{K}) \setminus B_{\mathbb{S}}$ there is some $\alpha_m \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$ and $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$.

If we were only interested in the fulfilment of condition (i) above, then it would be enough to define $B_{\mathbb{S}} = \{\alpha_i \in B : \forall \alpha_k \in B \setminus Cn(\emptyset)\ (f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i) \text{ or } f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|)\}$, however, if $B_{\mathbb{S}}$ was defined in that way then condition (ii) would not, in general, be satisfied. Indeed, as we have concluded with the help of Counterexample 7.1.4, given a set $B = \{\alpha_p, \alpha_q, \alpha_r\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$, if the sentences α_p, α_q and α_r are such that $f_{\mathbb{S}}(\neg\alpha_q) \subset \|\neg\alpha_p\|$, $f_{\mathbb{S}}(\neg\alpha_q) \neq f_{\mathbb{S}}(\neg\alpha_p)$, $f_{\mathbb{S}}(\neg\alpha_r) \subset \|\neg\alpha_q\|$, $f_{\mathbb{S}}(\neg\alpha_r) \neq f_{\mathbb{S}}(\neg\alpha_q)$, $f_{\mathbb{S}}(\neg\alpha_r) \not\subseteq \|\neg\alpha_p\|$ and $f_{\mathbb{S}}(\neg\alpha_p) \not\subseteq \|\neg\alpha_r\|$ then, if $B_{\mathbb{S}}$ was defined as above suggested, in this case, we would obtain $B_{\mathbb{S}} = \{\alpha_r\}$. Hence, it would hold that $\alpha_p \in (B \cap \mathbf{K}) \setminus B_{\mathbb{S}}$ and $f_{\mathbb{S}}(\neg\alpha_r) \not\subseteq \|\neg\alpha_p\|$ and, therefore, condition (ii) would not be satisfied.

We can observe that, in the above described situation, condition (ii) is not satisfied essentially because, among other facts, it holds that $f_{\mathbb{S}}(\neg\alpha_q) \subset \|\neg\alpha_p\|$ and $f_{\mathbb{S}}(\neg\alpha_r) \subset \|\neg\alpha_q\|$, while, on the other hand, $f_{\mathbb{S}}(\neg\alpha_r) \not\subseteq \|\neg\alpha_p\|$. Roughly speaking, in those conditions, what happens is that the sentence $\alpha_q \in B$, which is responsible for the non-inclusion of α_p in $B_{\mathbb{S}}$, does not belong to the set $B_{\mathbb{S}}$ either, and that fact (together with the remaining circumstances) causes that for all $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ it holds that $f_{\mathbb{S}}(\neg\alpha_m) \not\subseteq \|\neg\alpha_l\|$.

Now notice that, given two sentences α and β in $\mathcal{L} \setminus Cn(\emptyset)$, it follows immediately from Definition 3.3.5 that if $f_{\mathbb{S}}(\neg\alpha) \subset \|\neg\beta\|$ then $\mathbb{S}_{\neg\beta} \subseteq \mathbb{S}_{\neg\alpha}$. On the other hand, if $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$ and $f_{\mathbb{S}}(\neg\alpha) \subset \|\neg\beta\|$ then $f_{\mathbb{S}}(\neg\alpha) \subseteq f_{\mathbb{S}}(\neg\beta)$.

Hence, we can conclude that conditions $f_{\mathbb{S}}(\neg\alpha_q) \subset \|\neg\alpha_p\|$, $f_{\mathbb{S}}(\neg\alpha_r) \subset \|\neg\alpha_q\|$ and $f_{\mathbb{S}}(\neg\alpha_r) \not\subseteq \|\neg\alpha_p\|$ can only hold simultaneously as long as $\mathbb{S}_{\neg\alpha_q} \subset \mathbb{S}_{\neg\alpha_r}$.

Therefore, from all the above we can observe that:

1. Given a set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$, in order to assure that the set $B_{\mathbb{S}} \subseteq B$ satisfies conditions (i) and (ii) above, then it must be defined in a way which imposes that a sentence $\alpha_i \in B$ only belongs to $B_{\mathbb{S}}$ if $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|$, for all sentences α_k which are, themselves, elements of $B_{\mathbb{S}}$ (but not necessarily for all sentences of B).

2. If C is a finite set such that $C \cap Cn(\emptyset) = \emptyset$ and for some $\mathcal{D} \in \mathbb{S}$ it holds that $\mathbb{S}_{\neg\alpha_i} = \mathcal{D}$, for all $\alpha_i \in C \cap \mathbf{K}$, and $C_{\mathbb{S}} = \{\alpha_i \in C : \forall \alpha_k \in C\ (f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i) \text{ or } f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|)\}$ then the conditions obtained from (i) and (ii), by replacing, in each of those clauses, B by C, are satisfied.[123]

[123]Notice that a rigorous proof that this indeed holds can be trivially obtained by combining Observation 7.2.3 and Lemma 7.2.5

3. If α and β are two sentences such that $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ then $f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\|$.

Since it is based in the facts listed above, we are now in a position to explain the idea behind the definition of the $\mathbb{S}$-based filtration of a set B (Definition 7.2.2). For that purpose in what follows we assume that B is a finite set such that $B \setminus Cn(\emptyset) \neq \emptyset$ and describe how the set $B_{\mathbb{S}}$ is obtained as well as the motivations behind such procedure.

The first concern underlying such definition has to do with point 1. above and consists in constructing the set $B_{\mathbb{S}}$ a the result of a sequence of steps (rather than by a single step definition) in order to assure that at each stage of that construction we only prevent from being included in $B_{\mathbb{S}}$ those sentences $\alpha_i \in B$ such that $f_{\mathbb{S}}(\neg\alpha_k) \neq f_{\mathbb{S}}(\neg\alpha_i)$ and $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_i\|$, for some sentence $\alpha_k \in B$ which is somehow guaranteed to be such that it *will belong* to the set $B_{\mathbb{S}}$ when its *ongoing* construction is finished.

So, as it can be seen from its definition above, in order to construct the set $B_{\mathbb{S}}$ we start by partitioning B in several classes $B_0, C_1, \ldots, C_m$, where $B_0 = B \cap Cn(\emptyset)$ and for each $i \in \{1, \ldots, m\}$ the class C_i is such that if $\alpha_r \in C_i$ then for all $\alpha_s \in B \setminus Cn(\emptyset)$ it holds that $\alpha_s \in C_i$ if and only if $\mathbb{S}_{\neg\alpha_s} = \mathbb{S}_{\neg\alpha_r}$ (hence $m \geq 1$ is uniquely determined by the set B and the system of spheres $\mathbb{S}$). Furthermore, given two such classes C_i and C_j, if $i < j$, $\alpha_r \in C_i$ and $\alpha_s \in C_j$ then $\mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}$.

In the above conditions we have that, for each $i \in \{1, \ldots, m\}$ the class C_i is such that if $\alpha_r \in C_i$ and $i < j$, with $j \in \{1, \ldots, m\}$, then for all $\alpha_s \in C_j$ it holds that $f_{\mathbb{S}}(\neg\alpha_s) \not\subset \|\alpha_r\|$. Hence, if $\alpha_p \in C_1$ then $f_{\mathbb{S}}(\neg\alpha_q) = f_{\mathbb{S}}(\neg\alpha_p)$ or $f_{\mathbb{S}}(\neg\alpha_q) \not\subset \|\neg\alpha_p\|$, for all $\alpha_q \in B \setminus (B_0 \cup C_1)$. However, there can be some sentences $\alpha_l \in C_1$ for which it does not hold that $f_{\mathbb{S}}(\neg\alpha_m) = f_{\mathbb{S}}(\neg\alpha_l)$ or $f_{\mathbb{S}}(\neg\alpha_m) \not\subset \|\neg\alpha_l\|$, for all $\alpha_m \in C_1$.

So, naturally, the next step in the construction of the set $B_{\mathbb{S}}$ consists in obtaining the subset of C_1 defined by $C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}$, where $C_1' = C_1$.[124] It follows immediately from its construction and some of our remarks above that the set C_1'' is such that, on the one hand, if $\alpha_p \in C_1''$ then $f_{\mathbb{S}}(\neg\alpha_q) = f_{\mathbb{S}}(\neg\alpha_p)$ or $f_{\mathbb{S}}(\neg\alpha_q) \not\subset \|\neg\alpha_p\|$, for all $\alpha_q \in C_1'' \cup (B \setminus (B_0 \cup C_1))$ and, on the other hand, for all $\alpha_l \in C_1 \setminus C_1''$ there is some $\alpha_m \in C_1''$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$[123]. At this point we define the set $B_1 = C_1''$, which is the set of sentences of C_1 that we will include in the set $B_{\mathbb{S}}$. Indeed, if $B \setminus (B_0 \cup C_1) = \emptyset$ (or, equivalently, if $m = 1$) then the set $B_{\mathbb{S}}$ is simply defined by $B_{\mathbb{S}} = B_1$.

If, instead, $B \setminus (B_0 \cup C_1) \neq \emptyset$ (or, equivalently, if $m > 1$), the next phase consists in choosing which sentences of C_2 shall be included in the set $B_{\mathbb{S}}$. We do this in two steps. First we consider the set $C_2' = \{\alpha_i \in C_2 : \forall \alpha_j \in B_1 \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\}$[125] which consists of the subset of C_2 such that $\alpha_i \in C_2'$ if and only if $\alpha_i \in C_2$ and $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|$ (or $f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$), for all $\alpha_j \in B_1 \cup (B \setminus (B_0 \cup C_1 \cup C_2))$. Moreover, we also have that if $\alpha_l \in C_2 \setminus C_2'$ then there is some $\alpha_m \in B_1$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$. However there may be some sentence $\alpha_k \in C_2'$ such that $f_{\mathbb{S}}(\neg\alpha_h) \subset \|\neg\alpha_k\|$ and $f_{\mathbb{S}}(\neg\alpha_h) \neq f_{\mathbb{S}}(\neg\alpha_k)$ for some $\alpha_h \in C_2'$.

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[124] Notice that, according to Observation 7.2.3, $C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i) \text{ or } f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

[125] Notice that, according to Observation 7.2.3, $C_2' = \{\alpha_i \in C_2 : \forall \alpha_j \in B_1 \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i) \text{ or } f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

Hence, secondly we must obtain the subset $C_2'' \subseteq C_2'$ which represents to C_2' the same that the set C_1'' represented to C_1'', i.e. $C_2'' = \{\alpha_i \in C_2' : \forall \alpha_j \in C_2' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}$. Afterwards we define $B_2 = B_1 \cup C_2''$. Therefore B_2 is a subset of $C_1 \cup C_2$ such that (a) if $\alpha_i \in B_2$ then $f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|$, for all $\alpha_j \in B_2 \cup (B \setminus (B_0 \cup C_1 \cup C_2))$ and (b) for all $\alpha_l \in (C_1 \cup C_2) \setminus B_2$ there is some $\alpha_m \in B_2$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$. Now, if $B \setminus (B_0 \cup C_1 \cup C_2) = \emptyset$ (or, equivalently, if $m = 2$) then we just define $B_{\mathbb{S}} = B_2$. Otherwise, we obtain the set B_3 by applying to the class C_3 the same procedure (with the necessary obvious adaptations) that we have just described above regarding the class C_2, and we go on repeating this process until we have obtained the set B_m. Then, finally, we define $B_{\mathbb{S}} = B_m$ and we are done.

Next we will prove that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, if B is a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ and $B_{\mathbb{S}}$ is the above defined $\mathbb{S}$-based filtration of B then conditions (i) and (ii) of Conjecture 7.2.1 are indeed satisfied.

We start by showing, in the following observation, that the first of the two just mentioned conditions holds:

Observation 7.2.4 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences. If* $\alpha_i \in B_{\mathbb{S}}$*, then* $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$*, for all* $\alpha_t \in B_{\mathbb{S}}$*.*

Proof. A proof for this observation can be found in page 199. ∎

Given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ the above observation presents a property satisfied by all the elements in the $\mathbb{S}$-based filtration of a finite set of sentences B. From another point of view we can say that such property comprises a necessary condition for a sentence of B to belong to $B_{\mathbb{S}}$. Next we introduce a lemma which we will use immediately afterwards for (a) providing a necessary condition for an element of a finite set B, such that $B \cap Cn(\emptyset) = \emptyset$, to be left out from its $\mathbb{S}$-based filtration $B_{\mathbb{S}}$, and (b) proving that the second one of the above mentioned conditions is also satisfied.

Lemma 7.2.5 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. Assume* $C = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L} \setminus Cn(\emptyset)$*, with* $n \geq 1$ *is a non-empty finite set of sentences such that* $\mathbb{S}_{\neg\alpha_1} = \cdots = \mathbb{S}_{\neg\alpha_n}$*, and, for each* $\alpha_j \in C$ *let* C_{α_j} *be the set defined by* $C_{\alpha_j} = \{\alpha_k \in C : f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_j)\}$*.*

If $\alpha_j \in C$ *and* $C_{\alpha_j} \neq \emptyset$ *then there is some* $\alpha_l \in C_{\alpha_j}$ *such that* $C_{\alpha_l} = \emptyset$*.*

Proof. A proof for this lemma can be found in page 200. ∎

Given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a finite set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$, we are now in a position to prove the following observation, which on the one hand presents a property satisfied by all the elements of B that do not belong to its $\mathbb{S}$-based filtration $B_{\mathbb{S}}$, and on the other hand immediately yields that the above mentioned condition (ii) of Conjecture 7.2.1 holds.

Observation 7.2.6 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$*. If* $\alpha_j \in B \setminus B_{\mathbb{S}}$*, then there is some* $\alpha_l \in B_{\mathbb{S}}$ *such that* $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ *and* $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$*.*

Proof. A proof for this observation can be found in page 201. ■

We may see that the above observation yields a sufficient condition for an element of a finite set of sentences B, such that $B \cap Cn(\emptyset) = \emptyset$, to be kept in the set $B_{\mathbb{S}}$. More precisely, from the above result we can conclude that, given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and a set B in the mentioned conditions, the property satisfied by the elements of $B_{\mathbb{S}}$ presented in Observation 7.2.4 is not only a necessary but also a sufficient condition for an element in B to belong also to its $\mathbb{S}$-based filtration $B_{\mathbb{S}}$. This is more formally stated in the following Corollary (of Observations 7.2.4 and 7.2.6).

Corollary 7.2.7 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*,* B *be a (non-empty) finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$ *and* α_i *be a sentence in* B*. Then* $\alpha_i \in B_{\mathbb{S}}$ *if and only if for all* $\alpha_t \in B_{\mathbb{S}}$ *it holds that* $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$*.*

At this point it is also worth to remark that the fact exposed in Observation 7.2.6 assures that, whenever $B \cap Cn(\emptyset) = \emptyset$, the set $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$ contains some $\neg\alpha_j$-world for all sentences $\alpha_j \in B$ and, consequently, if the operation $\div_{\mathbb{S}}$ is defined as suggested in Conjecture 7.2.1 then the set $\mathbf{K}\div_{\mathbb{S}}B$ does not contain any elements from B.

Now, having in mind the above mentioned conjecture and noticing that it follows from Observations 7.2.4 and 7.2.6 that, for any set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$, if $B_{\mathbb{S}}$ is the $\mathbb{S}$-based filtration of B then $B_{\mathbb{S}} \subseteq B$ and conditions (i) and (ii) of the mentioned conjecture are satisfied, we are naturally led to suggest the following definition for the *system of spheres-based multiple contractions* by finite sets:

Definition 7.2.8 (System of Spheres-based multiple contractions) *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. The* $\mathbb{S}$-based multiple contraction on $\mathbf{K}$ *by finite sets is the multiple contraction function* $\div_{\mathbb{S}}$ *defined by:*

$$\mathbf{K}\div_{\mathbb{S}}B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

for any finite set of sentences B *and where* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B*.*

An operator $\div$ *on* $\mathbf{K}$ *is a* system of spheres-based multiple contraction on $\mathbf{K}$ *by finite sets if and only if there is a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$ *such that* $\mathbf{K}\div B = \mathbf{K}\div_{\mathbb{S}}B$*, for any finite set of sentences* B*.*

Shortly we will show that the above defined system of spheres-based multiple contractions by finite sets fulfil our initial main goal for this chapter, i.e. that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, the $\mathbb{S}$-based multiple contraction on $\mathbf{K}$ by finite sets $\div_{\mathbb{S}}$ is a partial meet multiple contraction and, on the other hand, it is such that for all sentences $\alpha \in \mathcal{L}$ it holds that $\mathbf{K}\div_{\mathbb{S}}\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction.

However, before proceeding to prove the above mentioned facts, we need to introduce some auxiliary results.

We start by presenting the following result which is an obvious consequence of Observations 7.2.4 and 7.2.6.

Observation 7.2.9 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$*. If* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B*, then:*

(i) If $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_i) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_i) \not\subseteq \|\neg\alpha_j\|$*, for all* $\alpha_i \in B_{\mathbb{S}}$*;*

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_k \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$*.*

Proof. A proof for this observation can be found in page 202. ■

Now we present a lemma which shall be very useful in some of the following proofs.

Lemma 7.2.10 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$*. If* $B_{\mathbb{S}}$ *is a subset of* B *such that:*

(i) If $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_k) \not\subseteq \|\neg\alpha_j\|$*, for all* $\alpha_k \in B_{\mathbb{S}}$*, and*

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$*,*

then $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$*, and* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$*.*

Proof. A proof for this lemma can be found in page 202. ■

Combining Observation 7.2.9 with the above lemma we can obtain the following result which interrelates the set $\mathbb{W}_{\mathbf{K}\perp B}$ and the $\mathbb{S}$-based filtration $B_{\mathbb{S}}$ of B, for any finite set of sentences B.

Observation 7.2.11 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$*. If* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B*, then* $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$*, and* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$*.*

Proof. A proof for this observation can be found in page 203. ■

Given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ the following lemmas present some properties concerning the normalization and the $\mathbb{S}$-based filtration of a set of sentences as well as their interrelation.

Lemma 7.2.12 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* G *and* H *be two finite sets of sentences. If every subset* X *of* $\mathbf{K}$ *implies some element of* G *if and only if* X *implies some element of* H*, then for all* $\beta_i \in (G_N \cap \mathbf{K})_{\mathbb{S}}$ *there is some* $\varsigma_i \in (H_N \cap \mathbf{K})_{\mathbb{S}}$ *such that* $\|\beta_i\| = \|\varsigma_i\|$*.*

Proof. A proof for this lemma can be found in page 203. ■

Lemma 7.2.13 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences. Then:*

(i) $B_N \cap \mathbf{K} = (B \cap \mathbf{K})_N$.

(ii) $B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$.

(iii) $(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$.

(iv) $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.

Proof. A proof for this lemma can be found in page 203. ■

We are finally in a position to present the following main result of the present chapter which confirms that every system of spheres-based multiple contraction is a partial meet multiple contraction.

Theorem 7.2.14 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. Then the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$ *by finite sets is a partial meet multiple contraction on* $\mathbf{K}$ *by finite sets.*

Proof. A proof for this theorem can be found in page 204. ■

At this point we recall Conjecture 7.2.1 to say that, by making some small modifications to the proof of the above presented theorem, we can easily obtain a proof for the following result whose statement is a slightly modified version of the statement of such conjecture.

Theorem 7.2.15 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. Suppose that for any finite set of sentences* B *satisfying* $B \cap Cn(\emptyset) = \emptyset$ *the set* $B_{\mathbb{S}}$ *is a subset of* B *such that the following conditions hold:*[126]

(i) If $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_k) \not\subseteq \|\neg\alpha_j\|$*, for all* $\alpha_k \in B_{\mathbb{S}}$*;*

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$*;*

(iii) For any finite set of sentences C *satisfying* $C \cap Cn(\emptyset) = \emptyset$*, if every subset* X *of* $\mathbf{K}$ *implies some element of* B *if and only if* X *implies some element of* C*, then for all* $\beta_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}$ *there is some* $\varsigma_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}$ *such that* $\|\beta_i\| = \|\varsigma_i\|$*;*

[126] We notice that condition (ii) below coincides with condition (ii) of Observation 7.2.9 rather than with condition (ii) of Conjecture 7.2.1. Furthermore, we remark that it is easy to check (cf. Proof of Observation 7.2.9) that if a set $B_{\mathbb{S}}$ satisfies condition (ii) of the mentioned conjecture then it also satisfies condition (ii) of the statement of the present theorem.

(iv) $B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$;

(v) $(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$;

(vi) $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.

Then the multiple contraction function $\div_{\mathbb{S}}$ *on* $\mathbf{K}$ *defined by:*

$$\mathbf{K}\div_{\mathbb{S}}B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

for any finite set of sentences B*, is a partial meet multiple contraction on* $\mathbf{K}$ *by finite sets.*

Proof. A proof for this theorem can be found in page 205. ∎

It is now time to highlight that the above defined system of spheres-based multiple contractions fulfil the other main requirement that we wished from such operations, namely that they would generalize Grove's system of spheres-based (singleton) contractions. That fact is more detailedly and formally stated in the following observation, which follows immediately from Definitions 3.3.7 and 7.2.8.

Observation 7.2.16 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. If* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction by finite sets and* $-_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based (singleton) contraction, then:*

- *For any finite set of sentences* B *such that* $B \cap Cn(\emptyset) = \emptyset$ *it holds that:*

$$\mathbf{K}\div_{\mathbb{S}}B = \bigcap_{\alpha_i \in B_{\mathbb{S}}} \mathbf{K}\div_{\mathbb{S}}\{\alpha_i\} = \bigcap_{\alpha_i \in B_{\mathbb{S}}} \mathbf{K}-_{\mathbb{S}}\alpha_i,$$

where $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B*.*

- *The identity* $\mathbf{K}\div_{\mathbb{S}}\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$ *is satisfied for any sentence* $\alpha \in \mathcal{L}$*.*

Now, with the goal of clarifying the idea behind Definitions 7.2.2 and 7.2.8, we present an example illustrating the obtention of the set $\mathbf{K}\div_{\mathbb{S}}B$, where $\mathbf{K}$ is a belief set, $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$, B is a finite set such that $B \cap Cn(\emptyset) = \emptyset$ and $\div_{\mathbb{S}}$ is the $\mathbb{S}$-based multiple contraction on $\mathbf{K}$ by finite sets.

Example 7.2.17 *Let* $\mathbf{K}$ *be a belief set and* $B = \{\alpha_1, \alpha_2, \alpha_3\}$ *be a set of sentences such that* $B \subseteq \mathbf{K} \setminus Cn(\emptyset)$*. Assume also that the sentences* α_1, α_2 *and* α_3 *are such that* $\|\neg\alpha_1\| \setminus \|\neg\alpha_3\| \neq \emptyset$*,* $\|\neg\alpha_2\| \setminus \|\neg\alpha_3\| \neq \emptyset$ *and* $\|\neg\alpha_3\| \cap (\|\neg\alpha_2\| \setminus \|\neg\alpha_1\|) \neq \emptyset$*. Furthermore, let* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *such that* $\mathbb{S}_{\neg\alpha_1} = \mathbb{S}_{\neg\alpha_2} \subset \mathbb{S}_{\neg\alpha_3}$*,* $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$ *and* $f_{\mathbb{S}}(\neg\alpha_3) \not\subset \|\neg\alpha_1\|$*.* [127]

Having this conditions in mind, our next step will be the construction of the $\mathbb{S}$*-based filtration* $B_{\mathbb{S}}$ *of the set* B*. We start by remarking that* $B \setminus Cn(\emptyset) = B = \{\alpha_1, \alpha_2, \alpha_3\} \neq \emptyset$*. So, according to the process of construction of* $B_{\mathbb{S}}$ *described in Definition 7.2.2, in this specific case we have that:*

[127] Notice that all these conditions are subsumed by the conditions imposed on $\mathbf{K}$, B, $\alpha_1, \alpha_2, \alpha_3$ and $\mathbb{S}$ in the context of Counterexample 7.1.4. That is, if $\mathbf{K}$, B, $\alpha_1, \alpha_2, \alpha_3$ and $\mathbb{S}$ satisfy the conditions stated in the mentioned counterexample then they also satisfy the above requirements.

- $B_0 = B \cap Cn(\emptyset) = \emptyset$.
- $(B \setminus Cn(\emptyset))/_{\smile}$ *is composed of two equivalence classes (and, therefore,* $m = 2$*):* $C_1 = \{\alpha_3\}$ *and* $C_2 = \{\alpha_1, \alpha_2\}$.
- $B_1 = C_1'' = C_1' = C_1 = \{\alpha_3\}$
- $C_2' = \{\alpha_1\}$ *(notice that* $\alpha_2 \notin C_2'$ *because* $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$*).*
- $C_2'' = C_2' = \{\alpha_1\}$.
- $B_{\mathbb{S}} = B_2 = B_1 \cup C_2'' = \{\alpha_1, \alpha_3\}$.

Hence, according to Definition 7.2.8, in the above conditions we have that $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_1) \cup f_{\mathbb{S}}(\neg\alpha_3))$*, where* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$ *by finite sets.*

In Figure 7.4 we present a possible graphical representation of the situation described in the present example. The shaded region there represents the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)$*. Therefore, as we have mentioned above, according to Definition 7.2.8, the set* $\mathbf{K} \div_{\mathbb{S}} B$ *is given by the intersection of all the worlds in that region.*

Notice also that, having in mind that if $\mathbf{K}$*,* B*,* $\alpha_1, \alpha_2, \alpha_3$ *and* $\mathbb{S}$ *satisfy the conditions stated in Counterexample 7.1.4 then they also satisfy the requirements of the present example (cf. Footnote 127), in the referred figure we have represented the sets* $\|\neg\alpha_1\|$*,* $\|\neg\alpha_2\|$*,* $\|\neg\alpha_3\|$ *and* $\|\mathbf{K}\|$ *as well as the underlying system of spheres* $\mathbb{S}$ *in precisely the same positions that in Figure 7.3 so that the differences in the constructions of each of the sets* $\mathbf{K} \div^3_{\mathbb{S}} B$ *and* $\mathbf{K} \div_{\mathbb{S}} B$ *are easier to observe just by comparing those two figures. In particular we can notice that, in Figure 7.3 the shaded region represents the set* $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_3)$*, while the highlighted region in Figure 7.4 corresponds to the set* $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_1) \cup f_{\mathbb{S}}(\neg\alpha_3)$.

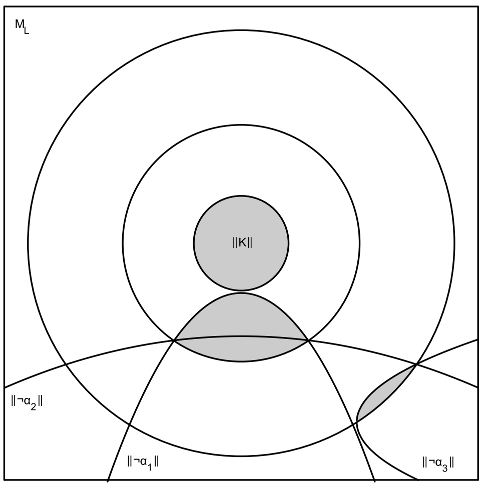

Figure 7.4: Graphical representation of a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ for a belief set $\mathbf{K} \neq Cn(\emptyset)$, where the shaded region highlights the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)$, with $B = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbf{K} \setminus Cn(\emptyset)$.

7.3 Summary

In the present chapter we have introduced, in Definition 7.2.8, a new class of multiple contraction functions − the *system of spheres-based multiple contractions* − which are a generalization (cf. Observation 7.2.16) to the case of contraction by (possibly non-singleton) sets of sentences of Grove's system of spheres-based contraction functions.

Having in mind that in the singleton contraction level the S.S.-based contractions are transitively relational partial meet contractions (in fact, those two classes of singleton contraction function are identical and constitute a subclass of the class of partial meet contractions), we are led to expect that the S.S.-based multiple contraction functions that we have presented are *transitively relational partial meet multiple contractions*.

Nevertheless, in the present thesis we have only focused in assuring that the new class of multiple contractions proposed was a subclass of the class of partial meet multiple contractions (cf. Theorem 7.2.14). We leave it as subject of future work to investigate if (as it is suggested by the analogy with the singleton case) it holds, additionally, that the S.S.-based multiple contractions are indeed transitively relational partial meet multiple contractions.

Chapter 8

Epistemic Entrenchment-based Multiple Contraction Functions

In this chapter our main goal is to introduce a new class of multiple contraction functions, namely the *epistemic entrenchment-based multiple contractions* which generalize the epistemic entrenchment-based (singleton) contractions [Gär88, GM88] to the case of contractions by (possibly non-singleton) sets of sentences rather than by a single sentence. Furthermore, having in mind that in the singleton contraction level the class of S.S.-based contractions coincides with the class of E.E.-based contractions (cf. Corollary 4.2.12), it is also our intention that the class of E.E.-based multiple contractions that we shall introduce coincides with the class of S.S.-based multiple contractions that has been presented in the previous chapter.

Before moving on to the search for an explicit definition for such a class of multiple contraction functions we notice that such investigation shall be essentially based on the results presented in Section 4.2 which show the very tight interconnection between Systems of Spheres and Epistemic Entrenchment relations and, furthermore, between S.S.-based (singleton) contractions and E.E.-based (singleton) contractions, as well as on the definition of the S.S.-based multiple contractions introduced in Section 7.2.

We start by noticing that it follows from Observations 4.2.1 and 4.2.11 that for every epistemic entrenchment relation $\leq$ there is a system of spheres $\mathbb{S}$ such that condition ($\leq -\mathbb{S}$) is satisfied and, conversely, for every system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ for which that same condition holds. Therefore, in order to assure that the class of epistemic entrenchment-based multiple contractions that we shall define coincides with the class of system of spheres-based multiple contractions it is enough to assure that given an arbitrary E.E. relation $\leq$ the definition proposed for the $\leq$-based multiple contraction $\div_{\leq}$ is such that it coincides with the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$, where $\mathbb{S}$ is any system of spheres such that $\leq$ and $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$).

Now, let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ such that $\leq$ and $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$) (cf. Observation 4.2.11). Having in mind the argument exposed in the previous paragraph, in what follows we investigate how the $\mathbb{S}$-based multiple contraction can be defined (exclusively) by means of $\leq$.

In other words, we want to define an operation $\div_{\leq}$ on $\mathbf{K}$ whose construction is based uniquely on the E.E. relation $\leq$ (instead of on the system of spheres $\mathbb{S}$) but which, on the other hand, is such that for all finite sets B:

$$\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\mathbb{S}}B$$

So, let B be an arbitrary finite set of sentences. Recalling the definition of the $\mathbb{S}$-based multiple contraction (see Definition 7.2.8) we immediately observe that in order to define the set $\mathbf{K}\div_{\mathbb{S}}B$ in terms of the E.E. relation $\leq$ our first step must be to find a way of defining the set $B_{\mathbb{S}}$ − the $\mathbb{S}$-based filtration of B (see Definition 7.2.2) − by means of the relation $\leq$ rather than by means of the system of spheres $\mathbb{S}$. If B is such that $B \subseteq Cn(\emptyset)$ then $B_{\mathbb{S}} = \emptyset$. So, in what follows we assume $B \setminus Cn(\emptyset) \neq \emptyset$.

Now, recalling the definition of the sets $C_1, \ldots, C_m, C'_1, \ldots, C'_m$, $C''_1, \ldots, C''_m$, $B_1, \ldots, B_m$ − the subsets of B considered in the process of construction of $B_{\mathbb{S}}$ described in Definition 7.2.2 − and having in mind Lemma 4.2.5 we can see that the following lemma will be helpful in order to reformulate the definition of the above mentioned subsets of B in terms of $\leq$.

Lemma 8.0.1 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$. *Then the following statements hold:*

(i) If $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$ *then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\| \textit{ iff } f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|.$$

(ii) If $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$ *then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset f_{\mathbb{S}}(\neg\beta) \textit{ iff } (f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\| \textit{ or } f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|).$$

Proof. A proof for this lemma can be found in page 207. ■

From Lemmas 4.2.5 and 8.0.1 we can conclude that the following observation holds:

Observation 8.0.2 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *such that condition* ($\leq -\mathbb{S}$) *holds. Furthermore, let* B *be a finite set of sentences such that* $B \setminus Cn(\emptyset) \neq \emptyset$ *and* $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ *be the subsets of* B *constructed as described in Definition 7.2.2. Then:*

(a) For all $l \in \{1, \ldots, m\}$ *it holds that*

$$\begin{aligned} C''_l &= \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ f_{\mathbb{S}}(\neg\alpha_j) \not\subseteq \|\neg\alpha_i\| \textit{ or } f_{\mathbb{S}}(\neg\alpha_i) \subseteq \|\neg\alpha_j\|\} \\ &= \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \textit{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\}. \end{aligned}$$

(b) If $m > 1$ *then for all* $l \in \{2, \ldots, m\}$ *the following identities are satisfied:*

$$\begin{aligned} C'_l &= \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ f_{\mathbb{S}}(\neg\alpha_j) \not\subseteq \|\neg\alpha_i\|\}; \\ &= \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ \alpha_j \vee \neg\alpha_i \leq \alpha_j\}. \end{aligned}$$

Having in mind the definition of the $\mathbb{S}$-based filtration of B (see Definition 7.2.2) $-$ $B_{\mathbb{S}}$ $-$ the above presented facts suggest that such set can be alternatively defined as presented in the following definition (provided that $\leq$ is an epistemic entrenchment relation such that $\mathbb{S}$ and $\leq$ satisfy condition ($\leq -\mathbb{S}$)).

Definition 8.0.3 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$.

Consider a finite set of sentences $B = \{\alpha_1, \dots, \alpha_n\}$ *such that* $B \setminus Cn(\emptyset) \neq \emptyset$.

Denote by $C_1, \dots, C_m$ *the (different) equivalence classes in the quotient set of* $(B \setminus Cn(\emptyset))$ *by* $\backsim$, *i.e.* $\{C_1, \dots, C_m\} = (B \setminus Cn(\emptyset))/_{\backsim}$, *where* $\backsim$ *is the equivalence relation on* $B \setminus Cn(\emptyset)$ *defined by:*

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset),\ \alpha \backsim \beta \ \ \textit{iff} \ \ \alpha =_{E.E.} \beta,$$

Moreover, assume that the equivalence classes $C_1, \dots, C_m$ *are ordered according to the following condition:*

$$\textit{If } 1 \leq i < j \leq m \textit{ then } \forall \alpha_r \in C_i\ \forall \alpha_s \in C_j\ \ \alpha_s < \alpha_r.$$

Now consider the following list of subsets of B*:*

$$\begin{array}{rcl} B_0 & = & B \cap Cn(\emptyset) \\ C_1' & = & C_1 \\ C_1'' & = & \{\alpha_i \in C_1' : \forall \alpha_j \in C_1'\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \textit{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\} \\ B_1 & = & C_1'' \end{array} \tag{8.1}$$

Moreover, if $m > 1$ *for all* $l \in \{2, \dots, m\}$, *let* C_l', C_l'' *and* B_l *be the sets defined by:*

$$\begin{array}{rcl} C_l' & = & \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1}\ \alpha_j \vee \neg\alpha_i \leq \alpha_j\}; \\ C_l'' & = & \{\alpha_i \in C_l' : \forall \alpha_j \in C_l'\ \alpha_j \vee \neg\alpha_i \leq \alpha_j \textit{ or } \alpha_i < \alpha_i \vee \neg\alpha_j\}; \\ B_l & = & B_{l-1} \cup C_l''. \end{array}$$

The $\leq$-based filtration of B *is the set* $B_{\leq} = B_m$.

If D *is a set of sentences such that* $D \subseteq Cn(\emptyset)$ *then the* $\leq$-based filtration of D *is the empty set, i.e.* $D_{\leq} = \emptyset$.

It follows immediately from Observation 8.0.2 and condition ($\leq -\mathbb{S}$) that the identity $B_{\mathbb{S}} = B_{\leq}$ holds. This fact is more formally stated in the following observation:

Observation 8.0.4 *Let* $\mathbf{K}$ *be a belief set,* B *be a finite set of sentences and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *such that condition (*$\leq -\mathbb{S}$*) holds. Then*

$$B_{\mathbb{S}} = B_{\leq},$$

where $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B *(cf. Definition 7.2.2) and* $B_{\leq}$ *is the* $\leq$*-based filtration of* B *(cf. Definition 8.0.3).*

Now that we have seen how to define the set $B_{\mathbb{S}}$ in terms of the epistemic entrenchment relation $\leq$, it only remains to investigate how the $\leq$-based multiple contraction $\div_{\leq}$ on $\mathbf{K}$ shall be defined in order that $\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\mathbb{S}}B$, where $\div_{\mathbb{S}}$ is the $\mathbb{S}$-based multiple contraction on $\mathbf{K}$.

If $B \cap Cn(\emptyset) \neq \emptyset$ then it follows immediately from the definition of $\div_{\mathbb{S}}$ that $\div_{\leq}$ must be such that $\mathbf{K}\div_{\leq}B = \mathbf{K}$. Now assume B is a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. In order to gain some insight on how the $\mathbf{K}\div_{\leq}B$ shall be defined in this case, we start by observing that, having in mind the definitions of the $\leq$-based singleton contraction (Definition 3.4.4) and of the $\mathbb{S}$-based singleton contraction (Definition 3.3.7), it follows from Theorem 4.2.6 that, if φ is a sentence such that $\nvdash \varphi$, then $\{\beta \in \mathbf{K} : \varphi < \varphi \vee \beta\} = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi))$.

Taking this into account and reminding the definition of the $\mathbb{S}$-based multiple contraction $\div_{\mathbb{S}}$ (Definition 7.2.8), we can observe that, for such a set B, it holds that $\mathbf{K}\div_{\mathbb{S}}B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_j)\right)\right) = \bigcap\{Th\left(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_j)\right) : \alpha_j \in B_{\mathbb{S}}\}$ $= \bigcap\{\{\beta \in \mathbf{K} : \alpha_j < \alpha_j \vee \beta\} : \alpha_j \in B_{\mathbb{S}}\} = \{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\}$ (notice that in this last equality we have made use of the above proven fact that $B_{\mathbb{S}} = B_{\leq}$).

Having this in mind we are led to define the $\leq$-based multiple contraction as follows:

Definition 8.0.5 (Epistemic Entrenchment-based multiple contractions) *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. The* $\leq$-based multiple contraction on $\mathbf{K}$ *by finite sets is the multiple contraction function* $\div_{\leq}$ *defined by:*

$$\mathbf{K}\div_{\leq}B = \begin{cases} \{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}, \qquad (CM_{\div_{\leq}})$$

for any finite set of sentences B *and where* $B_{\leq}$ *is the* $\leq$*-based filtration of* B*.*

An operator $\div$ *on* $\mathbf{K}$ *is an* epistemic entrenchment-based multiple contraction on $\mathbf{K}$ *by finite sets if and only if there is an epistemic entrenchment relation* $\leq$ *with respect to* $\mathbf{K}$*, such that* $\mathbf{K}\div B = \mathbf{K}\div_{\leq}B$*, for any finite set of sentences* B*.*

Next we prove that the above definition is adequate in the sense that the class of multiple contraction functions there introduced satisfies the required properties. More precisely, in what follows we show that the class of epistemic entrenchment-based multiple contractions:

- Generalizes to the multiple contraction level the class of epistemic entrenchment-based singleton contractions (cf. Definition 3.4.4).

- Coincides with the class of the system of spheres-based multiple contractions (cf. Definition 7.2.8).

The following theorem asserts that if $\mathbb{S}$ and $\leq$ satisfy condition ($\leq -\mathbb{S}$) then the $\mathbb{S}$-based multiple contraction coincides with the $\leq$-based multiple contraction.

Theorem 8.0.6 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If* $\leq$ *and* $\mathbb{S}$ *satisfy condition* ($\leq -\mathbb{S}$) *then, for any finite set of sentences* B*,*

$$\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\mathbb{S}}B,$$

where $\div_{\leq}$ *is the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *and* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$*.*

Proof. A proof for this theorem can be found in page 208. ∎

The next observation, which is is an immediate consequence of Definitions 3.4.4 and 8.0.5, formally states that that the $\leq$-based multiple contraction of a belief set $\mathbf{K}$ by a finite set of sentences B consists of the intersection of the results of the $\leq$-based singleton contractions of $\mathbf{K}$ by each of the sentences of $B_{\leq}$. Therefore, in particular, we can conclude that, as desired, the epistemic entrenchment-based multiple contractions are indeed a generalization of the epistemic entrenchment-based (singleton) contractions.

Observation 8.0.7 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If* $\div_{\leq}$ *is the* $\leq$*-based multiple contraction by finite sets and* $-_{\leq}$ *is the* $\leq$*-based (singleton) contraction, then:*

- *For any finite set of sentences* B *such that* $B \cap Cn(\emptyset) = \emptyset$ *it holds that:*

$$\mathbf{K}\div_{\leq}B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div_{\leq}\{\alpha_i\} = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}-_{\leq}\alpha_i,$$

  where $B_{\leq}$ *is the* $\leq$*-based filtration of* B*.*

- *The identity* $\mathbf{K}\div_{\leq}\{\alpha\} = \mathbf{K}-_{\leq}\alpha$ *is satisfied for any sentence* α*.*

At this point we can observe that the converse of the statement of Theorem 8.0.6 is also satisfied and, therefore, the following stronger result, which generalizes (to the multiple contractions case) Theorem 4.2.6 also holds.

Corollary 8.0.8 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then, for any finite set of sentences* B*,*

$$\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\mathbb{S}}B,$$

if and only if $\leq$ *and* $\mathbb{S}$ *satisfy condition* ($\leq -\mathbb{S}$)*.*

Proof. A proof for this corollary can be found in page 208. ∎

The next theorem points out that the class of epistemic entrenchment-based multiple contractions coincides with the class of systems of spheres-based multiple contractions. In fact, having Theorem 8.0.6 in mind and recalling that given a system of spheres $\mathbb{S}$ there is an epistemic entrenchment relation $\leq$ such that condition ($\leq -\mathbb{S}$) holds, and vice versa (see Observations 4.2.1 and 4.2.11), we can conclude that the following theorem holds:

Theorem 8.0.9 *Let* **K** *be a belief set. A multiple contraction function on* **K** *by finite sets of sentences is a system of spheres-based multiple contraction on* **K** *if and only if it is an epistemic entrenchment-based multiple contraction on* **K**.

Combining the above theorem with Theorem 7.2.14 we can immediately conclude that all epistemic entrenchment-based multiple contractions are also partial meet multiple contractions, as stated in the following corollary:

Corollary 8.0.10 *Let* **K** *be a belief set. An epistemic entrenchment-based multiple contraction on* **K** *by finite sets is a partial meet multiple contraction on* **K** *by finite sets.*

In summary, in the present chapter we have introduced, in Definition 8.0.5 a new class of multiple contraction operations − the *E.E.-based multiple contractions* − which: (a) is formed by a kind of multiple contraction functions which are, as we have clarified in Observation 8.0.7, a generalization of the E.E.-based (singleton) contractions introduced in [Gär88, GM88] (and which we have presented in Section 3.4) to the case of contraction by sets of sentences and (b) coincides with the class of S.S.-based multiple contractions (cf. Theorem 8.0.9), analogously to what is the case in what concerns the interrelation between the singleton contraction counterparts of such two classes of functions.

Chapter 9

An Axiomatic Characterization for the System of Spheres-based and the Epistemic Entrenchment-based Multiple Contraction Functions

Our main goal in the present chapter is to obtain an axiomatic characterization for the epistemic entrenchment-based multiple contraction and, by means of Theorem 8.0.9, also for the system of spheres-based multiple contraction.

First of all we recall the following condition, proposed by Gärdenfors and Makinsson [Gär88, GM88] to define an epistemic entrenchment relation $\leq$ with respect to a given belief set $\mathbf{K}$ by means of a given contraction function $-$ on $\mathbf{K}$:

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K}-\alpha \wedge \beta \text{ or } \vdash \alpha \wedge \beta. \tag{$C_{\leq}$}$$

Such condition plays a central role in the obtention of the axiomatic characterization presented in [GM88] for the epistemic entrenchment-based (singleton) contraction. More precisely, we note that the proof of the postulates-to-construction part of such characterization (cf. Observation 3.4.6) presented in the mentioned paper consists essentially of the following two phases: (i) showing that whenever $-$ satisfies all the basic and supplementary AGM postulates for belief set contraction the ordering $\leq$ defined by ($C_{\leq}$) satisfies (EE1)-(EE5) and (ii) proving that the $\leq$-based (singleton) contraction coincides with $-$.

Based on Gärdenfors and Makinsson's proposal, given a belief set $\mathbf{K}$ we introduce the following condition which defines a binary relation on $\mathcal{L}$ by means of a given multiple (rather than singleton) contraction function $\div$ on $\mathbf{K}$:

$$\forall \alpha, \beta \in \mathcal{L},\ \alpha \leq \beta \text{ iff } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta. \tag{$CM_{\leq}$}$$

As it can be easily foreseen, such condition will play, in the process of obtaining an axiomatic characterization for the epistemic entrenchment-based multiple contractions, an analogous role to the one that has been played by condition ($C_{\leq}$) in the derivation of the homologous result at the singleton level.

The remaining of this chapter is organized as follows: In Section 9.1 we introduce all the postulates that shall be necessary for the desired characterization and show that the epistemic entrenchment-based multiple contractions satisfy such postulates. Afterwards, in Section 9.2, we show that any multiple contraction function that satisfies those postulates is an epistemic entrenchment-based multiple contraction. Finally, we close the chapter presenting, in Section 9.3, the pretended representation theorem for the epistemic entrenchment-based multiple contraction and the system of spheres-based multiple contraction and some concluding remarks.

9.1 Construction-to-postulates

We start this section by recalling the following list of multiple contraction postulates that have been introduced in Section 5.2 and which will be mentioned and used in what follows:

- **Package closure:** $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$.
- **Package inclusion:** $\mathbf{K}\div B \subseteq \mathbf{K}$.
- **Package vacuity:** If $B \cap \mathbf{K} = \emptyset$, then $\mathbf{K}\div B = \mathbf{K}$.
- **Package success:** If $B \cap Cn(\emptyset) = \emptyset$, then $B \cap \mathbf{K}\div B = \emptyset$.
- **Package extensionality:** If for every sentence α in B there is a sentence β in C such that $\vdash \alpha \leftrightarrow \beta$, and vice versa, then $\mathbf{K}\div B = \mathbf{K}\div C$.
- **Package recovery:** $\mathbf{K} \subseteq Cn((\mathbf{K}\div B) \cup B)$.
- **Package uniformity:** If every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C, then $\mathbf{K}\div B = \mathbf{K}\div C$.
- **Package relevance:** If $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K}\div B$, then there is a set K' such that $\mathbf{K}\div B \subseteq K' \subseteq \mathbf{K}$ and $B \cap Cn(K') = \emptyset$ but $B \cap Cn(K' \cup \{\beta\}) \neq \emptyset$.

We notice that the above postulates consist of properties (requirable) of a multiple contraction $\div$ on a belief set $\mathbf{K}$. Furthermore, since throughout the present chapter we will consider only multiple contraction functions by finite sets, we will assume that the sets B and C occurring in the statements of the above postulates represent arbitrary *finite* sets of sentences.

Before moving on it is convenient to remark here that the present chapter we will frequently make use Observation 5.2.3, according to which, if a multiple contraction function $\div$ on $\mathbf{K}$ (by finite sets) satisfies *P-inclusion*, *P-success*, *P-uniformity* and *P-relevance* then it satisfies all the above introduced postulates.[128]

At this point it is convenient to notice that, making use of some of the results included in Chapters 5 and 8 (some of which are, on their turn, a consequence of some results from Chapter 7), we can easily conclude that all the above introduced postulates are satisfied by any epistemic entrenchment-based multiple contraction.

[128]Notice that, throughout the present chapter the expression *P-recovery* refers to the same postulate that was designated by *finite P-recovery* in Section 5.2 and, in particular, in the statement of Observation 5.2.3.

To see this, bearing in mind the above paragraph, it is enough to observe that, since all epistemic entrenchment-based multiple contractions are partial meet multiple contractions (cf. Corollary 8.0.10) it follows immediately from the axiomatic characterization of the partial meet multiple contraction presented in Observation 5.5.1 that the following observation holds:

Observation 9.1.1 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-inclusion*,* P-success*,* P-uniformity *and* P-relevance*.*

The fact that all epistemic entrenchment-based multiple contractions satisfy the postulates mentioned in the statement of Observation 9.1.1 is essential in all that follows towards the presentation of an axiomatic characterization for such operations.

For that matter, since above we have asserted only in an indirect way that the mentioned observation holds, with the goal of clarifying (that and) how the representation theorem for the epistemic entrenchment-based multiple contractions that we present further below can be obtained in a direct way, we devote Section *F*.2 (of the Appendix) to the presentation of a proof of Observation 9.1.1 which does not rely on the results concerning the interrelations between the partial meet multiple contractions and the system of spheres-based multiple contractions and between these latter operations and the epistemic entrenchment-based multiple contractions (presented in Chapters 7 and 8, respectively).

In what follows we introduce a few more multiple contraction postulates which will be needed for the axiomatic characterization (that we shall present further ahead) of the epistemic entrenchment-based multiple contractions and show that such operations satisfy those postulates.

Since the epistemic entrenchment-based multiple contractions are a generalization of their singleton contractions counterparts (in the sense of Observation 8.0.7) it is natural to seek inspiration for the axiomatic characterization of these multiple contraction operations in the analogous characterization for the epistemic entrenchment-based singleton contraction proposed in [Gär88, GM88].

Thus, we start by observing that, apart from the basic AGM postulates (whose multiple contraction analogues we have already proven above to be satisfied by all epistemic entrenchment-based multiple contractions), the axiomatic characterization proposed by Gärdenfors and Makinsson [Gär88, GM88] included the supplementary AGM postulates:

- **Conjunctive overlap/Intersection:** $\mathbf{K}{-}\alpha \cap \mathbf{K}{-}\beta \subseteq \mathbf{K}{-}\alpha \wedge \beta$ (for any $\alpha, \beta \in \mathcal{L}$);

- **Conjunctive inclusion/Conjunction:** If $\alpha \notin \mathbf{K}{-}\alpha \wedge \beta$, then $\mathbf{K}{-}\alpha \wedge \beta \subseteq \mathbf{K}{-}\alpha$, for any belief set $\mathbf{K}$ (for any $\alpha, \beta \in \mathcal{L}$);

and, moreover, in the proof of the *postulates-to-construction* part of such representation theorem, those two postulates are fundamental to assure that the binary relation $\leq$ defined from a contraction function $-$ (satisfying the basic and supplementary

AGM postulates), by means of condition ($C_\leq$), is an epistemic entrenchment relation with respect to $\mathbf{K}$.

Furthermore, we also recall that in the beginning of the present chapter, based on condition ($C_\leq$) (proposed in [Gär88, GM88]), we have introduced condition ($CM_\leq$) with the intention that the binary relation $\leq$ defined, by means of it, from a given multiple (rather than from a singleton) contraction operation $\div$, is an epistemic entrenchment relation with respect to $\mathbf{K}$, regarded that the multiple contraction function $\div$ satisfies certain properties.

Therefore, having in mind that:

(a) Condition ($CM_\leq$) can be obtained from condition ($C_\leq$) simply by replacing in this latter condition $\mathbf{K}-\alpha \wedge \beta$ by $\mathbf{K}\div\{\alpha \wedge \beta\}$ (for any $\alpha, \beta \in \mathcal{L}$).

(b) In [GM88] to prove that the binary relation $\leq$ defined by ($C_\leq$) was an epistemic entrenchment relation it has been necessary to require that the contraction function $-$ satisfied, among others, the postulates of *Conjunctive overlap/Intersection* and *Conjunctive inclusion/Conjunction*.

we are led to expect that, in order to assure that the binary relation on $\mathcal{L}$ defined by ($CM_\leq$) is an epistemic entrenchment relation, it may be necessary (or, at least, useful) to impose that the multiple contraction function $\div$ satisfies the following postulates of *Package conjunctive overlap/Package intersection* and *Package conjunctive inclusion/Package conjunction*:[129]

- **Package conjunctive overlap/Package intersection:** $\mathbf{K}\div\{\alpha\}\cap\mathbf{K}\div\{\beta\} \subseteq \mathbf{K}\div\{\alpha \wedge \beta\}$ (for any $\alpha, \beta \in \mathcal{L}$).

- **Package conjunctive inclusion/Package conjunction:** If $\alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\}$, then $\mathbf{K}\div\{\alpha \wedge \beta\} \subseteq \mathbf{K}\div\{\alpha\}$ (for any $\alpha, \beta \in \mathcal{L}$).

The following observation, which can be straightforwardly proven, states more formally some of the above exposed facts and presents some results which materialize the intuition that has driven the proposal of the postulates of *Package conjunctive overlap/Package intersection* and *Package conjunctive inclusion/Package conjunction* as well as their formulations.

Observation 9.1.2 *Let* $\mathbf{K}$ *be a belief set and* $-$ *and* $\div$ *be, respectively, a singleton contraction function on* $\mathbf{K}$ *and a multiple contraction function on* $\mathbf{K}$ *such that for all sentences* α *it holds that* $\mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}$*. Then:*

(i) Conditions ($C_\leq$) *and* ($CM_\leq$) *are equivalent, that is, they define the same binary relation* ($\leq$) *on* $\mathcal{L}$*.*

[129]Notice that the statements of *Package conjunctive overlap/Package Intersection* and *Package conjunctive inclusion/Package Conjunction* are obtained simply by replacing in the statements of the postulates of *Conjunctive overlap/Intersection* and *Conjunctive inclusion/Conjunction*, respectively, every reference to a result of a (singleton) contraction of the form $\mathbf{K}-\delta$ by $\mathbf{K}\div\{\delta\}$, for any sentence $\delta \in \mathcal{L}$.

(ii) $-$ *satisfies* Conjunctive overlap/Intersection *(respectively,* Conjunctive inclusion/Conjunction*) if and only if* $\div$ *satisfies* Package conjunctive overlap/ Package intersection *(respectively,* Package conjunctive inclusion/Package conjunction*).*

(iii) If $\div$ *satisfies* P-closure*,* P-inclusion*,* P-vacuity*,* P-success*,* P-recovery*,* P-extensionality*, then* $-$ *satisfies* closure*,* inclusion*,* vacuity*,* success*,* recovery*,* extensionality*, respectively.*

Combining the above observation with Observations 8.0.7 and 3.4.5 we can easily conclude that the epistemic entrenchment-based multiple contractions satisfy the two latter introduced multiple contraction postulates as well as condition $(CM_{\leq})$.

Observation 9.1.3 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *as well as the condition* $(CM_{\leq})$*.*

Proof. A proof for this observation can be found in page 209. ■

We have thus shown that the epistemic entrenchment-based multiple contractions satisfy the *multiple contraction analogues* of all the basic and supplementary AGM postulates for belief set contraction. Nevertheless, to be able to provide an axiomatic characterization for the epistemic entrenchment-based multiple contractions we still need to introduce one more multiple contraction postulate.

To be in a position to formulate such postulate we must start by showing that the binary relation $\leq$ defined by condition $(CM_{\leq})$ is an epistemic entrenchment relation with respect to $\mathbf{K}$ as long as the multiple contraction $\div$ on $\mathbf{K}$ satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection* and *P-conjunctive inclusion/P-conjunction*.

Observation 9.1.4 *Let* $\mathbf{K}$ *be a belief set. If* $\div$ *is a multiple contraction function on* $\mathbf{K}$ *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection *and* P-conjunctive inclusion/P-conjunction*, then the binary relation* $\leq$ *defined from* $\div$ *by means of condition* $(CM_{\leq})$ *is an epistemic entrenchment relation with respect to* $\mathbf{K}$*.*

Proof. A proof for this observation can be found in page 209. ■

Having the above observation in mind we can now proceed to formulate the last postulate which will be necessary for the axiomatic characterization of the epistemic entrenchment-based multiple contractions that we shall present further ahed. Such postulate is somehow suggested by the fact that $\mathbf{K}\div_{\leq}B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div_{\leq}\{\alpha_i\}$, for any finite set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$ (cf. Observation 8.0.7), and its formulation is the following:

- **Package singleton reduction:** For any finite set B such that $B \cap Cn(\emptyset) = \emptyset$, if $\div$ satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection* and *P-conjunctive inclusion/P-conjunction* then $\mathbf{K} \div B = \bigcap_{\alpha_i \in B_{\leq_\div}} \mathbf{K} \div \{\alpha_i\}$, where $\leq_\div$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

$$\alpha \leq_\div \beta \text{ if and only if } \alpha \notin \mathbf{K} \div \{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta,$$

and $B_{\leq_\div}$ is the $\leq_\div$-based filtration of B.

Remark 9.1.5 *Notice that the need to include in the above formulation of the postulate of* P-singleton reduction *the condition that $\div$ satisfies some other multiple contraction postulates is due to the fact that, since we have only assured that the binary relation $\leq$ defined from $\div$ by means of condition ($CM_{\leq}$) is an epistemic entrenchment relation as long as $\div$ satisfies those other postulates (cf. Observation 9.1.4), it only makes sense to refer to the $\leq_\div$-based filtration of a finite set of sentences (cf. Definition 8.0.3), which is used in the formulation of such postulate, if the multiple contraction operation $\div$ satisfies those* extra *postulates.*

However, since throughout the remainder of this text whenever we refer to (or make use of) the postulate of P-singleton reduction *that shall be done in settings where the remaining above mentioned postulates are (or are assumed to be) also satisfied, to lighten the writing we can (abdicating of some rigour) use the following* lighter *formulation of that postulate:*

- **Package singleton reduction:** *For any finite set B such that $B \cap Cn(\emptyset) = \emptyset$, it holds that $\mathbf{K} \div B = \bigcap_{\alpha_i \in B_{\leq_\div}} \mathbf{K} \div \{\alpha_i\}$, where $\leq_\div$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by*

$$\alpha \leq_\div \beta \textit{ if and only if } \alpha \notin \mathbf{K} \div \{\alpha \wedge \beta\} \textit{ or } \vdash \alpha \wedge \beta,$$

and $B_{\leq_\div}$ is the $\leq_\div$-based filtration of B.

Loosely speaking to say that a certain multiple contraction function $\div$ satisfies the postulate of *P-singleton reduction* is to say that the result of the (multiple) contraction of $\mathbf{K}$ by any finite set B can be obtained by intersecting the results of the singleton contractions (by means of that same multiple contraction function) of $\mathbf{K}$ by some appropriately chosen sentences of B (more precisely, the sentences in the $\leq_\div$-based filtration of B).

Having introduced such postulate for multiple contraction function, we now proceed to show that such property is satisfied by all epistemic entrenchment-based multiple contractions.

Observation 9.1.6 *Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Then the $\leq$-based multiple contraction $\div$ on $\mathbf{K}$ satisfies* P-singleton reduction*.*

Proof. A proof for this observation can be found in page 210. ∎

9.2 Postulates-to-construction

In this section we show that if a multiple contraction function $\div$ on $\mathbf{K}$ satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction* and *P-singleton reduction* then it is an epistemic entrenchment-based multiple contraction. To be more precise, we will see that, in those conditions, $\div$ coincides with the $\leq$-based multiple contraction $\div_{\leq}$ on $\mathbf{K}$ where $\leq$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by condition $(CM_{\leq})$.[130]

The following result settles the *second step* of the proof of the postulates to construction part of the axiomatic characterization for the epistemic entrenchment-based multiple contractions that we will present in the next section.[131]

Observation 9.2.1 *Let* $\mathbf{K}$ *be a belief set,* $\div$ *be a multiple contraction function on* $\mathbf{K}$ *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *and* P-singleton reduction *and* $\leq$ *be the epistemic entrenchment relation with respect to* $\mathbf{K}$ *defined by condition* $(CM_{\leq})$*.*[130] *Then* $\mathbf{K}\div B = \mathbf{K}\div_{\leq}B$*, for any finite set* B*, where* $\div_{\leq}$ *is the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *defined by* $(CM_{\div_{\leq}})$*.*

Proof. A proof for this observation can be found in page 210. ■

9.3 Representation Theorems and Concluding Remarks

In this section we essentially summarize and reorganize the main results obtained in the previous ones in order to present more clearly and explicitly an axiomatic characterization for the epistemic entrenchment-based multiple contractions and, consequently, also for the system of spheres-based multiple contractions.

Thus, Theorems 9.3.1 and 9.3.2 below show that the epistemic entrenchment-based multiple contractions can be axiomatically charaterized by the postulates of *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction* and *P-singleton reduction* and, moreover, they show how to define an epistemic entrenchment relation with respect to a belief set $\mathbf{K}$ by means of a multiple contraction function $\div$ on $\mathbf{K}$ satisfying the above mentioned postulates, and vice versa.[132]

The following theorem basically groups in one single statement the facts already exposed in Observations 9.1.1, 9.1.3 and 9.1.6.

[130] Notice that Observation 9.1.4 assures that whenever the above conditions hold the binary relation $\leq$ thus defined is indeed an epistemic entrenchment relation with respect to $\mathbf{K}$.

[131] It is worth clarifying at this point that the *first step* of such proof consists of demonstrating that the above mentioned binary relation $\leq$ (defined from $\div$ by means of condition $(CM_{\leq})$) is in fact an epistemic entrenchment relation with respect to $\mathbf{K}$, which is taken care of by Observation 9.1.4.

[132] Notice that Theorems 9.3.1 and 9.3.2, can be seen as generalizations to the case of multiple contraction of Observations 3.4.5 and 3.4.6, respectively

Theorem 9.3.1 *Let* **K** *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* **K**. *Then the* $\leq$*-based multiple contraction* $\div$ *on* **K** *satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction *as well as* Condition $(CM_{\leq})$*.*

Now, gathering Observations 9.1.4 and 9.2.1 we obtain the theorem below, which exposes that any multiple contraction function $\div$ satisfying the postulates mentioned in the formulation of Theorem 9.3.1 is an epistemic entrenchment-based multiple contraction, by presenting an explicit definition (based on $\div$) of an epistemic entrenchment relation $\leq$ such that $\div$ coincides with the $\leq$-based multiple contraction $\div_{\leq}$.

Theorem 9.3.2 *Let* **K** *be a belief set and* $\div$ *be a multiple contraction function on* **K** *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *and* P-singleton reduction*. Then the binary relation* $\leq$ *defined from* $\div$ *by means of condition* $(CM_{\leq})$ *is an epistemic entrenchment relation with respect to* **K**, *and* $\mathbf{K}\div B = \mathbf{K}\div_{\leq}B$, *for any finite set* B, *where* $\div_{\leq}$ *is the* $\leq$*-based multiple contraction* $\div_{\leq}$ *on* **K** *defined by* $(CM_{\div_{\leq}})$*.*

At this point it is worth remarking that it follows immediately from Theorems 9.3.1 and 9.3.2 that conditions $(CM_{\div_{\leq}})$ and $(CM_{\leq})$ are *equivalent* in the sense of the following Corollary (which can be seen as a generalization to the case of multiple contraction of Corollary 3.4.7):

Corollary 9.3.3 *Let* **K** *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* **K** *and* $\div$ *be a multiple contraction function on* **K** *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *and* P-singleton reduction*. Then:*

(i) If $\leq$ *is the binary relation defined from* $\div$ *by* $(CM_{\leq})$ *then the* $\leq$*-based multiple contraction function* $\div_{\leq}$*, determined by* $(CM_{\div_{\leq}})$*, is identical to* $\div$*.*

(ii) If $\div$ *coincides with the* $\leq$*-based contraction function* $\div_{\leq}$*, determined by* $(CM_{\div_{\leq}})$*, then the epistemic entrenchment relation that is obtained from* $\div$ *by means of* $(CM_{\leq})$ *is identical to* $\leq$*.*

We close this chapter with the following theorem, which essentially interconnects Theorems 8.0.9, 9.3.1 and 9.3.2. Roughly speaking, such theorem results of gathering and reformulating the main results of the present and the previous chapter, namely that: the class of epistemic entrenchment-based multiple contractions coincides with the class of system of spheres-based multiple contractions and both those classes of multiple contraction functions can be axiomatically characterized by the postulates of *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction*, *P-singleton reduction*.

Theorem 9.3.4 *Let* **K** *be a belief set and* ÷ *be a multiple contraction function on* **K**. *Then the following statements are equivalent:*

1. ÷ *is a system of spheres-based multiple contraction on* **K**.
2. ÷ *is an epistemic entrenchment-based multiple contraction on* **K**.
3. ÷ *satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction*.*

In summary, in the present chapter we have obtained a representation theorem for the class of E.E.-based multiple contractions (which we introduced in Chapter 8) and also for the class of S.S.-based multiple contractions (presented in Chapter 7), given that, according to Theorem 8.0.9, those two classes of multiple contraction functions are identical. Thus, as stated in Theorem 9.3.4, it holds that those functions can be axiomatically characterized by the multiple contraction postulates of *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction* and *P-singleton reduction*. The first four of such postulates are precisely the postulates included in the well known axiomatic characterization of the partial meet multiple contraction that we have recalled in Observation 5.5.1. The postulates of *P-conjunctive overlap/P-intersection* and *P-conjunctive inclusion/P-conjunction* are straightforward adaptations of the supplementary AGM postulates for contraction to the case of multiple contraction. Finally, the postulate of *P-singleton reduction* is a less intuitive property which, however, was necessary in our proof of the postulates-to-construction part of the mentioned representation theorem. Nevertheless, in spite of its arguably unnatural formulation, such postulate was intuitively motivated by our observations clarifying that and how both the E.E.-based multiple contractions and the S.S.-based multiple contractions can be reduced to (or, in other words, defined by means of) the corresponding singleton contraction operations. Furthermore, in support of that postulate we may yet remark that it can be straightforwardly noticed that its *translation* into a singleton contraction postulate results in a property which is trivially satisfied by any AGM contraction which satisfies both the basic and the supplementary AGM postulates for contraction.

Chapter 10

Conclusion

This final chapter of the present text is devoted to the presentation of a brief summary of the main results achieved in this work and of some potential topics for future investigation.

In Section 10.1 we highlight the main contributions of this thesis and situate them in the context of the *general picture* of the Logic of Theory Change.

Then, in Section 10.2 we list some (potential) topics for future investigation which arise naturally from the research reported in the present thesis and in some related works that can be found in the literature (namely the ones that we have described in Section 5.6).

10.1 Summary

The AGM model of belief change is considered the standard model in the Logic of Theory Change. In such framework essentially three kinds of changes are considered, namely expansions, contractions and revisions. However, in their seminal paper [AGM85], Alchourrón, Gärdenfors and Makinson have considered only expansions and contractions as basic operations and treated revisions as being operations which are derived from those two. Such option was taken in accordance with the arguments presented by Levi [Lev77] supporting that all acceptable revision functions can be obtained by a two steps procedure consisting of a contraction followed by an expansion, as described in equation (2.1), which is, nowadays, commonly known as the Levi identity.

Therefore, in particular, the class of contraction functions proposed in the AGM framework has acquired a crucial role in the study of the Logic of Theory Change. In fact such class of functions has been characterized in several different (however equivalent) ways. Among those different characterizations we emphasize the following three constructive models of AGM contraction functions: partial meet contractions ([AGM85]), system of spheres-based contractions ([Gro88]) and epistemic entrenchment-based contractions ([Gär88, GM88]). Each of those models was described in Chapter 3 and, furthermore, in Chapter 4 we exposed the well known facts that the two latter mentioned models can be seen as two alternative ways of defining the same class of contraction functions and that, on the other hand, the class of S.S.-based contractions (or E.E.-based contractions) coincides with the class of *transitively relational* partial meet contractions, which is a (proper) subclass of the class of partial meet contractions.

In Figure 10.1 we present a diagram exposing all the above mentioned interrelations among the three models and indicating the references where such interrelations have been proven to hold (in a direct way).

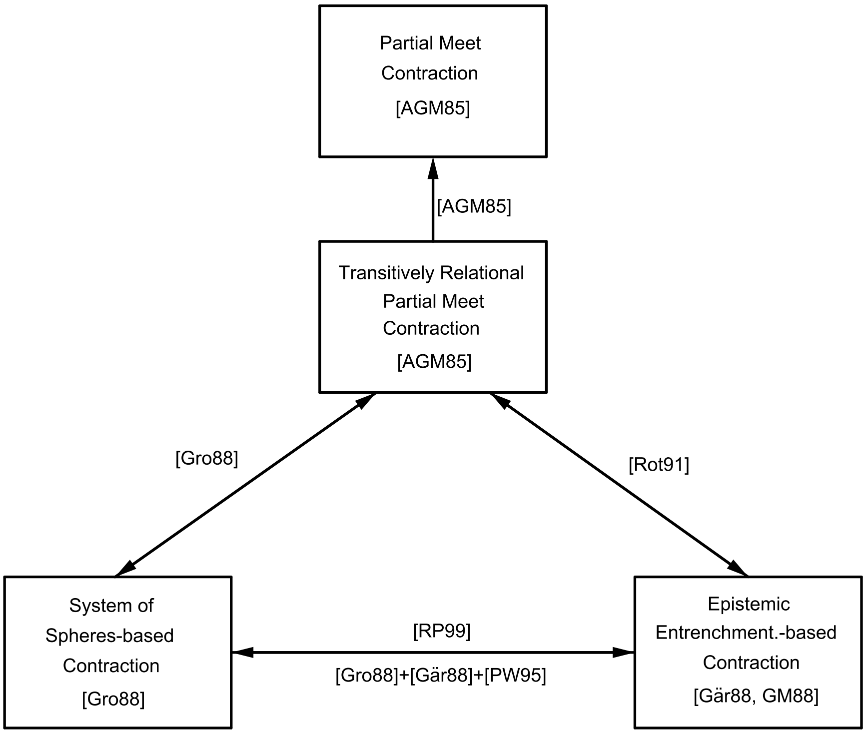

Figure 10.1: Diagram of the known interrelations among the models of partial meet contraction, system of spheres-based contraction and epistemic entrenchment-based contraction with the indication of the references where each of those interconnections was originally proven to hold, in a direct way.

At this point we remark that the direct proof for the fact that the class of system of spheres-based contractions is identical to the class of epistemic entrenchment-based contractions, that we have presented in Section 4.2 is an original (although minor) contribution of the present thesis (which we have obtained by combining some results from each of the following three works [Gro88, Gär88, PW95]). We must, however, notice here that such result itself was already very well known and, furthermore, even a direct proof for it was already presented in [RP99]. Thus, the originality of the contribution of this thesis in that regard rests exclusively in the fact that the (direct) proof for that result that is presented here differs from any other proof for that fact which was, until now, presented in the literature.

Not very long after the AGM model was introduced several researchers pointed out the need for the obtention of generalizations and extensions of such model. In particular, some studies regarding the generalization of the operation of theory contraction to the case of contractions of theories by sets of sentences (rather than only by one single sentence) have been subsequently presented in the literature (e.g.

[Fuh88, Han91a, Nie91, FH94, FSS03]).

However, until know, only one of the three constructive models of singleton contraction functions that we have mentioned above, had been generalized to the case of multiple contraction, namely the model of partial meet contraction, whose generalization to the case of contractions by sets of sentences is known as partial meet multiple contraction and was presented in [Han89, FH94]. Taking that into account, our main goal in the present thesis was to generalize the other two models, i.e. the S.S.-based contractions and the E.E.-based contractions models to the case of multiple contraction. Having in mind the known interrelation between those models in the singleton contraction level, it was our intention that the new models of multiple contraction functions that we would present, namely the system of spheres-based multiple contractions and the epistemic entrenchment-based multiple contractions, were such that:

1. The class of S.S.-based multiple contractions coincided with the class of E.E.-based multiple contractions.

2. Every S.S.-based multiple contraction (and, equivalently, every E.E.-based multiple contraction) was a partial meet multiple contraction.

3. Given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|K\|$ (resp. an epistemic entrenchment $\leq$ with respect to $\mathbf{K}$) the $\mathbb{S}$-based (resp. $\leq$-based) multiple contraction on $\mathbf{K}$ was a generalization of the $\mathbb{S}$-based (resp. $\leq$-based) contraction on $\mathbf{K}$ in the sense that, on the one hand it accounted for contractions of $\mathbf{K}$ by sets of sentences rather than only by a single sentence and, on the other hand, for any sentence α, the $\mathbb{S}$-based (resp. $\leq$-based) multiple contraction of $\mathbf{K}$ by the singleton set $\{\alpha\}$ coincided with the $\mathbb{S}$-based (resp. $\leq$-based) contraction of $\mathbf{K}$ by the (single sentence) α.

With the above exposed goal in mind we started our quest for an adequate definition for each of those two new classes of functions by obtaining, in Chapter 6, a characterization of the partial meet multiple contractions in terms of possible worlds. To be more precise, in the mentioned chapter we first identified which propositions (i.e. sets of possible worlds) correspond to the set of possible worlds of a remainder (cf. Theorem 6.1.6) and afterwards, based on that result, we exposed, in Theorem 6.3.6, how the partial meet multiple contractions can be described in terms of possible worlds. Such characterization of the partial meet multiple contraction generalizes the possible worlds semantics for partial meet (singleton) contraction presented in [Gro88, Han99b] (which we exposed in Section 4.1). Another result obtained in that same chapter which is also worth to be mentioned here is the characterization of the full meet multiple contraction that we have presented in Corollary 6.2.9.

As we have stated above, the main motivation that underlaid the obtention of the above mentioned description of the partial meet multiple contractions in terms of possible worlds was the intention of subsequently using such a characterization as a starting point towards the finding of a suitable way of constructing a partial meet multiple contraction by means of systems of spheres. Thus, as it was expectable, the main contribution of the succeeding Chapter 7 was precisely the presentation, in Definition 7.2.8, of a way of constructing system of spheres-based multiple contractions which satisfy the requirements that we had in mind. However, such definition is quite unintuitive and is based on the very technical concept of $\mathbb{S}$-based filtration of a set of sentences (where $\mathbb{S}$ is a system of spheres) which we have introduced in Definition 7.2.2. For that reason, in that chapter, with the goals of motivating the construction of system of spheres-based multiple contractions that we have there presented and of clarifying how that definition was reached, we have introduced such definition as the result of successive modifications to previous (and more intuitive) constructions (with those modifications being naturally suggested by the reasons why each of the previously proposed constructions give rise to operations which are not partial meet multiple contractions). Regarding the main results of Chapter 7 we must mention here Theorem 7.2.14 and Observation 7.2.16 which confirm that the multiple contraction operations proposed in Definition 7.2.8 indeed satisfy the requirements 2. − 3. that we have stated in the beginning of the present section. Moreover, in Observation 7.2.16 it is also exposed that, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, the $\mathbb{S}$-based multiple contraction of $\mathbf{K}$ by a finite set of sentences B can be reduced to the intersection of the results of some $\mathbb{S}$-based (singleton) contractions of $\mathbf{K}$ by certain (adequately chosen single) sentences of B.

Having presented a definition for the S.S.-based multiple contractions satisfying the required conditions, in order to fulfil the goals that we have stated above, it remained only to present a definition for the E.E.-based multiple contractions which was such that conditions 1. and 3. listed above were satisfied. Thus, in Chapter 8, making use of the tight interconnection between systems of spheres and epistemic entrenchment relations (that we exposed in Section 4.2) we presented a definition for the epistemic entrenchment-based multiple contractions (namely, Definition 8.0.5) which we have proven, cf. Theorem 8.0.9, to be such that the class of those functions coincides with the class of system of spheres-based multiple contractions. Additionally we have shown that, as stated in Observation 8.0.7, for any epistemic entrenchment relation $\leq$, the $\leq$-based multiple contraction satisfies property 3. above and, moreover, analogously to what we have mentioned to be the case in what concerns S.S.-based multiple contractions, the $\leq$-based multiple contraction of $\mathbf{K}$ by a finite set of sentences B can be reduced to the intersection of the results of some $\leq$-based (singleton) contractions of $\mathbf{K}$ by certain (adequately chosen single) sentences of B.

Thus, in Chapters 7 and 8 we have obtained, as we wished, the generalization to the case of multiple contraction of the classes of S.S.-based and E.E.-based contractions and, furthermore, we have assured that the definitions proposed for such generalizations were such that the interrelation between the corresponding singleton contraction classes was preserved in the multiple contractions level. In Figure 10.2 we present a diagram which represents the main definitions and results presented in chapters 7 and 8. We notice that such diagram highlights those among the definitions and results illustrated by the diagram in Figure 10.1 which we have generalized to the multiple contraction level in Chapters 7 and 8.

Finally, we dedicated Chapter 9 to the presentation of a representation theorem for the classes of E.E.-based multiple contraction and of S.S.-based multiple

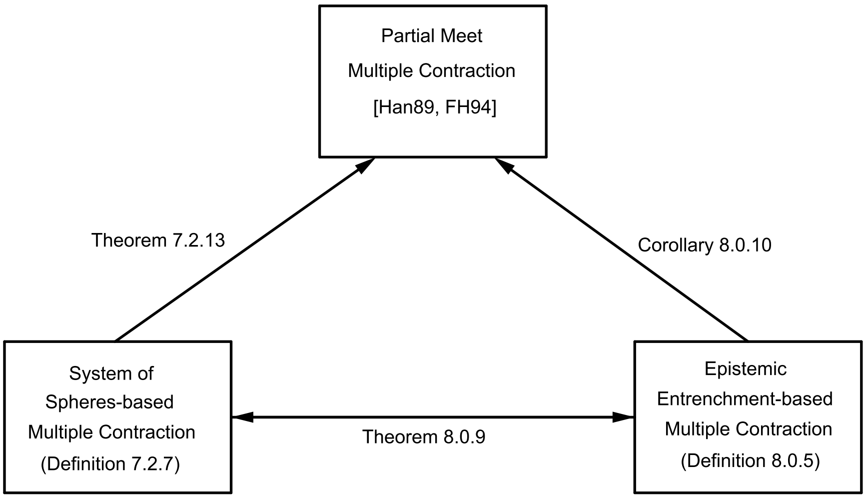

Figure 10.2: Diagram highlighting the results that we have proven to hold concerning the interrelations among the newly proposed classes of system of spheres-based multiple contraction and epistemic entrenchment-based multiple contraction and the standard model of partial meet multiple contraction.

contractions. Such an axiomatic characterization is exposed in Theorems 9.3.1 and 9.3.2. We close that chapter with the presentation of Theorem 9.3.4, which follows from the combination of the two above mentioned theorems and Theorem 8.0.9 and, therefore, summarizes the central results of chapters 8 and 9, namely by stating that the newly introduced classes of E.E.-based multiple contractions and of S.S.-based multiple contractions are identical and providing an axiomatic characterization for such functions.

10.2 Future Work

In this section we present a list of topics of (potential) future research which arise naturally in the sequence of the investigation reported in the present thesis and of the comparison of the definitions and results here obtained with the analogous ones presented in the main related works that can be found in the literature (namely the ones that we have described in Section 5.6). Thus, the mentioned list of (plausible) subsequent future work topics is the following:

1. To investigate if (as it is suggested by the analogy with the singleton case) the S.S.-based multiple contractions (or, equivalently, the E.E.-based multiple contractions) are *transitively relational partial meet multiple contractions*. That is, to verify if, given a belief set $\mathbf{K}$ and an arbitrary S.S.-based multiple contraction $\div_{\mathbb{S}}$ on $\mathbf{K}$, it is possible to define a transitive relation $\sqsubseteq$ on the set $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ such that the multiple contraction $\div_{\mathbb{S}}$ coincides with the partial meet multiple contraction generated by the package selection function γ defined, for all sets of sentences B, by $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ if $B \cap Cn(\emptyset) \neq \emptyset$, and

by the *marking-off identity*:

$$\gamma(\mathbf{K}\bot B) = \{X \in \mathbf{K}\bot B : X' \sqsubseteq X \text{ for all } X' \in \mathbf{K}\bot B\},$$

if $B \cap Cn(\emptyset) = \emptyset$.

In that regard we anticipate here that the first intuitive step towards the definition of such a binary relation $\sqsubseteq$ shall be the *translation* of the definition of the S.S.-based multiple contractions to the language of remainders by means of the relation between sets of possible worlds and remainders that we have exposed in Theorem 6.1.6.

At this point it is worth to remark also that proving that the S.S.-based multiple contractions (and, equivalently, the E.E.-based multiple contractions) are transitively relational partial meet multiple contractions can be seen as generalizing to the case of contraction by (possibly non-singleton) sets of sentences Grove's direct proof of the fact that every S.S.-based (singleton) contraction is a transitively relational partial meet (singleton) contraction (cf. Section 4.1).

2. To investigate the converse relation to the one mentioned in the previous point. That is, to find out if every transitively relational partial meet multiple contraction by finite sets is a S.S.-based multiple contraction (and an E.E.-based multiple contraction). We remark here that a possible first step for investigating if that interrelation between classes of multiple contraction functions holds (as well as the one mentioned in the previous point) is to obtain an axiomatic characterization for the transitively relational partial meet multiple contractions by finite sets, which is also an interesting topic for future research on its own right.

3. If one of the two previous points is proven not to hold, then another interesting topic of research arises, namely to find a definition and an axiomatic characterization for a (different) class of multiple contraction functions which:

    - Generalizes the class of S.S.-based singleton contraction functions (resp. the class of E.E.-based singleton contraction functions) to the case of contractions by sets of sentences (rather than by a single sentence).
    - Coincides with the class of transitively relational partial meet multiple contractions.

4. An adequate start for the potential future quest for a different definition of S.S.-based multiple contractions may be the attempt to obtain a proof for the converse of Theorem 7.2.15 or some similar result.

5. To find a more natural postulate than the postulate of P-singleton reduction which can be used instead of that one in an axiomatic characterization for S.S.-based multiple contractions (and E.E.-based multiple contractions) alternative to the one that we have presented in Theorem 9.3.4.

6. Additionally (or alternatively) to the research suggested in the previous point it would be interesting to obtain a more general class than the class of S.S.-based (and E.E.-based) multiple contractions which is axiomatically characterized by all the postulates included in the axiomatic characterization presented in Theorem 9.3.4 except for the postulate of P-singleton reduction.

7. To find a definition and an axiomatic characterization for a class of multiple contraction functions which generalizes the class of S.S.-based singleton contraction functions to the case of contractions by possibly infinite sets of sentences (rather than only by finite sets).

   Regarding this goal, on the one hand, we must remark that it may be convenient to start the search for such generalization (of the definitions and results concerning the S.S.-based multiple contraction that we have presented in Chapter 7) by the investigation of the possible world semantics of the partial meet multiple contractions by (possibly) infinite sets, since the characterization of the partial meet multiple contractions in terms of possible worlds that we have presented in Chapter 6 only accounts for partial meet multiple contractions by finite sets.

   On the other hand, we notice that such a definition may be based on the one that we have presented for S.S.-based multiple contractions by finite sets (i.e. Definition 7.2.8). However we must take the following fact into account: Given a finite set of sentences B such that $B \setminus Cn(\emptyset) \neq \emptyset$, the set C_1 which appears in Definition 7.2.2 (of the set $B_{\mathbb{S}}$ – the $\mathbb{S}$-based filtration of B) is the least element of the quotient set $(B \setminus Cn(\emptyset))$ by $\backsim$ (the binary relation introduced in the mentioned definition) when we consider (on that set) the total order $\preccurlyeq$ defined by:

$$\text{For all } U, V \in (B \setminus Cn(\emptyset))/_{\backsim}\ U \preccurlyeq V \text{ iff } \forall \alpha_r \in U\ \forall \alpha_s \in V\ \ \mathbb{S}_{\neg\alpha_s} \subseteq \mathbb{S}_{\neg\alpha_r}. \tag{10.1}$$

   On the other hand, if B is an infinite set then it may happen that the quotient set $(B \setminus Cn(\emptyset))/_{\backsim}$ is also infinite and, furthermore, it may be the case that there is not a least element of that set under the total order $\preccurlyeq$ defined by condition (10.1). Thus, since the first step of the process of construction of the $\mathbb{S}$-based filtration of a set B (presented in Definition 7.2.2) consists precisely in the determination of the least element (under $\preccurlyeq$) of the set $(B \setminus Cn(\emptyset))/_{\backsim}$, in that case it is impossible to obtain the set $B_{\mathbb{S}}$ (in fact, and for that reason, the $\mathbb{S}$-based filtration of a set is only defined for finite sets B).

   Having the above in mind, we notice that if $\mathbb{S}$ is a system of spheres which satisfies the following condition

$$\text{For all sets of sentences } C \subseteq \mathcal{L} \setminus Cn(\emptyset) \text{ the quotient set } C/_{\backsim} \text{ contains a least element under } \preccurlyeq, \tag{10.2}$$

   then the definition of $\mathbb{S}$-*based filtration* of a set of sentences can be *extended* to the case of infinite sets of sentences by making a slight modification to

Definition 7.2.2.[133] Additionally we may observe at this point that, as long as $B_{\mathbb{S}}$ is defined for any (possibly infinite) set of sentences B such that $B \setminus Cn(\emptyset) \neq \emptyset$, it holds that the function $\div_{\mathbb{S}}$ introduced in Definition 7.2.8 is also defined for any (possibly infinite) set of sentences B and, therefore, the function thus defined becomes a natural candidate to be the $\mathbb{S}$-based multiple belief contraction (by possibly infinite sets of sentences).

Notice that the idea of using only a subclass of the class of all the systems of spheres in the process of defining the S.S.-based multiple contractions by (possibly infinite) sets of sentences is analogous to the one followed in [Pep04] where the $\mathbb{S}$-based multiple revision $\oplus_{\mathbb{S}}$ is defined by equation ($\oplus_{\mathbb{S}}$) but only for systems of spheres $\mathbb{S}$ which satisfy conditions (SM) and (SD) (cf. Subsection 5.6.3).

8. (Assuming that the goal described in the previous point is achieved) To obtain a definition for a class of multiple contraction functions which generalizes the class of E.E.-based singleton contraction functions to the case of contractions by possibly infinite sets of sentences (rather than only by finite sets) and is identical to the class of S.S.-based multiple contraction functions obtained as a result of the investigation suggested in the previous point.

   We notice that such goal can be achieved by an analogous procedure to the one that we have used in Chapter 8 to obtain the definition of the E.E.-based multiple contractions, which was based essentially on the definition of the S.S.-based multiple contractions (introduced in Chapter 7) and on the tight interconnection between systems of spheres and epistemic entrenchment relations (which we have exposed in Section 4.2).

9. To obtain an explicit definition as well as an axiomatic characterization for the class consisting of functions $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ which are such that for any belief set $\mathbf{K}$ and any set of sentences B:

$$\mathbf{K}\ominus B = \mathbf{K}\div\neg B, \tag{10.3}$$

   where $\div$ is a S.S.-based multiple contraction on $\mathbf{K}$ (or, equivalently, an E.E.-based multiple contraction on $\mathbf{K}$) and $\neg B$ is the negation of B introduced in Definition 5.6.1. Furthermore, to investigate the interrelation between that class of functions and each one of the two classes of set contraction functions proposed in [ZF01], namely the *partial meet set contractions* and the *NOP-based contractions*, and with the class of *comparative possibility preorder-based set contractions* which was introduced in [Pep] (cf. Subsection 5.6.2).

   We notice that proceeding in such a way we would be somehow obtaining a possible-worlds semantics for set contractions. Or, more precisely, we would be characterizing the set contractions in terms of systems of spheres.

[133] We notice that, in particular, if $\mathcal{L}$ is a finitely generated propositional language (i.e. a propositional language generated from a finite alphabet) then the set of all possible worlds $\mathcal{M}_{\mathcal{L}}$ is finite and, consequently, in that case, any system of spheres is also finite. Therefore, in particular, in such a context condition (10.2) is satisfied by any system of spheres $\mathbb{S}$.

10. To obtain an explicit definition as well as an axiomatic characterization for the multiple contraction functions $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ which are such that, for any belief set $\mathbf{K}$ and any set of sentences B:

$$\mathbf{K}\div B = \mathbf{K}\ominus\neg B, \tag{10.4}$$

where $\ominus$ is a NOP-based contraction (resp. a comparative possibility preorder-based set contraction) and $\neg B$ is the negation of B introduced in Definition 5.6.1. And, additionally to study the interrelation between the class of such multiple contraction functions and the class of E.E.-based multiple contractions (and S.S.-based multiple contractions).

11. The obtention of the above mentioned axiomatic characterizations and interrelations might also be useful towards the achievement of a more general result interconnecting *multiple contractions* and *set contractions* which extends the results stated in Observations 5.6.5 and 5.6.6 presented in Subsection 5.6.2 (that were originally stated, without proof, in [ZCZC97]). In this respect we remark that an attentive analysis of the concepts of set contraction and of multiple contraction and of the definition of negation of a set of sentences (cf. Definition 5.6.1) as well as of the motivation that underlaid the suggestion of such definition leads to the following conjecture, which would be worth proving or refuting in some future research.

Conjecture 10.2.1 [134] *The following statements hold:*[135]

(i) If the function $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *satisfies the postulates of* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction*, then the function* $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *defined from* $\div$ *by means of equation* (10.3) *satisfies the postulates* $(\ominus 1)-(\ominus 8)$ *listed in [ZF01, Subsection* 2.2*] as well as the identity* (10.4)*.*

(ii) If the function $\ominus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *satisfies the postulates* $(\ominus 1)-(\ominus 8)$ *listed in [ZF01, Subsection* 2.2*], then the function* $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *obtained from* $\ominus$ *by means of equation* (10.4) *satisfies the postulates of* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction *and the identity* (10.3)*.*

The attempt to find a proof for the above conjecture seems to be an appropriate starting point for the search of a satisfactory result concerning the interrelation

[134] We notice that, however the statement of this conjecture is similar to the result that arises from the combination of Observations 5.6.5 and 5.6.6 (presented without proof in [ZCZC97]), it is considerably more embracing than that one.

[135] The list of postulates for multiple contraction included in each of the two following statements are the ones which occur in the axiomatic characterization that we have presented in Theorem 9.3.4. Nevertheless, the research that may be made in the sequence of some of the above listed points may suggest that a different set of multiple contraction postulates should be considered in the statement of this conjecture.

between the two mentioned kinds of multiple change operations (i.e. *multiple contractions* and *set contractions*).

Regarding such interrelation it would also be interesting to identify a multiple contraction postulate which: (a) is satisfied by any multiple contraction function obtained, by means of equation (10.4), from a set contraction which satisfies the limit postulate (for set contraction), and (b) is such that the set contraction defined from a multiple contraction function which satisfies it, by means of equation (10.3) satisfies the limit postulate (for set contraction).

12. To obtain an explicit definition as well as an axiomatic characterization for the functions $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ which are such that for any belief set $\mathbf{K}$ and any set of sentences B:

$$\mathbf{K}{\div}B = \mathbf{K} \cap (\mathbf{K}{\oplus}\neg B), \tag{10.5}$$

where $\oplus$ is a S.S.-based multiple revision $\oplus$ on $\mathbf{K}$ (cf. Subsection 5.6.3) and $\neg B$ is the negation of B introduced in Definition 5.6.1. And, subsequently, to investigate the relationship between that class of multiple contraction functions and the class of S.S.-based multiple contractions.

At this point we observe that equation (10.5) introduced above is a generalization to the case of changes by (possibly non-singleton) sets of sentences of the Harper identity (i.e. equation (2.2)). Having this in mind, in the sequence, we shall designate such equation by *generalized Harper identity*.

13. Vice versa, to obtain an explicit definition as well as an axiomatic characterization for the multiple revision functions $\oplus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ that are obtained from a S.S.-based multiple contraction $\div$ by means of the generalized Levi identity (equation (5.1)) and to investigate the relationship between that class of multiple revision functions and the class of S.S.-based multiple revisions introduced in [Pep04] (cf. Subsection 5.6.3).

14. A possible first step towards the obtention of the above mentioned axiomatic characterizations and interrelations (and which is, furthermore, an interesting topic of research on its own right) may be the clarification of whether the following conjecture is true or false:

Conjecture 10.2.2 *The following statements hold:*[135]

(i) If the function $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ satisfies the postulates of P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction*, then the function $\oplus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ defined from $\div$ by means of the generalized Levi identity (equation* (5.1)*) satisfies the the eight multiple revision postulates presented in [ZF01, Subsection* 2.1*] (and in [Pep04, Section* 5*]) and equation* (10.5)*.*

(ii) If the function $\oplus : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *satisfies the the eight multiple revision postulates presented in [ZF01, Subsection* 2.1*] (and in [Pep04, Section* 5*]), then the function* $\div : \mathcal{T}_{\mathcal{L}} \times \mathcal{P}(\mathcal{L}) \to \mathcal{T}_{\mathcal{L}}$ *obtained from* $\oplus$ *by means of the generalized Harper identity (equation* (10.5)*) satisfies the postulates of* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction*,* P-singleton reduction *as well as the identity* (5.1)*.*

We notice that the above conjecture is based on the results regarding the interrelation between set contraction and multiple revision that were stated in Observation 5.6.4 and on the assumption that Conjecture 10.2.1 above holds. In fact, it follows from Observation 5.6.4 that if Conjecture 10.2.1 is true then Conjecture 10.2.2 also holds. Therefore, on the one hand, the confirmation of Conjecture 10.2.1 immediately implies the confirmation of Conjecture 10.2.2 and, on the other hand, the refutation of the latter conjecture immediately entails the refutation of the former one.

Furthermore, we also notice here that the statement of Conjecture 10.2.2 is a generalization to the case of multiple change operations of some of the results that we have presented in Subsection 2.4.3 regarding the relationship between AGM (singleton) revisions and contractions that is highlighted by Levi and Harper identities (i.e. equations (2.1) and (2.2), respectively).

Appendix

Appendix A

Appendix of Chapter 3

A.1 Proofs of Chapter 3

In this section we present the proofs for the results presented throughout Chapter 3 which we have not found in the literature. We also present a proof for Observation 3.4.11 which differs from the one that was provided in [Foo90], where that result was originally presented.

For commodity of the reader, throughout this section, before each of those proofs we include the statement of the result that such proof refers to.

Observation 3.4.11 (**[Foo90]**) *Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE5). If $\alpha \in Cn(\emptyset)$ then for all $\delta \in \mathcal{L} \setminus Cn(\emptyset)$, $\delta < \alpha$, where $<$ denotes the strict part of $\leq$.*

Proof. Assume $\vdash \alpha$. It follows from (EE2) that $\forall \varepsilon \in \mathcal{L}, \varepsilon \leq \alpha$.

Now let $\delta \in \mathcal{L} \setminus Cn(\emptyset)$. Hence, $\delta \leq \alpha$. Now assume by *reductio* that $\alpha \leq \delta$, then, by (EE1), $\forall \varepsilon \in \mathcal{L}, \varepsilon \leq \delta$ and, from (EE5) we conclude that $\vdash \delta$, which contradicts $\delta \notin Cn(\emptyset)$. Hence, we have that $\alpha \not\leq \delta$ and we can conclude that $\forall \delta \in \mathcal{L} \setminus Cn(\emptyset), \delta < \alpha$ as we wished to prove. ∎

Observation 3.4.12 *Let $\mathbf{K}$ be a belief set and $\leq$ be a relation that satisfies (EE1) and (EE4). Then $\alpha \notin \mathbf{K}$ if and only if for all $\beta \in \mathbf{K}$, $\alpha < \beta$, where $<$ denotes the strict part of $\leq$.*

Proof. Left-to-right: Let $\alpha \notin \mathbf{K}$. It follows from (EE4) that $\alpha \leq \varphi$ for all $\varphi \in \mathcal{L}$. Now let $\beta \in \mathbf{K}$ and assume by *reductio* that $\beta \leq \alpha$. Then it follows from (EE1) that $\beta \leq \varphi$ for all $\varphi \in \mathcal{L}$ and by (EE4) we obtain that $\beta \notin \mathbf{K}$ which contradicts our hypothesis. Therefore we can conclude that, for all $\beta \in \mathbf{K}$, $\alpha < \beta$.

Right-to-left: Assume that for all $\beta \in \mathbf{K}$, $\alpha < \beta$. Then it follows immediately that $\alpha \notin \mathbf{K}$. Indeed, assume by *reductio* that $\alpha \in \mathbf{K}$. Then it follows from the hypothesis that $\alpha < \alpha$ which is absurd. ∎

Lemma 3.4.13 *Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE3) and let $<$ denote the strict part of $\leq$. Then the following statements hold:*

(i) If $\alpha < \beta \vee \delta$ and $\alpha < \varepsilon \vee \neg\beta$, then $\alpha < \varepsilon \vee \delta$.

(ii) If $\alpha < \alpha \vee \neg\beta$ then $\beta \leq \alpha$.

(iii) If $\delta \leq \alpha$, $\alpha < \alpha \vee \beta$ and $\delta < \delta \vee \neg\alpha$, then $\delta < \delta \vee \beta$.

Proof. (i) Let $\alpha < \beta \vee \delta$ and $\alpha < \varepsilon \vee \neg\beta$. From the former condition and the fact that $\beta \vee \delta \vdash \beta \vee \delta \vee \varepsilon$, making use of (EE2), (EE1) and Observation 3.4.8, we obtain that $\alpha < \beta \vee \delta \vee \varepsilon$. On the other hand, using the same arguments, $\alpha < \varepsilon \vee \neg\beta$ and $\varepsilon \vee \neg\beta \vdash \neg\beta \vee \varepsilon \vee \delta$ yield $\alpha < \neg\beta \vee \varepsilon \vee \delta$.

From the latter condition and $\alpha < \beta \vee \delta \vee \varepsilon$ it follows by Observation 3.4.10, that $\alpha < (\neg\beta \vee \varepsilon \vee \delta) \wedge (\beta \vee \delta \vee \varepsilon)$. Hence, from $\vdash (\neg\beta \vee \varepsilon \vee \delta) \wedge (\beta \vee \delta \vee \varepsilon) \leftrightarrow \varepsilon \vee \delta$ and intersubstitutivity (Observation 3.4.9) we obtain that $\alpha < \varepsilon \vee \delta$ and this finishes the proof.

(ii) Let $\alpha < \alpha \vee \neg\beta$. Then, since $\vdash (\alpha \vee \beta) \wedge (\alpha \vee \neg\beta) \leftrightarrow \alpha$, by intersubstitutivity (Observation 3.4.9) we have that $\alpha \vee \neg\beta \not\leq (\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$. Hence, from (EE3) we can conclude that $\alpha \vee \beta \leq (\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$. Then, again making use of intersubstitutivity (Observation 3.4.9), we obtain that $\alpha \vee \beta \leq \alpha$. On the other hand (EE2) gives us that $\beta \leq \alpha \vee \beta$. Finally, by (EE1), from the last two conditions we can conclude that $\beta \leq \alpha$, and this finishes the proof.

(iii) Assume $\delta \leq \alpha$, $\alpha < \alpha \vee \beta$ and $\delta < \delta \vee \neg\alpha$

From $\delta \leq \alpha$ and $\alpha < \alpha \vee \beta$ it follows, by (EE1) and Observation 3.4.8, that $\delta < \alpha \vee \beta$. From this last condition and $\delta < \delta \vee \neg\alpha$, it follows by Lemma 3.4.13-(i) that $\delta < \delta \vee \beta$. ∎

A.2 The Second Modelling in [Gro88] and Its Interrelation With the Epistemic Entrenchment-based Contractions

The goal of this section is to present the class of revision functions introduced by Grove in [Gro88, Sec. 3] and to expose the interrelation between such class and the class of the E.E.-based contractions (presented in Section 3.4). Namely, in the present section we shall show that the class of E.E.-based contraction functions coincides with the class of the contraction functions which can be obtained by applying the Harper Identity (i.e. equation (2.2)) to one of the revision functions of the mentioned class proposed by Grove.

Furthermore, after disclosing such interconnection, we will make use of it to clarify the remark that we have made in Footnote 63 on page 54, regarding the fact that part of the results stated in Observations 3.4.5 and 3.4.6 is in fact a consequence of [Gro88, Theorems 3 and 4](see Observation *A.2.3* below) and [Gär88, Theorems 3.2 - 3.5] (see Observations 2.4.4 – 2.4.7 (of the present thesis)).

In [Gro88, Sec. 3], for any given belief set $\mathbf{K}$, Grove suggested a way of ordering the sentences in $\mathcal{L}$ which can be used to decide what sentences shall be removed from or added to $\mathbf{K}$ whenever some belief change process occurs.

More precisely, for that purpose, Grove suggested the use of a relation $\preceq_{\mathrm{Gro}}$ on $\mathcal{L}$ satisfying the following conditions:[136]

($\preceq_{\mathrm{Gro}}1$) $\preceq_{\mathrm{Gro}}$ is connected.[137]

($\preceq_{\mathrm{Gro}}2$) $\preceq_{\mathrm{Gro}}$ is transitive.

($\preceq_{\mathrm{Gro}}3$) If $\vdash \alpha \to \beta \vee \delta$, then either $\beta \preceq_{\mathrm{Gro}} \alpha$ or $\delta \preceq_{\mathrm{Gro}} \alpha$.

($\preceq_{\mathrm{Gro}}4$) If $\mathbf{K} \neq \mathbf{K}_\perp$ then α is $\preceq_{\mathrm{Gro}}$-minimal ($\forall \beta \in \mathcal{L}\ \alpha \preceq_{\mathrm{Gro}} \beta$) if and only if $\neg\alpha \notin \mathbf{K}$.

($\preceq_{\mathrm{Gro}}5$) α is $\preceq_{\mathrm{Gro}}$-maximal ($\forall \beta \in \mathcal{L}\ \beta \preceq_{\mathrm{Gro}} \alpha$) if and only if $\vdash \neg\alpha$.

Remark A.2.1 *In [Gro88] the statement of property ($\preceq_{Gro}4$) does not include the condition that $\mathbf{K} \neq \mathbf{K}_\perp$. However, in [Mey99, Footnote 5] it is noted that without such condition [Gro88, Theorem 4] (see Observation A.2.3 below) does not hold for the case $\mathbf{K} = \mathbf{K}_\perp$. Furthermore, Meyer [Mey99, Footnote 5] also remarks that [Gär88, Lemma 4.27] (see Observation A.2.4-(ii) below) only holds if the above mentioned condition is included in such statement.*

Throughout this text, given a belief set $\mathbf{K}$, we will use the expression ***Grovean relation*** with respect to $\mathbf{K}$ to designate a binary relation $\preceq_{\mathrm{Gro}}$ on $\mathcal{L}$ that satisfies conditions ($\preceq_{\mathrm{Gro}}1$)–($\preceq_{\mathrm{Gro}}5$) above.

After introducing the above mentioned class of binary relations, Grove [Gro88, Sec. 3] remarked that, given a belief set $\mathbf{K}$, the Grovean relations with respect to $\mathbf{K}$ "are, in a strong sense, equivalent to systems of spheres centred on $\|\mathbf{K}\|$". Indeed the following fact is stated in [Gro88, p. 164] (as well as in [Gär88, p. 95]): Given a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ the relation $\preceq_{\mathbb{S}}$ defined by:[138]

$$\text{For all } \alpha, \beta \in \mathcal{L}\ \alpha \preceq_{\mathbb{S}} \beta \text{ iff } \mathbb{S}_\alpha \subseteq \mathbb{S}_\beta \tag{A.1}$$

is a Grovean relation with respect to $\mathbf{K}$.

Furthermore, given a belief set $\mathbf{K}$ and a Grovean relation $\preceq_{\mathrm{Gro}}$ with respect to $\mathbf{K}$, Grove proposed a way of defining a $\preceq_{\mathrm{Gro}}$-based revision on $\mathbf{K}$ and obtained an axiomatic characterization of the revision functions of that kind:

Definition A.2.2 ([Gro88]) *Let $\mathbf{K}$ be a belief set and $\preceq_{Gro}$ be a Grovean relation with respect to $\mathbf{K}$. The $\preceq_{\mathrm{Gro}}$-based revision on $\mathbf{K}$ is the operation $*_{\preceq_{Gro}}$ defined, for any $\alpha \in \mathcal{L}$, by:*[139]

$$\mathbf{K} *_{\preceq_{Gro}} \alpha = \begin{cases} \{\beta \in \mathcal{L} : (\alpha \wedge \beta) \prec_{Gro} (\alpha \wedge \neg\beta)\} & \text{, if } \nvdash \neg\alpha \\ \mathbf{K}_\perp & \text{, if } \vdash \neg\alpha \end{cases},$$

[136] The notation $\preceq_{\mathrm{Gro}}$ is not the one that has been used in [Gro88]. There, such relation has been denoted by $\leq$ but here we use an alternative notation to avoid confusion with the notation we have adopted to represent an E.E.-relation.

[137] It is noted in [Gro88, Footnote 1] that ($\preceq_{\mathrm{Gro}}1$) can be derived from ($\preceq_{\mathrm{Gro}}2$) and ($\preceq_{\mathrm{Gro}}3$).

[138] Here it must be noted that in [Gro88] if $\alpha \in \mathcal{L}$ is such that $\|\alpha\| = \emptyset$ then $\mathbb{S}_\alpha$ is taken to be $\mathcal{M}_{\mathcal{L}}$.

[139] In fact in his definition Grove did not consider the case $\vdash \neg\alpha$. However, as it is explained in further detail in [Mey99, Footnote 6], that is clearly a necessary part of the definition.

where $\prec_{Gro}$ denotes the strict part of the binary relation $\preceq_{Gro}$.

An operation $*$ on **K** *is a* Grovean relation-based revision *on* **K** *if and only if there is a Grovean relation $\preceq_{Gro}$ with respect to* **K** *such that, for all sentences $\alpha \in \mathcal{L}$, $\mathbf{K}{*}\alpha = \mathbf{K}{*}_{\preceq_{Gro}}\alpha$.*

Observation A.2.3 ([Gro88, Theorems 3 and 4]) [140]

Let **K** *be a belief set. An operation $*$ on* **K** *is a* Grovean relation-based revision *on* **K** *if and only if it satisfies the basic and the supplementary AGM postulates for belief set revision.*

To emphasize the strength of the interrelation between systems of spheres and Grovean relations it is worth noticing that the proof of the above observation (or, more precisely, of [Gro88, Theorems 3 and 4]), presented in [Gro88], is based on the axiomatic characterization of the S.S.-based revisions[141] which is provided in that same paper ([Gro88, Theorems 1 and 2]). More precisely, the proof of the *right-to-left* part of this observation is based on the possibility of defining, by means of condition (*A*.1), a Grovean relation from a system of spheres. And, on the other hand, the proof of the *left-to-right* part is based on the construction of a system of spheres $\mathbb{S}$ from a Grovean relation $\preceq_{\text{Gro}}$ in such a way that the $\preceq_{\text{Gro}}$-based revision coincides with the $\mathbb{S}$-based revision[142].

Now that we have seen how the Grovean relations and the Systems of Spheres are related, having in mind the well known interrelation between these latter structures and the Epistemic Entrenchment relations (cf. Section 4.2), it is not surprising that Grovean relations and epistemic entrenchment relations are very closely related as well. In fact, as we will see right away, Grovean relations and E.E. relations (cf. Definition 3.4.1) are, in a certain sense, dual to each other.[143]

To clarify this latter statement we start by presenting the following results of [Gär88, Sec. 4.8] which show that (and how) it is possible to define a Grovean relation from an Epistemic Entrenchment relation and vice versa.

Observation A.2.4 ([Gär88, Lemmas 4.26 and 4.27]) [144]

Let **K** *be a belief set. Then:*

(i) If $\preceq_{Gro}$ is a Grovean relation with respect to **K** *then the relation $\leq_{\preceq_{Gro}}$ defined by:*

$$\text{For all } \alpha, \beta \in \mathcal{L}\ \alpha \leq_{\preceq_{Gro}} \beta \text{ iff } \neg\alpha \preceq_{Gro} \neg\beta \tag{A.2}$$

is an epistemic entrenchment relation with respect to **K**.

[140] The *right-to-left* part of this observation corresponds to [Gro88, Theorem 3] and the *left-to-right* part corresponds to [Gro88, Theorem 4].

[141] Here, by S.S.-based revisions we mean the revision functions which can be seen as resulting from applying the Levi Identity (equation (2.1)) to some S.S.-based contraction.

[142] Here, by $\mathbb{S}$-based revision we mean the revision functions which results of applying the Levi Identity (equation (2.1)) to the $\mathbb{S}$-based contraction.

[143] Notice that we have already made reference to this fact in Footnote 61 on page 52.

[144] Statement (i) of this Observation corresponds to [Gär88, Lemma 4.26] and Statement (ii) corresponds to [Gär88, Lemma 4.27].

(ii) If $\leq$ is an epistemic entrenchment relation with respect to $\mathbf{K}$ *then the relation* $\preceq_{\leq}$ *defined by:*

$$\text{For all } \alpha, \beta \in \mathcal{L}, \ \ \alpha \preceq_{\leq} \beta \text{ iff } \neg\alpha \leq \neg\beta \tag{A.3}$$

is a Grovean relation with respect to $\mathbf{K}$.

The following couple of observations reveals the strong connection between the E.E.-based contractions and the Grovean relation-based revisions.

Observation A.2.5 ([Gär88, Lemma 4.28]) *Let* $\mathbf{K}$ *be a belief set,* $\preceq_{Gro}$ *be a Grovean relation with respect to* $\mathbf{K}$, $*_{\preceq_{Gro}}$ *be the* $\preceq_{Gro}$*-based revision and* $\leq_{\preceq_{Gro}}$ *be the E.E.-relation defined from* $\preceq_{Gro}$ *by condition* (A.2). *Then the contraction function* $-$ *on* $\mathbf{K}$ *defined from* $*_{\preceq_{Gro}}$ *by means of the Harper Identity (i.e. equation* (2.2)*) coincides with the* $\leq_{\preceq_{Gro}}$*-based contraction* $-_{\leq_{\preceq_{Gro}}}$*, i.e.,*

$$\text{For all } \alpha \in \mathcal{L}, \ \ \mathbf{K}-\alpha = \mathbf{K} -_{\leq_{\preceq_{Gro}}} \alpha.$$

Observation A.2.6 ([Gär88, Lemma 4.29]) *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$, $-_{\leq}$ *be the* $\leq$*-based contraction and* $\preceq_{\leq}$ *be the Grovean relation defined from* $\leq$ *by condition* (A.3). *Then the revision function* $*$ *on* $\mathbf{K}$ *defined from* $-_{\leq}$ *by means of the Levi Identity (i.e. equation* (2.1)*) coincides with the* $\preceq_{\leq}$*-based revision* $*_{\preceq_{\leq}}$*, i.e.,*

$$\text{For all } \alpha \in \mathcal{L}, \ \ \mathbf{K}*\alpha = \mathbf{K} *_{\preceq_{\leq}} \alpha.$$

Remark A.2.7 *Combining the two above observations with Observations A.2.4 and 2.4.8 we can immediately conclude that the class of epistemic entrenchment-based contractions coincides with the class of contraction functions which are obtained by applying the Harper Identity (i.e. equation* (2.2)*) to some Grovean relation-based revision.*

From the above remark, we can immediately conclude that, as we have stated in Footnote 63 on page 54, part of the facts stated in Observations 3.4.5 and 3.4.6 (namely, the part consisting of the axiomatic characterization of the E.E.-based contractions) follows essentially from [Gro88, Theorems 3 and 4] (see Observation *A.2.3* above) and [Gär88, Theorems 3.2 - 3.5] (see Observations 2.4.4 – 2.4.7 (of the present thesis)). Indeed, having in mind Remark *A.2.7*, as it is mentioned in [Gär88, p. 96], combining the just mentioned Observation *A.2.3* and Observations 2.4.4 – 2.4.8 it follows immediately that the following representation theorem for E.E.-based contractions holds:

Observation A.2.8 ([Gär88, Theorem 4.30]) *Let* $\mathbf{K}$ *be a belief set. An operation* $-$ *on* $\mathbf{K}$ *is an epistemic entrenchment-based contraction on* $\mathbf{K}$ *if and only if it satisfies the basic and the supplementary AGM postulates for belief set contraction.*

We close this section with the explanation of the remark that we have made in Footnote 74 on page 71 based on the results that we have introduced above.

Let $\mathbf{K}$ be a belief set and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. In the mentioned footnote we have stated that, based on the proof of [Gro88, Theorem 4] (i.e., the *left-to-right* part of Observation *A*.2.3), it is possible to prove that there is a S.S. $\mathbb{S}$ centred on $\|\mathbf{K}\|$ such that the $\leq$-based contraction on $\mathbf{K}$ and the $\mathbb{S}$-based contraction on $\mathbf{K}$ coincide. Now we show how that can be done. So, let $-_{\leq}$ be the $\leq$-based contraction and $*$ be the revision function on $\mathbf{K}$ defined from $-_{\leq}$ by means of the Levi Identity (i.e. equation (2.1)). According to Observation *A*.2.6, the revision function $*$ coincides with the $\preceq_{\leq}$-based revision $*_{\preceq_{\leq}}$, where $\preceq_{\leq}$ is the Grovean relation defined from $\leq$ by condition (*A*.3).

On the other hand, as we have mentioned in the paragraph immediately below Observation *A*.2.3, the proof of the *left-to-right* part of that theorem (i.e. of [Gro88, Theorem 4]) presents a way a of constructing a system of spheres $\mathbb{S}$ from a given Grovean relation $\preceq_{\text{Gro}}$ in such a way that the $\preceq_{\text{Gro}}$-based revision coincides with the $\mathbb{S}$-based revision[142]. So, we can conclude that there is a system of spheres, say $\mathbb{S}$, such that the $\preceq_{\leq}$-based revision $*_{\preceq_{\leq}}$ coincides with the $\mathbb{S}$-based revision[142] $*_{\mathbb{S}}$.

Hence, it follows that the revision function $*$ coincides with the $\mathbb{S}$-based revision[142] $*_{\mathbb{S}}$. Therefore, the contraction function defined from $*$ by means of the Harper Identity (i.e. equation (2.2)) coincides with the function that is obtained in the same way from $*_{\mathbb{S}}$.

But, since $*$ is the revision function defined from $-_{\leq}$ by means of the Levi Identity and (according to Footnote 142) $*_{\mathbb{S}}$ is the revision function which results of applying the Levi Identity to the $\mathbb{S}$-based contraction $-_{\mathbb{S}}$, recalling that, according to their respective axiomatic characterizations, $-_{\leq}$ and $-_{\mathbb{S}}$ satisfy the basic AGM postulates for contraction, it follows immediately from Observation 2.4.8 that the contraction function obtained from $*$ by means of the Harper Identity is the $\leq$-based contraction $-_{\leq}$ and the contraction function that is obtained in the same way from $*_{\mathbb{S}}$ is the $\mathbb{S}$-based contraction $-_{\mathbb{S}}$.

Therefore, we can conclude that the system of spheres $\mathbb{S}$ is such that such that the $\leq$-based contraction $-_{\leq}$ on $\mathbf{K}$ and the $\mathbb{S}$-based contraction $-_{\mathbb{S}}$ on $\mathbf{K}$ coincide, and this proves our point.

We must notice, however, that in the above reasoning we have made use of the axiomatic characterizations (more precisely of the construction-to-postulates part of those characterizations) of the system of spheres-based contractions and of the epistemic entrenchment-based contractions. Therefore, if we had used the above reasoning to prove that $\leq$-based contraction $-_{\leq}$ on $\mathbf{K}$ and the $\mathbb{S}$-based contraction $-_{\mathbb{S}}$ on $\mathbf{K}$ coincide instead of proceeding as we have done in Section 4.2, we would not have obtained a direct proof (in the sense of not making use of representation theorems at all) for that fact, as it was our intention.

Appendix B

Appendix of Chapter 4

B.1 Proofs of Chapter 4

In this section we present the proofs for the results presented throughout Chapter 4 which we have not found in the literature. We also present a proof for Observation 4.1.7 which differs from the one that was provided in [AM82], where that result was originally presented.

For commodity of the reader, throughout this section, before each of those proofs we include the statement of the result that such proof refers to.

Remark 4.1.6 *A contraction function $\sim$ is the full meet contraction on a belief set $\mathbf{K}$ if and only if, for any $\varphi \in \mathcal{L}$, $\mathbf{K} \sim \varphi = Cn(\mathbf{K} \sim \varphi)$ and:*

$$\|\mathbf{K} \sim \varphi\| = \begin{cases} \|\mathbf{K}\| \cup \|\neg\varphi\| & \text{, if } \varphi \in \mathbf{K} \setminus Cn(\emptyset) \\ \|\mathbf{K}\| & \text{, otherwise} \end{cases}.$$

Proof. Let $\mathbf{K}$ be a belief set and $\sim$ be the full meet contraction on $\mathbf{K}$. Then, according to Definition 3.1.7, $\mathbf{K} \sim \varphi = \mathbf{K}$, if $\vdash \varphi$, and $\mathbf{K} \sim \varphi = \bigcap \mathbf{K} \perp \varphi$, if $\nvdash \varphi$. Hence, if $\vdash \varphi$ or $\varphi \notin \mathbf{K}$ then $\mathbf{K} \sim \varphi = \mathbf{K}$ and, if $\nvdash \varphi$ and $\varphi \in \mathbf{K}$ then $\mathbf{K} \sim \varphi = \bigcap\{X : X \in \mathbf{K}\perp\varphi\}$.

Therefore, it follows trivially that $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\|$ for all sentences φ such that $\vdash \varphi$ or $\varphi \notin \mathbf{K}$.

Thus, it remains only to show that $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$ if $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. To see that this indeed holds, let $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. Then, it follows from Observation 4.1.1-3 (and Remark 4.1.2) that $\mathbf{K} \sim \varphi = \bigcap\{Th(\|\mathbf{K}\| \cup \{W\}) : W \in \|\neg\varphi\|\}$.

Then, since, according to Observation 4.1.1-2, for each $W \in \|\neg\varphi\|$ the set $X = Th(\|\mathbf{K}\| \cup \{W\})$ is a belief set such that $\|X\| = \|\mathbf{K}\| \cup \{W\}$, we have that $\mathbf{K} \sim \varphi$ is also a belief set such that $\mathbf{K} \sim \varphi \subseteq Th(\|\mathbf{K}\| \cup \{W\})$ for all $W \in \|\neg\varphi\|$ and it follows from Observation 3.3.3-(e) that $\bigcup_{W \in \|\neg\varphi\|}(\|\mathbf{K}\| \cup \{W\}) \subseteq \|\mathbf{K} \sim \varphi\|$, i.e., $\|\mathbf{K}\| \cup \|\neg\varphi\| \subseteq \|\mathbf{K} \sim \varphi\|$. To finish this part of the proof, next we show that $\|\mathbf{K} \sim \varphi\| \subseteq \|\mathbf{K}\| \cup \|\neg\varphi\|$. Let M be a world such that $M \in \mathcal{M}_{\mathcal{L}} \setminus (\|\mathbf{K}\| \cup \|\neg\varphi\|)$. We will show that $M \notin \|\mathbf{K} \sim \varphi\|$. Since $M \in \mathcal{M}_{\mathcal{L}}$, from $M \notin \|\neg\varphi\|$ it follows that $\varphi \in M$ and from $M \notin \|\mathbf{K}\|$ we can conclude that there is some $\beta \in \mathbf{K}$ such that $\neg\beta \in M$. Hence $\neg\varphi \vee \beta \notin M$. On the other hand, since $\beta \in \mathbf{K}$, it holds that $\neg\varphi \vee \beta \in H$, for every $H \in \mathbf{K}\perp\varphi$ and consequently $\neg\varphi \vee \beta \in \mathbf{K} \sim \varphi$. So, we can conclude that $M \notin \|\mathbf{K} \sim \varphi\|$. Thus, it holds that $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$, for any $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$.

Now we prove that the converse implication holds. Let $\sim$ be a contraction function on a belief set $\mathbf{K}$ such that, for any $\varphi \in \mathcal{L}$, $\mathbf{K} \sim \varphi = Cn(\mathbf{K} \sim \varphi)$ and $\|\mathbf{K} \sim \varphi\| = \begin{cases} \|\mathbf{K}\| \cup \|\neg\varphi\| & \text{, if } \varphi \in \mathbf{K} \setminus Cn(\emptyset) \\ \|\mathbf{K}\| & \text{, otherwise} \end{cases}$.

Then, since $\mathbf{K} \sim \varphi = Cn(\mathbf{K} \sim \varphi)$, it follows from Observation 3.3.3−(a) that $\mathbf{K} \sim \varphi = \begin{cases} Th(\|\mathbf{K}\| \cup \|\neg\varphi\|) & \text{, if } \varphi \in \mathbf{K} \setminus Cn(\emptyset) \\ \mathbf{K} & \text{, otherwise} \end{cases}$.

Finally, having in mind Observation 4.1.1-3 (and Remark 4.1.2) as well as Definition 3.1.1, from the above we can conclude that

$$\mathbf{K} \sim \varphi = \begin{cases} \bigcap \mathbf{K} \perp \varphi & \text{, if } \nvdash \varphi \\ \mathbf{K} & \text{, if } \vdash \varphi, \end{cases},$$

and, according to Definition 3.1.7, this proves that $\sim$ is the full meet contraction on $\mathbf{K}$, as we wished to show. ■

Observation 4.1.7 ([AM82, Observation 2.1]) *Let* $\mathbf{K}$ *be a belief set and* $\sim$ *be the operator of full meet contraction on* $\mathbf{K}$. *Then for all sentences* $\varphi \in \mathcal{L}$:

$$\mathbf{K} \sim \varphi = \begin{cases} \mathbf{K} \cap Cn(\neg\varphi) & \text{, if } \varphi \in \mathbf{K} \\ \mathbf{K} & \text{, if } \varphi \notin \mathbf{K} \end{cases}.$$

Proof. Let $\mathbf{K}$ be a belief set and $\sim$ be the operator of full meet contraction on $\mathbf{K}$.

First we notice that, if $\varphi \notin \mathbf{K}$ it follows immediately from Definition 3.1.7 that $\mathbf{K} \sim \varphi = \mathbf{K}$.

Hence, it remains to show that if $\varphi \in \mathbf{K}$ then $\mathbf{K} \sim \varphi = \mathbf{K} \cap Cn(\neg\varphi)$. So, let $\varphi \in \mathbf{K}$. We will consider separately the two cases $\varphi \in Cn(\emptyset)$ and $\varphi \notin Cn(\emptyset)$.

Case 1, $\varphi \in Cn(\emptyset)$. Then, on the one hand it follows from Definition 3.1.7 that $\mathbf{K} \sim \varphi = \mathbf{K}$ and, on the other hand we have that $Cn(\neg\varphi) = \mathcal{L}$. Therefore it holds that $\mathbf{K} \sim \varphi = \mathbf{K} \cap Cn(\neg\varphi)$, as we wished to prove.

Case 2, $\varphi \notin Cn(\emptyset)$. Then $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$ and, according to Remark 4.1.6, it holds that $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$. On the other hand, since $\mathbf{K} \sim \varphi$ is a belief set, it follows from Observation 3.3.3-(a) that $\mathbf{K} \sim \varphi = \bigcap \|\mathbf{K} \sim \varphi\|$. Hence, we have that $\mathbf{K} \sim \varphi = \bigcap(\|\mathbf{K}\| \cup \|\neg\varphi\|) = (\bigcap \|\mathbf{K}\|) \cap (\bigcap \|\neg\varphi\|) = \mathbf{K} \cap Cn(\neg\varphi)$, and this finishes the proof. ■

Observation 4.1.8 (Based on [Gro88]) *Let* $\mathbf{K}$ *be a belief set and* $-$ *be a partial meet contraction on* $\mathbf{K}$. *Then, for any* $\varphi \in \mathcal{L}$:

1. *If* $\vdash \varphi$ *or* $\varphi \notin \mathbf{K}$, *then* $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\|$
2. *If* $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$, *then* $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$, *where* $\mathcal{M}$ *is a non-empty subset of* $\|\neg\varphi\|$.

Proof. Let $\mathbf{K}$ be a belief set and $-$ be a partial meet contraction on $\mathbf{K}$. Then, according to Definition 3.1.6, $\mathbf{K}-\varphi = \mathbf{K}$, if $\vdash \varphi$, and $\mathbf{K}-\varphi = \bigcap\gamma(\mathbf{K}\bot\varphi)$, where $\gamma(\mathbf{K}\bot\varphi)$ is a non-empty subset of $\mathbf{K}\bot\varphi$, if $\nvdash \varphi$.

Hence, if $\vdash \varphi$ or $\varphi \notin \mathbf{K}$ then $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\|$, and we can conclude that point 1. in the statement of the observation holds.

Now we assume that $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$. In order to show that 2. also holds we only need to show that $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$, where $\mathcal{M}$ is a non-empty subset of $\|\neg\varphi\|$. In order to prove that such identity indeed holds we start by recalling that, in this case, according to Remark 4.1.6, if $\sim$ is the full meet contraction on $\mathbf{K}$ then $\|\mathbf{K} \sim \varphi\| = \|\mathbf{K}\| \cup \|\neg\varphi\|$. Therefore, since $\mathbf{K} \sim \varphi$ and $\mathbf{K}-\varphi$ are belief sets (because they are the result of the intersection of a family of belief sets), and observing that $\mathbf{K} \sim \varphi \subseteq \mathbf{K}-\varphi$, we can conclude from Observation 3.3.3-(e) that $\|\mathbf{K}-\varphi\| \subseteq \|\mathbf{K}\| \cup \|\neg\varphi\|$. On the other hand, there is at least one set H in $\mathbf{K}\bot\varphi$ such that $\mathbf{K}-\varphi \subseteq H$. Such a set H is a belief set (see Observation 3.1.4) and it holds that $\|H\| = \|\mathbf{K}\| \cup \{W\}$ for some $\neg\varphi$-world W (according to Observation 4.1.1-1). So, again from Observation 3.3.3-(e) it follows that the set $\|\mathbf{K}-\varphi\|$ contains $\|\mathbf{K}\|$ and at least one $\neg\varphi$-world. Therefore from all the above we can conclude that, if $-$ is a partial meet contraction and $\varphi \in \mathcal{L}$ is such that $\varphi \in \mathbf{K} \setminus Cn(\emptyset)$ then $\|\mathbf{K}-\varphi\| = \|\mathbf{K}\| \cup \mathcal{M}$, where $\mathcal{M}$ is a non-empty subset of $\|\neg\varphi\|$. ■

Observation 4.2.2 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then condition* ($\leq -\mathbb{S}$) *is satisfied if and only if*

$$\forall\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset),\ \alpha \leq \beta \textit{ iff } \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}. \qquad (\leq -\mathbb{S}')$$

Proof. Assume the conditions mentioned in the statement of the observation hold.

We will prove that condition ($\leq -\mathbb{S}$) is satisfied if and only if condition ($\leq -\mathbb{S}'$) is also satisfied.

First we assume that condition ($\leq -\mathbb{S}'$) holds and show that condition ($\leq -\mathbb{S}$) is also satisfied, i.e., that for any $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \beta$ if and only if either $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ or $\vdash \beta$. We will prove this by double implication.

($\Rightarrow$) Let α, β be any two sentences such that $\alpha \leq \beta$. We must show that either $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ or $\vdash \beta$. In order to do that, assume $\nvdash \beta$. Then it also holds that $\nvdash \alpha$. Indeed, assume by *reductio* that $\vdash \alpha$. It follows from (EE2) that $\forall\phi \in \mathcal{L},\ \phi \leq \alpha$ and, since $\alpha \leq \beta$, we obtain by (EE1) that $\forall\phi \in \mathcal{L},\ \phi \leq \beta$. Finally, according to (EE5), it follows that $\vdash \beta$ which is a contradiction. Hence, $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$ and we can conclude, by ($\leq -\mathbb{S}'$), that $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$.

($\Leftarrow$) Let α, β be any two sentences such that either $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ or $\vdash \beta$.

Case 1, $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$. Then $\alpha, \beta \notin Cn(\emptyset)$ and it follows from ($\leq -\mathbb{S}'$) that $\alpha \leq \beta$.

Case 2, $\vdash \beta$. Then it follows immediately from (EE2) that $\alpha \leq \beta$.

The other direction is obvious since it follows immediately from condition ($\leq -\mathbb{S}$) that, for any $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha \leq \beta$ if and only if $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$, i.e. that condition ($\leq -\mathbb{S}'$) is satisfied. ■

Lemma 4.2.4 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the following three conditions are equivalent:*

1. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha \leq \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$ *(notice that this is condition ($\leq -\mathbb{S}'$)).*
2. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha < \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ *and* $\alpha =_{E.E.} \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.
3. $\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$, $\alpha < \beta \Rightarrow \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ *and* $\alpha =_{E.E.} \beta \Rightarrow \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.

Proof. 1. $\Rightarrow$ 2. Assume 1. holds and let α, β be two arbitrary sentences in $\mathcal{L} \setminus Cn(\emptyset)$. We start by showing that $\alpha < \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$.

$$\begin{aligned} & \alpha < \beta \\ \Leftrightarrow\ & \alpha \leq \beta \text{ and } \beta \not\leq \alpha \\ \Leftrightarrow\ & \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta} \text{ and } \mathbb{S}_{\neg\beta} \not\subseteq \mathbb{S}_{\neg\alpha} \text{ (by 1.)} \\ \Leftrightarrow\ & \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}. \end{aligned}$$

Now we show that $\alpha =_{\text{E.E.}} \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$

$$\begin{aligned} & \alpha =_{\text{E.E.}} \beta \\ \Leftrightarrow\ & \alpha \leq \beta \text{ and } \beta \leq \alpha \\ \Leftrightarrow\ & \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta} \text{ and } \mathbb{S}_{\neg\beta} \subseteq \mathbb{S}_{\neg\alpha} \text{ (by 1.)} \\ \Leftrightarrow\ & \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}. \end{aligned}$$

2. $\Rightarrow$ 3. This implication is immediate.

3. $\Rightarrow$ 1. Assume 3. holds and let α, β be two arbitrary sentences in $\mathcal{L} \setminus Cn(\emptyset)$. We must show that $\alpha \leq \beta \Leftrightarrow \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$.

($\Rightarrow$)

$$\begin{aligned} & \alpha \leq \beta \\ \Leftrightarrow\ & \alpha < \beta \text{ or } \beta =_{\text{E.E.}} \alpha \\ \Rightarrow\ & \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta} \text{ or } \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta} \text{ (by 3.)} \\ \Leftrightarrow\ & \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}. \end{aligned}$$

($\Leftarrow$) Let $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$. Then either $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ or $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.
Case 1, $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$.

$$\begin{aligned} & \mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta} \\ \Rightarrow\ & \mathbb{S}_{\neg\beta} \not\subset \mathbb{S}_{\neg\alpha} \\ \Rightarrow\ & \beta \not< \alpha \text{ (by 3.)} \\ \Leftrightarrow\ & \alpha \leq \beta \text{ (by connectivity (Observation 3.4.3 - (i)))}. \end{aligned}$$

Case 2, $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.

$$\begin{aligned} & \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta} \\ \Rightarrow\ & \mathbb{S}_{\neg\alpha} \not\subset \mathbb{S}_{\neg\beta} \text{ and } \mathbb{S}_{\neg\beta} \not\subset \mathbb{S}_{\neg\alpha} \\ \Rightarrow\ & \alpha \not< \beta \text{ and } \beta \not< \alpha \text{ (by 3.)} \\ \Leftrightarrow\ & \beta \leq \alpha \text{ and } \alpha \leq \beta \text{ (by connectivity (Observation 3.4.3 - (i)))} \\ \Rightarrow\ & \alpha \leq \beta. \end{aligned}$$

■

Lemma 4.2.5 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If* $\mathbb{S}$ *and* $\leq$ *satisfy condition* ($\leq -\mathbb{S}$)*, then for any* $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$ *and any* $\beta \in \mathcal{L}$ *it holds that*

$$\alpha < \alpha \vee \beta \textit{ iff } f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|.$$

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, $\leq$, α and β are in the conditions introduced in the statement of the lemma. We must show that $\alpha < \alpha \vee \beta$ if and only if $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|$. To do that, we consider two cases separately.

Case 1, $\vdash \alpha \vee \beta$. Then, on the one hand, $\|\neg\alpha \wedge \neg\beta\| = \emptyset$ which is equivalent to $\|\neg\alpha\| \cap \|\neg\beta\| = \emptyset$. Hence $\|\neg\alpha\| \subseteq \|\beta\|$ and, consequently, in particular, $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|$. On the other hand it follows immediately from Observation 3.4.11 that $\alpha < \alpha \vee \beta$. Therefore we can trivially conclude that $\alpha < \alpha \vee \beta \Leftrightarrow f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|$.

Case 2, $\nvdash \alpha \vee \beta$.

($\Rightarrow$) Let $\alpha < \alpha \vee \beta$. It follows from ($\leq -\mathbb{S}$) and Lemma 4.2.4 that $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\alpha\wedge\neg\beta}$.

Now let $W \in f_{\mathbb{S}}(\neg\alpha)$. Then $W \in \|\neg\alpha\|$ and $W \in \mathbb{S}_{\neg\alpha}$. Assume, by *reductio*, that $W \in \|\neg\beta\|$, then $W \in \|\neg\alpha\| \cap \|\neg\beta\| = \|\neg\alpha \wedge \neg\beta\|$ and from this we can conclude that $\mathbb{S}_{\neg\alpha\wedge\neg\beta} \subseteq \mathbb{S}_{\neg\alpha}$, which contradicts $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\alpha\wedge\neg\beta}$. Hence $W \notin \|\neg\beta\|$, so $W \in \|\beta\|$. Therefore $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|$.

($\Leftarrow$) Let $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\beta\|$. We must show that $\alpha < \alpha \vee \beta$. But, since condition ($\leq -\mathbb{S}$) holds, according to Lemma 4.2.4, in order to prove that, it is enough to verify that $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\alpha\wedge\neg\beta}$. That $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\alpha\wedge\neg\beta}$ follows immediately from the fact that all $\{\neg\alpha \wedge \neg\beta\}$-worlds are $\{\neg\alpha\}$-worlds. Now assume by *reductio* that $\mathbb{S}_{\neg\alpha\wedge\neg\beta} \subseteq \mathbb{S}_{\neg\alpha}$, then $\mathbb{S}_{\neg\alpha\wedge\neg\beta} = \mathbb{S}_{\neg\alpha}$. Hence $\mathbb{S}_{\neg\alpha} \cap \|\neg\alpha \wedge \neg\beta\| \neq \emptyset$, which is equivalent to $\mathbb{S}_{\neg\alpha} \cap \|\neg\alpha\| \cap \|\neg\beta\| \neq \emptyset$, which contradicts $\mathbb{S}_{\neg\alpha} \cap \|\neg\alpha\| \subseteq \|\beta\|$. ■

Theorem 4.2.6 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based contraction on* $\mathbf{K}$*,* $-_{\leq}$ *(cf. condition* ($C_{-\leq}$) *in Definition 3.4.4), and the* $\mathbb{S}$*-based contraction on* $\mathbf{K}$*,* $-_{\mathbb{S}}$ *(cf. Definition 3.3.7), coincide, i.e.,*

$$\forall\varphi \in \mathcal{L},\ \mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi,$$

if and only if condition ($\leq -\mathbb{S}$) *is satisfied.*

Proof. Assume $\mathbf{K}$, $\mathbb{S}$ and $\leq$ are in the conditions introduced in the statement of the theorem. We must show that $\forall\varphi \in \mathcal{L}$, $\mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi$ if and only if condition ($\leq -\mathbb{S}$) holds.

Right-to-left: Assume that $\mathbb{S}$ and $\leq$ satisfy condition ($\leq -\mathbb{S}$) and let φ be an arbitrary sentence of $\mathcal{L}$. We will prove that $\mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi$.

Case 1, $\vdash \varphi$. Then, according to Definitions 3.4.4 and 3.3.7, $\mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi = \mathbf{K}$ and we are done.

Case 2, $\nvdash \varphi$. Then, according to Definitions 3.4.4 and 3.3.7, in this case, $\mathbf{K}-_{\leq}\varphi = \mathbf{K}-_{\mathbb{S}}\varphi$ if and only if

$$\{\beta \in \mathbf{K} : \varphi < \varphi \vee \beta\} = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)).$$

So, in what follows, we prove such equality of sets:

$$\begin{array}{ll} & \beta \in \{\beta \in \mathbf{K} : \varphi < \varphi \vee \beta\} \\ \text{iff} & \beta \in \mathbf{K} \text{ and } \varphi < \varphi \vee \beta \\ \text{iff} & \beta \in \bigcap \|\mathbf{K}\| \text{ and } f_{\mathbb{S}}(\neg\varphi) \subseteq \|\beta\| \text{ (by Observation 3.3.3-(a) and} \\ & \text{Lemma 4.2.5)} \\ \text{iff} & \beta \in \bigcap \|\mathbf{K}\| \text{ and } \beta \in \bigcap f_{\mathbb{S}}(\neg\varphi) \\ \text{iff} & \beta \in \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)) \\ \text{iff} & \beta \in Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)) \end{array}$$

Left-to-right: Assume that $\forall \varphi \in \mathcal{L}$, $\mathbf{K} -_{\leq} \varphi = \mathbf{K} -_{\mathbb{S}} \varphi$. We must prove that condition ($\leq -\mathbb{S}$) holds. In order to do that, according to Lemma 4.2.4, it is enough show that $\mathbb{S}$ and $\leq$ satisfy condition 3. of the mentioned lemma, which is equivalent to

$$\forall \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset),\ \mathbb{S}_{\neg\alpha} \not\subset \mathbb{S}_{\neg\beta} \Rightarrow \alpha \not< \beta \text{ and } \mathbb{S}_{\neg\alpha} \neq \mathbb{S}_{\neg\beta} \Rightarrow \alpha \neq_{\text{E.E.}} \beta.$$

Hence, let α and β be two arbitrary elements of $\mathcal{L} \setminus Cn(\emptyset)$. We start by proving that $\mathbb{S}_{\neg\alpha} \not\subset \mathbb{S}_{\neg\beta} \Rightarrow \alpha \not< \beta$. So, let $\mathbb{S}_{\neg\alpha} \not\subset \mathbb{S}_{\neg\beta}$. It follows that $\beta \notin \mathbf{K} -_{\leq} \beta \wedge \alpha$. Indeed:

$$\begin{array}{ll} & \mathbb{S}_{\neg\alpha} \not\subset \mathbb{S}_{\neg\beta} \\ \Rightarrow & \mathbb{S}_{\neg\beta} \subseteq \mathbb{S}_{\neg\alpha} \\ \Rightarrow & \mathbb{S}_{\neg\beta \vee \neg\alpha} = \mathbb{S}_{\neg\beta} \\ \Rightarrow & f_{\mathbb{S}}(\neg(\beta \wedge \alpha)) \not\subseteq \|\beta\| \\ \Rightarrow & \beta \notin \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg(\beta \wedge \alpha))) \\ \Rightarrow & \beta \notin \mathbf{K} -_{\mathbb{S}} \beta \wedge \alpha \\ \text{iff} & \beta \notin \mathbf{K} -_{\leq} \beta \wedge \alpha \text{ (by the hypothesis).} \end{array}$$

Therefore, according to Definition 3.4.4, we have that $\beta \notin \{\beta \in \mathbf{K} : (\beta \wedge \alpha) < (\beta \wedge \alpha) \vee \beta\}$.

Now we consider two cases:

Case 1, $\beta \notin \mathbf{K}$. Then it follows from (EE4) that $\beta \leq \alpha$. Consequently $\alpha \not< \beta$.

Case 2, $\beta \in \mathbf{K}$. Then $(\beta \wedge \alpha) \not< (\beta \wedge \alpha) \vee \beta$ and by making use of intersubstitutivity (Observation 3.4.9) we obtain that $\beta \wedge \alpha \not< \beta$. Hence, from condition (3.1) it follows that $\beta \leq \beta \wedge \alpha$. Finally, since $\beta \wedge \alpha \vdash \alpha$, from (EE2) and (EE1) we can conclude that $\beta \leq \alpha$. Therefore $\alpha \not< \beta$.

Having seen this, it remains to prove that $\mathbb{S}_{\neg\alpha} \neq \mathbb{S}_{\neg\beta} \Rightarrow \alpha \neq_{\text{E.E.}} \beta$. So, let $\mathbb{S}_{\neg\alpha} \neq \mathbb{S}_{\neg\beta}$. Then either $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ or $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$. We assume without loss of generality that $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$. Since $\|\mathbf{K}\| \subseteq \mathbb{S}_{\neg\beta}$ (according to Definition 3.3.5), it follows that $\|\mathbf{K}\| \subset \mathbb{S}_{\neg\alpha}$. Hence $\alpha \in \bigcap \|\mathbf{K}\|$. Moreover, given that $Th(\|\mathbf{K}\|) = \mathbf{K}$ (according to Observation 3.3.3-(a)), we can conclude that $\alpha \in \mathbf{K}$.

Finally we show that it follows from $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$ that $\alpha \neq_{\text{E.E.}} \beta$. Again, we need to consider separately the cases $\beta \notin \mathbf{K}$ and $\beta \in \mathbf{K}$.

Case 1, $\beta \notin \mathbf{K}$. Then, since $\alpha \in \mathbf{K}$, it follows immediately from Observation 3.4.12 that $\beta < \alpha$. Therefore we can conclude that $\alpha \not\leq \beta$. Consequently $\alpha \neq_{\text{E.E.}} \beta$.

Case 2, $\beta \in \mathbf{K}$.

$$
\begin{array}{ll}
 & \mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha} \\
\Rightarrow & \mathbb{S}_{\neg\beta\vee\neg\alpha} = \mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha} \\
\Rightarrow & f_{\mathbb{S}}(\neg\beta \vee \neg\alpha) \subseteq \|\neg\beta\| \text{ and } f_{\mathbb{S}}(\neg\beta \vee \neg\alpha) \cap \|\neg\alpha\| = \emptyset \\
\Rightarrow & \beta \notin \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg(\beta \wedge \alpha))) \text{ and } \alpha \in \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg(\beta \wedge \alpha))) \\
 & \text{(given that } \alpha \in \bigcap \|\mathbf{K}\|) \\
\Rightarrow & \beta \notin \mathbf{K} -_{\mathbb{S}} \beta \wedge \alpha \text{ and } \alpha \in \mathbf{K} -_{\mathbb{S}} \beta \wedge \alpha \\
\text{iff} & \beta \notin \mathbf{K} -_{\leq} \beta \wedge \alpha \text{ and } \alpha \in \mathbf{K} -_{\leq} \beta \wedge \alpha \text{ (by the hypothesis)} \\
\Rightarrow & (\beta \wedge \alpha) \not< (\beta \wedge \alpha) \vee \beta \text{ and } (\beta \wedge \alpha) < (\beta \wedge \alpha) \vee \alpha \text{ (given that } \beta \in \mathbf{K}) \\
\Rightarrow & \beta \wedge \alpha \not< \beta \text{ and } \beta \wedge \alpha < \alpha \text{ (by intersubstitutivity (Observation 3.4.9))} \\
\Rightarrow & \beta \leq \beta \wedge \alpha \text{ and } \beta \wedge \alpha < \alpha \text{ (by condition (3.1))} \\
\Rightarrow & \beta < \alpha \text{ (by (EE1) and Observation 3.4.8)} \\
\Rightarrow & \alpha \neq_{\text{E.E.}} \beta \text{ (by condition (3.1))}
\end{array}
$$

■

Remark 4.2.7 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an E.E. relation with respect to* $\mathbf{K}$*. Then:*

(i) If $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$ *then* $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K} \setminus \{\alpha \in \mathcal{L} : \alpha \leq \alpha_i\}$.

(ii) If $\alpha_i \in \mathcal{L} \setminus \mathbf{K}$ *then* $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K}$.

Proof. Let $\mathbf{K}$ be a belief set and $\leq$ be an E.E. relation with respect to $\mathbf{K}$. In what follows we prove that conditions (i) and (ii) are satisfied:

(i) Assume $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$. We start by showing that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \subseteq \mathbf{K}$. To see this, let φ be an arbitrary sentence in $\mathcal{L} \setminus \mathbf{K}$. It follows from (EE4) that $\varphi \leq \alpha_i$. Therefore $\alpha_i \not< \varphi$ and, consequently, $\varphi \notin \{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$. Hence, we can conclude that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \subseteq \mathbf{K}$.

Having seen this, we now show that it holds that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K} \setminus \{\alpha \in \mathcal{L} : \alpha \leq \alpha_i\}$. Indeed:

$$
\begin{array}{ll}
 & \varphi \in \mathbf{K} \setminus \{\alpha \in \mathcal{L} : \alpha \leq \alpha_i\} \\
\text{iff} & \varphi \in \mathbf{K} \text{ and } \varphi \notin \{\alpha \in \mathcal{L} : \alpha \leq \alpha_i\} \\
\text{iff} & \varphi \in \mathbf{K} \text{ and } \varphi \not\leq \alpha_i \\
\text{iff} & \varphi \in \mathbf{K} \text{ and } \alpha_i < \varphi \text{ (according to 3.1)} \\
\text{iff} & \varphi \in \mathbf{K} \cap \{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \\
\text{iff} & \varphi \in \{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \text{ (since, as we have seen above, } \{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \subseteq \mathbf{K}).
\end{array}
$$

(ii) Let $\alpha_i \in \mathcal{L} \setminus \mathbf{K}$, we must prove that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \mathbf{K}$. From (i) above we can conclude that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} \subseteq \mathbf{K}$. On the other hand, it follows from Observation 3.4.12 that $\mathbf{K} \subseteq \{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$ and this finishes the proof. ■

Lemma 4.2.8 *Let $\leq$ be an E.E. relation with respect to a belief set* $\mathbf{K}$ *and* $\alpha_i, \alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$. *Then the following statements hold:*

(i) If $\alpha_i < \alpha_j$ *then* $\alpha_j \in Th(\mathcal{W}_{\alpha_i})$ *or, which is the same,* $\mathcal{W}_{\alpha_i} \subseteq \|\alpha_j\|$.

(ii) $\mathcal{W}_{\alpha_j} \cap \|\neg\alpha_j\| \neq \emptyset$.

Proof. Assume the conditions introduced in the statement of the lemma hold.

(i) Let $\alpha_i < \alpha_j$. Then $\alpha_j \in \{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$ and, according to the construction (4.1) of the set $\mathcal{W}_{\alpha_i}$, it follows that $\alpha_j \in \bigcap \mathcal{W}_{\alpha_i}$ and this finishes the proof.

(ii) According to (4.1), $\mathcal{W}_{\alpha_j} = \|\{\alpha \in \mathcal{L} : \alpha_j < \alpha\}\|$. Assume by *reductio* that $\mathcal{W}_{\alpha_j} \cap \|\neg\alpha_j\| = \emptyset$. Then $\{\alpha \in \mathcal{L} : \alpha_j < \alpha\} \vdash \alpha_j$. Since $\alpha_j \notin Cn(\emptyset)$, we have that $\{\alpha \in \mathcal{L} : \alpha_j < \alpha\} \neq \emptyset$ and it follows by compactness that there is a non-empty finite set $D = \{\delta_1, \ldots, \delta_n\}$ such that $D \subseteq \{\alpha \in \mathcal{L} : \alpha_j < \alpha\}$ and $\delta_1 \wedge \cdots \wedge \delta_n \vdash \alpha_j$. Now, on the one hand it follows from (EE2) that $\delta_1 \wedge \cdots \wedge \delta_n \leq \alpha_j$. On the other hand, by making use of (EE1) and (EE3) we can easily obtain by induction that there is some $\delta_i \in D$ such that $\delta_i \leq \delta_1 \wedge \cdots \wedge \delta_n$. Hence, by making use of (EE1), we can conclude that there is some $\delta_i \in \{\alpha \in \mathcal{L} : \alpha_j < \alpha\}$ such that $\delta_i \leq \alpha_j$ which is a contradiction. Hence $\mathcal{W}_{\alpha_j} \cap \|\neg\alpha_j\| \neq \emptyset$, as we wished to prove. ■

Lemma 4.2.9 *Let $\leq$ be an E.E. relation with respect to a belief set* $\mathbf{K}$ *and* $\alpha_i, \alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$. *Then the following statements hold:*

(i) $\alpha_i =_{\text{E.E.}} \alpha_j$ *if and only if* $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$.

(ii) $\alpha_i < \alpha_j$ *if and only if* $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$.

Proof. Assume the conditions introduced in the statement of the lemma hold.

We start by proving the left to right part of both the statements (i) and (ii), i.e. the following conditionals:

(i $\Rightarrow$) If $\alpha_i =_{\text{E.E.}} \alpha_j$ then $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$.

(ii $\Rightarrow$) If $\alpha_i < \alpha_j$ then $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$.

(i $\Rightarrow$) Let $\alpha_i =_{\text{E.E.}} \alpha_j$. Then it follows immediately from (EE1) and Observation 3.4.8 that $\{\alpha \in \mathcal{L} : \alpha_i < \alpha\} = \{\alpha \in \mathcal{L} : \alpha_j < \alpha\}$ and, according to equation (4.1), we can conclude that $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$.

(ii $\Rightarrow$) Let $\alpha_i < \alpha_j$. Then it follows immediately from (EE1) and Observation 3.4.8 that $\{\alpha \in \mathcal{L} : \alpha_j < \alpha\} \subset \{\alpha \in \mathcal{L} : \alpha_i < \alpha\}$. Therefore, according to equation (4.1), it holds that $\mathcal{W}_{\alpha_i} \subseteq \mathcal{W}_{\alpha_j}$. On the other hand, since from Lemma 4.2.8 it follows that $\mathcal{W}_{\alpha_i} \subseteq \|\alpha_j\|$ and $\mathcal{W}_{\alpha_j} \cap \|\neg\alpha_j\| \neq \emptyset$, we can conclude that $\mathcal{W}_{\alpha_j} \not\subseteq \mathcal{W}_{\alpha_i}$. Hence $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$, as we wished to prove.

Now we prove:

(i $\Leftarrow$) If $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$ then $\alpha_i =_{\text{E.E.}} \alpha_j$.

(ii $\Leftarrow$) If $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$ then $\alpha_i < \alpha_j$.

(i $\Leftarrow$) Let $\mathcal{W}_{\alpha_i} = \mathcal{W}_{\alpha_j}$. Then, on the one hand, $\mathcal{W}_{\alpha_i} \not\subset \mathcal{W}_{\alpha_j}$ and it follows from (ii $\Rightarrow$), already proven above, that $\alpha_i \not< \alpha_j$. Hence, from connectivity (Observation 3.4.3 - (i)) we obtain that $\alpha_j \leq \alpha_i$. On the other hand $\mathcal{W}_{\alpha_j} \not\subset \mathcal{W}_{\alpha_i}$ and reasoning analogously we obtain $\alpha_i \leq \alpha_j$. Therefore $\alpha_i =_{\text{E.E.}} \alpha_j$, as required.

(ii $\Leftarrow$) Let $\mathcal{W}_{\alpha_i} \subset \mathcal{W}_{\alpha_j}$. Then $\mathcal{W}_{\alpha_j} \not\subseteq \mathcal{W}_{\alpha_i}$ and it follows from (i $\Rightarrow$) and (ii $\Rightarrow$) that $\alpha_j \not\leq \alpha_i$. Hence, by connectivity (Observation 3.4.3 - (i)), we obtain that $\alpha_i < \alpha_j$, and this finishes the proof. ■

Observation 4.2.11 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an E.E. relation with respect to* $\mathbf{K}$*, and* $\mathbb{S}'$ *be the class of subsets of* $\mathcal{M}_{\mathcal{L}}$ *defined by:*

$$\mathbb{S}' = \{\mathcal{W}_{\alpha_i} : \alpha_i \in \mathcal{L} \setminus Cn(\emptyset)\},$$

where, for any $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$*,* $\mathcal{W}_{\alpha_i}$ *is the set defined in (4.1), i.e.,*

$$\mathcal{W}_{\alpha_i} = \|\{\alpha \in \mathcal{L} : \alpha_i < \alpha\}\|,$$

then the following statements hold:

(i) If $\mathbf{K} \neq \mathcal{L}$ *(i.e.,* $\mathbf{K}$ *is a consistent belief set), then the set* $\mathbb{S} = \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$*.*

(ii) If $\mathbf{K} = \mathcal{L}$*, then the set* $\mathbb{S} = \{\emptyset\} \cup \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ *is a system of spheres centred on* $\|\mathbf{K}\|$*.*

Moreover, in both cases $\mathbf{K} \neq \mathcal{L}$ *and* $\mathbf{K} = \mathcal{L}$*, it holds that the (respective) thus constructed system of spheres* $\mathbb{S}$ *and the given E.E. relation* $\leq$ *satisfy condition* $(\leq -\mathbb{S})$*.*

Proof. Let $\mathbf{K}$, $\leq$ and $\mathbb{S}'$ be as above stated.

(i) Let $\mathbf{K}$ be a consistent belief set. In what follows we prove that the set $\mathbb{S} = \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ is a system of spheres centred on $\|\mathbf{K}\|$.

First of all we notice that, by construction, $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$. Now we must show that it satisfies conditions ($\mathbb{S}$1) - ($\mathbb{S}$4) of Definition 3.3.5.

($\mathbb{S}$1) First notice that $\mathcal{M}_{\mathcal{L}}$ is such that for any $\mathcal{X} \in \mathbb{S}$ it holds that $\mathcal{X} \subseteq \mathcal{M}_{\mathcal{L}}$. Now let $\mathcal{U}$ and $\mathcal{V}$ be any two elements of $\mathbb{S}'$. Then there are two sentences $\alpha_i, \alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$ such that $\mathcal{U} = \mathcal{W}_{\alpha_i}$ and $\mathcal{V} = \mathcal{W}_{\alpha_j}$. It follows from connectivity (Observation 3.4.3 - (i)) that $\alpha_i \leq \alpha_j$ or $\alpha_j \leq \alpha_i$. So, according to Lemma 4.2.9, $\mathcal{W}_{\alpha_i} \subseteq \mathcal{W}_{\alpha_j}$ or $\mathcal{W}_{\alpha_j} \subseteq \mathcal{W}_{\alpha_i}$, that is, $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$. Hence, for any $\mathcal{R}, \mathcal{S} \in \mathbb{S}$, it holds that $\mathcal{R} \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{R}$.

($\mathbb{S}$2) Since $\mathbf{K}$ is consistent there is some $\alpha_i \in \mathcal{L} \setminus Cn(\emptyset)$ such that $\alpha_i \notin \mathbf{K}$. And in that case, as we have observed in Remark 4.2.10, it holds that $\mathcal{W}_{\alpha_i} = \|\mathbf{K}\|$. Therefore we can conclude that $\|\mathbf{K}\| \in \mathbb{S}$.

Now we show that for any $\mathcal{U} \in \mathbb{S}$ it holds that $\|\mathbf{K}\| \subseteq \mathcal{U}$.

On the one hand it is immediate that $\|\mathbf{K}\| \subseteq \mathcal{M}_{\mathcal{L}}$. On the other hand, it follows from Remark 4.2.7-(i) that, for any $\alpha_j \in \mathcal{L} \setminus Cn(\emptyset)$, $\{\alpha \in \mathcal{L} : \alpha_j < \alpha\} \subseteq \mathbf{K}$. Therefore, $\|\mathbf{K}\| \subseteq \|\{\alpha \in \mathcal{L} : \alpha_j < \alpha\}\|$, which is to say that $\|\mathbf{K}\| \subseteq \mathcal{W}_{\alpha_j}$. Hence for any $\mathcal{V} \in \mathbb{S}'$, $\|\mathbf{K}\| \subseteq \mathcal{V}$.

($\mathbb{S}3$) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ by construction.

($\mathbb{S}4$) Let φ be any sentence of $\mathcal{L}$. And assume there is some element in $\mathbb{S}$ that intersects $\|\varphi\|$. We must show that there is a smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$.

First we notice that it follows from $\|\varphi\| \neq \emptyset$ that $\neg\varphi \in \mathcal{L} \backslash Cn(\emptyset)$ and consequently the set $\mathcal{W}_{\neg\varphi}$ (defined by (4.1)) is in $\mathbb{S}$.

Now we show that $\mathcal{W}_{\neg\varphi}$ is the smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$. In order to do that we first notice that it follows from Lemma 4.2.8-(ii) that $\mathcal{W}_{\neg\varphi} \cap \|\varphi\| \neq \emptyset$. It remains to show that any element of $\mathbb{S}$ that is (strictly) contained in $\mathcal{W}_{\neg\varphi}$ does not intersect $\|\varphi\|$. For that purpose, let $\mathcal{H}$ be an arbitrary element of $\mathbb{S}$ such that $\mathcal{H} \subset \mathcal{W}_{\neg\varphi}$. Since $\mathcal{M}_{\mathcal{L}} \not\subset \mathcal{W}_{\neg\varphi}$, we have that $\mathcal{H} \in \mathbb{S}'$, so there is some $\alpha_h \in \mathcal{L} \setminus Cn(\emptyset)$ such that $\mathcal{H} = \mathcal{W}_{\alpha_h}$. From $\mathcal{W}_{\alpha_h} \subset \mathcal{W}_{\neg\varphi}$ it follows, according to Lemma 4.2.9-(ii), that $\alpha_h < \neg\varphi$. Finally, applying Lemma 4.2.8-(i), we conclude that $\mathcal{W}_{\alpha_h} \subseteq \|\neg\varphi\|$. Therefore $\mathcal{H} \cap \|\varphi\| = \emptyset$ as we wished to prove.

(ii) Let $\mathbf{K} = \mathcal{L}$ be the inconsistent belief set. We must prove that the set $\mathbb{S} = \{\emptyset\} \cup \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$ is a system of spheres centred on $\|\mathbf{K}\|$.

Just as in the above proof of (i) we start by noticing that, by construction, $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$. Now we must show that it satisfies conditions ($\mathbb{S}1$) - ($\mathbb{S}4$) from Definition 3.3.5.

($\mathbb{S}1$) It was shown in the above proof of (i) that, for any two elements of $\mathcal{U}, \mathcal{V} \in \mathbb{S}' \cup \{\mathcal{M}_{\mathcal{L}}\}$, it holds that $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$. On the other hand, $\emptyset \subseteq \mathcal{X}$ for any $\mathcal{X} \in \mathbb{S}$. Therefore we can conclude that $\mathcal{R} \subseteq \mathcal{S}$ or $\mathcal{S} \subseteq \mathcal{R}$, for any $\mathcal{R}, \mathcal{S} \in \mathbb{S}$.

($\mathbb{S}2$) Since $\mathbf{K} = \mathcal{L}$ is the inconsistent belief set, we have that $\|\mathbf{K}\| = \emptyset$. Therefore $\|\mathbf{K}\| \in \mathbb{S}$ by construction, and it follows trivially that $\|\mathbf{K}\| \subseteq \mathcal{U}$, for any $\mathcal{U} \in \mathbb{S}$.

($\mathbb{S}3$) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ by construction.

($\mathbb{S}4$) Let φ be any sentence of $\mathcal{L}$. And assume there is some element in $\mathbb{S}$ that intersects $\|\varphi\|$. Reasoning just as we did in the proof of (i) above, we obtain that $\mathcal{W}_{\neg\varphi} \in \mathbb{S}$ and $\mathcal{W}_{\neg\varphi} \cap \|\varphi\| \neq \emptyset$. Now, in order to show that $\mathcal{W}_{\neg\varphi}$ is the smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$, we take an arbitrary element $\mathcal{H}$ of $\mathbb{S}$ such that $\mathcal{H} \subset \mathcal{W}_{\neg\varphi}$ and we prove that $\mathcal{H} \cap \|\varphi\| = \emptyset$. We have to consider two cases:

Case 1, $\mathcal{H} = \emptyset$. Then it follows trivially that $\mathcal{H} \cap \|\varphi\| = \emptyset$.

Case 2, $\mathcal{H} \neq \emptyset$. Then, it can be shown just as in the above proof of (i) that $\mathcal{H} \cap \|\varphi\| = \emptyset$.

Now we prove that in both cases $\mathbf{K} \neq \mathcal{L}$ and $\mathbf{K} = \mathcal{L}$, it holds that the (respective) system of spheres $\mathbb{S}$ and the given E.E. relation $\leq$ satisfy condition ($\leq -\mathbb{S}$).

As we have seen in the respective proofs of ($\mathbb{S}4$) for each case, in both cases, we have that, for any consistent sentence φ, $\mathbb{S}_{\varphi} = \mathcal{W}_{\neg\varphi}$ (according to the notation introduced in Definition 3.3.5).

Hence, it follows from Lemma 4.2.9 that, for any $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$ it holds that:

(i) $\alpha =_{\text{E.E.}} \beta$ if and only if $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$.

(ii) $\alpha < \beta$ if and only if $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$.

From this, taking Observation 4.2.2 and Lemma 4.2.4 into account, we can conclude that, in both cases $\mathbf{K} \neq \mathcal{L}$ and $\mathbf{K} = \mathcal{L}$, it holds that $\leq$ and (the respective) $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$), and this finishes the proof. ∎

B.2 Interrelation Between Theorem 4.2.6 and [PW95, Theorem 6.1]

In Footnote 73 on page 71 we have remarked that Theorem 4.2.6 is equivalent to Theorem 6.1 of [PW95]. The main purpose of the present section is to clarify that remark.

Peppas and Williams, in their paper [PW95], have shown (see [PW95, Theorem 6.1]) that, given a belief set $\mathbf{K}$, if $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$ and $\leq$ is an epistemic entrenchment relation with respect to $\mathbf{K}$, then the $\mathbb{S}$-*based revision* on $\mathbf{K}$ and the $\leq$-*based revision* on $\mathbf{K}$ coincide[145] if and only if $\leq$ and $\mathbb{S}$ satisfy the following condition:

$$\text{For every } \alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset), \text{ such that } \nvdash \neg\alpha \text{ and } \nvdash \neg\beta, \alpha \leq \beta \text{ iff } \mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}. \tag{B.1}$$

In order to show that the above mentioned Theorem 6.1 of [PW95] is indeed equivalent to Theorem 4.2.6 we start by observing, in the following lemma, that condition ($B.1$) is satisfied if and only if condition ($\leq -\mathbb{S}$) holds.

Lemma B.2.1 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Then the conditions ($B.1$) and ($\leq -\mathbb{S}$) are equivalent.*[146]

Proof. Assume the conditions mentioned in the statement of the lemma hold.

We start by noticing that it follows from Observation 4.2.2 that, in order to show that the conditions ($B.1$) and ($\leq -\mathbb{S}$) are equivalent it is enough to prove that condition ($B.1$) is satisfied if and only if condition ($\leq -\mathbb{S}'$) is also satisfied.

It is obvious that if condition ($\leq -\mathbb{S}'$) is satisfied then condition ($B.1$) also holds.

Now we show that the reverse implication is also true.

Assume condition ($B.1$) is satisfied, and let α, β be two arbitrary elements of $\mathcal{L} \setminus Cn(\emptyset)$. We must show that $\alpha \leq \beta$ if and only if $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$. When $\nvdash \neg\alpha$ and $\nvdash \neg\beta$, this follows immediately from condition ($B.1$). Now we consider the remaining cases: $\vdash \neg\alpha$ or $\vdash \neg\beta$.

Case 1, $\vdash \neg\alpha$. Then, on the one hand we have that $\neg\beta \vdash \neg\alpha$, which is equivalent to $\alpha \vdash \beta$, and it follows from (EE2) that $\alpha \leq \beta$. On the other hand, since $\vdash \neg\alpha$ it holds that $\neg\alpha \in U$ for any $U \in \mathcal{M}_{\mathcal{L}}$. Therefore, according to ($\mathbb{S}$2) and to the definition of $\mathbb{S}_{\neg\alpha}$ (see Definition 3.3.5), it holds that $\mathbb{S}_{\neg\alpha} \subseteq \mathcal{U}$ for any $\mathcal{U} \in \mathbb{S} \setminus \{\emptyset\}$, which is to say that $\mathbb{S}_{\neg\alpha}$ is the smallest non-empty sphere in $\mathbb{S}$. Hence, in particular, given that $\nvdash \beta$, we have that $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$. So, in this case, it trivially holds that $\alpha \leq \beta$ if and only if $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$.

[145]The revision functions on $\mathbf{K}$ here named $\mathbb{S}$-*based revision* and $\leq$-*based revision* on $\mathbf{K}$ are the ones that can be obtained from the $\mathbb{S}$-based contraction (see Definition 3.3.7) and the $\leq$-based contraction (see condition ($C_{-\leq}$) in Definition 3.4.4) on $\mathbf{K}$, respectively, by means of the Levi Identity (equation (2.1)).

[146]Notice, however, that condition ($B.1$) can not be used (instead of condition ($\leq -\mathbb{S}$)) for the purpose of defining (as it was done in Observation 4.2.1) an epistemic entrenchment relation by means of a system of spheres. Observe that the justification for this statement is similar to the one that we have presented in Remark 4.2.3 to explain why condition ($\leq -\mathbb{S}'$) can not be used to define an epistemic entrenchment relation by means of a system of spheres.

Case 2, $\vdash \neg\beta$. Since we have above taken care of the case $\vdash \neg\alpha$, in what follows we assume $\nvdash \neg\alpha$.

Given that $\vdash \neg\beta$, we start by noticing that, according to ($\mathbb{S}$2) and to the definition of $\mathbb{S}_{\neg\beta}$ (see Definition 3.3.5), it holds that $\mathbb{S}_{\neg\beta} \subseteq \mathcal{U}$ for any $\mathcal{U} \in \mathbb{S} \setminus \{\emptyset\}$, which is to say that $\mathbb{S}_{\neg\beta}$ is the smallest non-empty sphere in $\mathbb{S}$. On the other hand, since $\vdash \neg\beta$, we have that, for any $\phi \in \mathcal{L}$, $\beta \vdash \phi$, and, from (EE2) we obtain that $\beta \leq \phi$ for any $\phi \in \mathcal{L}$.

Now let $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$. We must show that $\alpha \leq \beta$. From the above remarks, and the fact that we are assuming $\nvdash \neg\alpha$, we can conclude that $\mathbb{S}_{\neg\beta} \subseteq \mathbb{S}_{\alpha}$. Hence, $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\alpha}$ and, since $\nvdash \alpha$ and $\nvdash \neg\alpha$, it follows from (B.1) that $\alpha \leq \neg\alpha$. Therefore, making use of (EE3) and (EE1), we obtain that $\alpha \leq \alpha \wedge \neg\alpha$. Finally, since $\alpha \wedge \neg\alpha \vdash \beta$, using (EE2) and (EE1) we can conclude that $\alpha \leq \beta$, as required.

For the other direction, let $\alpha \leq \beta$. Since, as we have seen above, $\beta \leq \phi$ for any $\phi \in \mathcal{L}$, by making use of (EE1) we obtain that $\alpha \leq \phi$ for any $\phi \in \mathcal{L}$. Therefore, in particular, we have that $\alpha \leq \neg\alpha$, and, since $\nvdash \alpha$ and $\nvdash \neg\alpha$, condition (B.1) gives us that $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\alpha}$. Now, recalling that $\mathbb{S}_{\neg\beta}$ is the smallest non-empty sphere in $\mathbb{S}$, we notice that at least one of the equalities $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$ and $\mathbb{S}_{\alpha} = \mathbb{S}_{\neg\beta}$ holds. But then, because $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\alpha}$, in either case we have that $\mathbb{S}_{\neg\alpha} \subseteq \mathbb{S}_{\neg\beta}$, and this finishes the proof. ■

Now, having the above lemma in mind, we can immediately conclude that the statement of [PW95, Theorem 6.1] can be rewritten in the following way:

Observation B.2.2 (Reformulation of [PW95, Theorem 6.1]) *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\mathbb{S}$-based revision *on* $\mathbf{K}$ *and the* $\leq$-based revision *on* $\mathbf{K}$ *coincide*[145] *if and only if* $\leq$ *and* $\mathbb{S}$ *satisfy condition* ($\leq -\mathbb{S}$).

Hence, in order to conclude that [PW95, Theorem 6.1] and Theorem 4.2.6 are equivalent, it is enough to note that, given a belief set $\mathbf{K}$, a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ and an epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$, the $\mathbb{S}$-based contraction coincides with the $\leq$-based contraction on $\mathbf{K}$ if and only if the $\mathbb{S}$-based revision coincides with the $\leq$-based revision[145] on $\mathbf{K}$. Indeed, on the one hand, as mentioned in Footnote 145, the $\mathbb{S}$-based revision and the $\leq$-based revision on $\mathbf{K}$ are obtained from the $\mathbb{S}$-based contraction and the $\leq$-based contraction on $\mathbf{K}$, respectively, by means of the Levi Identity (i.e. equation (2.1)). On the other hand, since, according to their respective axiomatic characterizations, the $\leq$-based contraction and the $\mathbb{S}$-based contraction satisfy the basic AGM postulates for contraction, it follows from Observation 2.4.8 that the $\mathbb{S}$-based contraction and the $\leq$-based contraction on $\mathbf{K}$ are in fact the contraction functions which result of applying the Harper Identity (i.e. equation (2.2)) to the $\mathbb{S}$-based revision and to the $\leq$-based revision on $\mathbf{K}$, respectively.

Therefore, we can conclude that Observation B.2.2 and Theorem 4.2.6 are equivalent, and this fact immediately yields that Theorem 4.2.6 and [PW95, Theorem 6.1] are also equivalent which is precisely what we intended to expose in the present section.

Appendix C

Proofs of Chapter 6

In this chapter we present proofs for most of the results presented throughout Chapter 6. For commodity of the reader before each of those proofs we include the statement of the result that it corresponds to.

Observation 6.1.1 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$*. If* $X \in \mathbf{K} \perp B$ *then*

$$\|X\| = \|\mathbf{K}\| \cup \mathcal{N},$$

where $\mathcal{N} \subseteq \mathcal{M}_{\mathcal{L}}$ *is such that:*

1. $\mathcal{N} \subseteq \bigcup \{ \|\neg \alpha_i\| : \alpha_i \in B \}$.

2. $\mathcal{N} \cap \|\neg \alpha_i\| \neq \emptyset$*, for all* $\alpha_i \in B \cap \mathbf{K}$.

Proof. Let $\mathbf{K}$ be a belief set, B be a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$, and X be an element of the remainder set $\mathbf{K} \perp B$. We must show that $\|X\| = \|\mathbf{K}\| \cup \mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{M}_{\mathcal{L}}$ is such that conditions 1. and 2. in the statement of the observation are satisfied.

We start by noticing that to prove that this is indeed the case it is enough to show that:

(i) $\|\mathbf{K}\| \subseteq \|X\|$.

(ii) $\|X\| \subseteq \|\mathbf{K}\| \cup (\bigcup \{ \|\neg \alpha_i\| : \alpha_i \in B \})$.

(iii) $\|X\| \cap \|\neg \alpha_i\| \neq \emptyset$, for all $\alpha_i \in B$.

If $B \cap \mathbf{K} = \emptyset$ then $X = \mathbf{K}$ and conditions (i)-(iii) are trivially satisfied. Hence, now we assume that $B \cap \mathbf{K} \neq \emptyset$ and show that conditions (i)-(iii) hold in this case too.

(i) Recall that, according to Observation 3.1.4, the remainder X is a belief set. Since $\mathbf{K}$ is also a belief set and recalling that it follows from the definition of $\mathbf{K} \perp B$ that $X \subseteq \mathbf{K}$, using Observation 3.3.3-(e), we can conclude that $\|\mathbf{K}\| \subseteq \|X\|$. Hence condition (i) holds.

(ii) Let M be an arbitrary element of $\|X\|$. We must prove that $M \in \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B\})$. Assume by *reductio* that $M \notin (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B\}))$, then there is some $\delta \in \mathbf{K}$ such that $\neg\delta \wedge \bigwedge B \in M$, where $\bigwedge B$ denotes the conjunction of all elements of B (recall that by hypothesis B is finite). Therefore, because M is consistent, it follows that $\delta \vee \neg \bigwedge B \notin M$. Then, on the one hand, from $M \in \|X\|$ it follows that $\delta \vee \neg \bigwedge B \notin X$ and, on the other hand, since $\delta \in \mathbf{K}$ and $\mathbf{K}$ is a belief set, we have that $\delta \vee \neg \bigwedge B \in \mathbf{K}$. So we it holds that $\delta \vee \neg \bigwedge B \in \mathbf{K} \setminus X$.

Now, we notice that, since $X \in \mathbf{K}\bot B$, it follows from Definition 3.1.1 that there is some $\alpha_j \in B$ such that $\alpha_j \in Cn(X \cup \{\delta \vee \neg \bigwedge B\})$. Then, by deduction it follows that $(\neg\delta \wedge \bigwedge B) \vee \alpha_j \in Cn(X)$. But, from $\alpha_j \in B$ it follows that $(\neg\delta \wedge \bigwedge B) \vee \alpha_j \leftrightarrow \alpha_j$. Therefore we obtain that $\alpha_j \in Cn(X)$, which contradicts the fact that $X \in \mathbf{K}\bot B$. Hence we can conclude that $M \in \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B\})$ as we wished to prove.

(iii) Let $\alpha_i \in B$ and assume by *reductio* that $\|X\| \cap \|\neg\alpha_i\| = \emptyset$. Then it follows that $\alpha_i \in Cn(X)$ which is a contradiction since, according to Definition 3.1.1, it holds that $Cn(X) \cap B = \emptyset$. Therefore we have that $\|X\| \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B$, which is to say that condition (iii) holds. ■

Lemma 6.1.3 *Let* $\mathbf{K}$ *be a belief set,* B *be a finite set of sentences and* $\mathcal{N}$ *be a subset of* $\mathcal{M}_{\mathcal{L}}$. *If* $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, *for all* $\alpha_i \in B \cap \mathbf{K}$, *then there is some set* $\mathcal{N}'$ *such that* $\mathcal{N}' \subseteq \mathcal{N}$ *and* $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\bot B}$.[102]

Proof. Assume $\mathbf{K}$ and B are in the conditions stated above and let $\mathcal{N} \subseteq \mathcal{M}_{\mathcal{L}}$ be such that $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$. We will prove that there is some set $\mathcal{N}'$ such that $\mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\bot B}$. In order to do that we will consider separately the cases $B \cap \mathbf{K} = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$.

Case 1, $B \cap \mathbf{K} = \emptyset$. Then it is enough to define $\mathcal{N}' = \emptyset$ and it follows immediately that, on the one hand $\mathcal{N}' \subseteq \mathcal{N}$ and, on the other hand, it also holds that $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\bot B}$. Indeed, if $\mathcal{N}' = \emptyset$ and $B \cap \mathbf{K} = \emptyset$, then the following conditions are vacuously true:

1. If $\alpha_i \in B \cap \mathbf{K}$ then $\mathcal{N}' \cap \|\neg\alpha_i\| \neq \emptyset$.

2. If $M \in \mathcal{N}'$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{N}' \cap \|\neg\alpha_j\| = M$.

Therefore, according to Definition 6.1.2, we can conclude that $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\bot B}$.

Case 2, $B \cap \mathbf{K} \neq \emptyset$. For each $\alpha_i \in B \cap \mathbf{K}$ let M_{α_i} be one (arbitrarily chosen) world in $\mathcal{N} \cap \|\neg\alpha_i\|$ (recall that $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$) and consider the set $\mathcal{N}_0 = \{M_{\alpha_i} : \alpha_i \in B \cap \mathbf{K}\}$.

Notice that, since B is finite, $B \cap \mathbf{K}$ also is and, therefore, we may assume without loss of generality that $\mathcal{N}_0 = \{M_1, \ldots, M_n\}$, for some natural number $n \geq 1$ (because $B \cap \mathbf{K} \neq \emptyset$).

Next, for each $m \in \{1, \ldots n\}$, let $\mathcal{N}_m$ be the set defined in the following way:

$$\mathcal{N}_m = \begin{cases} \mathcal{N}_{m-1} \setminus \{M_m\} & \text{, if } \mathcal{N}_{m-1} \cap \|\neg\alpha_p\| \neq \{M_m\} \text{ forall } \alpha_p \in B \cap \mathbf{K} \\ \mathcal{N}_{m-1} & \text{, if } \mathcal{N}_{m-1} \cap \|\neg\alpha_q\| = \{M_m\} \text{ forsome } \alpha_q \in B \cap \mathbf{K} \end{cases}.$$

It follows immediately from the above construction that the set $\mathcal{N}_n$ satisfies the following conditions:

- $\mathcal{N}_n \subseteq \mathcal{N} \subseteq \bigcup_{\alpha_i \in (B \cap \mathbf{K})} \|\neg\alpha_i\|$.

- $\mathcal{N}_n \cap \|\neg\alpha_k\| \neq \emptyset$, for all $\alpha_k \in B \cap \mathbf{K}$.
- If $M_i \in \mathcal{N}_n$ then there is some $\alpha_q \in B \cap \mathbf{K}$ such that $\mathcal{N}_n \cap \|\neg\alpha_q\| = \{M_i\}$.

Hence, $\mathcal{N}' = \mathcal{N}_n$ is a set in the required conditions, i.e., $\mathcal{N}'$ is such that $\mathcal{N}' \subseteq \mathcal{N}$ and, according to Definition 6.1.2, $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\perp B}$. ■

Observation 6.1.4 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then the following statements hold:*

1. $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.
2. $B \cap \mathbf{K} = \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$.
3. $B \cap Cn(\emptyset) = \emptyset$ *and* $B \cap \mathbf{K} \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ *and* $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$.

Proof. Assume $\mathbf{K}$ is a belief set and B is a finite set of sentences. We start by noticing that in order to prove that the three statements 1.-3. introduced in the above observation are satisfied it is enough to show that:

(i) If $B \cap Cn(\emptyset) \neq \emptyset$ then $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.

(ii) If $B \cap \mathbf{K} = \emptyset$ then $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$.

(iii) If $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$ then $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ and $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$.

Now we prove that (i)-(iii) hold.

(i) Let $B \cap Cn(\emptyset) \neq \emptyset$. We will prove that $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$. Let $\alpha_j \in B \cap Cn(\emptyset)$. Then $\alpha_j \in B \cap \mathbf{K}$ and $\|\neg\alpha_j\| = \emptyset$. Hence, there is no $\mathcal{W}$ such that $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$. Therefore $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.

(ii) Let $B \cap \mathbf{K} = \emptyset$. We will prove that $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$. From $B \cap \mathbf{K} = \emptyset$ it follows that $\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} = \emptyset$ and then, by Definition 6.1.2, the set $\mathbb{W}_{\mathbf{K}\perp B}$ is such that $\mathbb{W}_{\mathbf{K}\perp B} \subseteq \mathcal{P}(\emptyset) = \{\emptyset\}$. Therefore either $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$ or $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$. On the other hand, given that $B \cap \mathbf{K} = \emptyset$, we have that conditions:

1. $\emptyset \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.
2. If $M \in \emptyset$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = M$.

are trivially satisfied. Hence, according to Definition 6.1.2, $\emptyset \in \mathbb{W}_{\mathbf{K}\perp B}$. Therefore we can conclude that $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$, for every B such that $B \cap \mathbf{K} = \emptyset$.

(iii) Let $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$. We will first prove that $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$. Since $B \cap Cn(\emptyset) = \emptyset$, for every $\alpha_i \in B \cap \mathbf{K}$ we have that $\|\neg\alpha_i\| \neq \emptyset$. Having this in mind, for each $\alpha_i \in B \cap \mathbf{K}$ let M_{α_i} be one (arbitrarily chosen) $\neg\alpha_i$-world and consider the set $\mathcal{N} = \{M_{\alpha_i} : \alpha_i \in B \cap \mathbf{K}\}$. It follows immediately from Lemma 6.1.3 that there is some set $\mathcal{N}'$ such that $\mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}' \in \mathbb{W}_{\mathbf{K}\perp B}$. Hence $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$. Now we show that $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$. Let α_j be an arbitrary sentence of $B \cap \mathbf{K}$ (notice that, given that $B \cap \mathbf{K} \neq \emptyset$ there is such a sentence). Then, since $\emptyset \cap \|\neg\alpha_j\| = \emptyset$, it follows from Definition 6.1.2 that $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$. ■

Theorem 6.1.6 *Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. Then:*

1. *If $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ then $X = Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$.*
2. *If $X \in \mathbf{K}\perp B$ then there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.*

Proof. Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. Now we prove the two statements above.

1. We must prove that if $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$, then $X = Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$.

We will consider three cases separately:

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, according to Observation 6.1.4-1., it holds that $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$ and therefore it is vacuously true that if $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ then $Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$.

Case 2, $B \cap \mathbf{K} = \emptyset$. Then, by Observation 6.1.4-2., we have that $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$. On the other hand $\mathbf{K}\perp B = \{\mathbf{K}\}$ (according to Definition 3.1.1). Hence, it follows immediately from the fact that $Th(\|\mathbf{K}\| \cup \emptyset) = Th(\|\mathbf{K}\|) = \mathbf{K}$ (by Observation 3.3.3-(a)) that if $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ then $Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$.

Case 3, $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$. We start by noticing that, in this case, according to Observation 6.1.4-3., $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ and $\emptyset \notin \mathbb{W}_{\mathbf{K}\perp B}$. Having this in mind, let $\mathcal{W}$ be an arbitrary element of $\mathbb{W}_{\mathbf{K}\perp B}$. We have that $Th(\|\mathbf{K}\| \cup \mathcal{W}) = \bigcap(\|\mathbf{K}\| \cup \mathcal{W}) = (\bigcap \|\mathbf{K}\|) \cap (\bigcap \mathcal{W}) = \mathbf{K} \cap (\bigcap \mathcal{W})$. Hence, we have to prove that $\mathbf{K} \cap (\bigcap \mathcal{W}) \in \mathbf{K}\perp B$. Therefore, according to Definition 3.1.1, we have to prove that

(i) $\mathbf{K} \cap (\bigcap \mathcal{W}) \subseteq \mathbf{K}$

(ii) $Cn(\mathbf{K} \cap (\bigcap \mathcal{W})) \cap B = \emptyset$

(iii) If $\beta \in \mathbf{K} \setminus (\mathbf{K} \cap (\bigcap \mathcal{W}))$ then $Cn((\mathbf{K} \cap (\bigcap \mathcal{W})) \cup \{\beta\}) \cap B \neq \emptyset$.

Condition (i) is trivially satisfied. Regarding (ii), let α_i be an arbitrary element of B, we will show that $\alpha_i \notin Cn(\mathbf{K} \cap (\bigcap \mathcal{W}))$. In order to do that we consider two cases separately:

Case 3.1, $\alpha_i \in B \setminus \mathbf{K}$. Then $\alpha_i \notin Cn(\mathbf{K})(= \mathbf{K})$ and from $\mathbf{K} \cap (\bigcap \mathcal{W}) \subseteq \mathbf{K}$ we can immediately conclude that $\alpha_i \notin Cn(\mathbf{K} \cap (\bigcap \mathcal{W}))$.

Case 3.2, $\alpha_i \in B \cap \mathbf{K}$. Then, according to Definition 6.1.2, there is some world $M \in \mathcal{W}$ such that $M \in \|\neg\alpha_i\|$. Therefore $\alpha_i \notin Cn(M)(= M)$ and from $\mathbf{K} \cap (\bigcap \mathcal{W}) \subseteq M$ we obtain that $\alpha_i \notin Cn(\mathbf{K} \cap (\bigcap \mathcal{W}))$.

Finally we prove (iii). Let $\beta \in \mathbf{K} \setminus (\mathbf{K} \cap (\bigcap \mathcal{W}))$. Then $\beta \in \mathbf{K}$ and $\beta \notin (\bigcap \mathcal{W})$. Hence there is some $M \in \mathcal{W}$ such that $\beta \notin M$. Therefore, because $M \in \mathcal{M}_{\mathcal{L}}$, it holds that $\neg\beta \in M$. Since $M \in \mathcal{W}$, according to Definition 6.1.2, there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$. Now, from $\neg\beta \in M$, $\alpha_j \in \mathbf{K}$ and $\alpha_j \in W_i$, for every $W_i \in \mathcal{W} \setminus M$, it follows that $\neg\beta \vee \alpha_j \in (\mathbf{K} \cap (\bigcap \mathcal{W}))$. Therefore $\alpha_j \in Cn((\mathbf{K} \cap (\bigcap \mathcal{W})) \cup \{\beta\})$, and then $Cn((\mathbf{K} \cap (\bigcap \mathcal{W})) \cup \{\beta\}) \cap B \neq \emptyset$ as required.

2. We must prove that if $X \in \mathbf{K}\perp B$ then there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.

Again we consider three cases separately:

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then it follows from Definition 3.1.1 that $\mathbf{K} \perp B = \emptyset$. Therefore it is vacuously true that if $X \in \mathbf{K} \perp B$ then there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B}$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.

Case 2, $B \cap \mathbf{K} = \emptyset$. Let $X \in \mathbf{K} \perp B$. Since, according to Definition 3.1.1, in this case, it holds that $\mathbf{K} \perp B = \{\mathbf{K}\}$, then $X = \mathbf{K}$. On the other hand, by Observation 6.1.4-2., we have that $\mathbb{W}_{\mathbf{K} \perp B} = \{\emptyset\}$. Observing that $Th(\|\mathbf{K}\| \cup \emptyset) = Th(\|\mathbf{K}\|) = \mathbf{K}$ (by Observation 3.3.3-(a)), we can conclude that $\mathcal{W} = \emptyset$ is such that $\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B}$ and $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$, as required.

Case 3, $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$. Let $X \in \mathbf{K} \perp B$. It follows from Observation 6.1.1 that $\|X\| = \|\mathbf{K}\| \cup \mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{M}_{\mathcal{L}}$ is such that $\mathcal{N} \cap \|\neg \alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.

Therefore, noticing that it follows from Observations 3.1.4 and 3.3.3-(a) that $X = Th(\|X\|)$, we can conclude that $X = Th(\|\mathbf{K}\| \cup \mathcal{N})$.

On the other hand, in the above conditions, according to Lemma 6.1.3, there is some set $\mathcal{N}'$ such that $\mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}' \in \mathbb{W}_{\mathbf{K} \perp B}$.

Let $X' = Th(\|\mathbf{K}\| \cup \mathcal{N}')$. To finish the proof it is enough to show that $X = X'$. Now, to see that this is indeed the case, we start by observing that, since $\|\mathbf{K}\| \cup \mathcal{N}' \subseteq \|\mathbf{K}\| \cup \mathcal{N}$, it follows from Observation 3.3.3-(d) that $X = Th(\|\mathbf{K}\| \cup \mathcal{N}) \subseteq Th(\|\mathbf{K}\| \cup \mathcal{N}') = X'$. Furthermore, according to point 1. proven above, because $\mathcal{N}' \in \mathbb{W}_{\mathbf{K} \perp B}$, we also have that $X' \in \mathbf{K} \perp B$. Hence, in particular it holds that $X' \subseteq \mathbf{K}$ and $Cn(X') \cap B = \emptyset$. From this and the fact that $X \in \mathbf{K} \perp B$ it follows that $X \not\subset X'$. Therefore, since $X \subseteq X'$, we can conclude that $X = X'$, and this finishes the proof. ■

Observation 6.2.1 *Let* $\mathbf{K}$ *be a belief set and* $\tilde{\div}$ *be the full meet multiple contraction on* $\mathbf{K}$. *Then*

$$\mathbf{K} \tilde{\div} B = \bigcap \left(\|\mathbf{K}\| \cup \left(\bigcup \mathbb{W}_{\mathbf{K} \perp B} \right) \right) = \mathbf{K} \cap \left(\bigcap \left(\bigcup \mathbb{W}_{\mathbf{K} \perp B} \right) \right),$$ [103]

for all finite sets of sentences B.

Proof. Let $\mathbf{K}$ be a belief set and $\tilde{\div}$ be the full meet multiple contraction on $\mathbf{K}$.

To show that the identities $\mathbf{K} \tilde{\div} B = \bigcap (\|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K} \perp B})) = \mathbf{K} \cap (\bigcap (\bigcup \mathbb{W}_{\mathbf{K} \perp B}))$ hold for all finite sets of sentences B we will consider two cases separately.

Case 1, $B \cap Cn(\emptyset) = \emptyset$. Then, according to Corollary 6.1.8 it follows that $\mathbf{K} \tilde{\div} B = \bigcap \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B}\} = \bigcap \{\bigcap (\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B}\} = \bigcap (\|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K} \perp B})) = (\bigcap \|\mathbf{K}\|) \cap (\bigcap (\bigcup \mathbb{W}_{\mathbf{K} \perp B})) = \mathbf{K} \cap (\bigcap (\bigcup \mathbb{W}_{\mathbf{K} \perp B}))$.[103]

Case 2, $B \cap Cn(\emptyset) \neq \emptyset$. Then we have on the one hand that $\mathbf{K} \tilde{\div} B = \mathbf{K}$ (according to Definition 5.3.3), and on the other hand that $\mathbb{W}_{\mathbf{K} \perp B} = \emptyset$ (according to Observation 6.1.4-1.) and, consequently, $\bigcup \mathbb{W}_{\mathbf{K} \perp B} = \emptyset$.

Therefore, in this case, we only need to show to show that $\mathbf{K} = \bigcap (\|\mathbf{K}\| \cup \emptyset)$ and $\mathbf{K} = \mathbf{K} \cap (\bigcap \emptyset)$. But, the former identity follows immediately from Observation 3.3.3-(a), and the latter equality is an obvious consequence of our assumption that $Th(\emptyset) = \bigcap \emptyset = \mathcal{L}$ (see Definition 3.3.2). ■

Observation 6.2.2 *Let* $\mathbf{K}$ *be a belief set and* B *a set of sentences. Then:*

1. *If* $B \cap Cn(\emptyset) \neq \emptyset$ *or* $B \cap \mathbf{K} = \emptyset$ *then* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \emptyset$.
2. $\bigcup \mathbb{W}_{\mathbf{K}\perp B} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.
3. *If* $B \cap \mathbf{K} = \emptyset$, *then* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.
4. *If* $\mathbf{K} = Cn(\emptyset)$ *then* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

Proof. 1. Follows immediately from Observation 6.1.4-1. and 2..

2. According to Definition 6.1.2, $\mathbb{W}_{\mathbf{K}\perp B} \subseteq \mathcal{P}\left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}\right)$, which implies immediately that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

3. Follows immediately from 1..

4. Assume $\mathbf{K} = Cn(\emptyset)$ and let B be any finite set. Then, on the one hand, it follows immediately from Remark 6.1.5 that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \emptyset$ and, on the other hand, $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} = \emptyset$ (since if $\alpha_j \in B \cap \mathbf{K}$ then $\alpha_j \in Cn(\emptyset)$ and therefore $\|\neg\alpha_j\| = \emptyset$). Therefore $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$. ■

Observation 6.2.3 *Let* $\mathbf{K}$ *a belief set and* B *be a set of sentences such that* $\mathbf{K} \neq Cn(\emptyset)$ *and* $B \cap \mathbf{K} \neq \emptyset$. *Then:*

1. *It may not hold that* $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ *(whether or not* $B \cap Cn(\emptyset) = \emptyset$[106]*).*
2. *If* M *is a world such that*
   (i) $M \in \|\neg\alpha_j\|$ *for some* $\alpha_j \in B \cap \mathbf{K}$,
   (ii) $M \notin \|\neg\alpha_l\|$ *for every* $\alpha_l \in B \setminus \{\alpha_j\}$, *and*
   (iii) there is some $\alpha_r \in B \cap \mathbf{K}$ *such that* $\|\neg\alpha_r\| \subset \|\neg\alpha_j\|$

   then $M \notin \bigcup \mathbb{W}_{\mathbf{K}\perp B}$.

Proof. To prove that the statement 1. in the above observation holds, in what follows we will present an example of a world M such that $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ but $M \notin \bigcup \mathbb{W}_{\mathbf{K}\perp B}$, for a belief set $\mathbf{K}$ and a finite set of sentences B such that $B \cap Cn(\emptyset) = \emptyset$.

Hence, let $\mathbf{K}$ be a belief set and $\alpha_1, \alpha_2, \alpha_3$ be three sentences such that $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{K} \setminus Cn(\emptyset)$ (notice that in that case it follows immediately that $\mathbf{K} \neq Cn(\emptyset)$[105]). Furthermore, assume that $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$, $\|\neg\alpha_2\| \not\subset \|\neg\alpha_3\|$, $\|\neg\alpha_3\| \not\subset \|\neg\alpha_2\|$ and $\|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|) \neq \emptyset$ and let M be an arbitrary element of $\|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$.

Now we notice that, in the above conditions, if we consider the set of sentences $B = \{\alpha_1, \alpha_2, \alpha_3\}$ then it is obvious that $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$. However, on the other hand, $M \notin \bigcup \mathbb{W}_{\mathbf{K}\perp B}$. Indeed assume by *reductio* that there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ such that $M \in \mathcal{W}$. Then, according to Definition 6.1.2, there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$. Therefore, since $B \cap \mathbf{K} = B = \{\alpha_1, \alpha_2, \alpha_3\}$ and $M \notin \|\neg\alpha_2\| \cup \|\neg\alpha_3\|$, it must be the case that $\mathcal{W} \cap \|\neg\alpha_1\| = \{M\}$. But, also from Definition 6.1.2 it follows that there must be some world $N \in \mathcal{W} \cap \|\neg\alpha_2\|$ and then, because $\|\neg\alpha_2\| \subset \|\neg\alpha_1\|$, we conclude that $N \in \mathcal{W} \cap \|\neg\alpha_1\|$ which contradicts $\mathcal{W} \cap \|\neg\alpha_1\| = \{M\}$ (since from $N \in \|\neg\alpha_2\|$ and $M \notin \|\neg\alpha_2\|$ it follows immediately that $N \neq M$).

Therefore, for a belief set $\mathbf{K}$ and a set of sentences B in the above conditions it holds that $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\} \not\subseteq \bigcup \mathbb{W}_{\mathbf{K}\perp B}$ and consequently $\bigcup \mathbb{W}_{\mathbf{K}\perp B} \neq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$.

It is worth to mention here that Figure 6.5 contains a possible graphical representation of the situation presented above, highlighting a world $M \in \|\neg\alpha_1\| \setminus (\|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$ which, as we have seen above, is such that $M \in (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}) \setminus (\bigcup \mathbb{W}_{\mathbf{K}\perp B})$.

To finish this proof we just remark that from the example presented above we can immediately conclude that point 2. of the above observation holds. ∎

Lemma 6.2.5 *Let B be a finite set of sentences. If $\alpha_l \in B$ and $\alpha_l \notin B_N$ then there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_l\|$.*

Proof. Let B be a finite set of sentences and α_l be a sentence such that $\alpha_l \in B \setminus B_N$. It follows from the definition of B_N that there is some $\alpha_s \in B$ such that $\|\neg\alpha_s\| \subset \|\neg\alpha_l\|$. If $\alpha_s \in B_N$ this ends the proof. Assume $\alpha_s \notin B_N$, then using the same arguments, there is some $\alpha_{s_1} \in B$ such that $\|\neg\alpha_{s_1}\| \subset \|\neg\alpha_s\|$. Now, on the one hand, if $\alpha_{s_1} \in B_N$ this ends the proof because under this conditions we have that $\|\neg\alpha_{s_1}\| \subset \|\neg\alpha_l\|$. On the other hand, if $\alpha_{s_1} \notin B_N$, reasoning in an analogous way we can find a $\alpha_{s_2} \in B$ which is related to α_{s_1} in precisely in the same way that α_{s_1} was related to α_s and so on.

Now consider the sequence of sentences (δ_n) defined in the following way $\delta_1 = \alpha_s$, $\delta_2 = \alpha_{s_1}$, $\delta_3 = \alpha_{s_2}$, ..., where the next terms of this sequence are the consecutive elements of B obtained using the process just described.

Note that $\|\neg\delta_j\| \not\subseteq \|\neg\delta_i\|$ whenever $j < i$.

Since B is finite there must be some $\delta_m \in B$ such that for all $\alpha_j \in B$, if $\|\neg\alpha_j\| \subseteq \|\neg\delta_m\|$, then $\|\neg\delta_m\| \subseteq \|\neg\alpha_j\|$, which means that $\delta_m \in B_N$.

And it follows from the construction of the sequence (δ_n) that $\|\neg\delta_m\| \subset \|\neg\alpha_l\|$, and this ends the proof. ∎

Lemma 6.2.6 *Let $\mathbf{K}$ be a belief set, B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ and $f : \{\|\alpha\| : \alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha\} \to \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ be a function such that if $\alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$.*

If $B_f \subseteq B$ is such that:

(i) If $\alpha_j \in B_f$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_f$;

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_f \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$;

then

1. *For every world $M \in \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ there is some subset $\mathcal{W}_M \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$, such that $M \in \mathcal{W}_M$ and $\mathcal{W}_M \in \mathbb{W}_{\mathbf{K}\perp B}$;*

2. $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} \neq \emptyset$ *and* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} = \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$.

Proof. Let $\mathbf{K}$, B, and f be a belief set, a finite set of sentences and a function, respectively, satisfying the conditions mentioned in the statement of the lemma. And assume that $B_f \subseteq B$ is such that conditions (i) and (ii) are satisfied. In what follows we show that 1. and 2. hold.

1. Case 1, $B \cap \mathbf{K} = \emptyset$. Then, since $B_f \subseteq B$, it holds that $B_f \cap \mathbf{K} = \emptyset$ and, consequently, $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} = \emptyset$. Hence 1. is vacuously true.

Case 2, $B \cap \mathbf{K} \neq \emptyset$. Let α_j be an arbitrary element of $B_f \cap \mathbf{K}$ and M be an arbitrary element of $f(\|\neg\alpha_j\|)$. We will now construct a set $\mathcal{W}_M$ that satisfies the above mentioned conditions.

First we prove that if $\alpha_l \in B \cap \mathbf{K} \cap M$ then $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} \cap \|\alpha_j\| \cap \|\neg\alpha_l\| \neq \emptyset$. In order to do that, let α_l be an arbitrary element of $B \cap \mathbf{K} \cap M$. Then from (ii), it follows that there is some $\alpha_m \in B_f \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$.

Furthermore, in these conditions, it holds that $f(\|\neg\alpha_m\|) \not\subseteq \|\neg\alpha_j\|$. To prove this, in what follows we will show first that (a) $f(\|\neg\alpha_m\|) \neq \|\neg\alpha_j\|$ and afterwards that (b) $f(\|\neg\alpha_m\|) \not\subset \|\neg\alpha_j\|$.

Regarding (a), to see that, in fact, $f(\|\neg\alpha_m\|) \neq \|\neg\alpha_j\|$ it is enough to notice that $M \notin f(\|\neg\alpha_m\|)$ and $M \in \|\neg\alpha_j\|$. Indeed, on the one hand, since $\alpha_l \in M$ it follows that $M \notin \|\neg\alpha_l\|$ and therefore, because $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$, we obtain that $M \notin f(\|\neg\alpha_m\|)$. On the other hand, from $M \in f(\|\neg\alpha_j\|)$ and $f(\|\neg\alpha_j\|) \subseteq \|\neg\alpha_j\|$ it follows that $M \in \|\neg\alpha_j\|$, and we are done.

Now we prove (b), i.e., that $f(\|\neg\alpha_m\|) \not\subset \|\neg\alpha_j\|$. First we notice that, since $M \in f(\|\neg\alpha_j\|)$, from $M \notin f(\|\neg\alpha_m\|)$ (already proven above) it follows that $f(\|\neg\alpha_m\|) \neq f(\|\neg\alpha_j\|)$. Then, observing that, because $\alpha_m \in B_f$, according to (i) it holds that $f(\|\neg\alpha_m\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_m\|) \not\subset \|\neg\alpha_j\|$, we can conclude that $f(\|\neg\alpha_m\|) \not\subset \|\neg\alpha_j\|$, as we wished to show.

Hence, from the above we can conclude that if $\alpha_l \in B \cap \mathbf{K} \cap M$ it holds that $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} \cap \|\alpha_j\| \cap \|\neg\alpha_l\| \neq \emptyset$.

Now we are in a position to show that there is a set $\mathcal{W}_M \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$, such that $M \in \mathcal{W}_M$ and $\mathcal{W}_M \in \mathbb{W}_{\mathbf{K}\perp B}$.

In order to see that, let $\mathcal{W}$ be the set defined by $\mathcal{W} = \{M\} \cup \{M_{\alpha_l} : \alpha_l \in B \cap \mathbf{K} \cap M\}$, where for each $\alpha_l \in B \cap \mathbf{K} \cap M$, M_{α_l} is an arbitrary element of $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} \cap \|\alpha_j\| \cap \|\neg\alpha_l\|$. Since $\mathcal{W} \subseteq \mathcal{M}_{\mathcal{L}}$ is such that $\mathcal{W} \cap \|\neg\alpha_k\| \neq \emptyset$ for all $\alpha_k \in B \cap \mathbf{K}$, it follows immediately from Lemma 6.1.3 that there is some set $\mathcal{W}'$ such that $\mathcal{W}' \subseteq \mathcal{W}$ and $\mathcal{W}' \in \mathbb{W}_{\mathbf{K}\perp B}$.

Furthermore, $\mathcal{W}' \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ and $M \in \mathcal{W}'$. Indeed, that $\mathcal{W}' \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ follows immediately from $\mathcal{W}' \subseteq \mathcal{W}$ and $\mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$. On the other hand, from $\mathcal{W}' \subseteq \mathcal{W}$ and the fact that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$, recalling that $\mathcal{W}' \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$ (according to Definition 6.1.2), we can conclude that $M \in \mathcal{W}'$.

Hence, if we define $\mathcal{W}_M = \mathcal{W}'$ then $\mathcal{W}_M \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$, $M \in \mathcal{W}_M$ and $\mathcal{W}_M \in \mathbb{W}_{\mathbf{K}\perp B}$, as required.

Thus, we have just shown that, for every world $M \in \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ there a set $\mathcal{W}_M \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$ such that $M \in \mathcal{W}_M$ and $\mathcal{W}_M \in \mathbb{W}_{\mathbf{K}\perp B}$.

2. Case 1, $B \cap \mathbf{K} = \emptyset$. Then, according to Observation 6.1.4-2., $\mathbb{W}_{\mathbf{K}\perp B} = \{\emptyset\}$. Hence, $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} = \{\emptyset\} \neq \emptyset$.

On the other hand, since $B_f \subseteq B$, it holds that $B_f \cap \mathbf{K} = \emptyset$ and, consequently, $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} = \bigcup\emptyset = \emptyset$. Therefore $\bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\} = \bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}(= \emptyset)$.

Case 2, $B \cap \mathbf{K} \neq \emptyset$. Then, from (ii) we can conclude that $B_f \cap \mathbf{K} \neq \emptyset$ and it follows immediately from 1. (already proven above) that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} \neq \emptyset$. To finish this proof it is enough to observe that it follows immediately from 1. that $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}\} = \bigcup\{f(\|\neg\alpha_i\|) : \alpha_i \in B_f \cap \mathbf{K}\}$. ■

Observation 6.2.7 *Let* $\mathbf{K}$ *be a belief set,* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *Then* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$.

Proof. Let $\mathbf{K}$ be a belief set and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. We must show that $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$.

As will see right away, such identity is an immediate consequence of Lemma 6.2.6.

Consider the function $f : \{\|\alpha\| : \alpha \in \mathcal{L} \text{ and } \not\vdash \neg\alpha\} \to \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ defined by $f(\|\alpha\|) = \|\alpha\|$, and let $B_f = B_N$ (where B_N is the normalization of B).

It obviously holds that if $\alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$ and that $B_f \subseteq B$. Now we prove that conditions (i) and (ii) of Lemma 6.2.6 are satisfied, i.e., that:

(i) If $\alpha_j \in B_N$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subseteq \|\neg\alpha_j\|$, for all $\alpha_k \in B_N$.

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_N \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$.

Indeed, condition (i) follows immediately from Definition 6.2.4. Now we show that condition (ii) also holds. Let α_l be an arbitrary element of $B \cap \mathbf{K}$, we must show that there is some $\alpha_m \in B_N \cap \mathbf{K}$ such that $\|\neg\alpha_m\| \subseteq \|\neg\alpha_l\|$ (since $f(\|\neg\alpha_m\|) = \|\neg\alpha_m\|$). To prove this we consider two cases:

Case 1, $\alpha_l \in B_N$. Then $\alpha_l \in B_N \cap \mathbf{K}$ and $\|\neg\alpha_l\| \subseteq \|\neg\alpha_l\|$, and we are done.

Case 2, $\alpha_l \notin B_N$. Then, it follows from Lemma 6.2.5 that there is some $\alpha_m \in B_N$ such that $\|\neg\alpha_m\| \subset \|\neg\alpha_l\|$. On the other hand, from $\alpha_l \in \mathbf{K}$ and $\|\neg\alpha_m\| \subset \|\neg\alpha_l\|$ we can conclude that $\alpha_m \in \mathbf{K}$. Hence, we have that $\alpha_m \in B_N \cap \mathbf{K}$ and $\|\neg\alpha_m\| \subseteq \|\neg\alpha_l\|$, as required.

Hence, we can conclude from Lemma 6.2.6 that $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$, and this finishes the proof. ■

Observation 6.2.8 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then*

$$\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}.$$

Proof. Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. We must show that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$.

First of all we show that the above identity is trivially satisfied whenever $B \cap Cn(\emptyset) \neq \emptyset$ or $B \cap \mathbf{K} = \emptyset$.

In order to do that we start by recalling that, according to Observation 6.1.4-1. and 2., in either of those cases it holds that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \emptyset$. Moreover, if $B \cap \mathbf{K} = \emptyset$ or $B \cap Cn(\emptyset) \neq \emptyset$, then it also holds that $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\} = \emptyset$. Indeed, on the one hand if $B \cap \mathbf{K} = \emptyset$ then, since $B_N \subseteq B$, we have that $B_N \cap \mathbf{K} = \emptyset$ and it follows immediately that $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\} = \emptyset$. On the other hand if $B \cap Cn(\emptyset) \neq \emptyset$ we obtain, according to Definition 6.2.4, that $B_N = B \cap Cn(\emptyset) \subseteq Cn(\emptyset)$, and therefore, since $Cn(\emptyset) \subseteq \mathbf{K}$, it follows that $B_N \cap \mathbf{K} = B_N \subseteq Cn(\emptyset)$. Hence, for all $\alpha_i \in B_N \cap \mathbf{K}$, $\|\neg\alpha_i\| = \emptyset$, and from this we can conclude that $\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\} = \emptyset$.

Hence, from the above we can conclude that, if $B \cap Cn(\emptyset) \neq \emptyset$ or $B \cap \mathbf{K} = \emptyset$ then $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\} = \emptyset$.

Having seen this, in what remains of this proof we assume B is a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ and $B \cap \mathbf{K} \neq \emptyset$, and we prove the equality of sets stated in the observation by double inclusion.

($\subseteq$) Let $M \in \bigcup \mathbb{W}_{\mathbf{K}\perp B}$. Then there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ such that $M \in \mathcal{W}$. So, according to Definition 6.1.2, it holds that:

(i) $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$

(ii) There is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$.

Now we consider two cases:

Case 1, $\alpha_j \in B_N$. Then, $\alpha_j \in B_N \cap \mathbf{K}$ and $M \in \|\neg\alpha_j\|$. Therefore we can conclude that $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$, and we are done.

Case 2, $\alpha_j \notin B_N$. Then, it follows from Lemma 6.2.5 that there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_j\|$. So, from (i) and (ii) we can conclude that $M \in \|\neg\alpha_k\|$. Furthermore, from $\alpha_j \in \mathbf{K}$ and $\|\neg\alpha_k\| \subset \|\neg\alpha_j\|$ we can conclude that $\alpha_k \in \mathbf{K}$ (since from $\|\neg\alpha_j\| \cap \|\mathbf{K}\| = \emptyset$ and $\|\neg\alpha_k\| \subset \|\neg\alpha_j\|$ it follows that $\|\neg\alpha_k\| \cap \|\mathbf{K}\| = \emptyset$). Hence, $M \in \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$, as required.

($\supseteq$) This inclusion is an immediate consequence of Observation 6.2.7, and we are done. ∎

Corollary 6.2.9 *Let* $\mathbf{K}$ *be a belief set and* $\dot{\sim}$ *be the operator of full meet multiple contraction on* $\mathbf{K}$*. Then, under the assumption that* $\bigcap \emptyset = \mathcal{L}$,[109] *it holds that*

$$\begin{aligned} \mathbf{K} \dot{\sim} B &= \bigcap \left(\|\mathbf{K}\| \cup \left(\bigcup \{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\} \right) \right) \\ &= \mathbf{K} \cap \left(\bigcap \{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\} \right), \end{aligned}$$

for all finite sets of sentences B.[110]

Proof. Let $\mathbf{K}$ be a belief set and $\dot{\sim}$ be the operator of full meet multiple contraction on $\mathbf{K}$.

That $\mathbf{K} \tilde{-} B = \bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$ follows immediately from the facts that $\mathbf{K} \tilde{-} B = \bigcap (\|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K}\perp B}))$ (according to Observation 6.2.1) and $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}$ (according to Observation 6.2.8).

It remains to show that the identity $\bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})) = \mathbf{K} \cap (\bigcap\{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\})$ also holds. To do that we start by noticing that $\bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})) = (\bigcap \|\mathbf{K}\|) \cap (\bigcap (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$ $= \mathbf{K} \cap (\bigcap (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$.

Hence, observing that $\bigcap (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}) = \bigcap \{(\bigcap \|\neg\alpha_i\|) : \alpha_i \in B_N \cap \mathbf{K}\} = \bigcap\{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\}$, we can conclude that $\bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})) = \mathbf{K} \cap (\bigcap\{Cn(\neg\alpha_i) : \alpha_i \in B_N \cap \mathbf{K}\})$ (under the assumption that $\bigcap \emptyset = \mathcal{L}$), as we wished to show. ■

Observation 6.2.10 *Let* $\mathbf{K}$ *be a belief set and* $\tilde{-}$ *be the operator of full meet multiple contraction on* $\mathbf{K}$. *Then*

$$\|\mathbf{K} \tilde{-} B\| = \|\mathbf{K}\| \cup \left(\bigcup \mathbb{W}_{\mathbf{K}\perp B}\right) = \|\mathbf{K}\| \cup \left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\right),$$

for all finite sets of sentences B.

Proof. Assume the conditions in the statement of the observation hold.

We start by noticing that the equality of sets $\|\mathbf{K}\| \cup (\bigcup \mathbb{W}_{\mathbf{K}\perp B}) = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$ follows immediately from Observation 6.2.8.

On the other hand, according to Corollary 6.2.9, it holds that $\mathbf{K} \tilde{-} B = \bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}))$. Therefore it follows that $\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}) \subseteq \|\mathbf{K} \tilde{-} B\|$.

Hence it only remains to prove that $\|\mathbf{K} \tilde{-} B\| \subseteq \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$. To show that this indeed holds we consider two cases.

Case 1, $B \cap \mathbf{K} = \emptyset$. Then, according to Definition 5.3.3, it holds that $\mathbf{K} \tilde{-} B = \mathbf{K}$. Therefore $\|\mathbf{K} \tilde{-} B\| = \|\mathbf{K}\|$ and it follows immediately that $\|\mathbf{K} \tilde{-} B\| \subseteq \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$.

Case 2, $B \cap \mathbf{K} \neq \emptyset$. We start by observing that, in this case it holds that $B_N \cap \mathbf{K} \neq \emptyset$. Indeed, since $B \cap \mathbf{K} \neq \emptyset$ there is some $\alpha_i \in B \cap \mathbf{K}$. Now, if $\alpha_i \in B_N$ then $\alpha_i \in B_N \cap \mathbf{K}$ and we can immediately conclude that $B_N \cap \mathbf{K} \neq \emptyset$. Otherwise, if $\alpha_i \in B_N$ then it follows from Lemma 6.2.5 that there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_i\|$. On the other hand, from $\alpha_i \in \mathbf{K}$ and $\|\neg\alpha_k\| \subset \|\neg\alpha_i\|$ we can conclude that $\alpha_k \in \mathbf{K}$. Hence, $\alpha_k \in B_N \cap \mathbf{K}$ and consequently $B_N \cap \mathbf{K} \neq \emptyset$.

Next we prove that $\|\mathbf{K} \tilde{-} B\| \subseteq \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$.

Let $M \in \|\mathbf{K} \tilde{-} B\|$ and assume by *reductio* that $M \notin \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$. Then there is some $\delta \in \mathbf{K}$ such that $\neg\delta \wedge \bigwedge(B_N \cap \mathbf{K}) \in M$ (where $\bigwedge(B_N \cap \mathbf{K})$ denotes the conjunction of all elements of $B_N \cap \mathbf{K}$[147]) and therefore, because M is consistent, $\delta \vee \neg \bigwedge(B_N \cap \mathbf{K}) \notin M$.

On the other hand, since $\delta \vee \neg \bigwedge(B_N \cap \mathbf{K}) \in \bigcap (\|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})) = \mathbf{K} \tilde{-} B$, it follows that $M \notin \|\mathbf{K} \tilde{-} B\|$, which contradicts our hypothesis.

Hence we can conclude that $M \in \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$, and this finishes the proof. ■

[147] Notice that since B is finite and $B_N \subseteq B$, it follows that $B_N \cap \mathbf{K}$ is finite.

Observation 6.3.1 *Let* $\mathbf{K}$ *be a belief set. If an operation* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *then*

$$\|\mathbf{K}\div B\| \subseteq \|\mathbf{K}\| \cup \left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\right),$$

for all finite sets of sentences B.

Proof. Let $\mathbf{K}$ be a belief set, B be any finite set of sentences and $\div$ be a partial meet multiple contraction on $\mathbf{K}$.

We must show that $\|\mathbf{K}\div B\| \subseteq \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$.

First we notice that it follows immediately from Definitions 5.3.2 and 5.3.3 that $\mathbf{K}\tilde{\div}B \subseteq \mathbf{K}\div B$, where $\tilde{\div}$ is the operator of full meet multiple contraction on $\mathbf{K}$. On the other hand, since, according to Observation 3.1.4, every remainder is a belief set, having in mind the above mentioned definitions and the fact that the intersection of belief sets is a belief set, it follows that $\mathbf{K}\tilde{\div}B$ and $\mathbf{K}\div B$ are belief sets. Hence, by Observation 3.3.3-(e), we can conclude that $\|\mathbf{K}\div B\| \subseteq \|\mathbf{K}\tilde{\div}B\|$.

Finally, since $\|\mathbf{K}\tilde{\div}B\| = \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$ (according to Observation 6.2.10), we can conclude that $\|\mathbf{K}\div B\| \subseteq \|\mathbf{K}\| \cup (\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\})$, as we wished to show. ■

Lemma 6.3.2 *Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then* $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})}$.

Proof. Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. We will show that $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})}$.

($\subseteq$) Let $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$. We must show that $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})}$. Hence, according to Definition 6.1.2, we must prove that:

(i) $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B_N \cap \mathbf{K}$.

(ii) If $M \in \mathcal{W}$ then there is some $\alpha_j \in B_N \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$.

Condition (i) follows immediately from $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$ and Definition 6.1.2. Now let M be an arbitrary element of $\mathcal{W}$. To show that (ii) holds it we must prove that there is some $\alpha_j \in B_N \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$. Since $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$, we have that there is some $\alpha_i \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_i\| = \{M\}$. If $\alpha_i \in B_N$ we are done. Now we assume that $\alpha_i \notin B_N$. Then, it follows from Lemma 6.2.5 that there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_i\|$. Hence, since $\mathcal{W} \cap \|\neg\alpha_k\| \neq \emptyset$ (according to Definition 6.1.2), from $\mathcal{W} \cap \|\neg\alpha_i\| = \{M\}$ we can conclude that $\mathcal{W} \cap \|\neg\alpha_k\| = \{M\}$.

($\supseteq$) Let $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})}$. Then conditions (i) and (ii) above are satisfied. Next, in order to show that $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B}$, we prove that:

1. $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.

2. If $M \in \mathcal{W}$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$.

Condition 1. follows immediately from (i) and Lemma 6.2.5. On the other hand condition 2. is an obvious consequence of (ii) and the fact that $B_N \cap \mathbf{K} \subseteq B \cap \mathbf{K}$. ■

Lemma 6.3.3 *Let* $\mathbf{K}$ *be a belief set and* G *and* H *be two finite sets of sentences. If every subset* X *of* $\mathbf{K}$ *implies some element of* G *if and only if* X *implies some element of* H*, then for all* $\beta_i \in G_N \cap \mathbf{K}$ *there is some* $\varsigma_i \in H_N \cap \mathbf{K}$ *such that* $\|\beta_i\| = \|\varsigma_i\|$.

Proof. Let $\mathbf{K}$ be a belief set and G and H be two finite sets of sentences. Assume that every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H and let $\beta_i \in G_N \cap \mathbf{K}$. We only need to show that there is an element of $H_N \cap \mathbf{K}$ which is logically equivalent to β_i. Since $\{\beta_i\} \subseteq \mathbf{K}$ and $\{\beta_i\} \vdash \beta_i \in G$, it follows from the hypothesis that there is some $\varsigma_i \in H$ such that $\{\beta_i\} \vdash \varsigma_i$. Then, by deduction we have that $\vdash \beta_i \rightarrow \varsigma_i$ which is equivalent to $\|\neg\varsigma_i\| \subseteq \|\neg\beta_i\|$. Since $\mathbf{K}$ is a belief set, we can conclude that $\varsigma_i \in \mathbf{K}$, hence, proceeding analogously, we can conclude that there is some $\beta_j \in G$ such that $\|\neg\beta_j\| \subseteq \|\neg\varsigma_i\|$. Then we have that $\|\neg\beta_j\| \subseteq \|\neg\beta_i\|$, and from the fact that $\beta_i \in G_N$ we can conclude that $\|\neg\beta_j\| = \|\neg\beta_i\|$. So, it holds that $\|\neg\beta_i\| = \|\neg\varsigma_i\|$.

It remains to prove that $\varsigma_i \in H_N \cap \mathbf{K}$. We already have that $\varsigma_i \in H \cap \mathbf{K}$. To show that $\varsigma_i \in H_N$ we must prove that for all $\varsigma_j \in H$ it holds that if $\|\neg\varsigma_j\| \subseteq \|\neg\varsigma_i\|$ then $\|\neg\varsigma_i\| \subseteq \|\neg\varsigma_j\|$. So, let $\varsigma_j \in H$ be such that $\|\neg\varsigma_j\| \subseteq \|\neg\varsigma_i\|$. Since $\|\neg\beta_i\| = \|\neg\varsigma_i\|$, it follows that $\|\neg\varsigma_j\| \subseteq \|\neg\beta_i\|$. Reasoning as above we can conclude that there is some $\beta_l \in G$ such that $\|\neg\beta_l\| \subseteq \|\neg\varsigma_j\|$. Then we have that $\|\neg\beta_l\| \subseteq \|\neg\beta_i\|$, and again from the fact that $\beta_i \in G_N$ we obtain that $\|\neg\beta_l\| = \|\neg\beta_i\|$, which allows us to conclude that $\|\neg\varsigma_i\| \subseteq \|\neg\varsigma_j\|$ as required. ■

Observation 6.3.4 *Let* $\mathbf{K}$ *be a belief set and* B *and* C *be two finite sets of sentences. Then:*

$$\mathbf{K}\perp B = \mathbf{K}\perp C \text{ iff } \mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}.$$

Proof. Let $\mathbf{K}$ be a belief set and B and C be two finite sets of sentences. We must show that $\mathbf{K}\perp B = \mathbf{K}\perp C$ iff $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$.

Left-to-right: Let $\mathbf{K}\perp B = \mathbf{K}\perp C$. We must show that $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$. But since, according to Lemma 6.3.2, this last equality is equivalent to $\mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})} = \mathbb{W}_{\mathbf{K}\perp(C_N\cap\mathbf{K})}$, it is enough to show the latter of these identities.

Since $\mathbf{K}\perp B = \mathbf{K}\perp C$, it holds that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C (according to Observation 3.1.3). Therefore, it follows from Lemma 6.3.3 and Definition 6.1.2 that $\mathbb{W}_{\mathbf{K}\perp(B_N\cap\mathbf{K})} = \mathbb{W}_{\mathbf{K}\perp(C_N\cap\mathbf{K})}$, and we are done.

Right-to-left: Let $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$. It follows immediately from Corollary 6.1.8 that $\mathbf{K}\perp B = \mathbf{K}\perp C$. ■

Theorem 6.3.6 *Let* $\mathbf{K}$ *be a belief set. An operation* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *by finite sets if and only if there is a propositional package selection function* f *for* $\|\mathbf{K}\|$ *such that:*

$$\mathbf{K}\div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})\right)\right),$$

for any finite set of sentences B.

Proof. Let $\mathbf{K}$ be a belief set.

Right-to-left We start by showing that if $\div$ is an operation such that, for all finite sets B, $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, where f is a propositional package selection function for $\|\mathbf{K}\|$, then $\div$ is a partial meet multiple contraction on $\mathbf{K}$ by finite sets.

So, let $\div$ be a contraction operation on $\mathbf{K}$ in the mentioned conditions. According to Definition 5.3.2, in order to show that $\div$ is a partial meet multiple contraction by finite sets we must show that there is some package selection function γ for $\mathbf{K}$, such that for all finite sets B, $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$.

Let γ be the operator defined, by $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ if $\mathbf{K}\perp B = \emptyset$, and $\gamma(\mathbf{K}\perp B) = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})\}$ if $\mathbf{K}\perp B \neq \emptyset$. In what follows we will show that:

(a) γ is a package selection function for $\mathbf{K}$, and

(b) $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, for all finite sets B.

In order to prove (a) we start by observing that γ is a (well-defined) function. Indeed, it follows immediately from Observation 6.3.4 and the definition of γ above that if $\mathbf{K}\perp C = \mathbf{K}\perp B$ then $\gamma(\mathbf{K}\perp C) = \gamma(\mathbf{K}\perp B)$, therefore we can conclude that γ is a function. Now we must prove that γ is a package selection function for $\mathbf{K}$.

Since, by the definition of γ, if $\mathbf{K}\perp B = \emptyset$ then $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$, it follows from Definition 5.3.1 that, in order to prove that (a) holds, it only remains to show that if $\mathbf{K}\perp B \neq \emptyset$ then $\emptyset \neq \gamma(\mathbf{K}\perp B) \subseteq \mathbf{K}\perp B$. Hence, let B be such that $\mathbf{K}\perp B \neq \emptyset$. We start by noticing that since $\mathbf{K}\perp B \neq \emptyset$, it follows from Observation 3.1.2 that $B \cap Cn(\emptyset) = \emptyset$. So, by Observation 6.1.4-1., we obtain that $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ and, therefore, according to Definition 6.3.5 it holds that $\emptyset \neq f(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$. On the other hand, by Theorem 6.1.6-1., it follows from $f(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$ that $Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\perp B$, for all $\mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})$. Hence we can conclude that $\gamma(\mathbf{K}\perp B) = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})\}$ is a non-empty subset of $\mathbf{K}\perp B$.

Therefore, γ is a package selection function for $\mathbf{K}$.

It is now time to prove that (b) also holds. Hence, let B be an arbitrary finite set. We will show that $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$. In order to that we consider two cases.

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, from Observation 6.1.4-1. we can conclude that $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$. Therefore, according to Definition 6.3.5 we have that $f(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$. Hence, it follows that $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B}))) = Th\,(\|\mathbf{K}\| \cup (\bigcup \emptyset)) = Th(\|\mathbf{K}\|)$. Then, since according to Observation 3.3.3-(a) $Th(\|\mathbf{K}\|) = \mathbf{K}$, we can conclude that, if $B \cap Cn(\emptyset) \neq \emptyset$ then $\mathbf{K}\div B = \mathbf{K}$. On the other hand, having in mind Observation 3.1.2, according to the definition of the function γ proposed above, we have that if $B \cap Cn(\emptyset) \neq \emptyset$ then $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$. Therefore, in this case, $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$.

Case 2, $B \cap Cn(\emptyset) = \emptyset$. Then, it follows from Observation 3.1.2 that $\mathbf{K}\perp B \neq \emptyset$, and, therefore, according to the definition of the function γ proposed above it holds that $\gamma(\mathbf{K}\perp B) = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})\}$.

On the other hand, we have that $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B}))) = \bigcap\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B}))) = \bigcap \{\bigcap\,(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})\} = \bigcap \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathbb{W}_{\mathbf{K}\perp B})\}$.

Hence, we can conclude that in this case it also holds that $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, as we wished to show.

Finally from (a) and (b) we can conclude that $\div$ is a partial meet multiple contraction on $\mathbf{K}$ by finite sets and this finishes the first part of the present proof.

Left-to-right Now we show that if $\div$ is a partial meet multiple contraction on $\mathbf{K}$ by finite sets, then there exists a propositional package selection function f for $\|\mathbf{K}\|$ such that $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, for every finite set B.

So, let $\div$ be a partial meet multiple contraction on $\mathbf{K}$ by finite sets. First of all we notice that it follows from Definition 5.3.2 that there is some package selection function γ for $\mathbf{K}$, such that for all finite sets B, $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$. Moreover, it follows from Definition 5.3.1 and Observation 3.1.2 that if $B \cap Cn(\emptyset) \neq \emptyset$ then $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$, and if $B \cap Cn(\emptyset) = \emptyset$ then $\emptyset \neq \gamma(\mathbf{K}\perp B) \subseteq \mathbf{K}\perp B$.

Having this in mind, let f be defined in the following way: $f(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$ if $B \cap Cn(\emptyset) \neq \emptyset$ and $f(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W}_X \in \mathbb{W}_{\mathbf{K}\perp B} : X = Th(\|\mathbf{K}\| \cup \mathcal{W}_X) \in \gamma(\mathbf{K}\perp B)\}$ if $B \cap Cn(\emptyset) = \emptyset$. We will now show that:

(i) f is a propositional package selection function for $\|\mathbf{K}\|$.

(ii) $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, for all finite sets B.

In order to prove (i) we start by observing that, since when $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$, it follows from Observation 6.3.4 and the definition of f above that $f(\mathbb{W}_{\mathbf{K}\perp B}) = f(\mathbb{W}_{\mathbf{K}\perp C})$, we can conclude that f is a (well-defined) function. Now, to show that f is a propositional package selection function for $\|\mathbf{K}\|$, we must prove that f satisfies conditions 1. and 2. of Definition 6.3.5. But 1., i.e. $f(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$, follows immediately from the definition of f. Now we show that 2. is also satisfied, i.e. that if $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ then $f(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$. Assume that $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$. Then it follows from Observation 6.1.4-1. that $B \cap Cn(\emptyset) = \emptyset$ and therefore, as we have already seen above we have that, on the one hand $f(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W}_X \in \mathbb{W}_{\mathbf{K}\perp B} : X = Th(\|\mathbf{K}\| \cup \mathcal{W}_X) \in \gamma(\mathbf{K}\perp B)\}$, and on the other hand $\gamma(\mathbf{K}\perp B) \neq \emptyset$. Hence, by Theorem, 6.1.6-2., we can conclude that $f(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

Therefore f is a propositional package selection function for $\|\mathbf{K}\|$.

Having seen this, it only remains to show that (ii) also holds. Let B be an arbitrary finite set. Since $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, we will prove that (ii) is satisfied by showing that $\bigcap \gamma(\mathbf{K}\perp B) = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$. In order to that we consider two cases.

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, as we have already seen above, $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ and, consequently, $\bigcap \gamma(\mathbf{K}\perp B) = \bigcap\{\mathbf{K}\} = \mathbf{K}$. On the other hand, from Observation 6.1.4-1. we can conclude that $\mathbb{W}_{\mathbf{K}\perp B} = \emptyset$. Therefore, since f is a propositional package selection function for $\|\mathbf{K}\|$ (already proven above), according to Definition 6.3.5, it holds that $f(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$. Hence, it follows that $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B}))) = Th\,(\|\mathbf{K}\| \cup (\bigcup \emptyset)) = Th(\|\mathbf{K}\|) = \mathbf{K}$ (where this last equality is justified by Observation 3.3.3-(a)). Therefore, in this case ($B \cap Cn(\emptyset) \neq \emptyset$) it holds that $\mathbf{K}\div B = Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$.

Case 2, $B \cap Cn(\emptyset) = \emptyset$. Then, it follows from Theorem 6.1.6-2. and from the definition of f that $\{X : X \in \gamma(\mathbf{K}\perp B)\} = \{Th(\|\mathbf{K}\| \cup \mathcal{W}_X) : \mathcal{W}_X \in f(\mathbb{W}_{\mathbf{K}\perp B})\}$. Therefore, we obtain that $\bigcap \gamma(\mathbf{K}\perp B) = \bigcap\{X : X \in \gamma(\mathbf{K}\perp B)\} = \bigcap\{Th(\|\mathbf{K}\| \cup \mathcal{W}_X) : \mathcal{W}_X \in f(\mathbb{W}_{\mathbf{K}\perp B})\} = \bigcap\{\bigcap(\|\mathbf{K}\| \cup \mathcal{W}_X) : \mathcal{W}_X \in f(\mathbb{W}_{\mathbf{K}\perp B})\} = \bigcap\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$ $= Th\,(\|\mathbf{K}\| \cup (\bigcup f(\mathbb{W}_{\mathbf{K}\perp B})))$, and this finishes the proof.

■

Appendix D

Proofs of Chapter 7

In this chapter we present proofs for most of the results presented throughout Chapter 7. For commodity of the reader before each of those proofs we include the statement of the result that it refers to.

Observation 7.2.3 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \setminus Cn(\emptyset) \neq \emptyset$. *Furthermore, let* $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ *be the subsets of* B *constructed as we described in the above definition. Then:*

(a) For all $l \in \{1, \ldots, m\}$ *it holds that* $C''_l = \{\alpha_i \in C'_l : \forall \alpha_j \in C'_l \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

(b) If $m > 1$ *then for all* $l \in \{2, \ldots, m\}$ *the following identity is satisfied:* $C'_l = \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1} \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

Proof. (a) follows immediately from the fact that, for all $l \in \{1, \ldots, m\}$, given $\alpha_i, \alpha_j \in C'_l$, it holds that if $f_{\mathbb{S}}(\neg\alpha_i) \not\subseteq f_{\mathbb{S}}(\neg\alpha_j)$ then $f_{\mathbb{S}}(\neg\alpha_i) \not\subset \|\neg\alpha_j\|$ (because $\mathbb{S}_{\neg\alpha_i} = \mathbb{S}_{\neg\alpha_j}$).

On the other hand, (b) is an immediate consequence of the fact that, if $m > 1$ and $l \in \{2, \ldots, m\}$ then from $\alpha_i \in C_l$ and $\alpha_j \in B_{l-1}$ it follows that $\mathbb{S}_{\neg\alpha_i} \subset \mathbb{S}_{\neg\alpha_j}$ and, consequently, $f_{\mathbb{S}}(\neg\alpha_j) \neq f_{\mathbb{S}}(\neg\alpha_i)$. ∎

Observation 7.2.4 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences. If* $\alpha_i \in B_{\mathbb{S}}$, *then* $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ *or* $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$, *for all* $\alpha_t \in B_{\mathbb{S}}$.

Proof. Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences.

We must show that if $\alpha_i \in B_{\mathbb{S}}$, then $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$, for all $\alpha_t \in B_{\mathbb{S}}$.

Case 1, $B \subseteq Cn(\emptyset)$. Then, according to Definition 7.2.2, $B_{\mathbb{S}} = \emptyset$ and it follows that it is vacuously true that if $\alpha_i \in B_{\mathbb{S}}$, then $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$, for all $\alpha_t \in B_{\mathbb{S}}$.

Case 2, $B \backslash Cn(\emptyset) \neq \emptyset$. Then, let $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\mathbb{S}}$ described in Definition 7.2.2.

Now we consider an arbitrary element α_i of $B_{\mathbb{S}}$ and show that for all $\alpha_t \in B_{\mathbb{S}}$ it holds that $f_{\mathbb{S}}(\neg\alpha_t) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_t) \not\subset \|\neg\alpha_i\|$.

In order to see that, assume by *reductio* that there is some $\alpha_t \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_t) \neq f_{\mathbb{S}}(\neg\alpha_i)$ and $f_{\mathbb{S}}(\neg\alpha_t) \subset \|\neg\alpha_i\|$. From the latter condition and Definition 3.3.5 it follows that $\mathbb{S}_{\neg\alpha_i} \subseteq \mathbb{S}_{\neg\alpha_t}$.

Case 2.1, $\mathbb{S}_{\neg\alpha_i} = \mathbb{S}_{\neg\alpha_t}$. Then, there is some class C_l such that $\alpha_i, \alpha_t \in C_l$. Since $\alpha_i, \alpha_t \in B_{\mathbb{S}}$ we must have $\alpha_i, \alpha_t \in C'_l$. But, in that case, from $f_{\mathbb{S}}(\neg\alpha_t) \neq f_{\mathbb{S}}(\neg\alpha_i)$ and $f_{\mathbb{S}}(\neg\alpha_t) \subset \|\neg\alpha_i\|$ we conclude that $\alpha_i \notin C''_l$ (cf. Observation 7.2.3−(a)), which contradicts $\alpha_i \in B_{\mathbb{S}}$.

Case 2.2, $\mathbb{S}_{\neg\alpha_i} \subset \mathbb{S}_{\neg\alpha_t}$. Then, there are two classes C_l and C_m, with $l < m$ such that $\alpha_t \in C_l$ and $\alpha_i \in C_m$. Since $\alpha_t \in B_{\mathbb{S}}$ we have that $\alpha_t \in C''_l$. Hence, from $C''_l \subseteq B_l \subseteq B_{m-1}$ we can conclude that $\alpha_t \in B_{m-1}$. But then, from $f_{\mathbb{S}}(\neg\alpha_t) \neq f_{\mathbb{S}}(\neg\alpha_i)$ and $f_{\mathbb{S}}(\neg\alpha_t) \subset \|\neg\alpha_i\|$ it follows that $\alpha_i \notin C'_m$ (cf. Observation 7.2.3−(b)), which contradicts $\alpha_i \in B_{\mathbb{S}}$. ∎

Lemma 7.2.5 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *Assume* $C = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L} \setminus Cn(\emptyset)$, *with* $n \geq 1$ *is a non-empty finite set of sentences such that* $\mathbb{S}_{\neg\alpha_1} = \cdots = \mathbb{S}_{\neg\alpha_n}$, *and, for each* $\alpha_j \in C$ *let* C_{α_j} *be the set defined by* $C_{\alpha_j} = \{\alpha_k \in C : f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_j)\}$.

If $\alpha_j \in C$ *and* $C_{\alpha_j} \neq \emptyset$ *then there is some* $\alpha_l \in C_{\alpha_j}$ *such that* $C_{\alpha_l} = \emptyset$.

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, C and C_{α_j}, for each $\alpha_j \in C$, are as mentioned in the statement of the lemma.

Let $\alpha_j \in C$ be such that $C_{\alpha_j} \neq \emptyset$. We must show that there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$.

First notice that it follows immediately from the definition of C_{α_i} for each $\alpha_i \in C$ that for any $\alpha_r \in C$, if $\alpha_r \in C_{\alpha_j}$ then $C_{\alpha_r} \subset C_{\alpha_j}$.

In what follows, given a finite set S, $\#S$ denotes the number of elements of S.

Now, in order to prove that there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$ we start by noticing that, since $\emptyset \neq C_{\alpha_j} \subset C$ (note that $\alpha_j \in C \setminus C_{\alpha_j}$), we have that $1 \leq \#C_{\alpha_j} < n$. Now we proceed according to the following (finite) sequence of steps:

Step 1: Pick some $\alpha_{j1} \in C_{\alpha_j}$. As we have seen above we have that $C_{\alpha_{j1}} \subset C_{\alpha_j}$, hence $0 \leq \#C_{\alpha_{j1}} < n - 1$.

Step 2: If $\#C_{\alpha_{j1}} = 0$ we have that $C_{\alpha_{j1}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j1}$ and this finishes the proof. Otherwise, pick some $\alpha_{j2} \in C_{\alpha_{j1}}$. Then $C_{\alpha_{j2}} \subset C_{\alpha_{j1}}$, hence $0 \leq \#C_{\alpha_{j2}} < n - 2$.

$\vdots$

Step i: If $\#C_{\alpha_{j(i-1)}} = 0$ we have that $C_{\alpha_{j(i-1)}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j(i-1)}$ and this finishes the proof. Otherwise, pick some $\alpha_{ji} \in C_{\alpha_{j(i-1)}}$. Then $C_{\alpha_{ji}} \subset C_{\alpha_{j(i-1)}}$, hence $0 \leq \#C_{\alpha_{ji}} < n - i$.

$\vdots$

Suppose after $n-2$ steps the above described process has not finished yet. Furthermore, assume that $\#C_{\alpha_{j(n-2)}} \neq 0$. Since, by construction, $0 \leq \#C_{\alpha_{j(n-2)}} < n-(n-2) = 2$, we can conclude that $C_{\alpha_{j(n-2)}}$ is a singleton set. Then, at step $n-1$ we pick the only element of $C_{\alpha_{j(n-2)}}$, which we denote by $\alpha_{j(n-1)}$, and from the fact that $C_{\alpha_{j(n-1)}} \subset C_{\alpha_{j(n-2)}}$ it must be the case that $C_{\alpha_{j(n-1)}} = \emptyset$.

Hence, after at most n steps the process must have finished. That is, at most at step n we must find some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$ as we wished to prove. ∎

Observation 7.2.6 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *If* $\alpha_j \in B \setminus B_{\mathbb{S}}$, *then there is some* $\alpha_l \in B_{\mathbb{S}}$ *such that* $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ *and* $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$.

Proof. Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$.

We must show that if $\alpha_j \in B \setminus B_{\mathbb{S}}$, then there is some $\alpha_l \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$.

We start by noticing that, if $B = \emptyset$ then the above condition is vacuously true.

Hence, in what remains of this proof we assume that $B \neq \emptyset$. So, it follows that $B \setminus Cn(\emptyset) \neq \emptyset$. Then, let $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m, B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\mathbb{S}}$ described in Definition 7.2.2.

Now we consider an arbitrary element α_j of $B \setminus B_{\mathbb{S}}$ and show that there is some $\alpha_l \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$.

We will consider separately the two possibilities $\alpha_j \in C_1$ or $\alpha_j \in C_n$, with $1 < n \leq m$.

Case 1, $\alpha_j \in C_1$. Then, since by construction $C_1 = C'_1$, it holds that $\alpha_j \in C'_1$. On the other hand, from $\alpha_j \notin B_{\mathbb{S}}$ it follows that $\alpha_j \notin C''_1$. Hence, there is some $\alpha_k \in C'_1$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_i)$.

Now we notice that, since B is finite, by construction we have that C'_1 is finite and for any $\alpha_r, \alpha_s \in C'_1$ it holds that $\mathbb{S}_{\neg\alpha_r} = \mathbb{S}_{\neg\alpha_s}$. Then, according to Lemma 7.2.5, there is some $\alpha_l \in C'_1$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset f_{\mathbb{S}}(\neg\alpha_j)$ and for all $\alpha_m \in C'_1$ it holds that $f_{\mathbb{S}}(\neg\alpha_m) \not\subset f_{\mathbb{S}}(\neg\alpha_l)$.

Taking the latter stated fact into account we can conclude, according to the definition of C''_1, that $\alpha_l \in C''_1$ and, consequently, $\alpha_l \in B_{\mathbb{S}}$. Finally we note that from $f_{\mathbb{S}}(\neg\alpha_l) \subset f_{\mathbb{S}}(\neg\alpha_j)$ it follows immediately that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$ as it was required.

Case 2, $\alpha_j \in C_n$, with $1 < n \leq m$. From $\alpha_j \notin B_{\mathbb{S}}$ it follows that $\alpha_j \notin C''_n$, and we have to consider the two possibilities $\alpha_j \notin C'_n$ or $\alpha_j \in C'_n$.

Case 2.1, $\alpha_j \notin C'_n$. Then there is some $\alpha_l \in B_{n-1}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$. Furthermore, since $B_{n-1} \subseteq B_{\mathbb{S}}$, it holds that $\alpha_l \in B_{\mathbb{S}}$.

On the other hand, it follows from the construction of B_{n-1} and C_n that $\mathbb{S}_{\neg\alpha_j} \subset \mathbb{S}_{\neg\alpha_l}$ and, therefore, according to Definition 3.3.5 it holds that $f_{\mathbb{S}}(\neg\alpha_l) \cap \mathbb{S}_{\neg\alpha_j} = \emptyset$. Hence, since $\emptyset \neq f_{\mathbb{S}}(\neg\alpha_j) \subseteq \mathbb{S}_{\neg\alpha_j}$, we can conclude that $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$, and this finishes the proof.

Case 2.2, $\alpha_j \in C'_n$. Then, since $\alpha_j \notin C''_n$, there is some $\alpha_k \in C'_n$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_j)$. Reasoning as we did in Case 1 above we can conclude that there is some $\alpha_l \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$ as we wished to prove. ∎

Observation 7.2.9 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *If* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based filtration of* B*, then:*

(i) If $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_i) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_i) \not\subseteq \|\neg\alpha_j\|$, *for all* $\alpha_i \in B_{\mathbb{S}}$*;*

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_k \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$.

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, C and B are as mentioned in the statement of the observation and let $B_{\mathbb{S}}$ be the $\mathbb{S}$-based filtration of B. We must prove that conditions (i) and (ii) above are satisfied.

We start by noticing that condition (i) is an immediate consequence of Observation 7.2.4.

Now we prove that condition (ii) also holds. Indeed, it follows from Observation 7.2.6 that if $\alpha_j \in B \setminus B_{\mathbb{S}}$, then there is some $\alpha_i \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$. On the other hand, according to Definition 3.3.5, for all $\alpha_m \in \mathcal{L} \setminus Cn(\emptyset)$ it holds that $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_m\|$. Hence we can conclude that for all $\alpha_p \in B$ there is some $\alpha_q \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_q) \subseteq \|\neg\alpha_p\|$.

Now let α_l be an arbitrary sentence such that $\alpha_l \in B \cap \mathbf{K}$, we must show that there is some $\alpha_k \in B_{\mathbb{S}} \cap \mathbf{K}$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$. To see that this indeed holds it is enough to notice that, on the one hand, according to the above paragraph, there is some $\alpha_k \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$ and, on the other hand, from $\alpha_l \in \mathbf{K}$ and $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$ it follows that $\alpha_k \in \mathbf{K}$. Therefore $\alpha_k \in B_{\mathbb{S}} \cap \mathbf{K}$ as required, and this finishes the proof. ∎

Lemma 7.2.10 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *If* $B_{\mathbb{S}}$ *is a subset of* B *such that:*

(i) If $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_k) \not\subseteq \|\neg\alpha_j\|$, *for all* $\alpha_k \in B_{\mathbb{S}}$*, and*

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$,

then $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$, *and* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$.

Proof. Assume B is a set of sentences satisfying $B \cap Cn(\emptyset) = \emptyset$ and let $B_{\mathbb{S}}$ be a subset of B such that conditions (i) and (ii) in the statement of the observation are satisfied.

Consider the function $f : \{\|\alpha\| : \alpha \in \mathcal{L}$ and $\nvdash \neg\alpha\} \rightarrow \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ defined by $f(\|\alpha\|) = f_{\mathbb{S}}(\alpha)$. Then it follows from Definition 3.3.5 that if $\alpha \in \mathcal{L}$ and $\nvdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$.

Notice also that, it follows from the hypothesis that $B_{\mathbb{S}} \subseteq B$ and that:

(i) If $\alpha_j \in B_{\mathbb{S}}$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subseteq \|\neg\alpha_j\|$, for all $\alpha_k \in B_{\mathbb{S}}$;

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$.

Hence, by Lemma 6.2.6 we can conclude that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$ and $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$. ∎

Observation 7.2.11 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. If $B_{\mathbb{S}}$ is the $\mathbb{S}$-based filtration of B, then $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$, and $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$.*

Proof. Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. According to Observation 7.2.9, the $\mathbb{S}$-based filtration $B_{\mathbb{S}}$ of B satisfies conditions (i)−(ii) of Lemma 7.2.10. Therefore, it follows immediately from the mentioned lemma that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$ and $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$. ∎

Lemma 7.2.12 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for all $\beta_i \in (G_N \cap \mathbf{K})_{\mathbb{S}}$ there is some $\varsigma_i \in (H_N \cap \mathbf{K})_{\mathbb{S}}$ such that $\|\beta_i\| = \|\varsigma_i\|$.*

Proof. Let $\mathbf{K}$, $\mathbb{S}$, G and H be as mentioned in the statement of the lemma. Assume every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H. According to Lemma 6.3.3 it holds that for all $\beta_i \in G_N \cap \mathbf{K}$ there is some $\varsigma_i \in H_N \cap \mathbf{K}$ such that $\|\beta_i\| = \|\varsigma_i\|$. Hence it follows trivially from the definition of $\mathbb{S}$-based filtration of a set of sentences that for all $\beta_i \in (G_N \cap \mathbf{K})_{\mathbb{S}}$ there is some $\varsigma_i \in (H_N \cap \mathbf{K})_{\mathbb{S}}$ such that $\|\beta_i\| = \|\varsigma_i\|$, and this finishes the proof. ∎

Lemma 7.2.13 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences. Then:*

(i) $B_N \cap \mathbf{K} = (B \cap \mathbf{K})_N$.

(ii) $B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$.

(iii) $(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$.

(iv) $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.

Proof. Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and B be a finite set of sentences. We will show that conditions (i)-(iv) in the statement of the lemma are satisfied.

(i)-(iii) Follow immediately from the definitions of normalization and of $\mathbb{S}$-based filtration of a set of sentences.

(iv) The inclusion $\bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i) \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$ follows immediately from the fact that $(B_{\mathbb{S}})_N \subseteq B_{\mathbb{S}}$. Now we show that the converse inclusion also holds. Assume $M \in \bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$. Then there is some $\alpha_j \in B_{\mathbb{S}}$ such that $M \in f_{\mathbb{S}}(\neg\alpha_j)$. If $\alpha_j \in (B_{\mathbb{S}})_N$ it follows immediately that $M \in \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$ and we are done. Now assume that $\alpha_j \notin (B_{\mathbb{S}})_N$. It follows from Lemma 6.2.5 that there is some $\alpha_k \in (B_{\mathbb{S}})_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_j\|$. Hence we have that $\alpha_j, \alpha_k \in B_{\mathbb{S}}$ and $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_j\|$. So we obtain from Observation 7.2.9-(i) that $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$. Therefore $M \in f_{\mathbb{S}}(\neg\alpha_k)$ and we can conclude that $M \in \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$. ∎

Theorem 7.2.14 *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$*. Then the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$ *by finite sets is a partial meet multiple contraction on* $\mathbf{K}$ *by finite sets.*

Proof. Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$, and $\div_{\mathbb{S}}$ be the $\mathbb{S}$-based multiple contraction on $\mathbf{K}$ by finite sets. We must show that $\div_{\mathbb{S}}$ is a partial meet multiple contraction by finite sets.

According to Theorem 6.3.6, in order to prove that, it is enough to show that there is a propositional package selection function h for $\|\mathbf{K}\|$ such that $\mathbf{K}\div_{\mathbb{S}}B = Th\,(\|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})))$, for all finite sets B.

Let h be such that $h(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$ if $B \cap Cn(\emptyset) \neq \emptyset$, and $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$ if $B \cap Cn(\emptyset) = \emptyset$.

Next will prove that:

(a) h is a propositional package selection function for $\|\mathbf{K}\|$.

(b) $\mathbf{K}\div_{\mathbb{S}}B = Th\,(\|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})))$, for all finite sets B.

(a) We start by showing that h is a function. Assume B and C are sets of sentences such that $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$. If $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C} = \emptyset$ then, according to Observation 6.1.4, $B \cap Cn(\emptyset) \neq \emptyset$ and $C \cap Cn(\emptyset) \neq \emptyset$ and, consequently, it follows from the definition of h that $h(\mathbb{W}_{\mathbf{K}\perp B}) = h(\mathbb{W}_{\mathbf{K}\perp C}) = \emptyset$. Now we consider the case $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C} \neq \emptyset$. Again by Observation 6.1.4 we have that $B \cap Cn(\emptyset) = C \cap Cn(\emptyset) = \emptyset$. Therefore, in order to demonstrate that $h(\mathbb{W}_{\mathbf{K}\perp B}) = h(\mathbb{W}_{\mathbf{K}\perp C})$ we must show that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp C} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$. To do that we start by proving that the following identity holds: $\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$.

Notice that, since $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$, it follows from Observation 6.3.4 that $\mathbf{K}\perp B = \mathbf{K}\perp C$. Therefore, according to Observation 3.1.3, it holds that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C.

Therefore, from Lemma 7.2.12 we can conclude that $\bigcup_{\alpha_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$. Hence, since $(B_N \cap \mathbf{K})_{\mathbb{S}} = ((B \cap \mathbf{K})_{\mathbb{S}})_N$ and $(C_N \cap \mathbf{K})_{\mathbb{S}} = ((C \cap \mathbf{K})_{\mathbb{S}})_N$ (according to Lemma 7.2.13-(i) and (iii)), we obtain that $\bigcup_{\alpha_i \in ((B \cap \mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in ((C \cap \mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$. So, Lemma 7.2.13-(iv) gives us that $\bigcup_{\alpha_i \in (B \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (C \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$. Finally, we can conclude from Lemma 7.2.13-(ii) that $\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$, as required.

To show that h is a propositional package selection function for $\|\mathbf{K}\|$ it only remains to prove that:

(i) $h(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$.

(ii) If $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ then $h(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

It follows immediately from the definition of h that condition (i) is satisfied. Now we show that (ii) also holds. Assume $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$. Then, by Observation 6.1.4, we have that $B \cap Cn(\emptyset) = \emptyset$. So, according to the definition of h, $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$, and we can conclude from Observation 7.2.11 that $h(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

(b) Let B be an arbitrary set of sentences. In what follows we show that $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)\right)$.

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, on the one hand, according to Definition 7.2.8, $\mathbf{K} \div_{\mathbb{S}} B = \mathbf{K}$. And, on the other hand, it follows from Observation 3.3.3−(a) and the definition of h that $Th\left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)\right) = \mathbf{K}$. Hence $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)\right)$, as required.

Case 2, $B \cap Cn(\emptyset) = \emptyset$. Then $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$ and, according to Definition 7.2.8, $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$. Hence, in order to show that the required identity holds, we will prove that, indeed, the following stronger identity is satisfied: $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)$.

First of all we must note that if $\alpha_i \notin \mathbf{K}$ then $f_{\mathbb{S}}(\neg\alpha_i) \subseteq \|\mathbf{K}\|$. Hence $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\right)$. Finally, observing that, according to Observation 7.2.11, $\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}) = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$, we can conclude that $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)$ as we wished to prove. ■

Theorem 7.2.15 *Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. Suppose that for any finite set of sentences B satisfying $B \cap Cn(\emptyset) = \emptyset$ the set $B_{\mathbb{S}}$ is a subset of B such that the following conditions hold:*[126]

(i) If $\alpha_j \in B_{\mathbb{S}}$ then $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_{\mathbb{S}}$;

(ii) For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ such that $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$;

(iii) For any finite set of sentences C satisfying $C \cap Cn(\emptyset) = \emptyset$, if every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C, then for all $\beta_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}$ there is some $\varsigma_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}$ such that $\|\beta_i\| = \|\varsigma_i\|$;

(iv) $B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$;

(v) $(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$;

(vi) $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.

Then the multiple contraction function $\div_{\mathbb{S}}$ on $\mathbf{K}$ defined by:

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

for any finite set of sentences B, is a partial meet multiple contraction on $\mathbf{K}$ by finite sets.

Proof. Let $\div_{\mathbb{S}}$ be the operation defined by

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

where, for any finite set of sentences B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $B_{\mathbb{S}}$ is a subset of B such that conditions (i)−(vi) in the statement of the theorem are satisfied. We must show that $\div_{\mathbb{S}}$ is a partial meet multiple contraction.

According to Theorem 6.3.6, in order to prove that, it is enough to show that there is a propositional package selection function h for $\|\mathbf{K}\|$ such that $\mathbf{K}\div_{\mathbb{S}}B = Th\,(\|\mathbf{K}\|\cup(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})))$, for all sets B.

Let h be such that $h(\mathbb{W}_{\mathbf{K}\perp B}) = \emptyset$ if $B \cap Cn(\emptyset) \neq \emptyset$, and $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$ if $B \cap Cn(\emptyset) = \emptyset$.

We start by showing that h is a function. Assume B and C are sets of sentences such that $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C}$. If $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C} = \emptyset$ then, according to Observation 6.1.4, $B \cap Cn(\emptyset) \neq \emptyset$ and $C \cap Cn(\emptyset) \neq \emptyset$ and, consequently, it follows from the definition of h that $h(\mathbb{W}_{\mathbf{K}\perp B}) = h(\mathbb{W}_{\mathbf{K}\perp C}) = \emptyset$. Now we consider the case $\mathbb{W}_{\mathbf{K}\perp B} = \mathbb{W}_{\mathbf{K}\perp C} \neq \emptyset$. Again by Observation 6.1.4 we have that $B \cap Cn(\emptyset) = C \cap Cn(\emptyset) = \emptyset$. Therefore, in order to demonstrate that $h(\mathbb{W}_{\mathbf{K}\perp B}) = h(\mathbb{W}_{\mathbf{K}\perp C})$ we must show that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp C} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in C_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$. To do that we start by proving that $\bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$. It follows from Observation 6.3.4 that $\mathbf{K}\perp B = \mathbf{K}\perp C$. Therefore, according to Observation 3.1.3, it holds that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C.

Then, from (iii) we can conclude that $\bigcup_{\alpha_i \in (B_N\cap\mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (C_N\cap\mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$. On the other hand it follows immediately from the definition of normalization of a set of sentences and from (v) that $(B_N \cap \mathbf{K})_{\mathbb{S}} = ((B\cap\mathbf{K})_{\mathbb{S}})_N$ and $(C_N \cap \mathbf{K})_{\mathbb{S}} = ((C\cap\mathbf{K})_{\mathbb{S}})_N$.

Hence, we obtain that $\bigcup_{\alpha_i \in ((B\cap\mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in ((C\cap\mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$. So, (vi) gives us that $\bigcup_{\alpha_i \in (B\cap\mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (C\cap\mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$. Finally, we can conclude from (iv) that $\bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$, as required.

To show that h is a propositional package selection function for $\|\mathbf{K}\|$ it only remains to prove that: (a) $h(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$, and (b) if $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ then $h(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

(a) follows immediately from the definition of h. For (b), assume $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$. Then, by Observation 6.1.4, we have that $B \cap Cn(\emptyset) = \emptyset$. So, according to the definition of h, $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$ and, since $B_{\mathbb{S}} \subseteq B$ and conditions (i) and (ii) are satisfied, we can conclude from Lemma 7.2.10 that $h(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

Now it only remains to show that $\mathbf{K}\div_{\mathbb{S}}B = Th\,(\|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})))$ for any set of sentences B.

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, on the one hand, according to the definition of $\div_{\mathbb{S}}$, it holds that $\mathbf{K}\div_{\mathbb{S}}B = \mathbf{K}$. And, on the other hand, it follows from the definition of h that $Th\,(\|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}))) = Th(\|\mathbf{K}\|)$. Hence, since $Th(\|\mathbf{K}\|) = \mathbf{K}$ (cf. Observation 3.3.3–(a)), we can conclude that $\mathbf{K}\div_{\mathbb{S}}B = Th\,(\|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})))$, as required.

Case 2, $B \cap Cn(\emptyset) = \emptyset$. Then $h(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\}$ and, according to the definition of $\div_{\mathbb{S}}$, we have that $\mathbf{K}\div_{\mathbb{S}}B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$. Now, in order to show that the required identity holds we will prove that, indeed, the following stronger identity is satisfied: $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}))$.

First of all we must note that if $\alpha_i \notin \mathbf{K}$ then $f_{\mathbb{S}}(\neg\alpha_i) \subseteq \|\mathbf{K}\|$. Hence $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\right)$. Finally, observing that (since $B_{\mathbb{S}} \subseteq B$ and satisfies conditions (i) and (ii)) it follows from Lemma 7.2.10 that $\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}) = \bigcup_{\alpha_i \in B_{\mathbb{S}}\cap\mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$, we can conclude that $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup (\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}))$ as we wished to prove. ■

Appendix E

Proofs of Chapter 8

In this chapter we present proofs for most of the results presented throughout Chapter 8. For commodity of the reader before each proof here presented we include the statement of the result that such proof refers to.

Lemma 8.0.1 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\alpha, \beta \in \mathcal{L} \setminus Cn(\emptyset)$*. Then the following statements hold:*

(i) If $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$ *then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\| \textit{ iff } f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|.$$

(ii) If $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$ *then*

$$f_{\mathbb{S}}(\neg\alpha) \not\subset f_{\mathbb{S}}(\neg\beta) \textit{ iff } (f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\| \textit{ or } f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|).$$

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, α and β are as mentioned in the statement of the lemma. Statement (i) is obviously true. Now we prove that (ii) also holds.

Assume $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$. Since $f_{\mathbb{S}}(\neg\alpha) \not\subset f_{\mathbb{S}}(\neg\beta) \Leftrightarrow f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta)$ or $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha)$, the thesis of the statement (ii) is equivalent to ($f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta)$ or $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha)$) $\Leftrightarrow$ ($f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|$ or $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$).

Then it is enough to show that $f_{\mathbb{S}}(\neg\alpha) \not\subseteq f_{\mathbb{S}}(\neg\beta) \Leftrightarrow f_{\mathbb{S}}(\neg\alpha) \not\subseteq \|\neg\beta\|$ and $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha) \Leftrightarrow f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$. Moreover, by symmetry it is indeed enough to prove one of the two mentioned equivalences. Hence, in what follows we will show that $f_{\mathbb{S}}(\neg\beta) \subseteq f_{\mathbb{S}}(\neg\alpha) \Leftrightarrow f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$.

($\Rightarrow$) Follows immediately from the fact that $f_{\mathbb{S}}(\neg\alpha) \subseteq \|\neg\alpha\|$.

($\Leftarrow$) Let $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\|$. Since $f_{\mathbb{S}}(\neg\beta) \subseteq \mathbb{S}_{\neg\beta}$ and $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$, we can conclude that $f_{\mathbb{S}}(\neg\beta) \subseteq \|\neg\alpha\| \cap \mathbb{S}_{\neg\alpha} (= f_{\mathbb{S}}(\neg\alpha))$. ■

Theorem 8.0.6 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. If* $\leq$ *and* $\mathbb{S}$ *satisfy condition* ($\leq -\mathbb{S}$) *then, for any finite set of sentences* B*,*

$$\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B,$$

where $\div_{\leq}$ *is the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *and* $\div_{\mathbb{S}}$ *is the* $\mathbb{S}$*-based multiple contraction on* $\mathbf{K}$*.*

Proof. Assume $\mathbf{K}$, $\mathbb{S}$ and $\leq$ are as mentioned in the statement of the theorem and let B be any finite set of sentences.

We must show that $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B$.

If $B \cap Cn(\emptyset) \neq \emptyset$ then $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B = \mathbf{K}$ and we are done.

Now we consider the case $B \cap Cn(\emptyset) = \emptyset$. In this case we must prove that

$$\{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_j)\right)\right)$$

We start by noticing that, according to Observation 8.0.4 it holds that $B_{\leq} = B_{\mathbb{S}}$. Now we prove the above mentioned required equality of sets.

$$\begin{array}{ll}
 & \beta \in \{\beta \in \mathbf{K} : \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta\} \\
\text{iff} & \beta \in \mathbf{K} \text{ and } \forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta \\
\text{iff} & \beta \in \bigcap \|\mathbf{K}\| \text{ and } \forall \alpha_j \in B_{\mathbb{S}}\ f_{\mathbb{S}}(\neg \alpha_j) \subseteq \|\beta\| \text{ (by Observation 3.3.3-(a),} \\
 & \text{Lemma 4.2.5 and the equality } B_{\leq} = B_{\mathbb{S}}) \\
\text{iff} & \beta \in \bigcap \|\mathbf{K}\| \text{ and } \forall \alpha_j \in B_{\mathbb{S}}\ \beta \in \bigcap f_{\mathbb{S}}(\neg \alpha_j) \\
\text{iff} & \beta \in \bigcap \left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_j)\right)\right) \\
\text{iff} & \beta \in Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_j \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_j)\right)\right)
\end{array}$$

■

Corollary 8.0.8 *Let $\mathbf{K}$ be a belief set, $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$ and $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$. Then, for any finite set of sentences B,*

$$\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B,$$

if and only if $\leq$ and $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$).

Proof. Let $\mathbf{K}$, $\mathbb{S}$ and $\leq$ be as stated above.

Now assume that $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B$, for any finite set of sentences B.

Then, in particular, $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B$, for any singleton set $B \subseteq \mathcal{L}$ and it follows from Observations 8.0.7 and 7.2.16 that $\mathbf{K} -_{\leq} \alpha = \mathbf{K} -_{\mathbb{S}} \alpha$, for any sentence $\alpha \in \mathcal{L}$. Hence, according to Theorem 4.2.6, condition ($\leq -\mathbb{S}$) is satisfied.

Hence we have just shown that if $\mathbf{K} \div_{\leq} B = \mathbf{K} \div_{\mathbb{S}} B$, for any finite set of sentences B, then condition ($\leq -\mathbb{S}$) holds.

The converse implication is given by Theorem 8.0.6. ■

Appendix F

Appendix of Chapter 9

F.1 Proofs of Chapter 9

In this section we present proofs for some of the results presented throughout Chapter 9. For commodity of the reader before each proof here presented we include the statement of the result to which such proof corresponds.

Observation 9.1.3 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *as well as the condition* ($CM_{\leq}$)*.*

Proof. Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, $\div$ be the $\leq$-based multiple contraction on $\mathbf{K}$ and $-$ be the $\leq$-based singleton contraction on $\mathbf{K}$.

We first remark that it follows from Observation 3.4.5 that $-$ satisfies *Conjunctive overlap/Intersection*, *Conjunctive inclusion/Conjunction* and condition ($C_{\leq}$).

On the other hand we have, according to Observation 8.0.7, that for all sentences α it holds that $\mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}$. Hence, by Observation 9.1.2-(i) and (ii), we can conclude that $\div$ satisfies *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction* and condition ($CM_{\leq}$). ■

Observation 9.1.4 *Let* $\mathbf{K}$ *be a belief set. If* $\div$ *is a multiple contraction function on* $\mathbf{K}$ *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection *and* P-conjunctive inclusion/P-conjunction*, then the binary relation* $\leq$ *defined from* $\div$ *by means of condition* ($CM_{\leq}$) *is an epistemic entrenchment relation with respect to* $\mathbf{K}$*.*

Proof. Let $\mathbf{K}$ be a belief set, $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection* and *P-conjunctive inclusion/P-conjunction* and $\leq$ be the binary relation defined by condition ($CM_{\leq}$).

We start by noticing that it follows from Observation 5.2.3 that $\div$ also satisfies *P-closure*, *P-vacuity*, *P-extensionality* and *P-recovery*.

Now we define a singleton contraction function on $\mathbf{K}$ in the following way:

$$\forall \alpha \in \mathcal{L},\ \mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}.$$

From the above we can conclude by Observation 9.1.2-(ii) and (iii) that $-$ satisfies all the basic and supplementary AGM postulates for belief set contraction (namely, *closure*, *inclusion*, *vacuity*, *success*, *recovery*, *extensionality*, *Conjunctive overlap/Intersection* and *Conjunctive inclusion/Conjunction*). Therefore Observation 3.4.6 allows us to conclude that the binary relation on $\mathcal{L}$ defined from $-$ by means of condition $(C_\leq)$ is an epistemic entrenchment relation with respect to $\mathbf{K}$.

Finally, since, according to Observation 9.1.2-(i) Conditions $(C_\leq)$ and $(CM_\leq)$ define the same binary relation $\leq$ on $\mathcal{L}$, the required conclusion follows immediately. ■

Observation 9.1.6 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-singleton reduction*.*

Proof. Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, $\div$ be the $\leq$-based multiple contraction on $\mathbf{K}$ and B be an arbitrary finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$.

We start by remarking that, according to Observations 9.1.1 and 9.1.3, it holds that $\div$ satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection* and *P-conjunctive inclusion/P-conjunction*.

Now let $\leq_\div$ be the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

$$\alpha \leq_\div \beta \text{ if and only if } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta.$$

We only need to show that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_\div}} \mathbf{K}\div\{\alpha_i\}$, where $B_{\leq_\div}$ is the $\leq_\div$-based filtration of B.

In order to do that, we first notice that, it follows from Observation 9.1.3 that condition $(CM_\leq)$ is satisfied. Hence the binary relation $\leq_\div$ introduced above coincides with the epistemic entrenchment relation $\leq$ (on which the contraction function $\div$ is based).

Therefore the equality that we need to prove is indeed equivalent to following one: $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_\leq} \mathbf{K}\div\{\alpha_i\}$. But this latter equality follows immediately from Observation 8.0.7, and the proof is complete. ■

Observation 9.2.1 *Let* $\mathbf{K}$ *be a belief set,* $\div$ *be a multiple contraction function on* $\mathbf{K}$ *that satisfies* P-inclusion*,* P-success*,* P-uniformity*,* P-relevance*,* P-conjunctive overlap/P-intersection*,* P-conjunctive inclusion/P-conjunction *and* P-singleton reduction *and* $\leq$ *be the epistemic entrenchment relation with respect to* $\mathbf{K}$ *defined by condition* $(CM_\leq)$*.*[130] *Then* $\mathbf{K}\div B = \mathbf{K}\div_\leq B$*, for any finite set* B*, where* $\div_\leq$ *is the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *defined by* $(CM_{\div_\leq})$*.*

Proof. Let $\mathbf{K}$ be a belief set, $\div$ be a multiple contraction function on $\mathbf{K}$ that satisfies *P-inclusion*, *P-success*, *P-uniformity*, *P-relevance*, *P-conjunctive overlap/P-intersection*, *P-conjunctive inclusion/P-conjunction* and *P-singleton reduction*. Furthermore, consider the epistemic entrenchment relation $\leq$ with respect to $\mathbf{K}$ defined by condition ($CM_{\leq}$) (notice that Observation 9.1.4 assures that under the above assumptions it holds that the binary relation $\leq$ thus defined is indeed an epistemic entrenchment relation with respect to $\mathbf{K}$) and let B be an arbitrary finite set of sentences.

We must show that $\mathbf{K}\div B = \mathbf{K}\div_{\leq} B$, where $\div_{\leq}$ is the $\leq$-based multiple contraction defined by ($CM_{\div_{\leq}}$). That is, we must prove that

$$\mathbf{K}\div B = \begin{cases} \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\} & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}.$$

If $B \cap Cn(\emptyset) \neq \emptyset$ then it follows immediately from *P-inclusion* and *P-relevance* that $\mathbf{K}\div B = \mathbf{K}$ and we are done.

So, it only remains to show that if $B \cap Cn(\emptyset) = \emptyset$ then $\mathbf{K}\div B = \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\}$. Hence, in what follows we assume that $B \cap Cn(\emptyset) = \emptyset$ and show that (under that assumption) the above equality is satisfied.

We start by noticing that it follows from *P-singleton reduction* that

$$\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq}} \mathbf{K}\div\{\alpha_i\}, \tag{9.1}$$

where $B_{\leq}$ is the $\leq$-based filtration of B. In fact, in the present conditions, according to that postulate, it holds that $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq_{\div}}} \mathbf{K}\div\{\alpha_i\}$, where $\leq_{\div}$ is the epistemic entrenchment relation with respect to $\mathbf{K}$ defined by

$$\alpha \leq_{\div} \beta \text{ if and only if } \alpha \notin \mathbf{K}\div\{\alpha \wedge \beta\} \text{ or } \vdash \alpha \wedge \beta,$$

and $B_{\leq_{\div}}$ is the $\leq_{\div}$-based filtration of B. So, since it follows from condition ($CM_{\leq}$) that $\leq_{\div}$ coincides with $\leq$, we can conclude that the equality (9.1) indeed holds.

Now let $-$ be the singleton contraction function on $\mathbf{K}$ defined in the following way:

$$\forall \alpha \in \mathcal{L},\ \mathbf{K}-\alpha = \mathbf{K}\div\{\alpha\}.$$

At this point we remark that it follows from Observation 5.2.3 that $\div$ also satisfies *P-closure*, *P-vacuity*, *P-extensionality* and *P-recovery*.

Therefore we can conclude by Observation 9.1.2 that: (a) Conditions ($C_{\leq}$) and ($CM_{\leq}$) define the same binary relation $\leq$ on $\mathcal{L}$, and (b) the singleton contraction $-$ satisfies all the basic and supplementary AGM postulates for belief set contraction (namely, *closure*, *inclusion*, *vacuity*, *success*, *recovery*, *extensionality*, *Conjunctive overlap/Intersection* and *Conjunctive inclusion/Conjunction*). Hence, it follows from Observation 3.4.6 that $\mathbf{K}-\alpha = \mathbf{K}-_{\leq}\alpha$ for any sentence α, where $-_{\leq}$ is the $\leq$-based contraction defined by condition ($C_{-_{\leq}}$).

Thus, we can conclude that

$$\forall \alpha_i \in B,\ \mathbf{K}\div\{\alpha_i\} = \{\beta \in \mathbf{K} : \alpha_i < \alpha_i \vee \beta\}. \tag{9.2}$$

Finally, combining (9.1) with (9.2) we get $\mathbf{K}\div B = \bigcap_{\alpha_i \in B_{\leq}} \{\beta \in \mathbf{K} : \alpha_i < \alpha_i \vee \beta\} = \{\beta \in \mathbf{K} : \forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta\}$ as required. ∎

F.2 Direct Proof of Observation 9.1.1

The purpose of this section is to provide an explicit proof of Observation 9.1.1 (which, so far, has only been assured to hold by means of some of the results included in Chapter 8 - some of which, on their turn, have been obtained as a consequence of some results from Chapter 7).

Thus, in order to be in a position to present such a proof, we first introduce some auxiliary results.

We start by presenting some observations regarding the composition of the $\leq$-based filtration of a set of sentences B. More precisely we will expose the main property satisfied by the elements of $B_{\leq}$ as well as the reason why an element of B is discarded along the construction of $B_{\leq}$. Such results will be particularly useful in the proofs that we shall present for the fact that all epistemic entrenchment-based multiple contractions satisfy *P-success* and *P-relevance*.

We must notice here that, due to the tight interconnection between systems of spheres and epistemic entrenchment relations (cf. Section 4.2) and, in particular, to the fact that, whenever an epistemic entrenchment relation $\leq$ and a system of spheres $\mathbb{S}$ satisfy condition ($\leq -\mathbb{S}$), the $\leq$-based filtration of a set of sentences coincides with its $\mathbb{S}$-based filtration (cf. Observation 8.0.4), such results can be seen as reformulations in terms of epistemic entrenchment relations of some of the ones that have been presented in Section 7.2 in the context of systems of spheres.[148] However, we provide direct proofs for each of the results stated below.

We start with an observation which presents a property satisfied by the elements of B that are kept in the set $B_{\leq}$:

Observation F.2.1 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$ *and* B *be a finite set of sentences. If* $\alpha_i \in B_{\leq}$, *then for any* $\alpha_t \in B_{\leq}$, $\alpha_t \vee \neg\alpha_i \leq \alpha_t$ *or* $\alpha_i < \alpha_i \vee \neg\alpha_t$.

Proof. Assume $\mathbf{K}$, $\leq$ and B are as mentioned in the statement of the Observation.

If $B \subseteq Cn(\emptyset)$ then, according to Definition 8.0.3, $B_{\leq} = \emptyset$ and we are done.

So, in what follows, assume $B \setminus Cn(\emptyset) \neq \emptyset$ and let $C_1, \ldots, C_m, C'_1, \ldots, C'_m$, $C''_1, \ldots, C''_m$, $B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\leq}$ described in Definition 8.0.3.

Now let α_i be an arbitrary element of $B_{\leq}$ and assume by *reductio* that there is some $\alpha_t \in B_{\leq}$ such that $\alpha_t < \alpha_t \vee \neg\alpha_i$ and $\alpha_i \not< \alpha_i \vee \neg\alpha_t$. It follows from Lemma 3.4.13-(ii) that $\alpha_i \leq \alpha_t$.

Case 1, $\alpha_i =_{\text{E.E.}} \alpha_t$. Then, there is some class C_l such that $\alpha_i, \alpha_t \in C_l$. Since $\alpha_i, \alpha_t \in B_{\leq}$ we must have $\alpha_i, \alpha_t \in C'_l$. But, in that case, from $\alpha_t < \alpha_t \vee \neg\alpha_i$ and $\alpha_i \not< \alpha_i \vee \neg\alpha_t$ we conclude that $\alpha_i \notin C''_l$, which contradicts $\alpha_i \in B_{\leq}$.

Case 2, $\alpha_i < \alpha_t$. Then, there are two classes C_l and C_n, with $l < n$ such that $\alpha_t \in C_l$ and $\alpha_i \in C_n$. Since $\alpha_t \in B_{\leq}$ we have that $\alpha_t \in C''_l$. Hence, from

[148] More precisely, we notice that Observation F.2.1, Lemma F.2.2, Observation F.2.3, and Corollary F.2.4 (below) are analogues (in the sense that they can be seen as restatements in terms of epistemic entrenchment relations) of Observation 7.2.4, Lemma 7.2.5, Observation 7.2.6 and Corollary 7.2.7 (all stated in terms of systems of spheres), respectively.

$C_l'' \subseteq B_l \subseteq B_{n-1}$ we can conclude that $\alpha_t \in B_{n-1}$. But then, from $\alpha_t < \alpha_t \vee \neg\alpha_i$ it follows that $\alpha_i \notin C_n'$, which contradicts $\alpha_i \in B_{\leq}$. ■

Now we introduce a lemma which we will use afterwards to demonstrate a partial converse of the above observation. More precisely, such lemma will be helpful for proving that the property that we have shown, in the above observation, to be satisfied by all sentences in $B_{\leq}$ is not only necessary but also sufficient for an element of B to be kept in the set $B_{\leq}$.

Lemma F.2.2 *Let $\leq$ be a relation that satisfies (EE1), (EE2) and (EE3). Assume that $C = \{\alpha_1, \ldots, \alpha_n\}$, with $n \geq 1$ is a non-empty finite set of sentences such that $\alpha_1 =_{E.E.} \cdots =_{E.E.} \alpha_n$, and, for each $\alpha_j \in C$ let C_{α_j} be the set defined by $C_{\alpha_j} = \{\alpha_k \in C : \alpha_k < \alpha_k \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_k\}$.*

If $\alpha_j \in C$ and $C_{\alpha_j} \neq \emptyset$ then there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$.

Proof. Assume $\mathbf{K}$, $\mathbb{S}$, C and C_{α_j}, for each $\alpha_j \in C$, are as mentioned in the statement of the lemma.

Let $\alpha_j \in C$ be such that $C_{\alpha_j} \neq \emptyset$. We must show that there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$.

First notice that given $\alpha_s, \alpha_t \in C$ if $\alpha_s \in C_{\alpha_j}$ and $\alpha_t < \alpha_t \vee \neg\alpha_s$, then $\alpha_t \in C_{\alpha_j}$. In order to verify this we take α_s and α_t in the mentioned conditions. From $\alpha_s \in C_{\alpha_j}$ it follows that $\alpha_s < \alpha_s \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_s$. Now, on the one hand, from $\alpha_s < \alpha_s \vee \neg\alpha_j$ and $\alpha_t < \alpha_t \vee \neg\alpha_s$ it follows by Lemma 3.4.13-(iii) that $\alpha_t < \alpha_t \vee \neg\alpha_j$. On the other hand, by the same lemma, from $\alpha_t < \alpha_t \vee \neg\alpha_s$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_s$, we obtain that $\alpha_j \not< \alpha_j \vee \neg\alpha_t$. Hence $\alpha_t \in C_{\alpha_j}$.

From the above it follows immediately that for any $\alpha_r \in C$, if $\alpha_r \in C_{\alpha_j}$ then $C_{\alpha_r} \subset C_{\alpha_j}$. Indeed, let $\alpha_r, \alpha_p \in C$ be such that $\alpha_r \in C_{\alpha_j}$ and $\alpha_p \in C_{\alpha_r}$, we will start by showing that $\alpha_p \in C_{\alpha_j}$. From $\alpha_p \in C_{\alpha_r}$ it follows that $\alpha_p < \alpha_p \vee \neg\alpha_r$ and, as we have seen above, from this condition and the fact that $\alpha_r \in C_{\alpha_j}$ we can conclude that $\alpha_p \in C_{\alpha_j}$. Hence $C_{\alpha_r} \subseteq C_{\alpha_j}$. Finally, since $\alpha_r \in C_{\alpha_j}$ but $\alpha_r \notin C_{\alpha_r}$ we can conclude that $C_{\alpha_r} \subset C_{\alpha_j}$.

Having seen this, to finish the proof it remains only to notice that it can be show as it was done in the proof of Lemma 7.2.5 that, as we wished to prove, there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$. ■

With the help of the above lemma we can prove the following observation, which presents property satisfied by all elements of a set of sentences B which are discarded along the process of construction of the $\leq$-based filtration of such set.

Observation F.2.3 *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and B be a finite set of sentences such that $B \cap Cn(\emptyset) = \emptyset$. If $\alpha_j \in B \setminus B_{\leq}$, then there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$.*

Proof. Let $\mathbf{K}$, $\leq$ and B be as mentioned in the statement of the observation.

If $B = \emptyset$ then the observation is trivially true. So, in what remains of this proof we assume that $B \neq \emptyset$ and let $C_1, \ldots, C_m, C_1', \ldots, C_m', C_1'', \ldots, C_m'', B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\leq}$ described in Definition 8.0.3 (notice that in the conditions of the observation it follows from $B \neq \emptyset$ that $B \setminus Cn(\emptyset) \neq \emptyset$).

Now assume α_j is such that $\alpha_j \in B$ but $\alpha_j \notin B_{\leq}$. We must show that there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$.

To do that, we consider separately the two possibilities $\alpha_j \in C_1$ or $\alpha_j \in C_n$, with $1 < n \leq m$.

Case 1, $\alpha_j \in C_1$. Then, since by construction $C_1 = C_1'$, it holds that $\alpha_j \in C_1'$. On the other hand, from $\alpha_j \notin B_{\leq}$ it follows that $\alpha_j \notin C_1''$. Therefore, there is some $\alpha_k \in C_1'$ such that $\alpha_k < \alpha_k \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_k$.

Now we notice that, since B is finite, by construction we have that C_1' is finite and for any $\alpha_r, \alpha_s \in C_1'$ it holds that $\alpha_r =_{\text{E.E.}} \alpha_s$. Then, according to Lemma F.2.2, there is some $\alpha_l \in C_1'$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$, $\alpha_j \not< \alpha_j \vee \neg\alpha_l$ and, for any $\alpha_m \in C_1'$ it holds that $\alpha_m \not< \alpha_m \vee \neg\alpha_l$ or $\alpha_l < \alpha_l \vee \neg\alpha_m$.

It remains to remark that, according to the definition of C_1'', it follows from the conditions above that $\alpha_l \in C_1''$. Consequently $\alpha_l \in B_{\leq}$, and we are done.

Case 2, $\alpha_j \in C_n$, with $1 < n \leq m$. From $\alpha_j \notin B_{\leq}$ it follows that $\alpha_j \notin C_n''$, and we have to consider the two possibilities $\alpha_j \notin C_n'$ or $\alpha_j \in C_n'$.

Case 2.1, $\alpha_j \notin C_n'$. Then there is some $\alpha_l \in B_{n-1}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$. Now, since $\alpha_l \in B_{n-1}$, on the one hand we have that $\alpha_l \in B_{\leq}$ and, on the other hand, it follows from the construction of B_{n-1} and C_n that $\alpha_j < \alpha_l$. So, $\alpha_l \not\leq \alpha_j$ and it follows from Lemma 3.4.13-(ii) that $\alpha_j \not< \alpha_j \vee \neg\alpha_l$. Hence α_l is such that $\alpha_l \in B_{\leq}$ and $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$ as we wished to prove.

Case 2.2, $\alpha_j \in C_n'$. Then, since $\alpha_j \notin C_n''$, there is some $\alpha_k \in C_n'$ such that $\alpha_k < \alpha_k \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_k$. Reasoning as we did in Case 1 above we can conclude that there is some $\alpha_l \in B_{\leq}$ such that $\alpha_l < \alpha_l \vee \neg\alpha_j$ and $\alpha_j \not< \alpha_j \vee \neg\alpha_l$, and this finishes the proof. ■

Finally, combining Observations F.2.1 and F.2.3 we obtain the following corollary stating a necessary and sufficient condition for an element of a set of sentences B to belong to the $\leq$-based filtration of such set.

Corollary F.2.4 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*,* B *be a (non-empty) finite set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$ *and* α_i *be a sentence in* B*. Then* $\alpha_i \in B_{\leq}$ *if and only if for any* $\alpha_t \in B_{\leq}$ *it holds that* $\alpha_t \vee \neg\alpha_i \leq \alpha_t$ *or* $\alpha_i < \alpha_i \vee \neg\alpha_t$*.*

Having seen this, before moving on towards the goal of the present section, it is worth to make here a very brief detour just to notice that, taking the above results into account we can conclude that the $\leq$-based filtration of a finite set of sentences can be described in one of the more concise forms presented in the following remark:

Remark F.2.5 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*,* B *be a finite set of sentences and* $B_{\leq}$ *be the* $\leq$*-based filtration of* B*. Then:*

1. $\alpha_i \in B_{\leq}$ *if and only if* $\alpha_i \in B$ *and for every* $\alpha_t \in B_{\leq}$ *the following conditions are satisfied:*
   - *(a) If* $\alpha_i < \alpha_t$*, then* $\alpha_t \vee \neg\alpha_i \leq \alpha_t$
   - *(b) If* $\alpha_i =_{\text{E.E.}} \alpha_t$*, then* $\alpha_t \vee \neg\alpha_i \leq \alpha_t$ or $\alpha_i < \alpha_i \vee \neg\alpha_t$

2. $B_{\leq} = \{\alpha_i \in B : \forall \alpha_t \in B_{\leq}(\alpha_t \vee \neg\alpha_i \leq \alpha_t \text{ or } \alpha_i < \alpha_i \vee \neg\alpha_t)\}$.

Now, returning to our path towards the presentation of a direct proof for Observation 9.1.1, we introduce one more auxiliary result which will be needed in the proofs that we shall present for the fact that all epistemic entrenchment-based multiple contractions satisfy *P-success* and *P-relevance*. Namely, the following lemma, which asserts that the epistemic entrenchment-based multiple contractions satisfy *P-closure* (and to which we provide a direct proof).

Lemma F.2.6 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-closure.

Proof. Let $\mathbf{K}$, $\leq$ and $\div$ be as mentioned in the statement of the lemma and assume B is an arbitrary finite sets of sentences. We must show that $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$.

If $B \cap Cn(\emptyset) \neq \emptyset$ it follows from ($CM_{\div_{\leq}}$) that $\mathbf{K}\div B = \mathbf{K}$. Then it follows from the hypothesis that $\mathbf{K}\div B = Cn(\mathbf{K}\div B)$. Hence, for the rest of this proof we assume that $B \cap Cn(\emptyset) = \emptyset$. That $\mathbf{K}\div B \subseteq Cn(\mathbf{K}\div B)$ follows from the fact that the consequence operation Cn satisfies *inclusion* (i.e. Cn is such that $A \subseteq Cn(A)$, for any set of sentences A).

To show the converse inclusion we take $\delta \in Cn(\mathbf{K}\div B)$ and we prove that $\delta \in \mathbf{K}\div B$. In order to do that, according to ($CM_{\div_{\leq}}$), and since we are assuming that $B \cap Cn(\emptyset) = \emptyset$, we must show that $\delta \in \mathbf{K}$ and $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$.

Case 1, $\mathbf{K}\div B \neq \emptyset$. Since we are assuming the consequence operation Cn is compact, there is a finite set $S \subseteq \mathbf{K}\div B$ such that $S \vdash \delta$. Furthermore, given that $\mathbf{K}\div B \neq \emptyset$, we can assume that $S \neq \emptyset$. Now we consider separately two possibilities.

Case 1.1, S is a singleton set, i.e. $S = \{\beta_1\}$ for some $\beta_1 \in \mathbf{K}\div B$. Then it follows from *P-inclusion* (already proven) that $\beta_1 \in \mathbf{K}$ and, since $\beta_1 \vdash \delta$ and $\mathbf{K}$ is a belief set we can conclude that $\delta \in \mathbf{K}$. On the other hand, it follows from $\beta_1 \in \mathbf{K}\div B$ and $B \cap Cn(\emptyset) = \emptyset$ that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \beta_1$. Finally, observing that from $\beta_1 \vdash \delta$ it follows that, for any $\alpha_j \in B_{\leq}$, it holds that $\alpha_j \vee \beta_1 \vdash \alpha_j \vee \delta$, by (EE2), (EE1) and Observation 3.4.8 we obtain that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$, as we wished to show.

Case 1.2, $S = \{\beta_1, \ldots, \beta_n\}$, with $n \geq 2$. Since $\beta_1, \ldots, \beta_n \in \mathbf{K}\div B \subseteq \mathbf{K}$ (by *P-inclusion* already proven), and $\{\beta_1, \ldots, \beta_n\} \vdash \delta$, it follows from the fact that $\mathbf{K}$ is a belief set that $\delta \in \mathbf{K}$. Next we must show that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$. From $\beta_1, \ldots, \beta_n \in \mathbf{K}\div B$ and $B \cap Cn(\emptyset) = \emptyset$ it follows that $\forall \beta_i \in S \forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \beta_i$. Hence, for any given $\alpha_l \in B_{\leq}$ we have that $\alpha_l < \alpha_l \vee \beta_1$ and $\alpha_l < \alpha_l \vee \beta_2$. From these two conditions it follows by Observation 3.4.10, that $\alpha_l < (\alpha_l \vee \beta_1) \wedge (\alpha_l \vee \beta_2)$. Then, since $\vdash (\alpha_l \vee \beta_1) \wedge (\alpha_l \vee \beta_2) \leftrightarrow (\alpha_l \vee (\beta_1 \wedge \beta_2))$, by intersubstitutivity (Observation 3.4.9) we obtain that $\alpha_l < \alpha_l \vee (\beta_1 \wedge \beta_2)$. Hence $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee (\beta_1 \wedge \beta_2)$. By iteration of the previous procedure we can conclude that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee (\beta_1 \wedge \ldots \wedge \beta_n)$. Now we note that, since $\beta_1 \wedge \ldots \wedge \beta_n \vdash \delta$ we have that for any $\alpha_l \in B_{\leq}$ it holds that $\alpha_l \vee (\beta_1 \wedge \ldots \wedge \beta_n) \vdash \alpha_l \vee \delta$. So, it follows from (EE2) that $\forall \alpha_j \in B_{\leq} \alpha_j \vee (\beta_1 \wedge \ldots \wedge \beta_n) \leq \alpha_j \vee \delta$. Finally by (EE1) and Observation 3.4.8 we obtain that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$ and this finishes the proof for this case.

Case 2, $\mathbf{K}\div B = \emptyset$. This means that $\vdash \delta$. Then, that $\delta \in \mathbf{K}$ follows from the fact that $\mathbf{K}$ is a belief set. Hence it remains to prove that $\forall \alpha_j \in B_{\leq} \alpha_j < \alpha_j \vee \delta$. Hence, let α_j be an arbitrary element of $B_{\leq}$. That $\alpha_j \leq \alpha_j \vee \delta$ follows immediately from

(EE2). It remains to prove that $\alpha_j \vee \delta \not\leq \alpha_j$. From $\vdash \delta$ it follows that $\vdash \alpha_j \vee \delta$. So, by (EE2), $\varepsilon \leq \alpha_j \vee \delta$ for all $\varepsilon \in \mathcal{L}$. On the other hand, since $\nvdash \alpha_j$, it follows from (EE5) that $\varepsilon \not\leq \alpha_j$ for some $\varepsilon \in \mathcal{L}$. Then, by (EE1) $\alpha_j \vee \delta \not\leq \alpha_j$ and this finishes the proof. ∎

At this point we remark that with the above lemma we finish the introduction of auxiliary results which we will need in the proofs that we shall present for the fact that all epistemic entrenchment-based multiple contractions satisfy each of the postulates included in the statement of Observation 9.1.1 except *P-uniformity*.

However, our direct proof of the fact that any epistemic entrenchment-based multiple contraction satisfies *P-uniformity* needs some more auxiliary lemmas which we introduce right away. In fact, as we shall clarify below, most of such results are indeed nothing more than reformulations (essentially for a matter of consistency with the notation and terminology used throughout this chapter) of some lemmas already introduced in previous chapters.

Thus, we start by stating the following obvious fact:

Observation F.2.7 *Let α_j, α_k be two sentences in $\mathcal{L}$. Then $\vdash \alpha_j \rightarrow \alpha_k$ if and only if $\|\neg\alpha_k\| \subseteq \|\neg\alpha_j\|$.*

Now, bearing in mind the above proposition, we notice that the normalization of a set of sentences B (introduced in Definition 6.2.4) can be alternatively defined, without making use of the notion of possible world, in the following equivalent way:

Definition F.2.8 (Reformulation of Definition 6.2.4) *Let B be a set of sentences. The* normalization of B, *denoted by B_N, is the following subset of B:*[149]

$$B_N = \{\alpha_i \in B : \forall \alpha_j \in B,\ \nvdash \alpha_i \rightarrow \alpha_j \text{ or } \vdash \alpha_j \rightarrow \alpha_i\}.$$

Furthermore, Observation F.2.7 yields that Lemmas 6.2.5 and 6.3.3 can be reformulated in the following (equivalent) way (that does not require the use of the concept of possible world):

Lemma F.2.9 (Reformulation of Lemma 6.2.5) *Let B be a finite set of sentences and $\alpha_l \in B$. If $\alpha_l \notin B_N$ then there is some $\alpha_k \in B_N$ such that $\vdash \alpha_l \rightarrow \alpha_k$ and $\nvdash \alpha_k \rightarrow \alpha_l$.*

Lemma F.2.10 (Reformulation of Lemma 6.3.3) *Let $\mathbf{K}$ be a belief set and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for every element of $G_N \cap \mathbf{K}$ there is a logically equivalent element of $H_N \cap \mathbf{K}$.*

[149] Notice that the set B_N can, alternatively, be defined in the following (equivalent) way:

$$B_N = \{\alpha_i \in B : \alpha_i \nvdash \alpha_j \text{ for all } \alpha_j \in B \text{ s.t. } Cn(\alpha_i) \neq Cn(\alpha_j)\}.$$

Apart from the ones above we yet need to introduce a few more lemmas which will be instrumental to prove that the epistemic entrenchment-based multiple contractions satisfy *P-uniformity*. In particular, the following couple of Lemmas presents some properties concerning the normalization and the $\leq$-based filtration of a set of sentences (where $\leq$ is an epistemic entrenchment relation with respect to a given belief set $\mathbf{K}$) as well as their interrelation. Again we must notice here that, due to the tight interconnection between systems of spheres and epistemic entrenchment relations exposed in Section 4.2, such two Lemmas can be seen as immediate consequences of the combination of some of the results of that section with some of the lemmas that have been presented in Section 7.2.[150] However, we provide direct proofs for each of them.

Lemma F.2.11 *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$ and G and H be two finite sets of sentences. If every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H, then for every element of $(G_N \cap \mathbf{K})_{\leq}$ there is a logically equivalent element of $(H_N \cap \mathbf{K})_{\leq}$.*

Proof. Let $\mathbf{K}$, $\leq$, G and H be as mentioned in the statement of the lemma. Assume every subset X of $\mathbf{K}$ implies some element of G if and only if X implies some element of H. According to Lemma F.2.10 it holds that for all $\beta_i \in G_N \cap \mathbf{K}$ there is some $\varsigma_i \in H_N \cap \mathbf{K}$ such that $\vdash \beta_i \leftrightarrow \varsigma_i$. Hence, it follows trivially from the definition of $\leq$-based filtration of a set of sentences (Definition 8.0.3) and intersubstitutivity (Observation 3.4.9) that for all $\beta_i \in (G_N \cap \mathbf{K})_{\leq}$ there is some $\varsigma_i \in (H_N \cap \mathbf{K})_{\leq}$ such that $\vdash \beta_i \leftrightarrow \varsigma_i$, as required. ■

Lemma F.2.12 *Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, B be a finite set of sentences and β be a sentence. Then:*

(i) $B_N \cap \mathbf{K} = (B \cap \mathbf{K})_N$.[151]

(ii) $B_{\leq} \cap \mathbf{K} = (B \cap \mathbf{K})_{\leq}$.

(iii) $(B_N)_{\leq} = (B_{\leq})_N$.

(iv) $\forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta$ *if and only if* $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$.

Proof. Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, B be a finite set of sentences and β be a sentence. We will show that conditions (i)-(iv) in the statement of the lemma are satisfied.

(i)-(iii) Follow immediately from the definitions of normalization and of $\leq$-based filtration of a set of sentences.

(iv) We must show that $\forall \alpha_i \in B_{\leq}\ \alpha_i < \alpha_i \vee \beta$ if and only if $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$.

[150] More precisely, Lemmas F.2.11 and F.2.12 are analogues (in the sense that they can be seen as reformulations in terms of epistemic entrenchment relations) of Lemmas 7.2.12 and 7.2.13 (both presented in Section 7.2 in the context of systems of spheres), respectively.

[151] Notice that this equality has already been stated in Lemma 7.2.13-(i). We only repeat it here for convenience.

The *left-to-right* implication follows immediately from the fact that $(B_{\leq})_N \subseteq B_{\leq}$. Now we prove that the converse implication also holds. Assume $\forall \alpha_i \in (B_{\leq})_N\ \alpha_i < \alpha_i \vee \beta$ and let α_j be an arbitrary element of $B_{\leq}$. We only need to show that $\alpha_j < \alpha_j \vee \beta$. If $\alpha_j \in (B_{\leq})_N$ then it follows from our above assumption that $\alpha_j < \alpha_j \vee \beta$, and we are done. Now assume that $\alpha_j \notin (B_{\leq})_N$. Then, according to Lemma F.2.9, there is some $\alpha_k \in (B_{\leq})_N$ such that $\vdash \alpha_j \to \alpha_k$ (and $\not\vdash \alpha_k \to \alpha_j$). Then (EE2) yields $\alpha_j \leq \alpha_k$. Furthermore, according to the above assumptions it holds that $\alpha_k < \alpha_k \vee \beta$.

On the other hand, according to the definition of $B_{\leq}$, from $\alpha_k \in B_{\leq}$ we can conclude that $\not\vdash \alpha_k$. Then, since $\vdash \alpha_k \vee \neg\alpha_j$, it follows from Observation 3.4.11 that $\alpha_k < \alpha_k \vee \neg\alpha_j$. Now, since $\alpha_j, \alpha_k \in B_{\leq}$, it follows from Observation F.2.1 that $\alpha_j < \alpha_j \vee \neg\alpha_k$.

Finally, from $\alpha_j \leq \alpha_k$, $\alpha_k < \alpha_k \vee \beta$ and $\alpha_j < \alpha_j \vee \neg\alpha_k$ we can conclude by Lemma 3.4.13-(iii) that $\alpha_j < \alpha_j \vee \beta$, and (iv) is proved. ■

To be in a position to present our direct proof for the fact that the epistemic entrenchment-based multiple contractions satisfy *P-Uniformity*, we need to introduce one last auxiliary result. Namely, the following lemma, which basically confirms that the following quite intuitive statement indeed holds: Given an epistemic entrenchment relation $\leq$ with respect to a certain belief set $\mathbf{K}$ the result of the $\leq$-based contraction of $\mathbf{K}$ by B coincides with the result of the removal of the set $(B_N \cap \mathbf{K})$ from $\mathbf{K}$ by means of that same operation.

Lemma F.2.13 *Let* $\mathbf{K}$ *be a belief set,* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*,* $\div_{\leq}$ *be the* $\leq$*-based multiple contraction on* $\mathbf{K}$ *and* B *be a finite set of sentences. Then:*

(i) $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$.

(ii) $\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\leq}(B_N \cap \mathbf{K})$.

Proof. Let $\mathbf{K}$, $\leq$, $\div_{\leq}$ and B be as mentioned in the statement of the lemma.

We start by showing that condition (i) in the statement of the lemma holds. The *right-to-left* implication follows trivially from the fact that $B_N \cap \mathbf{K} \subseteq B$. Now we prove that the converse implication also holds. Assume $B \cap Cn(\emptyset) \neq \emptyset$ and let $\beta \in B \cap Cn(\emptyset)$. Then $\beta \in B$ and $\beta \in Cn(\emptyset)$. Now, on the one hand, it follows immediately from Definition F.2.8 that $\beta \in B_N$ and, on the other hand, from the fact that $\mathbf{K}$ is a belief set we obtain that $\beta \in \mathbf{K}$, hence that $\beta \in B_N \cap \mathbf{K}$ and finally that $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$. So, (i) is proved.

Now we prove (ii), i.e. that $\mathbf{K}\div_{\leq}B = \mathbf{K}\div_{\leq}(B_N \cap \mathbf{K})$. Since (i) holds, according to Definition 8.0.5, to show that the above equality holds it suffices to prove that if $\beta \in \mathbf{K}$ then

$$\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \beta \text{ iff } \forall \alpha_l \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_l < \alpha_l \vee \beta. \tag{F.3}$$

Let $\beta \in \mathbf{K}$. We start by remarking that (F.3) is equivalent to

$$\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})\ \alpha_j < \alpha_j \vee \beta \text{ iff } \forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta. \tag{F.4}$$

To see that such equivalence indeed holds it is enough to notice that given $\alpha_j \in B_{\leq} \setminus \mathbf{K}$ it follows from the fact that $\mathbf{K}$ is a belief set that $\alpha_j \vee \beta \in \mathbf{K}$ and, therefore, by Observation 3.4.12 we can conclude that $\alpha_j < \alpha_j \vee \beta$.

Having seen this, to finish the proof we only need to show that (*F*.4) holds.

To prove that, we start by remarking that it follows from Lemma *F*.2.12-(i) and (iii) that $(B_N \cap \mathbf{K})_{\leq} = ((B \cap \mathbf{K})_{\leq})_N$. Hence, it holds that $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in ((B \cap \mathbf{K})_{\leq})_N\ \alpha_j < \alpha_j \vee \beta$. On the other hand, according to Lemma *F*.2.12-(iv) we have that $\forall \alpha_j \in ((B \cap \mathbf{K})_{\leq})_N\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$. Furthermore, *F*.2.12-(ii) yields $\forall \alpha_j \in (B \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})\ \alpha_j < \alpha_j \vee \beta$.

Thus, from all the above stated equivalences it follows immediately that $\forall \alpha_j \in (B_{\leq} \cap \mathbf{K})\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$, which is the desired conclusion. ∎

Now we are, finally, in a position to present a proof of Observation 9.1.1 which does not make use of the interrelations among epistemic entrenchment-based multiple contractions, system of spheres-based multiple contractions and partial meet multiple contractions (presented in Chapters 7 and 8).

Observation 9.1.1 *Let* $\mathbf{K}$ *be a belief set and* $\leq$ *be an epistemic entrenchment relation with respect to* $\mathbf{K}$*. Then the* $\leq$*-based multiple contraction* $\div$ *on* $\mathbf{K}$ *satisfies* P-inclusion*,* P-success*,* P-uniformity *and* P-relevance*.*

Proof. Let $\mathbf{K}$ be a belief set, $\leq$ be an epistemic entrenchment relation with respect to $\mathbf{K}$, $\div$ be the $\leq$-based multiple contraction on $\mathbf{K}$ and B and C be two arbitrary finite sets of sentences. In what follows we prove that $\div$ satisfies each of the postulates mentioned in the statement of the observation.

- **Package inclusion:** Follows immediately from the definition of the $\leq$-based multiple contraction on $\mathbf{K}$ (See Definition 8.0.5).

- **Package success:** Let $B \cap Cn(\emptyset) = \emptyset$. We must show that $B \cap \mathbf{K}\div B = \emptyset$. Let α be an arbitrary element of B. If $\alpha \notin \mathbf{K}$, it follows immediately from ($CM_{\div_{\leq}}$) that $\alpha \notin \mathbf{K}\div B$. Hence, we assume that $\alpha \in \mathbf{K}$. Since $B \cap Cn(\emptyset) = \emptyset$, it follows from ($CM_{\div_{\leq}}$) that it will only be the case that $\alpha \in \mathbf{K}\div B$ if $\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \alpha$.

  Case 1, $\alpha \in B_{\leq}$. Then, since $\alpha \vee \alpha \vdash \alpha$, it follows from (EE2) that $\alpha \vee \alpha \leq \alpha$. Hence $\alpha \not< \alpha \vee \alpha$ and condition $\forall \alpha_j \in B_{\leq}\ \alpha_j < \alpha_j \vee \alpha$ fails.

  Case 2, $\alpha \notin B_{\leq}$. Then, according to Observation *F*.2.3, there is some $\alpha_i \in B_{\leq}$ such that $\alpha_i < \alpha_i \vee \neg\alpha$. Assume by *reductio* that $\alpha_i < \alpha_i \vee \alpha$. From the last two conditions it follows by Observation 3.4.10, that $\alpha_i < (\alpha_i \vee \neg\alpha) \wedge (\alpha_i \vee \alpha)$. Finally, from $\vdash (\alpha_i \vee \neg\alpha) \wedge (\alpha_i \vee \alpha) \leftrightarrow \alpha_i$ and intersubstitutivity (Observation 3.4.9) we obtain that $\alpha_i < \alpha_i$ which is absurd.

- **Package uniformity:** Assume that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C. We must prove that $\mathbf{K}\div B = \mathbf{K}\div C$. According to Lemma *F*.2.13, such equality is equivalent to the following one $\mathbf{K}\div(B_N \cap \mathbf{K}) = \mathbf{K}\div(C_N \cap \mathbf{K})$. So, in what follows we will show this latter equality is satisfied. To do that, according to the Definition 8.0.5, it is enough to prove that the two following statements hold:

(a) $(B_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$ if and only if $(C_N \cap \mathbf{K}) \cap Cn(\emptyset) \neq \emptyset$.

(b) If $\beta \in \mathbf{K}$ then $\forall \alpha_j \in (B_N \cap \mathbf{K})_{\leq}\ \alpha_j < \alpha_j \vee \beta$ if and only if $\forall \alpha_l \in (C_N \cap \mathbf{K})_{\leq}\ \alpha_l < \alpha_l \vee \beta$

To show that (a) is satisfied we first notice that, since $\emptyset \subseteq \mathbf{K}$, it follows from the hypothesis that $B \cap Cn(\emptyset) \neq \emptyset$ if and only if $C \cap Cn(\emptyset) \neq \emptyset$. Now, combining this latter equivalence with Lemma F.2.13-(i), we can assert that (a) holds.

Finally, in order to prove that (b) also holds, we recall that it follows from Lemma F.2.11 that for every element of $(B_N \cap \mathbf{K})_{\leq}$ there is a logically equivalent element of $(C_N \cap \mathbf{K})_{\leq}$, and vice versa. From this it follows immediately, by intersubstitutivity (Observation 3.4.9), that (b) holds, and this finishes the proof.

- **Package relevance:** Let $\beta \in \mathbf{K}$ and $\beta \notin \mathbf{K} \div B$. Then it follows from Definition 8.0.5 that $B \cap Cn(\emptyset) = \emptyset$ and $\exists \alpha_j \in B_{\leq}$, such that

$$\alpha_j \vee \beta \leq \alpha_j \tag{9.5}$$

We must show that there is a set K' such that:

(i) $\mathbf{K} \div B \subseteq K' \subseteq \mathbf{K}$

(ii) $\forall \alpha_i \in B\ \alpha_i \notin Cn(K')$

(iii) $\exists \alpha_r \in B : \neg\beta \vee \alpha_r \in Cn(K')$

Consider $K' = \mathbf{K} \div B \cup \{\neg\beta \vee \alpha_j\}$, where α_j is an element of $B_{\leq}$ that satisfies condition (9.5). We have that K' trivially satisfies (iii) and $\mathbf{K} \div B \subseteq K'$. Now, in order to verify that it also satisfies $K' \subseteq \mathbf{K}$, the remaining inclusion stated in (i), first recall that it follows from *P-inclusion*, already proven above, that $\mathbf{K} \div B \subseteq \mathbf{K}$. It remains to show that $\neg\beta \vee \alpha_j \in \mathbf{K}$. Assume by *reductio* that $\alpha_j \notin \mathbf{K}$. It follows by (EE4) that $\forall \delta \in \mathcal{L}, \alpha_j \leq \delta$. Hence, using (EE2), (9.5) and (EE1) we obtain that $\forall \delta \in \mathcal{L}, \beta \leq \delta$. But, from this it follows by (EE4) that $\beta \notin \mathbf{K}$, which contradicts our assumption. Therefore $\alpha_j \in \mathbf{K}$ and, since $\mathbf{K}$ is a belief set, we can also conclude that $\neg\beta \vee \alpha_j \in \mathbf{K}$ as required.

It remains to prove (ii). Assume by *reductio* that $\alpha_i \in B$ is such that $K' \vdash \alpha_i$. Since $K' = \mathbf{K} \div B \cup \{\neg\beta \vee \alpha_j\}$, it follows by deduction that $\mathbf{K} \div B \vdash (\neg\beta \vee \alpha_j) \rightarrow \alpha_i$. Hence, by *P-closure* (which is attested to hold by Lemma F.2.6), $(\beta \wedge \neg\alpha_j) \vee \alpha_i \in \mathbf{K} \div B$. Then, according to the definition of $\div$, we have that

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < ((\beta \wedge \neg\alpha_j) \vee \alpha_i) \vee \alpha_k$$

and, since $\vdash ((\beta \wedge \neg\alpha_j) \vee \alpha_i) \vee \alpha_k \leftrightarrow (\beta \vee \alpha_i \vee \alpha_k) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_k)$, it follows from intersubstitutivity (Observation 3.4.9), (EE2), (EE1) and Observation 3.4.8 that

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < \beta \vee \alpha_i \vee \alpha_k \tag{9.6}$$

and

$$\forall \alpha_k \in B_{\leq}\ \alpha_k < \neg\alpha_j \vee \alpha_i \vee \alpha_k \tag{9.7}$$

Now we distinguish two cases.

Case 1, $\alpha_i \in B_{\leq}$. Since $\alpha_j \in B_{\leq}$, according to Observation F.2.1, we have that $\alpha_i \vee \neg\alpha_j \leq \alpha_i$ or $\alpha_j < \alpha_j \vee \neg\alpha_i$.

Case 1.1, $\alpha_i \vee \neg\alpha_j \leq \alpha_i$. This is a contradiction because, since $\alpha_i \in B_{\leq}$, it follows from (9.7), $\vdash \neg\alpha_j \vee \alpha_i \vee \alpha_i \leftrightarrow \neg\alpha_j \vee \alpha_i$, and intersubstitutivity (Observation 3.4.9), that $\alpha_i < \neg\alpha_j \vee \alpha_i$.

Case 1.2, $\alpha_j < \alpha_j \vee \neg\alpha_i$. From this and the fact that $\alpha_j \vee \neg\alpha_i \vdash \alpha_j \vee \neg\alpha_i \vee \beta$ making use of (EE2), (EE1) and Observation 3.4.8 we obtain that

$$\alpha_j < \alpha_j \vee \neg\alpha_i \vee \beta. \tag{9.8}$$

Since $\alpha_j \in B_{\leq}$, from (9.6) we have that $\alpha_j < \beta \vee \alpha_i \vee \alpha_j$. From the latter condition and (9.8) it follows by Observation 3.4.10, that $\alpha_j < (\alpha_j \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_j)$.

Finally from $\vdash (\alpha_j \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_j) \leftrightarrow \alpha_j \vee \beta$ and intersubstitutivity (Observation 3.4.9) we obtain that $\alpha_j < \alpha_j \vee \beta$, which contradicts (9.5).

Case 2, $\alpha_i \notin B_{\leq}$. Then, let $\alpha_x \in B_{\leq}$ be such that[152]

$$\alpha_x < \alpha_x \vee \neg\alpha_i \tag{9.9}$$

Since $\alpha_x \in B_{\leq}$ it follows from (9.6) and (9.7), respectively, that:

$$\alpha_x < \beta \vee \alpha_i \vee \alpha_x \tag{9.10}$$
$$\alpha_x < \neg\alpha_j \vee \alpha_i \vee \alpha_x \tag{9.11}$$

From (9.9) and the fact that $\alpha_x \vee \neg\alpha_i \vdash \alpha_x \vee \neg\alpha_i \vee \beta$ making use of (EE2), (EE1) and Observation 3.4.8 we obtain that $\alpha_x < \alpha_x \vee \neg\alpha_i \vee \beta$. From the latter condition and (9.10) it follows by Observation 3.4.10, that $\alpha_x < (\alpha_x \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_x)$. Hence, from $\vdash (\alpha_x \vee \neg\alpha_i \vee \beta) \wedge (\beta \vee \alpha_i \vee \alpha_x) \leftrightarrow \alpha_x \vee \beta$ and intersubstitutivity (Observation 3.4.9) we obtain that

$$\alpha_x < \alpha_x \vee \beta \tag{9.12}$$

From (9.9) and the fact that $\alpha_x \vee \neg\alpha_i \vdash \alpha_x \vee \neg\alpha_i \vee \neg\alpha_j$ making use of (EE2), (EE1) and Observation 3.4.8 we obtain that $\alpha_x < \alpha_x \vee \neg\alpha_i \vee \neg\alpha_j$. From the latter condition and (9.11) it follows by Observation 3.4.10, that $\alpha_x < (\alpha_x \vee \neg\alpha_i \vee \neg\alpha_j) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_x)$. Hence, from $\vdash (\alpha_x \vee \neg\alpha_i \vee \neg\alpha_j) \wedge (\neg\alpha_j \vee \alpha_i \vee \alpha_x) \leftrightarrow \alpha_x \vee \neg\alpha_j$ and intersubstitutivity (Observation 3.4.9) we obtain that

$$\alpha_x < \alpha_x \vee \neg\alpha_j \tag{9.13}$$

[152] Notice that it follows from Observation F.2.3 that there is some $\alpha_x \in B_{\leq}$ in such conditions.

According to Lemma 3.4.13-(ii), it follows from (9.13) that $\alpha_j \leq \alpha_x$.

Now we note that, since $\alpha_j, \alpha_x \in B_{\leq}$, according to Corollary F.2.4, we have that either $\alpha_x \vee \neg\alpha_j \leq \alpha_x$ or $\alpha_j < \alpha_j \vee \neg\alpha_x$.

Case 2.1, $\alpha_x \vee \neg\alpha_j \leq \alpha_x$. This contradicts (9.13).

Case 2.2, $\alpha_j < \alpha_j \vee \neg\alpha_x$. From this and the fact that $\alpha_j \vee \neg\alpha_x \vdash \alpha_j \vee \neg\alpha_x \vee \beta$, it follows by (EE2), (EE1) and Observation 3.4.8 that

$$\alpha_j < \alpha_j \vee \neg\alpha_x \vee \beta \tag{9.14}$$

From $\alpha_j \leq \alpha_x$, (9.12), (EE1) and Observation 3.4.8 we obtain that $\alpha_j < \alpha_x \vee \beta$. Hence, from $\alpha_x \vee \beta \vdash \alpha_x \vee \beta \vee \alpha_j$, (EE2), (EE1) and Observation 3.4.8 we obtain that

$$\alpha_j < \alpha_x \vee \beta \vee \alpha_j \tag{9.15}$$

From (9.14) and (9.15) it follows by Observation 3.4.10, that $\alpha_j < (\alpha_j \vee \neg\alpha_x \vee \beta) \wedge (\alpha_x \vee \beta \vee \alpha_j)$. Hence, from $\vdash (\alpha_j \vee \neg\alpha_x \vee \beta) \wedge (\alpha_x \vee \beta \vee \alpha_j) \leftrightarrow \alpha_j \vee \beta$ and intersubstitutivity (Observation 3.4.9) we obtain that $\alpha_j < \alpha_j \vee \beta$, which contradicts (9.5).

■

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