Nonautonomous Difference Equations with Applications

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To Daniela and Helena

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Resumo

Esta dissertação está dividida em duas partes. Na primeira parte apresentamos alguns conceitos e noções básicas que levam à construção de sistemas dinâmicos não autónomos discretos. Utiliza-se o conceito de "skew-product" para estudar algumas propriedades destes sistemas no que concerne à sua periodicidade e estabilidade. É feita uma breve abordagem à variedade central e à bifurcação dos pontos periódicos nas equações de diferenças não-autónomas unidimensionais.

Na segunda parte faz-se um estudo de alguns modelos aplicados à ecologia/biologia e à economia. Devido à dificuldade em manipular certas relações que surgem no estudo que se faz aos modelos, apresentamos cálculo computacional por forma a ilustrar e a descrever a dinâmica do modelo.

Uma das principais contribuições desta tese é o estudo da estabilidade dos sistemas não-lineares quando o valor próprio se encontra no círculo unitário. Outra é o estudo das bifurcações, em particular, o diagrama de bifurcação no espaço dos parâmetros, do modelo de competição entre duas espécies definido por equações de Ricker.

Constatou-se que a dinâmica do modelo de competição de Ricker é semelhante à do modelo logístico de competição. Assim, acreditamos que deve existir uma certa classe de mapas bidimensionais para os quais se poderá generalizar os nossos resultados.

Finalmente, não podemos deixar de realçar as técnicas que se utiliza na prova da existência de solução positiva do sistema não-autónomo bidimensional de Ricker .

Key Words

Sistemas não-autónomos periódicos, Periodicidade, Estabilidade, Bifurcação, Modelos de competição, Efeito Allee.

Abstract

This work is divided in two parts. In the first part we develop the theory of discrete nonautonomous dynamical systems. In particular, we investigate skew-product dynamical system, periodicity, stability, center manifold, and bifurcation.

In the second part we present some concrete models that are used in ecology/biology and economics. In addition to developing the mathematical theory of these models, we use simulations to construct graphs that illustrate and describe the dynamics of the models.

One of the main contributions of this dissertation is the study of the stability of some concrete nonlinear maps using the center manifold theory. Moreover, the second contribution is the study of bifurcation, and in particular the construction of bifurcation diagrams in the parameter space of the autonomous Ricker competition model.

Since the dynamics of the Ricker competition model is similar to the logistic competition model, we believe that there exists a certain class of two-dimensional maps with which we can generalize our results.

Finally, using the Brouwer's fixed point theorem and the construction of a compact invariant and convex subset of the space, we present a proof of the existence of a positive periodic solution of the nonautonomous Ricker competition model.

Key Words

Nonautonomous periodic systems, Periodicity, Stability, Bifurcation, Competition Models, Allee effect.

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Introduction

In population dynamics, and from the mathematical point of view, there are two major modeling strategies: (i) The **continuous time approach** using theory of differential equations; and (ii) The **discrete time approach** which is more related with the structure of data of a population. Hence, many laws of the nature are intrinsically discrete.

In particular, when we study a mathematical model, one of the main objectives is to give relevant information/contribution for the importance of the model. Moreover, different modeling strategies with different assumptions to describe particular features of the population, is frequently used among researchers.

Throughout this work we use the discrete time analysis. Hence, this work deals with difference equations, or in other equivalent terminology, recursions, iterations and discrete dynamical systems. Such kind of iterative procedures are omnipresent in mathematics, as well in a couple of related sciences such as Biology, Ecology, Economic, Engineering, etc.

There are two main kind of discrete time approach: (i) The **autonomous model** in which the dynamics of the system is given by the same map. This kind of systems predicts the evolution without considering many changes along the time. In other words, the equation that generate the system has a constant parameters. (ii) The **nonautonomous model**, i.e., difference equations whose right-hand side, explicitly depend on time or change. Hence, seasonal influences, controlling, external effects, and other mechanisms are allow in nonautonomous models. In concrete equations the constant parameters are replaced by time-dependent sequence of parameters.

In this work we have both, autonomous and nonautonomous models. Explicitly, an autonomous difference equation is defined by

$$x_{n+1} = f(x_n), n = 0, 1, 2, \dots. \tag{1}$$

If x_0 is the initial population, its evolution along the time n is given by

$$\begin{aligned} x_1 &= f(x_0), \\ x_2 = f(x_1) &= f \circ f(x_0) = f^2(x_0), \\ &\vdots \\ x_{n+1} &= f^n(x_0), \\ &\vdots \end{aligned}$$

As it is clearly seen, the evolution of the population is generated by the same map.

The nonautonomous difference equation is governed by the rule

$$x_{n+1} = f_n(x_n), n = 0, 1, 2, \dots. \tag{2}$$

Here the evolution of the population is generated by the composition of the sequence of maps

$$f_0, f_1, f_2, \ldots.$$

Explicitly,

$$\begin{aligned} x_1 &= f_0(x_0), \\ x_2 &= f_1(x_1) = f_1 \circ f_0(x_0), \\ &\vdots \\ x_{n+1} &= f_n \circ f_{n-1} \circ \ldots \circ f_1 \circ f_0(x_0), \\ &\vdots \end{aligned}$$

If the sequence of maps is periodic, i.e., $f_{n+p} = f_n$, for all $n = 0, 1, 2\ldots$ and some positive integer $p > 1$, then we talk about nonautonomous periodic difference equations. Systems where the sequence of maps is periodic model population with fluctuation habitat, and they are commonly called periodically forced systems. Under this scenario, the evolution of the population is given by

$$\begin{aligned} x_1 &= f_0(x_0), \\ x_2 &= f_1 \circ f_0(x_0), \\ &\vdots \\ x_p &= f_{p-1} \circ \ldots \circ f_1 \circ f_0(x_0), \\ x_{p+1} &= f_0 \circ f_{p-1} \circ \ldots \circ f_1 \circ f_0(x_0), \\ &\vdots \end{aligned}$$

Throughout this work, the nonautonomous part is about periodic forced systems.

Notice that the nonautonomous periodic difference equation (2) does not generate a discrete (semi)dynamical system [30] as it may not satisfy the (semi)group property. One of the most effective ways of converting the nonautonomous difference equation (2) into a genuine discrete (semi)dynamical system is the construction of the associated skew-product system as described in a series of papers by Elaydi and Sacker [28, 29, 30, 31]. It is noteworthy to mention that this idea was originally used to study nonautonomous differential equations by Sacker and Sell [68].

Most of the contents in Chapter 1 is devoted to this theory. We present all the necessary machinery to develop the skew-product system. The principal results and new developments concerning the periodicity of the system and its stability is present. These new developments include results associated with the introduction of the notion of the derivative in the skew-product. Hence a natural extension to nonautonomous periodic systems of the main stability results of autonomous systems is developed. One entire section is about the center manifold and the stable and unstable manifold as well for any nonlinear periodic system. Finally we end the chapter presenting a brief extension of the main results of bifurcation of one-dimensional autonomous maps to periodic equations, i.e., maps with one variable and p parameters. Moreover, degeneracy conditions for both, variables and parameters is studied.

In Chapter 2 and in Chapter 3 we study some concrete models trying to apply, whenever need, the theory developed in Chapter 1. So Chapter 2 is totally dedicated to the study of the properties of some competition models for two species. Stability, bifurcation and attenuance/resonance are the main focus here. We study in deep the autonomous Ricker competition map. We found out that the new model that we propose, the logistic competition model, exhibits a similar behavior than the Ricker competition model. We also turn our attention to the nonautonomous Ricker competition model and to the Leslie-Gower competition model.

In Chapter 3 we study the properties of unimodal Allee maps for both autonomous and nonautonomous systems. The first model is devoted to population dynamics while the second is in economic theory.

Finally, in Chapter 4 we present some ideas of the possible continuation of our current research. Open problems and conjectures are given.

Chapter 1

Nonautonomous periodic systems

Dynamical systems occur in all branches of science. “The main goal of the study of dynamical systems is to understand the long behavior of states in a system for which there is a deterministic rule for how a state evolves” (Martin Rasmussen [62]). For this reason “an understanding of the asymptotic behavior of a dynamical systems is probably one of the most relevant problems in sciences based on mathematical modeling” (Christian Potzsche [61]). Hence, one of the mathematical concept of a dynamical system is based on the simple fact that there are certain rules that governs our natural laws. These rules, in general, can be described by discrete mathematical models.

A good example that illustrates what is a discrete dynamical system is the following: “Take a scientific calculator and input any number whatsoever. Then start striking one of the function keys over and over again. … For example, if we repeatedly strike the *exp* key, given an initial input x, we are computing the sequence of numbers

$$x, e^x, e^{e^x}, e^{e^{e^x}}, \ldots .$$

That is, we are iterating the exponential function.” (Robert Devaney [24]).

In this example the evolution of the system is due to the same function. This is known as an autonomous discrete dynamical system and it may be defined by the difference equation

$$x_{n+1} = f(x_n), n = 0, 1, 2, \ldots .$$

The evolution here is based in the successive composition of the map f. Hence, a natural question in the theory of discrete dynamical systems is the following: given an initial value x_0, what happens to the sequence of iterates

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots ?$$

Notice that this law is time-independent.

However, in many concrete cases, this notion of discrete dynamical system, is not enough to describe real world phenomena. For instance in population dynamics is more realistic to consider evolutionary adaptations due the influence of the environment. The appropriate technique to solve such problem is the use of nonautonomous models or time-dependent equations. Such kind of models can be governed by the difference equation

$$x_{n+1} = f_n(x_n), n = 0, 1, 2, \ldots .$$

According to this law, the evolution is time-dependent because the sequence of functions f_i, $i = 0, 1, 2, \ldots$ usually is not equal. In general, this kind of models possesses a parameter which changes in time.

In real world, in general, most of the known phenomena have the tendency to repeat after some time. Here arises the nonautonomous periodic models. According to our terminology, in the sequence of functions we have for some integer $p > 1$, $f_{n+p} = f_n$, for all $n = 0, 1, 2, \ldots$. However, the nonautonomous periodic difference equation does not generate a discrete (semi)dynamical system as it may not satisfy the (semi)group property. This problem was positively solved by Elaydi and Sacker, in a series of articles [28, 29, 30, 31], where these mathematicians developed the theory of skew-product systems.

It should be noted that Ziyad AlSharawi, a former Elaydi's PhD student, used the skew-product theory in his dissertation [6]. He was able to extend the Sharkovsky's theorem and its converse to periodic difference equations. An extension of the well-known Singer's theorem is present in [6]. Moreover, AlSharawi did a relevant work in the construction of concrete periodic systems and almost periodic systems according to certain previous properties on the maps.

We are going to use the skew-product theory in another perspective. We focus here in the properties of the stability of the solutions of periodic difference equation. Moreover, we study the bifurcation for such kind of systems.

Hence in Section 1.1 we present the principal definitions and techniques that permit us the construction of a discrete (semi)dynamical system via the nonautonomous periodic difference equation.

In Section 1.2 we study some properties of the solutions of the nonautonomous periodic systems and in the next section we focus our attention in its stability. A natural extension of the more common properties of autonomous systems is introduced.

The theory of the center manifold and the stable and unstable manifold for these systems is study in Section 1.4. We introduce techniques that allow us to study the stability of a fixed point, for any non-linear map, when one (or more) of the eigenvalues are on the unit circle.

In Section 1.5 a natural extension to periodic difference equations of the main results of bifurcation is treated. The techniques are similar to differential equations and are known for maps with one variable and one parameter. Moreover, degeneracy conditions for both variable and parameters are developed.

1.1 Skew-product Systems

Let X be a topological space and $\mathbb{Z}$ be the set of integers. A discrete dynamical system (X, π) is defined as a map $\pi : X \times \mathbb{Z} \to X$ such that π is continuous and satisfies the following two properties

1. $\pi(z, 0) = z$ for all $z \in X$,

2. $\pi(\pi(z, s), t) = \pi(z, s + t)$, $s, t \in \mathbb{Z}$ and $z \in X$ (the group property).

We say (X, π) is a discrete semidynamical system if $\mathbb{Z}$ is replaced by $\mathbb{Z}^+$, the set of nonnegative integers, and the group property is replaced by the semigroup property.

Notice that (X, π) can be generated by a map F defined as $\pi(z, n) = F^n(z)$, where F^n denotes the n^{th} composition of the map F. We observe that the crucial property here is the semigroup property.

A difference equation is called autonomous if it is generated by one map such as

$$z_{n+1} = F(z_n), z \in X, n \in \mathbb{Z}^+. \tag{1.1}$$

Notice that for any $z_0 \in X, z_n = F^n(z_0)$. Hence, the orbit $O(z_0) = \{z_0, z_1, z_2, \ldots\}$ in Eq. (1.1) is the same as the set $O(z_0) = \{z_0, F(z_0), F^2(z_0), \ldots\}$ under the map F.

A difference equation is called nonautonomous if it is governed by the rule

$$z_{n+1} = F(n, z_n), n \in \mathbb{Z}^+, \tag{1.2}$$

which may be written in the friendlier form

$$z_{n+1} = F_n(z_n), n \in \mathbb{Z}^+, \tag{1.3}$$

where $F_n(z) = F(n, z)$, $z \in X$. Here the orbit of a point z_0 is generated by the composition of the sequence of maps $\{F_n\}$. Explicitly,

$$\begin{aligned} O(z_0) &= \{z_0, F_0(z_0), F_1(F_0(z_0)), F_2(F_1(F_0(z_0))), ...\} \\ &= \{z_0, z_1, z_2, ...\}. \end{aligned}$$

When $F_i = F_{i \bmod p}$, $i \in \mathbb{Z}^+$ then we say that Eq. (1.3) is $p-$periodic, where p is the minimal period.

It should be pointed out here that the nonautonomous difference equation may not generate a discrete semidynamical system as it may not satisfy the semigroup property. The following example illustrates this point.

Example 1 *Consider the two-dimensional nonautonomous difference equation*

$$(x_{n+1}, y_{n+1}) = \left((-1)^n \left(\frac{n+1}{n+2}\right) x_n, \frac{1}{n+1} y_n\right), n \in \mathbb{Z}^+. \tag{1.4}$$

The solution of Eq. (1.4) is

$$(x_n, y_n) = \left((-1)^{\frac{n(n-1)}{2}} \frac{x_0}{n+1}, \frac{y_0}{n!}\right).$$

Let $\pi((x_0, y_0), n) = (x_n, y_n)$. Then

$$\begin{aligned} \pi(\pi((x_0, y_0), m), n) &= \pi\left(((-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{m+1}, \frac{y_0}{m!}), n\right) \\ &= \left((-1)^{\frac{n(n-1)}{2}} (-1)^{\frac{m(m-1)}{2}} \cdot \frac{x_0}{(n+1)(m+1)}, \frac{1}{n!} \frac{1}{m!} y_0\right). \end{aligned}$$

However,

$$\pi((x_0, y_0), m+n) = \left((-1)^{\frac{(n+m)(n+m-1)}{2}} \frac{x_0}{m+n+1}, \frac{1}{(m+n)!} y_0\right).$$

Now we are going to present the appropriate framework in which the nonautonomous difference equation generate a discrete semidynamical system.

Consider the nonautonomous difference equation (1.2) where $F(n, \cdot) \in C(\mathbb{Z}^+ \times X, X) = C$, C is the space of continuous functions. The space C is equipped with the topology of uniform convergence on compact subsets of $\mathbb{Z}^+ \times X$. Let $F_t(n, \cdot) = F(t+n, \cdot)$ and $\mathcal{A} = \{F_t(n, \cdot) : t \in \mathbb{Z}^+\}$

be the set of translates, denoted by H, in C. Then $G(n,\cdot) \in \omega(\mathcal{A})$, the omega limit set of $\mathcal{A}$, if for each $n \in \mathbb{Z}^+$,

$$|F_t(n,z) - G(n,z)| \to 0$$

uniformly for z in compact subsets of X, as $t \to \infty$ along some subsequence $\{t_{n_i}\}$. The closure of $\mathcal{A}$ in C is called the hull of $F(n,\cdot)$ and is denoted by $Y = cl(\mathcal{A}) = \mathcal{H}(F)$.

On the space Y, we define a discrete semidynamical system $\sigma : Y \times \mathbb{Z}^+ \to Y$ by $\sigma(H(n,\cdot),t) = H_t(n,\cdot)$; that is σ is the shift map.

Define the composition operator Φ as follows

$$\Phi_n^i = F_{i+n-1} \circ \ldots \circ F_{i+1} \circ F_i \equiv \Phi_n(F(i,\cdot)).$$

When $i = 0$, we write Φ_n^0 as Φ_n.

The skew-product system is now defined as

$$\pi : X \times Y \times \mathbb{Z}^+ \to X \times Y,$$

with

$$\pi((z,G),n) = (\Phi_n(G(i,z)), \sigma(G,n)).$$

If $G = f_i$, then $\pi((z,f_i),n) = (\Phi_n^i(z), f_{i+n})$.

The following commuting diagram illustrates the notion of skew-product systems where $\mathcal{P}(a,b) = b$ is the projection map in the second component.

$$\begin{array}{ccc} X \times Y \times \mathbb{Z}^+ & \xrightarrow{\pi} & X \times Y \\ \Big\downarrow{\scriptstyle \mathcal{P}\times id} & & \Big\downarrow{\scriptstyle \mathcal{P}} \\ Y \times \mathbb{Z}^+ & \xrightarrow[\sigma]{} & Y \end{array}$$

For each $G(n,\cdot) \equiv g_n \in Y$, we define the fiber $\mathcal{F}_g$ over G as $\mathcal{F}_g = \mathcal{P}^{-1}(G)$. If $g = f_i$, we write $\mathcal{F}_g$ as $\mathcal{F}_i$.

Theorem 2 *[30] π is a discrete semidynamical system.*

Example 3 (Example 1 revisited) *Let us reconsider the nonautonomous difference equation*

$$(x_{n+1}, y_{n+1}) = \left((-1)^n \left(\frac{n+1}{n+2}\right) x_n, \frac{1}{n+1} y_n\right), (x(0), y(0)) = (x_0, y_0), n \in \mathbb{Z}^+.$$

Hence,

$$F(n,x,y) = \left((-1)^n \frac{n+1}{n+2} x, \frac{1}{n+1} y\right) = f_n(x,y) = (f_{1,n}(x), f_{2,n}(y)).$$

Its omega limit set is given by $G(n,x,y) = ((-1)^n x, 0)$, that is, g_n is a periodic sequence given by $g_0 = g_{2n}$, $g_1 = g_{2n+1}$, for all $n \in \mathbb{Z}^+$, in which $g_0(x,y) = (x,0)$, and $g_1(x,y) = (-x,0)$.

It is easy to verify that π defined as $\pi((x,y,f_i),n) = (\Phi_n^i(x,y), f_{i+n})$ is a semidynamical system, i.e, the map

$$\pi((x,y,f_i),n) = \left(\left((-1)^{in+\frac{n(n-1)}{2}} \frac{i+1}{i+n+1} x, \frac{1}{\prod_{j=0}^{n-1}(i+1+j)} y\right), \left((-1)^{(n+i)} \frac{n+i+1}{n+i+2} x, \frac{1}{n+i+1} y\right)\right)$$

satisfy the semigroup property.

1.2 Periodicity

In this section our focus will be on the $p-$periodic nonautonomous difference equation of the form

$$z_{n+1} = F_n(z_n), n \in \mathbb{Z}^+, \tag{1.5}$$

where $z \in X$, $F_i = F_i \text{mod} p$, $i \in \mathbb{Z}^+$ with $F_i \in X$, where p is the minimal period of the equation.

Recall that z^* is a fixed point of the autonomous equation $z_{n+1} = F(z_n)$ if $F(z^*) = z^*$ and z^* is a fixed point of the nonautonomous equation (1.5) if it is a fixed point for all the maps, i.e., $F_i(z^*) = z^*$, $\forall i \in \mathbb{Z}^+$.

We begin by defining an $r-$periodic cycle.

Definition 4 *An ordered set of points $C_r = \{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}\}$ is an $r-$periodic cycle in X if*

$$F_{(i+nr) \text{ mod } p}(\bar{z}_i) = \bar{z}_{(i+1)\ mod\ r}, n \in \mathbb{Z}^+.$$

In particular,

$$F_i(\bar{z}_i) = \bar{z}_{i+1}, 0 \leq i \leq r-2,$$

and

$$F_t(\bar{z}_{t \text{ mod } r}) = \bar{z}_{(t+1) \text{ mod } r}, r-1 \leq t \leq p-1.$$

It should be noted that the $r-$periodic cycle C_r in X generates an $s-$periodic cycle on the skew-product $X \times Y$ ($Y = \{F_0, F_1, \ldots, F_{p-1}\}$) of the form

$$\widehat{C_s} = \{(\bar{z}_0, F_0), (\bar{z}_1, F_1), ..., (\bar{z}_{(s-1) \text{ mod } r}, F_{(s-1) \text{ mod } p})\},$$

where $s = lcm[r, p]$ is the least common multiple of r and p.

To distinguish these two cycles, the $r-$periodic cycle C_r on X is called an $r-$**geometric cycle** (or simply $r-$periodic cycle when there is no confusion), and the $s-$periodic cycle $\widehat{C_s}$ on $X \times Y$ is called an $s-$**complete cycle**.

Example 5 *Let $\alpha, \beta \in (0, 1)$ with $\alpha \neq \beta$ and define*

$$f_n(x, y) = \begin{cases} 1 + \alpha(1 + x - y) \text{ if } n = 0 \text{ mod } 4 \\ \alpha(-1 + x + y) \text{ if } n = 1 \text{ mod } 4 \\ 1 + \beta(1 + x - y) \text{ if } n = 2 \text{ mod } 4 \\ \beta(-1 + x + y) \text{ if } n = 3 \text{ mod } 4 \end{cases},$$

and

$$g_n(x, y) = \begin{cases} \beta(-1 + x + y) \text{ if } n = 0 \text{ mod } 4 \\ 1 + \beta(1 - x + y) \text{ if } n = 1 \text{ mod } 4 \\ \alpha(-1 + x + y) \text{ if } n = 2 \text{ mod } 4 \\ 1 + \alpha(1 - x + y) \text{ if } n = 3 \text{ mod } 4 \end{cases},$$

$n \in \mathbb{Z}^+$. This leads to a $4-$periodic two-dimensional nonautonomous difference equation. There is, however, a $2-$periodic geometric cycle, namely, $C_2 = \{(0, 1), (1, 0)\}$ (see Fig. 1.1). This periodic cycle in the space $\mathbb{R}^2_+$ generates the following $4-$complete cycle in the skew-product $\mathbb{R}^2_+ \times Y$

$$\widehat{C_4} = \{((0, 1), (f_0, g_0)), ((1, 0), (f_1, g_1)), ((0, 1), (f_2, g_2)), ((1, 0), (f_3, g_3))\},$$

where $Y = \{(f_0, g_0), (f_1, g_1), (f_2, g_2), (f_3, g_3)\}$.

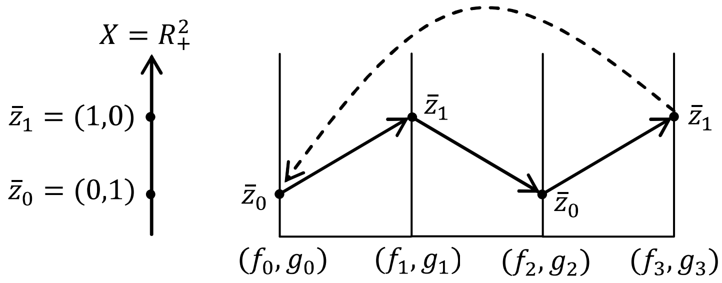

Figure 1.1: A 2−periodic geometric cycle on $X = \mathbb{R}^2_+$ and a 4−periodic complete cycle on the skew product $X \times Y$ where $Y = \{F_0, F_1, F_2, F_3\}$, $F_i = (f_i, g_i)$, $i = 0, 1, 2, 3$ for the difference equation defined in example 5.

We are going to provide a deeper analysis of the preceding example. Let $d = \gcd(r, p)$ be the greatest common divisor of r and p, $s = lcm[r, p]$ be the least common multiple of r and p, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. The following result is crucial in the setting of periodic cycles. It is published in [29] for a general metric space.

Lemma 6 *[29] Let $C_r = \{\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_{r-1}\}$ be a set of points in X. Then the following statements are equivalent.*

1. *C_r is a periodic cycle of minimal period r.*
2. *For $0 \le i \le r-1$, $F_{(i+nd) \bmod p}(\overline{z}_i) = \overline{z}_{(i+1) \bmod r}$.*
3. *For $0 \le i \le r-1$, we have that*

   $F_{i \bmod p}\left(\overline{z}_{i \bmod r}\right) = F_{(i+d) \bmod p}\left(\overline{z}_{i \bmod r}\right) = \ldots = F_{(i+(m-1)d) mod\, p}\left(\overline{z}_{i \bmod r}\right) = \overline{z}_{(i+1) \bmod r}$;
   $F_{i \bmod p}\left(\overline{z}_{(i+d) \bmod r}\right) = F_{(i+d) \bmod p}\left(\overline{z}_{(i+d) \bmod r}\right) = \ldots = F_{(i+(m-1)d) mod\, p}\left(\overline{z}_{(i+d) \bmod r}\right) = \overline{z}_{(i+d+1) \bmod r}$;
   $\vdots$
   $F_{i \bmod p}\left(\overline{z}_{(i+(\ell-1)d) \bmod r}\right) = F_{(i+d) \bmod p}\left(\overline{z}_{(i+(\ell-1)d) \bmod r}\right) = \ldots = F_{(i+(m-1)d) mod\, p}\left(\overline{z}_{(i+(\ell-1)d) \bmod r}\right) = \overline{z}_{(i+(\ell-1)d+1) \bmod r}$.

Let $\mathbb{K}$ denote either $\mathbb{R}^q$ or $\mathbb{C}^q$ and let us write the parameter family of maps $F_i(z)$ as $F_i(\alpha, z)$, $i \in \mathbb{Z}^+$, where $\alpha \in \mathbb{K}$ is a parameter vector and $z \in X$. The following definition is used in the sequel.

Definition 7 *$F_i : \mathbb{K} \times X \to X$, $i \in \mathbb{Z}^+$, is one to one with respect to the parameter vector α if for $\alpha \neq \widehat{\alpha}$ one has $F(\alpha, z) \neq F(\widehat{\alpha}, z)$, $\forall z \in X$ (with exception of the fixed points).*

As an example, the nonautonomous Ricker competition model given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n e^{K_n - x_n - a_n y_n} \\ y_n e^{L_n - y_n - b_n x_n} \end{pmatrix}, n \in \mathbb{Z}^+,$$

where $K_n > 0$ and $L_n > 0$ are the carrying capacities of species x and y, respectively, and $a_n > 0$ and $b_n > 0$ are the competition parameters of species y and x, respectively, is one to one with respect to the carrying capacities parameter vector (fixing the competition parameters). For instance, from the following identity

$$\begin{pmatrix} xe^{K_0 - x - ay} \\ ye^{L_0 - y - bx} \end{pmatrix} = \begin{pmatrix} xe^{K_1 - x - ay} \\ ye^{L_1 - y - bx} \end{pmatrix}$$

we conclude that $K_0 = K_1$ and $L_0 = L_1$. If the carrying capacities are fixed and the competition parameters vary, then the model still is one to one with respect to the competition parameter vector.

As a consequence of Definition 7 the map is one to one with respect to the parameter vector if it is a one parameter vector family of maps. This observation motivates the following result.

Theorem 8 *Suppose that $F(\alpha, z)$, $\alpha \in \mathbb{K}, z \in X$, $n \in \mathbb{Z}^+$ is a one parameter vector family of maps one to one in α and write the nonautonomous difference equations as*

$$z_{n+1} = F_n(\alpha_n, z_n), \tag{1.6}$$

in which $F_n = F_{n \bmod p}$, $n \in \mathbb{Z}^+$. If the $p-$periodic difference equation (1.6) with minimal period p has a nontrivial periodic cycle of minimal period r, then $r = tp$, $t \in \mathbb{Z}^+$.

Proof. Suppose that the Eq. (1.6) has a periodic cycle

$$C_r = \{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}\}$$

of period $r \neq tp$, $t \in \mathbb{Z}^+$. Let $d = \gcd(r, p)$, $s = lcm[r, p]$, $m = \frac{p}{d}$, and $\ell = \frac{s}{p}$. Then $d < p$. By Lemma 6, we have that

$$\begin{aligned}
F_0(\alpha_0, \bar{z}_0) = F_d(\alpha_d, \bar{z}_0) = \ldots = F_{(m-1)d}(\alpha_{(m-1)d}, \bar{z}_0) = \bar{z}_1;\\
F_0(\alpha_0, \bar{z}_d) = F_d(\alpha_d, \bar{z}_d) = \ldots = F_{(m-1)d}(\alpha_{(m-1)d}, \bar{z}_d) = \bar{z}_{d+1};\\
\vdots\\
F_0(\alpha_0, \bar{z}_{(\ell-1)d}) = F_d(\alpha_d, \bar{z}_{(\ell-1)d}) = \ldots = F_{(m-1)d}(\alpha_{(m-1)d}, \bar{z}_{(\ell-1)d}) = \bar{z}_{(\ell-1)d+1}.
\end{aligned}$$

Since the maps are one to one in the parameter vector, they do not intersect, unless they are equal. Similarly, one may show that $F_i = F_{i+d} = \ldots = F_{i+(m-1)d}$, $i \in \mathbb{Z}^+$. This implies that Eq. (1.6) is of minimal period d, a contradiction. ■

As we remarked earlier if the $p-$periodic nonautonomous difference equation has a periodic cycle of minimal period r, then the associated skew-product system π has a periodic cycle of period $s = lcm[r, p]$ ($s-$complete cycle). There are p fibers $\mathcal{F}_i = \mathcal{P}^{-1}(F_i)$. Are the s periodic points equally distributed on the fibers?

Before giving the definitive answer to this question, let us examine the diagram presented in Fig. 1.2 in which $p = 9$, and $r = 6$.

There are two points on each fiber. The graphs F_0, F_3, and F_6 intersect at the two points $(\bar{z}_0, \bar{z}_1)$, $(\bar{z}_3, \bar{z}_4)$; the graphs F_1, F_4, and F_7 intersect at the two points $(\bar{z}_1, \bar{z}_2)$, $(\bar{z}_4, \bar{z}_5)$; and the graphs F_2, F_5, F_8 intersect at the points $(\bar{z}_2, \bar{z}_3)$, $(\bar{z}_5, \bar{z}_0)$.

Note that the number of periodic points on each fiber is 2, which is $\ell = \frac{lcm[r,p]}{p}$. The following crucial lemma proves this observation.

Lemma 9 *[28] The orbit of $(\bar{z}_i, F_i)$ in the skew-product system intersects each fiber $\mathcal{F}_j$, $j = 0, 1, \ldots, p-1$, in exactly $\ell = s/p$ points and each of these points is periodic under the skew-product π with period s.*

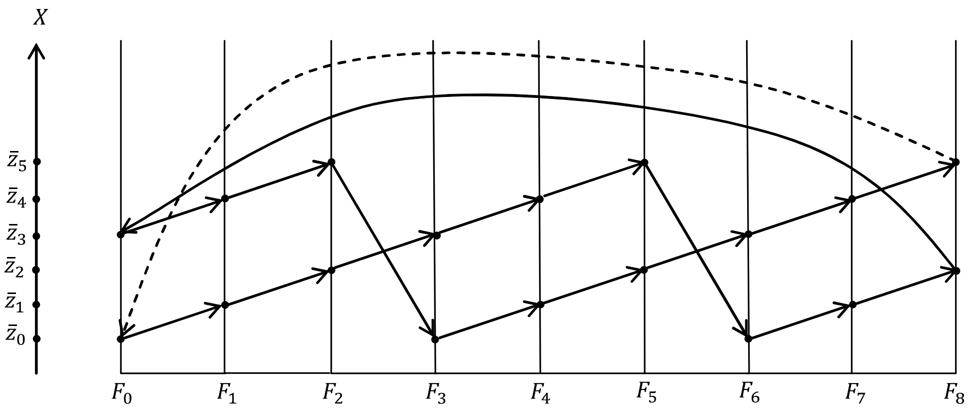

Figure 1.2: A 6−periodic cycle in a 9−periodic system.

1.3 Stability

Let $\lambda_1, \lambda_2, \ldots, \lambda_q$ be the (real or complex - not necessary different) eigenvalues of the real or complex matrix $A_{q\times q}$. The spectral radius of the matrix A is given by

$$\rho(A) = \max\{|\lambda_i| : 1 \leq i \leq q\}.$$

It is well known that the fixed point z^* of the autonomous equation (1.1) is asymptotically stable (respectively unstable) if $\rho\,(JF(z^*)) < 1$ (respectively $\rho\,(JF(z^*)) > 1$), where $JF(z^*)$ is the Jacobian of F evaluated at z^*.

In particular, when $q = 2$, i.e., when (x^*, y^*) is a fixed point of the two dimensional autonomous equation (1.1), it is known [26, pp 200] that $\rho\,(JF(x^*, y^*)) < 1$ if and only if

$$|tr\,(JF(x^*, y^*))\,| - 1 < det\,(JF(x^*, y^*)) < 1, \tag{1.7}$$

where tr and det denote the trace and the determinant of the matrix, respectively, and $JF(x^*, y^*)$ is the Jacobian evaluated at the fixed point (x^*, y^*). Using the trace-determinant analysis, in [26] the author presents a complete stability analysis for 2− dimensional systems.

In Section 1.4 we will present the techniques necessary to determine the stability of the fixed point z^* when $\rho\,(JF(z^*)) = 1$ for non-linear maps.

On the space X if $C_r = \{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}\}$, $r > 1$, is an r−periodic cycle of Eq. (1.1), then it is asymptotically stable if

$$\rho\left(\prod_{i=s-1}^{0} JF(\bar{z}_{i \bmod r})\right) < 1, \tag{1.8}$$

where $JF(\bar{z}_{i \bmod r})$ is the Jacobian evaluated along the periodic orbit C_r.

From linear algebra, recall that $\rho(A) \leq \|A\|$ for any norm, where A is a $q \times q$ real or complex matrix, but equality does not necessarily hold. Indeed, for a given norm there are matrices A such that $\rho(A)$ and $\|A\|$ may be arbitrarily far apart. On the other hand, there is a result that shows that, for a given matrix A we can always find some norm such that $\|A\|$ is arbitrarily close to $\rho(A)$ (for more details about this point see [59]). In our context it says that for any $\epsilon > 0$ there is a norm $|||\cdot|||$ on $\mathbb{R}^q$ (or $\mathbb{C}^q$) such that

$$|||\prod_{i=s-1}^{0} JF(\bar{z}_{i \bmod r})||| \leq \rho\left(\prod_{i=s-1}^{0} JF(\bar{z}_{i \bmod r})\right) + \epsilon.$$

Taking ϵ so small as we want, under condition (1.8), it is possible to find a norm $|||\cdot|||$ such that

$$|||\prod_{i=s-1}^{0} JF(\overline{z}_{i \bmod r})||| < 1.$$

Notice that if $A_1, A_2, \ldots, A_n$ are $q \times q$ matrices, then $A_1A_2 \ldots A_n$ and the cyclic permutations $A_{j+1} \ldots A_nA_1 \ldots A_j$ have the same set of eigenvalues, where $1 \leq j \leq n$ (see [4]). Thus each of the r matrices of the permuted Jacobian products has the same set of eigenvalues. Consequently, the order of the products in $\prod_{i=r-1}^{0} JF(\overline{x}_i, \overline{y}_i)$ is irrelevant for the spectral radius.

We now study stability of the nonautonomous equation. We start by the basic definitions of stability.

Definition 10 *Let $C_r = \{\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_{r-1}\}$ be an $r-$periodic cycle in the $p-$periodic equation (1.5) in a metric space $(X, \widehat{d})$, and $s = lcm[r, p]$ be the least common multiple of p and r. Then*

1. *C_r is stable if given $\epsilon > 0$, there exists $\delta > 0$ such that*
$$\widehat{d}(z_0, \overline{z}_{i \bmod r}) < \delta \text{ implies } \widehat{d}(\Phi^i_{ns}(z_0), \Phi^i_{ns}(\overline{z}_{i \bmod r})) < \epsilon,$$
for all $n \in \mathbb{Z}^+$, and $0 \leq i \leq p-1$. Otherwise, C_r is said unstable.

2. *C_r is attracting if there exists $\eta > 0$ such that*
$$\widehat{d}(z_0, \overline{z}_{i \bmod r}) < \eta \text{ implies } \lim_{n\to\infty} \Phi^i_{ns}(z_0) = \overline{z}_{i \bmod r},$$
for all $n \in \mathbb{Z}^+$, and $0 \leq i \leq p-1$.

3. *We say that C_r is asymptotically stable if it is both stable and attracting. If in addition, $\eta = \infty$, C_r is said to be globally asymptotically stable.*

Before we present an immediate (basic) consequence of this definition let us recall the chain rule for vector maps which is well known but not used frequently. It says that the Jacobian matrix of the composition of maps is the product of the Jacobian matrices of the maps. In our notation, let us to write $z \in X$ as $(z^1, z^2, \ldots, z^q)$ and each map $F_n \in X$ as $(F^1_n, F^2_n, \ldots, F^q_n)$, $n \in \mathbb{Z}^+$. For each $0 \leq i \leq p-1$ we write the Jacobian of the composition operator Φ^i_m as

$$J\Phi^i_m(z) = \frac{\partial(\Phi^{1,i}_m, \Phi^{2,i}_m, \ldots, \Phi^{q,i}_m)}{\partial(z^1, z^2, \ldots, z^q)} = \begin{pmatrix} \frac{\partial(\Phi^{1,i}_m)}{\partial z^1} & \cdots & \frac{\partial(\Phi^{1,i}_m)}{\partial z^q} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\Phi^{q,i}_m)}{\partial z^1} & \cdots & \frac{\partial(\Phi^{q,i}_m)}{\partial z^q} \end{pmatrix}.$$

Using the chain rule this last matrix is equivalent to

$$\begin{pmatrix} \frac{\partial F^1_{m+i-1}}{\partial \Phi^{1,i}_{m-1}} & \cdots & \frac{\partial F^1_{m+i-1}}{\partial \Phi^{q,i}_{m-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^q_{m+i-1}}{\partial \Phi^{1,i}_{m-1}} & \cdots & \frac{\partial F^q_{m+i-1}}{\partial \Phi^{q,i}_{m-1}} \end{pmatrix} \times \ldots \times \begin{pmatrix} \frac{\partial F^1_{i+1}}{\partial \Phi^{1,i}_{1}} & \cdots & \frac{\partial F^1_{i+1}}{\partial \Phi^{q,i}_{1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^q_{i+1}}{\partial \Phi^{1,i}_{1}} & \cdots & \frac{\partial F^q_{i+1}}{\partial \Phi^{q,i}_{1}} \end{pmatrix} \times \begin{pmatrix} \frac{\partial F^1_{i}}{\partial z^1} & \cdots & \frac{\partial F^1_{i}}{\partial z^q} \\ \vdots & \ddots & \vdots \\ \frac{\partial F^q_{i}}{\partial z^1} & \cdots & \frac{\partial F^q_{i}}{\partial z^q} \end{pmatrix},$$

or

$$JF_{m+i-1}\left(\Phi^i_{m-1}(z)\right) \times \ldots \times JF_{i+1}\left(\Phi^i_1(z)\right) \times JF_i(z).$$

Thus,

$$J\Phi^i_m(z) = \prod_{j=m-1}^{0} JF_{j+i}\left(\Phi^i_j(z)\right), \Phi^i_0(z) = z.$$

Theorem 11 *Let X be a metric space induced by the norm $|||\cdot|||$, $C_r = \{\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{r-1}\}$ be an $r-$periodic cycle of the $p-$periodic equation (1.5), and $s = lcm[r, p]$ be the least common multiple of p and r. Then C_r is asymptotically stable if $||| \prod_{i=s-1}^{0} JF_{i\ mod\ p}(\bar{z}_{i \bmod r})||| < 1$.*

Proof. From the hypothesis the norm of the Jacobian of the composition operator is upper bounded by one, i.e., there exists an open sphere $S_\epsilon(\bar{z}_{i \bmod r})$ with center $\bar{z}_{i \bmod r}$ and radius ϵ such that

$$|||J\Phi_s^i(z)||| \leq M < 1, \forall z \in S_\epsilon(\bar{z}_{i \bmod r}).$$

Then using the mean value theorem for vector-valued function we have

$$|||\Phi_s^i(z_0) - \Phi_s^i(\bar{z}_{i \bmod r})||| \leq M||z_0 - \bar{z}_{i \bmod r}|||,$$

for $0 \leq i \leq p-1$, $z_0 \in S_\epsilon(\bar{z}_{i \bmod r})$, or equivalently,

$$|||\Phi_s^i(z_0) - \bar{z}_{i \bmod r}||| \leq M|||z_0 - \bar{z}_{i \bmod r}|||.$$

Since $M < 1$ the last inequality implies that $\Phi_s^i(z_0) \in S_\epsilon(\bar{z}_{i \bmod r})$, $0 \leq i \leq p-1$. Using the same argument one obtain

$$|||\Phi_{2s}^i(z_0) - \bar{z}_{i \bmod r}||| \leq M^2|||Z_0 - \bar{z}_{i \bmod r}|||.$$

By induction we can prove that

$$|||\Phi_{ns}^i(z_0) - \bar{z}_{i \bmod r}||| \leq M^n|||z_0 - \bar{z}_{i \bmod r}|||, \forall n \in \mathbb{Z}^+. \tag{1.9}$$

This implies that $\lim_{n\to\infty} \Phi_{ns}^i(z_0) = \bar{z}_{i \bmod r}$, $0 \leq i \leq p-1$. Thus C_r is attracting.

To see the stability of C_r note that the following relation holds

$$\bar{z}_{i \bmod r} = \Phi_{ns}^i(\bar{z}_{i \bmod r}), \forall n \in \mathbb{Z}^+.$$

Consequently the relation (1.9) is equivalent to

$$|||\Phi_{ns}^i(z_0) - \Phi_{ns}^i(\bar{z}_{i \bmod r})||| \quad \leq \quad M^n|||z_0 - \bar{z}_{i \bmod r}|||,$$

$0 < M < 1$ and $0 \leq i \leq p-1$. Taking $\delta < \epsilon$, for any $\epsilon > 0$ we obtain

$$\begin{aligned} |||\Phi_{ns}^i(z_0) - \Phi_{ns}^i(\bar{z}_{i \bmod r})||| &\leq M^n|||Z_0 - \overline{Z}_{i \bmod r}||| \\ &< M^n\delta \\ &< M^n\epsilon < \epsilon. \end{aligned}$$

Thus C_r is stable. Consequently, C_r is asymptotically stable. ■

Remark 12 *Notice that if z^* is a fixed point of the sequence of maps F_i, $i \geq 0$, i.e., z^* is a fixed point of each individual map, then one can adapt the precedent result and show that z^* is asymptotically stable if $||| \prod_{i=0}^{n} JF_i(z^*)||| < 1$, for all $n \geq 0$.*

Consider the skew-product system π on $X \times Y$ with X a metric space with metric $\widehat{d}$, $Y = \{F_0, F_1, \ldots, F_{p-1}\}$ equipped with the discrete metric $\widetilde{d}$, where

$$\widetilde{d}(F_i, F_j) = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}.$$

Define a metric d^π on $X \times Y$ as

$$d^\pi = \Big((z, F_i), (u, F_j)\Big) = \widehat{d}(z, u) + \widetilde{d}(F_i, F_j).$$

Let $\pi^1(z, F) = \pi((z, F), 1)$, then $\pi^n(z, F) = \pi((z, F), n)$. Thus $\pi^1 : X \times Y \to X \times Y$ is a continuous map which generates an autonomous system on $X \times Y$. Consequently, the stability definitions of fixed points and periodic cycles follow the standard ones that may be found in [24, 26].

Now we give a definition of stability for a complete periodic cycle in the skew-product system.

Definition 13 *A complete periodic cycle* $\widehat{C_s} = \{(\overline{z}_0, F_0), (\overline{z}_1, F_1), ..., (\overline{z}_{(s-1) \bmod r}, F_{(s-1)\, mod\, p})\}$ *is*

1. *stable if given* $\epsilon > 0$*, there exits* $\delta > 0$*, such that*

$$d^\pi((z_0, F_i), (\overline{z}_0, F_0)) < \delta \textit{ implies } d^\pi(\pi^{ns}(z_0, F_i), \pi^{ns}(\overline{z}_0, F_0)) < \epsilon, \forall n \in \mathbb{Z}^+.$$

   Otherwise, $\widehat{C_s}$ *is said unstable;*

2. *attracting if there exists* $\eta > 0$ *such that*

$$d^\pi((z_0, F_i), (\overline{z}_0, F_0)) < \eta \textit{ implies } \lim_{n\to\infty} \pi^{ns}(z_0, F_i) = (\overline{z}_0, f_0);$$

3. *asymptotically stable if it is both stable and attracting. If in addition,* $\eta = \infty$*,* $\widehat{C_s}$ *is said to be globally asymptotically stable.*

Since $F_{i+ns} = F_i$ for all n, it follows from the above convergence that $F_i = F_i \bmod p$. Hence, stability can occur only on each fiber $X \times \{F_i\}, 0 \le i \le p - 1$.

It should be noted that one may reformulate Theorem 11 in the setting of the skew-product system. However, to do so, one needs to develop the notion of derivative in the space $X \times Y$.

Definition 14 *Let* $\pi^m : X \times Y \to X \times Y$ *defined as* $\pi^m(z, F_i) = (\Phi^i_m(z), F_{i+m})$*. The generalized derivative of* π^m *is defined as*

$$D(\pi^m(z, F_i)) = J\Phi^i_m(z) = \prod_{j=m-1}^{0} JF_{j+i}(\Phi^i_j(z)),$$

with $\Phi^i_0(z) = z$*. In particular, when* $X = \mathbb{R}$ *we write*

$$D(\pi^m(x, f_i)) = \frac{d}{dx}(\Phi^i_m(x)) = \left(\Phi^i_m\right)'(x) = \prod_{j=m-1}^{0} f'_{j+i}(\Phi^i_j(x)), \Phi^i_0(x) = x.$$

Theorem 15 *A complete periodic cycle* $\widehat{C_s} = \{(\overline{z}_0, F_0), (\overline{z}_1, F_1), ..., (\overline{z}_{(s-1) \bmod r}, F_{(s-1)\, mod\, p})\}$ *of the skew-product system* π *on* $X \times Y$ *is asymptotically stable if* $|||D(\pi^s(\overline{z}_{i \bmod r}, F_i))||| < 1$.

Proof. The proof is analogous to the proof of Theorem 11 and it will be omitted. ■

We now present a crucial result in the settings of periodic discrete dynamical systems.

Theorem 16 *[28] Assume that X is a connected metric space and each $F_i \in Y$ is a continuous map on X, with $F_{i+p} = F_i$. Let $C_r = \{\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_{r-1}\}$ be a periodic cycle of minimal period r of the $p-$periodic nonautonomous difference equation (1.5). If C_r is globally asymptotically stable, then r divides p. Moreover, $r = p$ if the sequence $\{F_n\}$ is a one-parameter family of maps $F(\alpha_n, z)$ and F is one to one with respect to α.*

Obviously, the stability of the periodic cycle can be studied via the spectral radius because for any matrix A one has $\rho(A) \leq \|A\|$ for any norm. Thus if

$$\|\prod_{i=s-1}^{0} JF_i(\overline{z}_{i \bmod r})\| < 1,$$

then it follows that all the eigenvalues of $\prod_{i=s-1}^{0} JF_i(\overline{z}_{i \bmod r})$ lie inside the unit disc.

In general, it is not easy and in most of the concrete cases it is an unknown problem, to determine the product of the Jacobians along the periodic orbit for nonautonomous difference equation in higher dimension. If it is possible to determine this product, then we can speak about stability of the cycle when $\rho\left(\prod_{i=s-1}^{0} JF_i(\overline{z}_{i \bmod r})\right) < 1$.

We now consider the particular case when $X = \mathbb{R}$. If $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ is an $r-$periodic cycle of the one-dimensional nonautonomous $p-$periodic difference equation (1.5), then C_r is asymptotically stable if

$$|\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r})| < 1,$$

and it is unstable if

$$|\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r})| > 1,$$

where s is the least common multiple of r and p. Note that $\overline{x}_{i \bmod r}$ is a fixed point of the composition operator Φ_s^i.

When the periodic cycle is nonhyperbolic the stability criteria are more involved. These criteria will be summarized in the next two results. The first treats the case when $\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r}) = 1$ while the second treats the case $\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r}) = -1$. Since the proof follows the same techniques as in [26] for one map we will omit the proof here.

Theorem 17 *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of the one-dimensional nonautonomous $p-$periodic difference equation (1.5) such that $\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r}) = 1$. If $\left(\Phi_s^i\right)'(x)$, $\left(\Phi_s^i\right)''(x)$ and $\left(\Phi_s^i\right)'''(x)$ are continuous at $\overline{x}_{i \bmod r}$, then the following statements hold:*

1. *if $\left(\Phi_s^i\right)''(\overline{x}_{i \bmod r}) \neq 0$, then C_r is unstable;*
2. *if $\left(\Phi_s^i\right)''(\overline{x}_{i \bmod r}) = 0$ and $\left(\Phi_s^i\right)'''(\overline{x}_{i \bmod r}) > 0$, then C_r is unstable;*
3. *if $\left(\Phi_s^i\right)''(\overline{x}_{i \bmod r}) = 0$ and $\left(\Phi_s^i\right)'''(\overline{x}_{i \bmod r}) < 0$, then C_r asymptotically stable.*

Before present the second case, we need to introduce the notion of the **Schwarzian derivative**.

Definition 18 *The **Schwarzian derivative**, Sf, of a C^3 function f is defined by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

Theorem 19 *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of the one-dimensional nonautonomous $p-$periodic difference equation (1.5) such that $\left(\Phi_s^i\right)'(\overline{x}_{i \bmod r}) = -1$. If $\left(\Phi_s^i\right)'(x)$, $\left(\Phi_s^i\right)''(x)$ and $\left(\Phi_s^i\right)'''(x)$ are continuous at $\overline{x}_{i \bmod r}$, then the following statements hold:*

1. *if $S\Phi_s^i(\overline{x}_{i \bmod r}) < 0$, then C_r is asymptotically stable;*
2. *if $S\Phi_s^i(\overline{x}_{i \bmod r}) > 0$, then C_r is unstable.*

In [22] the authors present a complete classification of nonhyperbolic fixed points for one-dimensional autonomous maps. This classification is still valid for the $r-$periodic cycle C_r, i.e., when $\overline{x}_{i \bmod r}$ is a fixed point of the one-dimensional map Φ_s^i. Hence, replacing the map f by Φ_s^i and x^* by $\overline{x}_{i \bmod r}$ in Fig. 1.20 in [26, page 33] one has a complete classification of nonhyperbolic periodic points for one-dimensional periodic systems.

1.4 Invariant manifold

In this section we present the appropriate tools that allow us, to compute analytically, the center manifold and the stable and unstable manifold for any nonlinear map near the fixed point. It should mention that this section is based on our article [48].

Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a map such that $F \in C^2$ and $F(0) = 0$. Then one may write F as a perturbation of a linear map L,

$$F(X) = LX + R(X), \tag{1.10}$$

where L is a $k \times k$ matrix defined by $L = D(F(0))$, $R(0) = 0$ and $DR(0) = 0$, where D denotes the Jacobian matrix. Now we will introduce special subspaces of $\mathbb{R}^k$, called invariant manifold [77], that will play a central role in our study of stability and bifurcation.

An invariant manifold is a manifold embedded in its phase space with the property that it is invariant under the dynamical system generated by F. A subspace M of $\mathbb{R}^k$ is an invariant manifold if whenever $X \in M$, then $F^n(X) \in M$, for all $n \in \mathbb{Z}^+$. For the linear map L, one may split its spectrum $\sigma(L)$ into three sets σ_s, σ_u, and σ_c, for which $\lambda \in \sigma_s$ if $|\lambda| < 1$, $\lambda \in \sigma_u$ if $|\lambda| > 1$, and $\lambda \in \sigma_c$ if $|\lambda| = 1$.

Corresponding to these sets, we have three invariant manifold (linear subspaces) E^s, E^u, and E^c which are the generalized eigenspaces corresponding to σ_s, σ_u, and σ_c, respectively. It should be noted that some of these subspaces may be trivial.

The main question here is how to extend this linear theory to nonlinear maps. Corresponding to the linear subspaces E^s, E^u, and E^c, we will have the invariant manifold the stable manifold W^s, the unstable manifold W^u, and the center manifold W^c.

The center manifold theory [9, 10, 40, 52, 76, 77] is interesting only if $W^u = \{0\}$. For in this case, the dynamics on the center manifold W^c determines the dynamics of the system. The other interesting case is when $W^c = \{0\}$ and we have a saddle.

Let $E^s \subset \mathbb{R}^s$, $E^u \subset \mathbb{R}^u$, and $E^c \subset \mathbb{R}^t$, with $s + u + t = k$. Then one may formally define the above mentioned invariant manifold as follows:

$$W^s(0) = \{x \in \mathbb{R}^k | F^n(x) \to 0, n \to \infty\},$$

and

$$W^u(0) = \{x \in \mathbb{R}^k | \exists \{q_n\}_{n=0}^{\infty}, q_0 = x, \text{ and } F(q_{k+1}) = q_k, q_n \to 0, n \to \infty\}.$$

It is noteworthy to mention that the center manifold is not unique, while the stable and unstable manifold are unique.

The next result summarizes the basic invariant manifold theory

Theorem 20 (Invariant manifold theorem) *[39, 52] Suppose that $F \in C^2$. Then there exist C^2 stable W^s and unstable W^u manifold tangent to E^s and E^u, respectively, at $X = 0$ and C^1 center manifold W^c tangent to E^c at $X = 0$. Moreover, the manifold W^c, W^s and W^u are all invariant.*

Before embark in the main aim of this section, notice that the techniques that we are going to develop in the next subsections can be applied to nonautonomous periodic systems in higher dimension. To do that is enough to replace the map F by the composition operator Φ_s^i, where s is the least common multiple of p and r, with p the minimum period of the system and r the minimum period of the cycle.

In working with concrete maps, as we remarked earlier, is not easy to find explicitly the product of the Jacobians along the periodic orbit.

1.4.1 Center manifold

Let us first focus on the case when $\sigma_u = \emptyset$. Hence the absolute value of the eigenvalues of L are less or equal to one. By a suitable change of variables, one may represent the map F as the following system of difference equations

$$\begin{cases} x_{n+1} = Ax_n + f(x_n, y_n) \\ y_{n+1} = By_n + g(x_n, y_n) \end{cases}. \tag{1.11}$$

First we assume that all eigenvalues of $A_{t\times t}$ are on the unit circle and all the eigenvalues of $B_{s\times s}$ are inside the unit circle, with $t + s = k$. Moreover,

$$f(0,0) = 0,\ g(0,0) = 0,\ Df(0,0) = 0 \text{ and } Dg(0,0) = 0.$$

Since W^c is tangent to $E^c = \{(x, y) \in \mathbb{R}^t \times \mathbb{R}^s | y = 0\}$, it may be represented locally as the graph of a function $h : \mathbb{R}^t \to \mathbb{R}^t$ such that

$$W^c = \{(x, y) \in \mathbb{R}^t \times \mathbb{R}^s | y = h(x), h(0) = 0, Dh(0) = 0, |x| < \delta\},$$

for a sufficiently small δ.

Furthermore, the dynamics restricted to W^c is given locally by the equation

$$x_{n+1} = Ax_n + f(x_n, h(x_n)), x \in R^t. \tag{1.12}$$

The main feature of Eq. (1.12) is that its dynamics determine the dynamics of Eq. (1.11). So if $x^* = 0$ is a stable, asymptotically stable, or unstable fixed point of Eq. (1.12), then the fixed point $(x^*, y^*) = (0, 0)$ of Eq. (1.11) possesses the corresponding property.

To find the map $y = h(x)$, we substitute for y in Eq. (1.11) and obtain

$$\begin{cases} x_{n+1} = Ax_n + f(x_n, h(x_n)) \\ y_{n+1} = h(x_{n+1}) = h(Ax_n + f(x_n, h(x_n))) \end{cases}. \tag{1.13}$$

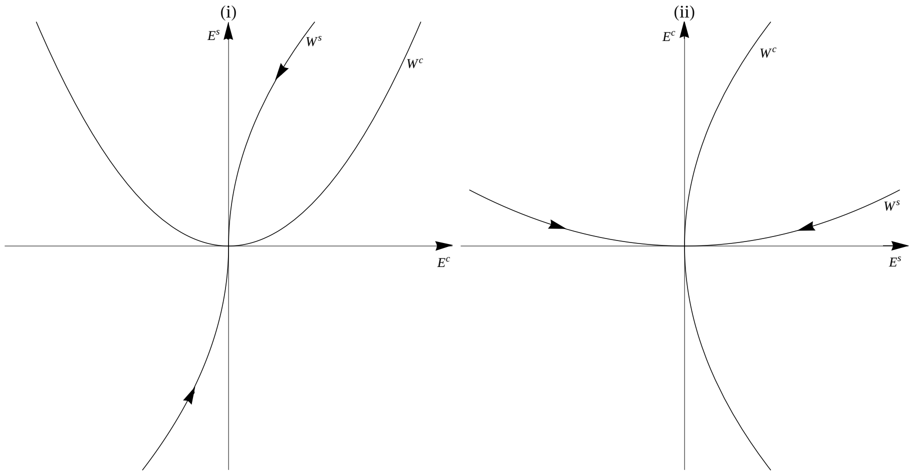

Figure 1.3: Stable and center manifold. In (i) we have $\sigma(A) = \sigma_c$ and $\sigma(B) = \sigma_s$ while in (ii) we have $\sigma(A) = \sigma_s$ and $\sigma(B) = \sigma_c$.

But

$$\begin{aligned} y_{n+1} &= By_n + g(x_n, y_n) \\ &= Bh(x_n) + g(x_n, h(x_n)). \end{aligned} \tag{1.14}$$

Equating (1.13) and (1.14) yields the center manifold equation

$$h[Ax_n + f(x_n, h(x_n))] = Bh(x_n) + g(x_n, h(x_n)). \tag{1.15}$$

Analogously if $\sigma(A) = \sigma_s$ and $\sigma(B) = \sigma_c$, one may define the center manifold W^c, and obtain the equation

$$y_{n+1} = By_n + g(h(y_n), y_n),$$

where $x = h(y)$.

1.4.2 An upper (lower) triangular System

In working with concrete maps, it is beneficial in certain cases to deal with the system without diagonalization.

Let us now consider the case when L is a block upper triangular matrix

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}. \tag{1.16}$$

There are two cases to consider:

1. Assume that $\sigma(A) = \sigma_s$, $\sigma(B) = \sigma_c$, and $\sigma_u = \emptyset$.

   The matrix L can be block diagonalizable. Hence there exists, a nonsingular matrix P of the form

   $$P = \begin{bmatrix} P_1 & P_3 \\ 0 & P_2 \end{bmatrix},$$

   such that

   $$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = P \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} P^{-1}.$$

Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.17}$$

Then $x = P_1 u + P_3 v$, and $y = P_2 v$. Thus one has

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + P^{-1} \begin{pmatrix} f(P_1 u + P_3 v, P_2 v) \\ g(P_1 u + P_3 v, P_2 v) \end{pmatrix}. \tag{1.18}$$

Applying the center manifold theorem to Eq. (1.18) yields a map $u = \tilde{h}(v)$ with $\tilde{h}(0) = 0 = \tilde{h}'(0)$. Moreover, the dynamics of Eq. (1.18) is completely determined by the dynamics of the equation

$$v_{n+1} = B v_n + \tilde{P}_2 g(P_1 \tilde{h}(v_n) + P_3 v_n, P_2 v_n),$$

where $\tilde{P}_1$ and $\tilde{P}_3$ are elements of the matrix

$$P^{-1} = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_3 \\ 0 & \tilde{P}_2 \end{bmatrix}.$$

We now have the relation

$$u = \tilde{P}_1 x - \tilde{P}_2 P_3 \tilde{P}_2 y = \tilde{h}(\tilde{P}_2 y).$$

Hence $x = h(y)$, where h is given by

$$h(y) = P_3 \tilde{P}_2 y + \tilde{P}_1^{-1} \tilde{h}_2(\tilde{P}_2 y).$$

Notice that $Dh(0) = P_3 \tilde{P}_2 I$, where I is the identity matrix.

2. Assume that $\sigma(A) = \sigma_c$, $\sigma(B) = \sigma_s$, and $\sigma_u = \emptyset$. We start from Eq. (1.18) and apply the center manifold theorem to obtain a map $v = \tilde{h}(u)$ with $\tilde{h}(0) = 0 = \tilde{h}'(0)$. The dynamics of Eq. (1.18) is completely determined by the dynamics of the equation

$$u_{n+1} = A u_n + \tilde{P}_1 f(P_1 u_n + P_3 \tilde{h}(u), P_2 \tilde{h}(u)) + \tilde{P}_3 g(P_1 u_n + P_3 \tilde{h}(u), P_2 \tilde{h}(u)), \tag{1.19}$$

where $\tilde{P}_1$, $\tilde{P}_2$, and $\tilde{P}_3$ are entries of the matrix

$$P^{-1} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_3 \\ 0 & \tilde{P}_2 \end{pmatrix}.$$

From (1.17) we have $u = \tilde{P}_1 x - \tilde{P}_1 P_3 \tilde{P}_2 y$ and $v = \tilde{P}_2 y$. Then $v = \tilde{h}(u)$ and thus

$$\tilde{P}_2 y = \tilde{h}(\tilde{P}_1 x - \tilde{P}_1 P_3 \tilde{P}_2 y).$$

Let $Q(x, y) = \tilde{P}_2 y - \tilde{h}(\tilde{P}_1 x - \tilde{P}_1 P_3 \tilde{P}_2 y)$. Then $Q(0, 0) = 0$, $DQ(0, 0)$ is of rank t. Hence by the implicit function theorem [65] there exits an open neighborhood $\Omega \subset \mathbb{R}^k$ of 0 and a unique function $h \in C^1(\Omega)$ such that $h(0) = 0 = Dh(0)$ and $Q(x, h(x)) = 0$, for all $x \in \Omega$.

Hence the curve $y = h(x)$ is the implicit solution of Eq. (1.19) and is the equation of the center manifold. To find the map h we use the center manifold equation

$$h[Ax + Ch(x) + f(x, h(x))] = Bh(x) + g(x, h(x)).$$

A final remark is in order. If we let $y = h(x)$ in (1.19) we obtain

$$h(x) = P_2 \tilde{h}(\tilde{P}_1 x - \tilde{P}_1 P_3 \tilde{P}_2 h(x)).$$

Note that $Dh(0) = D\tilde{h}(0) = 0$.

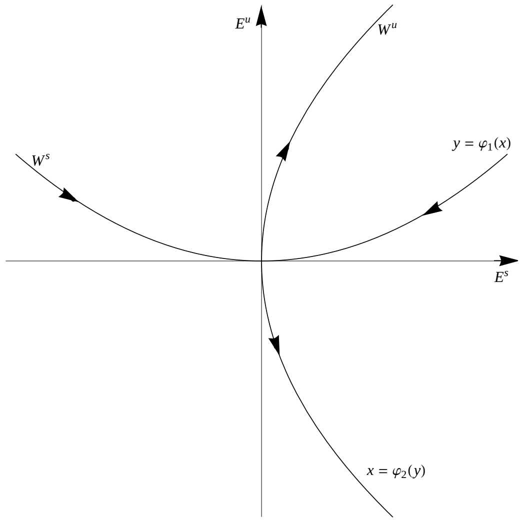

Figure 1.4: Stable and unstable manifold.

1.4.3 Stable and Unstable manifold

Suppose now that the map F is hyperbolic, that is $\sigma_c = \emptyset$. Then by Theorem 20, there are two unique invariant manifold W^s and W^u, both tangent to E^s and E^u at $X = 0$, which are graphs of the maps

$$\varphi_1 : E_1 \to E_2 \text{ and } \varphi_2 : E_2 \to E_1,$$

such that

$$\varphi_1(0) = \varphi_2(0) = 0 \text{ and } D(\varphi_1(0)) = D(\varphi_2(0)) = 0.$$

Letting $y_n = \varphi_1(x_n)$ yields

$$y_{n+1} = \varphi_1(x_{n+1}) = \varphi_1(Ax_n + C\varphi_1(x_n) + f(x_n, \varphi_1(x_n))).$$

But

$$y_{n+1} = B\varphi_1(x_n) + g(x_n, \varphi_1(x_n)).$$

Equating these two equations yields

$$\varphi_1(Ax_n + C\varphi_1(x_n) + f(x_n, \varphi_1(x))) = B\varphi_1(x_n) + g(x_n, \varphi_1(x_n)), \tag{1.20}$$

where we can take, without loss of generality, $\varphi_1(x) = \alpha_1 x^2 + \beta_1 x^3 + O(|x|^4)$.

Similarly, letting $x_n = \varphi_2(y_n)$ yields

$$x_{n+1} = \varphi_2(y_{n+1}) = \varphi_2(By_n + g(\varphi_2(y_n), y_n)),$$

where we can take, without loss of generality, $\varphi_2(x) = \alpha_2 x + \beta_2 x^2 + O(|x|^4)$.

But

$$x_{n+1} = A\varphi_2(y_n) + Cy_n + f(\varphi_2(y_n), y_n),$$

and hence

$$\varphi_2(By_n + g(\varphi_2(y_n), y_n)) = A\varphi_2(y_n) + Cy_n + f(\varphi_2(y_n), y_n). \tag{1.21}$$

Using Eq. (1.20) and Eq. (1.21), one can find the stable manifold

$$W^s = \{(x, y) \in \mathbb{R}^t \times \mathbb{R}^s | y = \varphi_1(x)\},$$

and the unstable manifold

$$W^u = \{(x, y) \in \mathbb{R}^t \times \mathbb{R}^s | x = \varphi_2(y)\}.$$

Bifurcation	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial \mu}$	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial \mu \partial x}$	$\frac{\partial^3 f}{\partial x^3}$	Sf
Saddle-node	1	$\neq 0$	$\neq 0$			
Transcritical	1	0	$\neq 0$	$\neq 0$		
Pitchfork	1	0	0	$\neq 0$	$\neq 0$	
Period-doubling	-1			$\neq 0$		$\neq 0$

Table 1.1: Types of bifurcation of nonhyperbolic fixed points in one-dimensional autonomous maps

1.5 Bifurcation in one-dimensional systems

The bifurcation theory for one-dimensional maps is well known and it may be found in any book of discrete dynamical systems. (See for instance [16, 26, 64, 37, 77]). Our aim here is to present a brief extension of this theory to nonautonomous periodic systems in which the sequence of maps arise from a one parameter family of maps.

It is noteworthy to mention that the techniques that we are going to follow are known for maps $f(x,\mu)$ with one variable x and one parameter μ. A good tutorial (in the sense of the language of difference equations) may be found in [60].

One can speak about bifurcation when a small change in the parameter (the bifurcation parameter) of a system causes a sudden qualitative change in its behavior.

- A saddle node bifurcation occurs if, near the bifurcation point (x^*,μ^*), the model possesses a unique curve of fixed points in the (μ, x) plane which passes through the bifurcation point and lies on one side of the line $\mu = \mu^*$. A saddle-node bifurcation has the normal form $x_{n+1} = x_n \pm \mu \pm x_n^2$.

- A transcritical bifurcation occurs if, near the bifurcation point (x^*,μ^*), the model possesses two curves of fixed points in the (μ, x) plane both of which pass through the bifurcation point and lies on both sides of the line $\mu = \mu^*$. An exchange of stability takes place at the bifurcation point. A transcritical bifurcation has the normal form $x_{n+1} = x_n \pm \mu x_n \pm x_n^2$.

- A pitchfork bifurcation occurs if, near the bifurcation point (x^*,μ^*), the model possesses two curves of fixed points in the (μ, x) plane both of which pass through the bifurcation point and one of which lies on both sides of the line $\mu = \mu^*$. A pitchfork bifurcation has the normal form $x_{n+1} = x_n \pm \mu x_n - a x_n^3$, where $a > 0$ is supercritical and $a < 0$ is subcritical.

- A period doubling bifurcation occurs if, near the bifurcation point (x^*,μ^*), the model possesses a single curve of fixed points in the (μ, x) plane, while the second iterate f^2 undergoes a pitchfork bifurcation at the bifurcation point.

In Table 1.1 we summarize the bifurcation conditions for a parameter family of unidimensional maps $f(x,\mu)$ in which (x^*,μ^*) is a nonhyperbolic fixed point.

1.5.1 Degeneracy

Before studying the bifurcation of periodic systems we present the following result which will be needed in the sequel.

Lemma 21 *Consider the set of maps $\mathcal{F} = \{f_0, f_1, \ldots, f_{p-1}\}$ and let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of the one-dimensional $p-$periodic difference equation*

$$x_{n+1} = f_n(x_n), f_{n+p} = f_p.$$

Then

$$\frac{\partial \Phi_s}{\partial x}(\overline{x}_0) = \frac{\partial \Phi_s^i}{\partial x}(\overline{x}_{i \bmod r}), \textit{ for any } i = 0, 1, 2, \ldots,$$

where s is the least common multiple of p and r.

Proof. If $i = 0$ then there is nothing to prove. Using the chain rule it follows

$$\begin{aligned}
\frac{\partial \Phi_s}{\partial x}(\overline{x}_0) &= \frac{\partial f_{s-1}}{\partial x}(\Phi_{s-1}(\overline{x}_0)) \times \frac{\partial f_{s-2}}{\partial x}(\Phi_{s-2}(\overline{x}_0)) \times \ldots \times \frac{\partial f_0}{\partial x}(\Phi_0(\overline{x}_0)) \\
&= \frac{\partial f_{s-1}}{\partial x}(\overline{x}_{s-1 \bmod r}) \times \frac{\partial f_{s-2}}{\partial x}(\overline{x}_{s-2 \bmod r}) \times \ldots \times \frac{\partial f_1}{\partial x}(\overline{x}_{1 \bmod r}) \times \frac{\partial f_0}{\partial x}(\overline{x}_0) \\
&= \prod_{j=0}^{s-1} \frac{\partial f_j}{\partial x}\left(\overline{x}_{j \bmod r}\right).
\end{aligned}$$

Since $\overline{x}_0 = \overline{x}_{s \bmod r}$ and $f_0 = f_s$ one has

$$\begin{aligned}
\frac{\partial \Phi_s}{\partial x}(\overline{x}_0) &= \frac{\partial f_0}{\partial x}(\overline{x}_0) \times \prod_{j=1}^{s-1} \frac{\partial f_j}{\partial x}\left(\overline{x}_{j \bmod r}\right) \\
&= \frac{\partial f_s}{\partial x}(\overline{x}_{s \bmod r}) \times \prod_{j=1}^{s-1} \frac{\partial f_j}{\partial x}\left(\overline{x}_{j \bmod r}\right) \\
&= \prod_{j=1}^{s} \frac{\partial f_j}{\partial x}\left(\overline{x}_{j \bmod r}\right) \\
&= \frac{\partial \Phi_s^1}{\partial x}(\overline{x}_{1 \bmod r}).
\end{aligned}$$

Similarly, $\overline{x}_{1 \bmod r} = \overline{x}_{1+s \bmod r}$ and $f_1 = f_{1+s}$. Hence

$$\frac{\partial \Phi_s}{\partial x}(\overline{x}_0) = \prod_{j=2}^{s+1} \frac{\partial f_j}{\partial x}\left(\overline{x}_{j \bmod r}\right) = \frac{\partial \Phi_s^2}{\partial x}(\overline{x}_{2 \bmod r}).$$

Continuing this process, for any $i = 0, 1, 2, \ldots$ one has

$$\frac{\partial \Phi_s}{\partial x}(\overline{x}_0) = \frac{\partial \Phi_s^1}{\partial x}(\overline{x}_{1 \bmod r}) = \ldots = \frac{\partial \Phi_s^i}{\partial x}(\overline{x}_{i \bmod r}).$$

■

Hence, by the above observation, for the degeneracy conditions in the variable, it is enough to work with the derivative $\frac{\partial \Phi_s}{\partial x}(\overline{x}_0)$.

Recall that we have three possible situations: $r < p$, $r = p$ or $r > p$. In any case $\overline{x}_{i \bmod r}$ is a fixed point of the composition operator Φ_s^i, where s is the least common multiple of r and p.

In [27] we presented a result for higher degeneracy conditions in the variable when there exits an eigenvalue 1. This result is based on the fact that $\overline{x}_{i \bmod p}$ is a fixed point of the composition operator Φ_p^i. In other words, when the maps are one to one in the parameter. Hence, by Theorem 8 the minimal period of the cycle is $r = tp$, $t = 1, 2, \ldots$. A simple extension gives the same result using the composition operator Φ_s^i and the cycle $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$.

Lemma 22 *[27] Consider the set of maps $\mathcal{F} = \{f_0, f_1, \ldots, f_{p-1}\}$ and let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of the one-dimensional $p-$periodic difference equation*

$$x_{n+1} = f_n(x_n), f_{n+p} = f_p.$$

Assume that the maps have a sufficient number of derivatives satisfying the conditions:

1. *The first bifurcation condition holds*

$$\left.\frac{\partial \Phi_s}{\partial x}\right|_{x=\overline{x}_0} = \prod_{j=0}^{s-1} \frac{df_j}{dx}\left(\overline{x}_{j \bmod r}\right) = 1.$$

2. *Higher degeneracy conditions hold for Φ_s*

$$m \geq 2: \left.\frac{\partial^n \Phi_s}{\partial x^n}(x)\right|_{x=\overline{x}_{i \bmod r}} = 0, \qquad 2 \leq n \leq m.$$

Then the composition operator Φ_s^i satisfies

$$\left.\frac{\partial^n \Phi_s^i}{\partial x^n}(x)\right|_{x=\overline{x}_{i \bmod r}} = 0, i = 0, 1, 2, \ldots \text{ and } 2 \leq n \leq m.$$

From now own, in this section, we assume that the sequence of maps f_i in the set $\mathcal{F} = \{f_0, f_1, \ldots, f_{p-1}\}$ arises from a one parameter family of maps, i.e, each map is defined as $f_i(x) = f(x, \mu_i)$ where $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Moreover, we assume that the sequence of parameters is $p-$periodic, i.e, $\mu_i = \mu_{i \bmod p}$. Under this assumption the one-dimensional difference equation

$$x_{n+1} = f_n(x_n), \tag{1.22}$$

is $p-$periodic because one has $f_n = f_{n+p}$ for all $n \in \mathbb{Z}^+$.

Lemma 23 *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of equation (1.22) and assume that the sequence of maps f_i in $\mathcal{F}$ has a sufficient number of derivatives satisfying the conditions:*

1. $$\left.\frac{\partial \Phi_s}{\partial x}(x)\right|_{x=\overline{x}_0} = \prod_{j=0}^{s-1} \frac{df_j}{dx}\left(\overline{x}_{j \bmod r}\right) = 1.$$

2. $$\left.\frac{\partial \Phi_s}{\partial \mu_0}(x)\right|_{x=\overline{x}_0} = 0.$$

Then

$$\left.\frac{\partial \Phi_s^i}{\partial \mu_0}(x)\right|_{x=\overline{x}_{i \bmod r}} = 0, i = 0, 1, 2, \ldots.$$

Proof. Let s be the lest common multiple of p and r. By lemma 9 the orbit of C_r, in the skew-product system, intersects each fiber $\mathcal{F}_j$, $j = 0, 1, \ldots, p-1$, in exactly $\ell = s/p$ points. Hence

$$\begin{gathered} f_0 = f_p = f_{2p} = \ldots = f_{(\ell-1)p}, \\ f_1 = f_{p+1} = f_{2p+1} = \ldots = f_{(\ell-1)p+1}, \\ \ldots \\ f_{p-1} = f_{2p-1} = f_{3p-1} = \ldots = f_{\ell p-1}. \end{gathered}$$

Note that $\ell p - 1 = s - 1$.

Using the chain rule it follows

$$\begin{aligned} \frac{\partial \Phi_s}{\partial \mu_0}(x)\Big|_{x=\bar{x}_0} = & \frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{\ell p-1 \bmod r}\right) \times \ldots \times \frac{\partial f_1}{\partial x}\left(\bar{x}_{(\ell-1)p+1 \bmod r}\right) \times \frac{\partial f_0}{\partial \mu_0}\left(\bar{x}_{(\ell-1)p \bmod r}\right) \times \\ & \frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(\ell-1)p-1 \bmod r}\right) \times \ldots \times \frac{\partial f_1}{\partial x}\left(\bar{x}_{(\ell-2)p+1 \bmod r}\right) \times \frac{\partial f_0}{\partial \mu_0}\left(\bar{x}_{(\ell-2)p \bmod r}\right) \times \\ & \vdots \\ & \frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{p-1 \bmod r}\right) \times \ldots \times \frac{\partial f_1}{\partial x}\left(\bar{x}_{1 \bmod r}\right) \times \frac{\partial f_0}{\partial \mu_0}\left(\bar{x}_0\right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial \Phi_s}{\partial \mu_0}(x)\Big|_{x=\bar{x}_0} = & \left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right) \times \ldots \times \prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right) \\ & \times \prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\bar{x}_{ip \bmod r}\right)\right). \end{aligned}$$

But $f_0 = f_p$ and $\bar{x}_0 = \bar{x}_{s \bmod r} = \bar{x}_{\ell p \bmod r}$. Consequently,

$$\begin{aligned} & \prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\bar{x}_{(i+1)p \bmod r}\right)\right) \times \prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right) \\ & \qquad \times \ldots \times \prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right) = \frac{\partial \Phi_s^1}{\partial \mu_0}(x)\Big|_{x=\bar{x}_{1 \bmod r}}. \end{aligned}$$

Now we can see that $f_1 = f_{p+1}$ and $\bar{x}_{1 \bmod r} = \bar{x}_{\ell p+1 \bmod r}$. Then

$$\begin{aligned} & \prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p+1}}{\partial x}\left(\bar{x}_{(i+1)p+1 \bmod r}\right)\right) \prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\bar{x}_{ip+s \bmod r}\right)\right) \times \\ & \prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right) \times \ldots \times \prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right) = \frac{\partial \Phi_s^2}{\partial \mu_0}(x)\Big|_{x=\bar{x}_{2 \bmod r}}. \end{aligned}$$

Continuing this process, one can see that

$$\frac{\partial \Phi_s}{\partial \mu_0}(x)\Big|_{x=\bar{x}_0} = \frac{\partial \Phi_s^i}{\partial \mu_0}(x)\Big|_{x=\bar{x}_{i \bmod r}},$$

for any $i = 0, 1, 2, \ldots$

Since

$$\left.\frac{\partial \Phi_s}{\partial x}(x)\right|_{x=\overline{x}_0} = \left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right) \times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial x}\left(\overline{x}_{ip \bmod r}\right)\right) = 1,$$

it follows that

$$\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right) \neq 0. \tag{1.23}$$

Consequently, by the fact that $\left.\frac{\partial \Phi_s}{\partial \mu_0}(x)\right|_{x=\overline{x}_0} = 0$ it follows that

$$\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right) = 0. \tag{1.24}$$

Thus $\left.\frac{\partial \Phi_s^i}{\partial \mu_0}(x)\right|_{x=\overline{x}_{i \bmod r}} = 0$, $i = 0, 1, 2, \ldots$. ■

Remark 24 *1. Notice that the following relation holds true*

$$\left.\frac{\partial \Phi_s}{\partial \mu_0}(x)\right|_{x=\overline{x}_0} = \left.\frac{\partial \Phi_s^i}{\partial \mu_0}(x)\right|_{x=\overline{x}_{i \bmod r}},$$

for any $i = 0, 1, 2, \ldots$, without any condition in the derivatives.

2. It is obvious that other degeneracy conditions on the parameters $\mu_1, \ldots, \mu_{p-1}$ are similar.

Lemma 25 *Let $C_r = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{r-1}\}$ be an $r-$periodic cycle of equation $x_{n+1} = f_n(x_n)$, $f_{n+p} = f_n$, with $f_i = f(x, \mu_i)$ and assume that the sequence of maps f_i in $\mathcal{F}$ has a sufficient number of derivatives satisfying the conditions:*

1.

$$\left.\frac{\partial \Phi_s}{\partial x}(x)\right|_{x=\overline{x}_0} = \prod_{j=0}^{s-1}\frac{df_j}{dx}\left(\overline{x}_{j \bmod r}\right) = 1.$$

2.

$$\left.\frac{\partial^n \Phi_s}{\partial \mu_0^n}(x)\right|_{x=\overline{x}_0} = 0, \textit{ for any } n \geq 2.$$

Then

$$\left.\frac{\partial^n \Phi_s^i}{\partial \mu_0^n}(x)\right|_{x=\overline{x}_{i \bmod r}} = 0, \textit{ for all } i = 0, 1, 2, \ldots.$$

Proof. Since

$$\begin{aligned}\left.\frac{\partial\Phi_s}{\partial\mu_0}(x)\right|_{x=\bar{x}_0} &= \left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right)\\ &\quad\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial\mu_0}\left(\bar{x}_{ip \bmod r}\right)\right),\end{aligned}$$

it follows that

$$\begin{aligned}\left.\frac{\partial^2\Phi_s}{\partial\mu_0^2}(x)\right|_{x=\bar{x}_0} &= \frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right]\\ &\quad\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial\mu_0}\left(\bar{x}_{ip \bmod r}\right)\right)+\\ &\quad\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right)\times\\ &\quad\frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial\mu_0}\left(\bar{x}_{ip \bmod r}\right)\right)\right].\end{aligned}$$

By (1.23) and (1.24) we have that

$$\left.\frac{\partial^2\Phi_s}{\partial\mu_0^2}(x)\right|_{x=\bar{x}_0} = 0 \; iff \; \frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial\mu_0}\left(\bar{x}_{ip \bmod r}\right)\right)\right] = 0.$$

But $f_0 = f_p$ and $\bar{x}_0 = \bar{x}_{s \bmod r} = \bar{x}_{\ell p \bmod r}$, and thus we have that $\left.\frac{\partial^2\Phi_s^1}{\partial\mu_0^2}(x)\right|_{x=\bar{x}_0}$ is given by

$$\begin{aligned}&\frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial\mu_0}\left(\bar{x}_{(i+1)p \bmod r}\right)\right)\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right]=\\ &\frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial\mu_0}\left(\bar{x}_{(i+1)p \bmod r}\right)\right)\right]\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)+\\ &\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial\mu_0}\left(\bar{x}_{(i+1)p \bmod r}\right)\right)\times\frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right].\end{aligned}$$

Hence

$$\begin{aligned}\left.\frac{\partial^2\Phi_s^1}{\partial\mu_0^2}(x)\right|_{x=\bar{x}_0} &= 0\times\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right)+\\ &\quad+0\times\left(\frac{\partial}{\partial\mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\bar{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\bar{x}_{ip+1 \bmod r}\right)\right)\right]\right)\\ &= 0.\end{aligned}$$

Continuing this process, one can see that

$$\left.\frac{\partial^2\Phi_s}{\partial^2\mu_0}(x)\right|_{x=\bar{x}_0} = \left.\frac{\partial^2\Phi_s^i}{\partial^2\mu_0}(x)\right|_{x=\bar{x}_{i \bmod r}} = 0,$$

for any $i = 0, 1, 2, \ldots$

Now $\frac{\partial^3 \Phi_s}{\partial \mu_0^3}(x)\Big|_{x=\overline{x}_0}$ is given by

$$\begin{aligned}
&\frac{\partial^2}{\partial \mu_0^2}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\\
&+\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\right)\\
+\quad&\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\frac{\partial}{\partial \mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\right]\\
&+\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\frac{\partial^2}{\partial \mu_0^2}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\right].
\end{aligned}$$

Hence

$$\frac{\partial^3 \Phi_s}{\partial \mu_0^3}(x)\Big|_{x=\overline{x}_0} = 0 \;\; iff \;\; \frac{\partial^2}{\partial \mu_0^2}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\right] = 0.$$

Again $f_0 = f_p$ and $\overline{x}_0 = \overline{x}_{s \bmod r} = \overline{x}_{\ell p \bmod r}$. Consequently, $\frac{\partial^3 \Phi_s^1}{\partial \mu_0^3}(x)\Big|_{x=\overline{x}_0}$ is now

$$\begin{aligned}
&\frac{\partial^2}{\partial \mu_0^2}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\overline{x}_{(i+1)p \bmod r}\right)\right)\\
&+\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\overline{x}_{(i+1)p \bmod r}\right)\right)\right)\\
+\quad&\frac{\partial}{\partial \mu_0}\left(\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\right)\times\frac{\partial}{\partial \mu_0}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\overline{x}_{(i+1)p \bmod r}\right)\right)\right]\\
&+\prod_{i=0}^{\ell-1}\left(\frac{\partial f_{p-1}}{\partial x}\left(\overline{x}_{(i+1)p-1 \bmod r}\right)\right)\times\ldots\times\prod_{i=0}^{\ell-1}\left(\frac{\partial f_1}{\partial x}\left(\overline{x}_{ip+1 \bmod r}\right)\right)\frac{\partial^2}{\partial \mu_0^2}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_p}{\partial \mu_0}\left(\overline{x}_{(i+1)p \bmod r}\right)\right)\right].
\end{aligned}$$

Hence

$$\frac{\partial^3 \Phi_s^1}{\partial \mu_0^3}(x)\Big|_{x=\overline{x}_0} = 0.$$

Similarly, for any $i = 0, 1, 2, \ldots$ one has

$$\frac{\partial^3 \Phi_s}{\partial^3 \mu_0}(x)\Big|_{x=\overline{x}_0} = \frac{\partial^3 \Phi_s^i}{\partial^3 \mu_0}(x)\Big|_{x=\overline{x}_{i \bmod r}} = 0.$$

By induction we show that

$$\frac{\partial^n \Phi_s}{\partial \mu_0^n}(x)\Big|_{x=\overline{x}_0} = 0 \;\; iff \;\; \frac{\partial^n}{\partial \mu_0^n}\left[\prod_{i=0}^{\ell-1}\left(\frac{\partial f_0}{\partial \mu_0}\left(\overline{x}_{ip \bmod r}\right)\right)\right] = 0.$$

Using the same argument as before, one can see that for any $i = 0, 1, 2, \ldots$ one has

$$\frac{\partial^n \Phi_s}{\partial^n \mu_0}(x)\Big|_{x=\overline{x}_0} = \frac{\partial^n \Phi_s^i}{\partial^n \mu_0}(x)\Big|_{x=\overline{x}_{i \bmod r}} = 0.$$

■

1.5.2 Saddle-node

Note that the composition operator Φ_s is defined from $\mathbb{R}\times\mathbb{R}^p$ to $\mathbb{R}$. Since $\overline{x}_0$ is a fixed point of $\Phi_s(x,\mu_0,\mu_1,...,\mu_{p-1})$ we can work with the autonomous difference equation

$$y_{n+1} = \Phi_s(y_n,\mu_0,\mu_1,...,\mu_{p-1}),$$

which contains one variable and p parameters.

Assume that $\Phi_s(y,\mu_0,\mu_1,...,\mu_{p-1})$ has a nonhyperbolic fixed point at $(\overline{x}_0,\overline{\mu}) = (\overline{x}_0,\overline{\mu}_0,\overline{\mu}_1,...,\overline{\mu}_{p-1})$, i.e., $\Phi_s(\overline{x}_0,\overline{\mu}) = \overline{x}_0$, such that

$$\frac{\partial\Phi_s}{\partial y}(\overline{x}_0,\overline{\mu}) = 1.$$

Shifting this equilibrium point to the origin $(0,0) = \overrightarrow{0}\in\mathbb{R}^{p+1}$ it follows that the nonhyperbolicity conditions will be

$$\Phi_s(\overrightarrow{0}) = 0 \text{ and } \frac{\partial\Phi_s}{\partial y}(\overrightarrow{0}) = 1. \tag{1.25}$$

The Taylor series of $\Phi_s(y,\overrightarrow{\mu})$ is given by

$$\Phi_s(y,\overrightarrow{\mu}) = \Phi_s(\overrightarrow{0}) + \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i + \frac{\partial\Phi_s}{\partial y}(\overrightarrow{0})y + \sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i y + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})y^2 + O(y^3,\mu_i^2).$$

Using (1.25) it follows

$$\Phi_s(y,\overrightarrow{\mu}) = y + \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i + \sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i y + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})y^2 + O(y^3,\mu_i^2).$$

In the difference equations notation

$$y_{n+1} = y_n + \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i + \sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i y_n + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})y_n^2 + O(y^3,\mu_i^2).$$

(The higher terms will be omitted).

Introducing a new variable $v = y-\delta$, where δ is a parameter, it follows that $v_n = y_n-\delta$ and thus

$$\begin{aligned} v_{n+1} &= v_n + \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i + \sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i\,(\delta+v_n) + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\,(\delta+v_n)^2 \\ &= v_n + \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i + \sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i\delta + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\delta^2 + \\ &\left(\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i + \frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\delta\right)v_n + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})v_n^2. \end{aligned}$$

To remove the linear terms $\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i + \frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\delta$ we require that

$$\delta = -\frac{\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i}{\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})}.$$

Obviously we need to assume that $\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0}) \neq 0$. Using this value of δ, our map becomes

$$v_{n+1} = v_n + \sum_{i=0}^{p-1} \frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i - \frac{1}{2}\frac{\left(\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial y\partial\mu_i}(\overrightarrow{0})\mu_i\right)^2}{\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})} + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})v_n^2.$$

Now we introduce a new parameter β such that

$$\begin{cases} \beta = \sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i - \frac{1}{2}\frac{\left(\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i\right)^2}{\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})} \text{ if } \prod_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0}) > 0 \\ \beta = -\left(\sum_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0})\mu_i - \frac{1}{2}\frac{\left(\sum_{i=0}^{p-1}\frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i\right)^2}{\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})}\right) \text{ if } \prod_{i=0}^{p-1}\frac{\partial\Phi_s}{\partial\mu_i}(\overrightarrow{0}) < 0 \end{cases}.$$

This assumption yields

$$\frac{\partial\Phi_s}{\partial\mu_0}(\overrightarrow{0}) \neq 0, \frac{\partial\Phi_s}{\partial\mu_1}(\overrightarrow{0}) \neq 0, ..., \frac{\partial\Phi_s}{\partial\mu_{p-1}}(\overrightarrow{0}) \neq 0. \tag{1.26}$$

Hence, the new map is now

$$v_{n+1} = v_n \pm \beta + \frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})v_n^2.$$

Making the changes of variable $z = \frac{1}{2}\left|\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\right| v$, our equation becomes

$$\begin{aligned} z_{n+1} &= \frac{1}{2}\left|\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\right| v_{n+1} = z_n \pm \beta\frac{1}{2}\left|\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\right| + \frac{\frac{1}{2}\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})}{\frac{1}{2}\left|\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\right|}\eta_n^2 \\ &= z_n \pm \sigma \pm z_n^2, \end{aligned} \tag{1.27}$$

where $\sigma = \beta\frac{1}{2}\left|\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})\right|$. Eq. (1.27) is the normal form of the saddle-node bifurcation.

Conclusion:

If the difference equation $y_{n+1} = \Phi_s(y_n, \mu_0, \mu_1, ..., \mu_{p-1})$ has a fixed point at $\overrightarrow{0} \in \mathbb{R}^{p+1}$ such that

$$\frac{\partial\Phi_s}{\partial y}(\overrightarrow{0}) = 1, \frac{\partial\Phi_s}{\partial\mu_0}(\overrightarrow{0}) \neq 0, \frac{\partial\Phi_s}{\partial\mu_1}(\overrightarrow{0}) \neq 0, ..., \frac{\partial\Phi_s}{\partial\mu_{p-1}}(\overrightarrow{0}) \neq 0\ ,$$

and

$$\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0}) \neq 0,$$

then in a small neighborhood of the fixed point $\overrightarrow{0}$ the map Φ_s is locally equivalent to one of the following normal forms

$$z_{n+1} = z_n \pm \sigma \pm z_n^2,$$

and the saddle-node bifurcation takes place.

Notice that the sign before z^2 is the same as the sign of $\frac{\partial^2\Phi_s}{\partial y^2}(\overrightarrow{0})$.

Example 26 *Let us consider the nonautonomous one-dimensional Ricker difference equation given by*

$$x_{n+1} = x_n e^{\lambda_n - x_n}, n \in \mathbb{Z}^+, \tag{1.28}$$

where $\lambda_n = \lambda_{n \bmod p}$*. Write Eq. (1.28) as* $x_{n+1} = R_n(x_n)$ *where* $R_n(x) = xe^{\lambda_n - x}$ *and let* $p = 2$ *be its minimal period.*

Since each map $R_n = R_{n \bmod 2}$*,* $n \in \mathbb{Z}^+$ *is one to one with the respect to parameter* $\lambda_n = \lambda_{n \bmod 2} > 0$*, it follows by Theorem 8 that the only possible cycles for Eq. (1.28) are cycles with minimal period r,* $r = 2t$*,* $t = 1, 2, \ldots$*. Let* $C_2 = \{\overline{x}_0, \overline{x}_1\}$ *be a nontrivial* $2-$*periodic cycle of Eq. (1.28), i.e.* $\overline{x}_0 = R_1 \circ R_0(\overline{x}_0)$*. Simplifying this equation one obtains*

$$\lambda_0 + \lambda_1 - \overline{x}_0 = \overline{x}_1 = \overline{x}_0 e^{\lambda_0 - \overline{x}_0}. \tag{1.29}$$

Since the derivative along the periodic cycle C_2 *is given by*

$$(1 - \overline{x}_0)(1 - \overline{x}_1),$$

one has a bifurcation when $(1 - \overline{x}_0)(1 - \overline{x}_1) = 1$*, or equivalently*

$$(1 - \lambda_0 - \lambda_1 + \overline{x}_0)(1 - \overline{x}_0) = 1. \tag{1.30}$$

Eq. (1.30) has the solutions

$$\overline{x}_0 = \overline{x}^{sn}_{\pm} = \frac{1}{2}\left(\lambda_0 + \lambda_1 \pm \sqrt{(\lambda_0 + \lambda_1)(\lambda_0 + \lambda_1 - 4)}\right), \lambda_0 + \lambda_1 > 4.$$

Notice that the map is given by

$$\Phi_2(x) = R_1 \circ R_0(x) = xe^{\lambda_0 + \lambda_1 - x(1 + e^{\lambda_0 - x})}.$$

At the fixed point $\overline{x}_0$ *of* Φ_2 *one has* $\frac{\partial}{\partial x}(\Phi_2(x))|_{x=\overline{x}_0} = 1$*,* $\frac{\partial}{\partial \lambda_1}(\Phi_2(x))|_{x=\overline{x}_0} = 1 \neq 0$*, and*

$$\frac{\partial}{\partial \lambda_0}(\Phi_2(x))|_{x=\overline{x}_0} = 1 - \lambda_0 - \lambda_1 + \overline{x}_0 < 0,$$

for all $\lambda_0 > 0$ *and* $\lambda_1 > 0$ *such that* $\lambda_0 + \lambda_1 > 4$*. So* $\frac{\partial}{\partial \lambda_0}(\Phi_2(x))|_{x=\overline{x}_0} \neq 0$*.*

Substituting the value of the bifurcation solutions in (1.29) yields two equations of the form

$$\lambda_0 + \lambda_1 - y = y \times e^{\lambda_0 - y}, \tag{1.31}$$

where y is given by $\overline{x}^{sn}_{+}$ *or* $\overline{x}^{sn}_{-}$*.*

The implicit function theorem guarantees the existence of solutions of the two equations given in (1.31). The exact solutions of these equations are represented implicitly, in the parameter space λ_0 *and* λ_1 *in Fig. 1.5 (grey curves). The stability region of* C_2 *is given by* $A_1 \cup A_2$*.*

Hence, from the implicit solutions, it is enough to consider the values of the parameters in the interval $(2, 3)$ *when the derivative equals 1. Since*

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(\Phi_2(x))|_{x=\overline{x}_0} &= -1 - e^{\lambda_0 - \overline{x}_0} + (3 - \overline{x}_0)(\lambda_0 + \lambda_1 - \overline{x}_0) + \\ &\quad (1 - \overline{x}_0)(1 - \lambda_0 - \lambda_1 + \overline{x}_0)(-1 - e^{\lambda_0 - \overline{x}_0} + \lambda_0 + \lambda_1 - \overline{x}_0),\end{aligned}$$

it follows that $\frac{\partial^2}{\partial x^2}(\Phi_2(x))|_{x=\overline{x}^{sn}_{+}} < 0$ *for all* $\lambda_0, \lambda_1 \in (2, 3) \times (2, 3)$ *and* $\frac{\partial^2}{\partial x^2}(\Phi_2(x))|_{x=\overline{x}^{sn}_{-}} > 0$ *for all* $\lambda_0, \lambda_1 \in (2, 3) \times (2, 3)$*. Consequently, on these bifurcation curves occurs saddle-node bifurcation.*

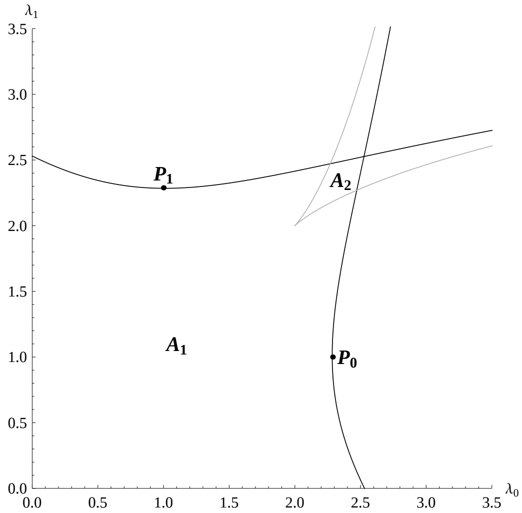

Figure 1.5: The bifurcation curves of a 2−periodic cycle, in the parameter space, of the 2−periodic Ricker equation.

1.5.3 Transcritical

Assume that the map $\Phi_s(y, \mu_0, \mu_1, ..., \mu_{p-1})$ has a nonhyperbolic fixed point at $\overrightarrow{0} \in \mathbb{R}^{p+1}$ (after shifting the fixed point $(\overline{x}_0, \overline{\mu}_0, \overline{\mu}_1, ..., \overline{\mu}_{p-1})$ to the origin) such that

$$\Phi_s(\overrightarrow{0}) = 0 \text{ and } \frac{\partial \Phi_s}{\partial y}(\overrightarrow{0}) = 1, \tag{1.32}$$

and

$$\frac{\partial \Phi_s}{\partial \mu_0}(\overrightarrow{0}) = 0, \frac{\partial \Phi_s}{\partial \mu_1}(\overrightarrow{0}) = 0, ..., \frac{\partial \Phi_s}{\partial \mu_{p-1}}(\overrightarrow{0}) = 0. \tag{1.33}$$

Let us write the vector parameter $\left(\mu_0, \mu_1, ..., \mu_{p-1}\right)$ as $\overrightarrow{\mu}$ and write the equation as

$$y_{n+1} = \Phi_s\left(y_n, \overrightarrow{\mu}\right). \tag{1.34}$$

Using (1.32) and (1.33), the Taylor series of (1.34) is given by

$$\Phi_s\left(y, \overrightarrow{\mu}\right) = y + \sum_{i=0}^{p-1} \frac{\partial^2 \Phi_s}{\partial \mu_i \partial y}(\overrightarrow{0})\mu_i y + \frac{1}{2}\frac{\partial^2 \Phi_s}{\partial y^2}(\overrightarrow{0})y^2 + O(x^3, \mu_i^2),$$

Introducing a new parameter β such that

$$\begin{cases} \beta = \sum_{i=0}^{p-1} \frac{\partial^2 \Phi_s}{\partial \mu_i \partial y}(\overrightarrow{0})\mu_i \text{ if } \prod_{i=0}^{p-1} \frac{\partial^2 \Phi_s}{\partial \mu_i \partial y}(\overrightarrow{0}) > 0 \\ \beta = -\sum_{i=0}^{p-1} \frac{\partial^2 \Phi_s}{\partial \mu_i \partial y}(\overrightarrow{0})\mu_i \text{ if } \prod_{i=0}^{p-1} \frac{\partial^2 \Phi_s}{\partial \mu_i \partial y}(\overrightarrow{0}) < 0 \end{cases},$$

and rescaling we obtain

$$v_{n+1} = (1 \pm \beta)\, v_n \pm v_n^2,$$

where $\frac{1}{2}\left|\frac{\partial^2 \Phi_s}{\partial y^2}(\overrightarrow{0})\right| y = v$. The non-degeneracy conditions for normalization are

$$\frac{\partial^2 \Phi_s}{\partial y^2}(\overrightarrow{0}) \neq 0, \frac{\partial^2 \Phi_s}{\partial \mu_0 \partial y}(\overrightarrow{0}) \neq 0, \frac{\partial^2 \Phi_s}{\partial \mu_1 \partial y}(\overrightarrow{0}) \neq 0, ..., \frac{\partial^2 \Phi_s}{\partial \mu_{p-1} \partial y}(\overrightarrow{0}) \neq 0.$$

Conclusion:

Let $\overrightarrow{0} \in \mathbb{R}^{p+1}$ be a nonhyperbolic fixed point of the difference equation

$$y_{n+1} = \Phi_s\left(y_n, \overrightarrow{\mu}\right)$$

such that $\Phi_s(\overrightarrow{0}) = 0$, $\frac{\partial \Phi_s}{\partial y}(\overrightarrow{0}) = 1$, and

$$\frac{\partial \Phi_s}{\partial \mu_0}(\overrightarrow{0}) = 0, \frac{\partial \Phi_s}{\partial \mu_1}(\overrightarrow{0}) = 0, ..., \frac{\partial \Phi_s}{\partial \mu_{p-1}}(\overrightarrow{0}) = 0.$$

If

$$\frac{\partial^2 \Phi_s}{\partial \mu_0 \partial y}(\overrightarrow{0}) \neq 0, \frac{\partial^2 \Phi_s}{\partial \mu_1 \partial y}(\overrightarrow{0}) \neq 0, ..., \frac{\partial^2 \Phi_s}{\partial \mu_{p-1} \partial y}(\overrightarrow{0}) \neq 0, \frac{\partial^2 \Phi_s}{\partial y^2}(\overrightarrow{0}) \neq 0,$$

then in a neighborhood of the fixed point $\overrightarrow{0}$ the map $\Phi_s\left(y, \overrightarrow{\mu}\right)$ is locally equivalent to one of the following normal forms

$$v_{n+1} = (1 \pm \beta)\, v_n \pm v_n^2,$$

and the transcritical bifurcation takes place.

- Notice that the sign before v^2 is the same as the sign of $\frac{\partial^2 \Phi_s}{\partial y^2}(\overrightarrow{0})$.

Example 27 *Consider the difference equation*

$$x_{n+1} = x_n(a_n - x_n)(a_{n+1} - x_n^2(a_n - x_n)),$$

with $a_n > 0$ for all $n = 0, 1, 2, \ldots$. Let $a_n = a_{n \bmod 2}$, for all $n = 0, 1, 2, \ldots$. This leads to a $2-$periodic difference equation. The composition operator Φ is given by

$$\Phi(x) = x(a_0 - x)(a_1 - x^2(a_0 - x)).$$

Consider the zero fixed point of Φ. (We do not shift the parameter vector since the derivatives equals). This fixed point is asymptotically stable when $a_0a_1 < 1$. Let $a_0a_1 = 1$. Hence one has $\frac{\partial \Phi}{\partial x}(0, a_0, a_1) = 1$. Moreover, $\frac{\partial \Phi}{\partial a_0}(0, a_0, a_1) = \frac{\partial \Phi}{\partial a_1}(0, a_0, a_1) = 0$, $\frac{\partial^2 \Phi}{\partial x^2}(0, a_0, a_1) = -2a_1 < 0$, $\frac{\partial^2 \Phi}{\partial a_0 \partial x}(0, a_0, a_1) = a_1 > 0$, and $\frac{\partial^2 \Phi}{\partial a_1 \partial x}(0, a_0, a_1) = a_0 > 0$. Hence the require conditions for the transcritical bifurcation are satisfied. Thus at the curve $a_0a_1 = 1$ in the parameter space (a_0, a_1) an exchange of stability takes place.

1.5.4 Pitchfork

Consider that the map $y_{n+1} = \Phi_s\left(y_n, \overrightarrow{\mu}\right)$ has a nonhyperbolic fixed point at $\overrightarrow{0} \in \mathbb{R}^{p+1}$ (after shifting the fixed point $(\overline{x}_0, \overline{\mu}_0, \overline{\mu}_1, ..., \overline{\mu}_{p-1})$ to the origin) such that

$$\Phi_s(\overrightarrow{0}) = 0 \text{ and } \frac{\partial \Phi_s}{\partial y}(\overrightarrow{0}) = 1, \tag{1.35}$$

and

$$\frac{\partial \Phi}{\partial \mu_0}(\overrightarrow{0}) = \frac{\partial \Phi}{\partial \mu_1}(\overrightarrow{O}) = ... = \frac{\partial \Phi}{\partial \mu_{p-1}}(\overrightarrow{0}) = \frac{\partial^2 \Phi}{\partial y^2} = 0. \tag{1.36}$$

Hence, the Taylor series is given by

$$\Phi\left(y,\overrightarrow{\mu}\right) = y + \sum_{i=0}^{p-1} \frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i y + \frac{1}{6}\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0})y^3.$$

Let β be a new parameter such that

$$\begin{cases} \beta = \sum\limits_{i=0}^{p-1} \frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i \text{ if } \prod\limits_{i=0}^{p-1} \frac{\partial^2\Phi_s}{\partial\mu_i\partial y} > 0 \\ \beta = -\sum\limits_{i=0}^{p-1} \frac{\partial^2\Phi_s}{\partial\mu_i\partial y}(\overrightarrow{0})\mu_i \text{if } \prod\limits_{i=0}^{p-1} \frac{\partial^2\Phi_s}{\partial\mu_i\partial y} < 0 \end{cases}.$$

Hence, the equation now becomes

$$y_{n+1} = y_n \pm \beta y_n + \frac{1}{6}\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0})y_n^3, \tag{1.37}$$

and by the change of variable $v = \sqrt{\frac{1}{6}\left|\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0})\right|}y$ we obtain

$$v_{n+1} = v_n \pm \beta v_n \pm v_n^3.$$

Conclusion

Consider that the map $y_{n+1} = \Phi_s\left(y_n,\overrightarrow{\mu}\right)$ has a nonhyperbolic fixed point at $\overrightarrow{0}$ such that

$$\Phi_s(\overrightarrow{0}) = 0 \text{ and } \frac{\partial\Phi_s}{\partial y}(\overrightarrow{0}) = 1,$$

and

$$\frac{\partial\Phi}{\partial\mu_0}(\overrightarrow{0}) = \frac{\partial\Phi}{\partial\mu_1}(\overrightarrow{O}) = ... = \frac{\partial\Phi}{\partial\mu_{p-1}}(\overrightarrow{0}) = \frac{\partial^2\Phi}{\partial y^2}(\overrightarrow{0}) = 0.$$

If

$$\frac{\partial^2\Phi_s}{\partial\mu_0\partial y}(\overrightarrow{0}) \neq 0\ ,\ \frac{\partial^2\Phi_s}{\partial\mu_1\partial y}(\overrightarrow{0}) \neq 0, ..., \frac{\partial^2\Phi_s}{\partial\mu_{p-1}\partial y}(\overrightarrow{0}) \neq 0$$

and

$$\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0}) \neq 0,$$

then in a neighborhood of the fixed point $\overrightarrow{0}$ this equation is equivalent to one of the following normal forms

$$v_{n+1} = v_n \pm \beta v_n \pm v_n^3,$$

and the pitchfork bifurcation takes place.

- Notice that if $\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0}) < 0$ then the pitchfork is supercritical and if $\frac{\partial^3\Phi_s}{\partial y^3}(\overrightarrow{0}) > 0$ the pitchfork is subcritical.

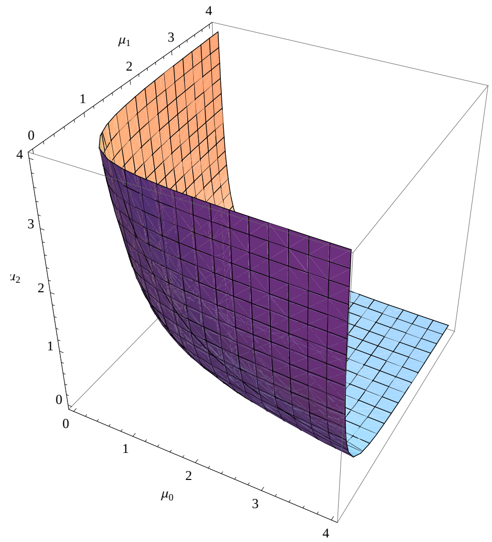

Figure 1.6: The bifurcation surface where occurs the pitchfork bifurcation of the zero fixed point of the equation $x_{n+1} = f_n(x_n)$ where $f_n(x) = \mu_n x - x^3$, $\mu_n > 0$ and $f_n = f_{n+3}$ for all $n = 0, 1, 2, \ldots$

Example 28 *Consider the difference equation $x_{n+1} = f_n(x_n)$ where $f_n(x) = \mu_n x - x^3$, $\mu_n > 0$ for all $n = 0, 1, 2, \ldots$. Let $\mu_n = \mu_n \bmod 3$. This leads to a 3−periodic difference equation since $f_n = f_{n+3}$, for all $n = 0, 1, 2, \ldots$. The composition operator is given by*

$$\Phi_3(x) = f_2 \circ f_1 \circ f_0(x).$$

Consider the zero fixed point of Φ_3. Direct computations show that

$$\frac{\partial \Phi_3}{\partial \mu_0}(0, \mu_0, \mu_1, \mu_2) = \frac{\partial \Phi_3}{\partial \mu_1}(0, \mu_0, \mu_1, \mu_2) = \frac{\partial \Phi_3}{\partial \mu_2}(0, \mu_0, \mu_1, \mu_2) = \frac{\partial^2 \Phi_3}{\partial x^2}(0, \mu_0, \mu_1, \mu_2) = 0.$$

The fixed point zero is asymptotically stable when $\mu_0\mu_1\mu_2 < 1$ since

$$\frac{\partial \Phi_3}{\partial x}(0, \mu_0, \mu_1, \mu_2) = \mu_0\mu_1\mu_2.$$

When $\mu_0\mu_1\mu_2 = 1$ it takes place a bifurcation. Computations show that when $\mu_0\mu_1\mu_2 = 1$ one has, $\frac{\partial^2 \Phi_3}{\partial \mu_0 \partial x}(0, \mu_0, \mu_1, \mu_2) = \mu_1\mu_2 > 0$, $\frac{\partial^2 \Phi_3}{\partial \mu_1 \partial x}(0, \mu_0, \mu_1, \mu_2) = \mu_0\mu_2 > 0$ and $\frac{\partial^2 \Phi_3}{\partial \mu_2 \partial x}(0, \mu_0, \mu_1, \mu_2) = \mu_0\mu_1 > 0$. Moreover,

$$\frac{\partial^3 \Phi_3}{\partial x^3}(0, \mu_0, \mu_1, \mu_2) = -6(\mu_0^3\mu_1^3 + \mu_0^3\mu_2 + \mu_1\mu_2) \neq 0.$$

Hence, at $\mu_0\mu_1\mu_2 = 1$ the pitchfork bifurcation takes place. (Notice that it is a supercritical bifurcation). In Figure 1.6 is presented in the parameter space (μ_0, μ_1, μ_2) the bifurcation surface.

1.5.5 Period-doubling

Let us assume that the map $y_{n+1} = \Phi_s(y_n, \mu_0, \mu_1, ..., \mu_{p-1})$ has a nonhyperbolic fixed point such that

$$\Phi_s(\overline{x}_0, \overline{\mu}_0, \overline{\mu}_1, ..., \overline{\mu}_{p-1}) = \overline{x}_0 \text{ and } \frac{\partial \Phi_s}{\partial y}(\overline{x}_0, \overline{\mu}_0, \overline{\mu}_1, ..., \overline{\mu}_{p-1}) = -1.$$

By shifting this equilibrium point to the origin $\overrightarrow{0} \in \mathbb{R}^{p+1}$, it follows that the nonhyperbolicity conditions becomes

$$\Phi_s(\overrightarrow{0}) = 0 \text{ and } \frac{\partial \Phi_s}{\partial y}(\overrightarrow{0}) = -1. \tag{1.38}$$

Because the slope of Φ_s is -1 the fixed point $\overrightarrow{0}$ must exist for all $(\mu_0, \mu_1, ..., \mu_{p-1})$ close to $(0, 0, ..., 0)$. In fact, by the relation

$$\Phi_s(y, \mu_0, \mu_1, ..., \mu_{p-1}) = y,$$

it follows that

$$\Phi_s(\overrightarrow{0}) + \frac{\partial \Phi_s}{\partial y}(\overrightarrow{0})y + \sum_{i=0}^{p-1} \frac{\partial \Phi_s}{\partial \mu_i}(\overrightarrow{0})\mu_i = y, \tag{1.39}$$

and consequently

$$y = \frac{1}{2} \sum_{i=0}^{p-1} \frac{\partial \Phi_s}{\partial \mu_i}(\overrightarrow{0})\mu_i.$$

This implies that we have a line of fixed points $y = Y(\mu_0, \mu_1, ..., \mu_{p-1})$. Notice that we cannot use this technique close to the other nonhyperbolic fixed point where $\frac{\partial \Phi_s}{\partial y}(\overrightarrow{0}) = 1$, because in that situation the variable y vanishes in Eq. (1.39) and we will not find any solution.

By the change of variables $z = y - Y$ we find a new map

$$z_{n+1} = \Psi(z_n, \overrightarrow{\mu}), \tag{1.40}$$

which has fixed points at $z = 0$ for all $\mu_0, \mu_1, ..., \mu_{p-1}$, i.e., $\Psi(0, \mu_0, \mu_1, ..., \mu_{p-1}) = 0$.

Therefore we now assume, without loss of generality, that we have a map $z_{n+1} = \Psi(z_n, \overrightarrow{\mu})$ with the following properties

$$\Psi(0, \overrightarrow{\mu}) = 0 \text{ and } \frac{\partial \Psi}{\partial z}(\overrightarrow{0}) = -1. \tag{1.41}$$

The Taylor expansion of Ψ is given by

$$\begin{aligned} \Psi(z, \overrightarrow{\mu}) &= \Psi(0, \overrightarrow{\mu}) + \frac{\partial \Psi}{\partial z}(0, \overrightarrow{\mu})z + \frac{\partial^2 \Psi}{\partial z^2}(0, \overrightarrow{\mu})\frac{z^2}{2} + \frac{\partial^3 \Psi}{\partial z^3}(0, \overrightarrow{\mu})\frac{z^3}{3!} + ... \\ &\approx A_1(\overrightarrow{\mu})z + A_2(\overrightarrow{\mu})z^2 + A_3(\overrightarrow{\mu})z^3, \end{aligned} \tag{1.42}$$

where

$$A_1(\overrightarrow{\mu}) = \frac{\partial \Psi}{\partial z}(0, \overrightarrow{\mu}), A_2(\overrightarrow{\mu}) = \frac{1}{2}\frac{\partial^2 \Psi}{\partial z^2}(0, \overrightarrow{\mu}) \text{ and } A_3(\overrightarrow{\mu}) = \frac{1}{6}\frac{\partial^3 \Psi}{\partial z^3}(0, \overrightarrow{\mu}).$$

Because it is convenient to remove the quadratic terms we use the quadratic change of variable

$$z = u + \alpha u^2. \tag{1.43}$$

The inverse change of variables is given by

$$u = \frac{-1 \pm \sqrt{1 + 4\alpha z}}{2\alpha}. \tag{1.44}$$

Notice that the graph of (1.43) is a parabola in the $(u, z)-$plane open upwards with two roots $u = 0$ and $u = -\alpha$. Hence we chose to work with the right branch of this parabola, because it passers on the point $z = 0$ and $u = 0$. Therefore it maps our equilibrium point $z = 0$ to the point $u = 0$. This branch corresponds to the sign + in (1.44) and therefore the inverse change of variables is

$$u = \frac{-1 + \sqrt{1 + 4\alpha z}}{2\alpha}. \tag{1.45}$$

Since $\sqrt{1 + 4\alpha z} = 1 + 2\alpha z - 2\alpha^2 z^2 + 4\alpha^3 z^3 + ...$, it follows that

$$u \approx z - \alpha z^2 + 2\alpha^2 z^3. \tag{1.46}$$

So, in the difference equation notation we have

$$u_{n+1} = z_{n+1} - \alpha z_{n+1}^2 + 2\alpha^2 z^3.$$

Using (1.42) and removing the terms up to the third order yields

$$u_{n+1} = A_1 z_n + (A_2 - \alpha A_1^2) z_n^2 + (A_3 - 2\alpha A_1 A_2 + 2\alpha^2 A_1^3) z_n^3.$$

By (1.43) it follows that $z^2 \approx u^2 + 2\alpha\mu^3$ and $z^3 \approx \mu^3$. Hence

$$u_{n+1} = A_1 u_n + (\alpha A_1 + A_2 - \alpha A_1^2) u_n^2 + (2\alpha A_2 - 2\alpha^2 A_1^2 + A_3 - 2\alpha A_1 A_2 + 2\alpha^2 A_1^3) u_n^3.$$

Now we can remove the quadratic term and it follows that

$$\alpha = \frac{A_2(\overrightarrow{\mu})}{A_1^2(\overrightarrow{\mu}) - A_1(\overrightarrow{\mu})}.$$

At the point $\overrightarrow{\mu} = \overrightarrow{0}$ we know that $A_1(\overrightarrow{0}) = -1$ and thus $\alpha = A_2(\overrightarrow{0})/2$. Consequently, at the origin the coefficient of u^3 is now

$$A = A_3 + A_2^2 = \frac{1}{6}\frac{\partial^3 \Psi}{\partial z^3}(\overrightarrow{0}) + \frac{1}{4}\left(\frac{\partial^2 \Psi}{\partial z^2}(\overrightarrow{0})\right)^2.$$

Hence, the new map is now

$$u_{n+1} = A_1 u_n + A u_n^3,$$

and by the change of variables $v = \sqrt{|A|}u$ we have

$$v_{n+1} = A_1 v_n \pm v_n^3. \tag{1.47}$$

The second iterate map is given by

$$v_{n+2} = A_1(A_1 v_n \pm v_n^3) \pm (A_1 v_n \pm v_n^3)^3.$$

Removing the terms up to the third order we have

$$v_{n+2} = A_1^2 v_n \pm (A_1 + A_1^3) v_n^3.$$

The Taylor expansion of $A_1(\overrightarrow{\mu})$ around the point $\overrightarrow{0}$ is

$$A_1\left(\overrightarrow{\mu}\right) = A_1(\overrightarrow{0}) + \sum_{i=0}^{p-1} \frac{\partial A_1}{\partial \mu_i}(\overrightarrow{0})\mu_i.$$

Because $A_1(\overrightarrow{\mu}) = \frac{\partial \Psi}{\partial z}(0, \overrightarrow{\mu})$ it follows that $\frac{\partial A_1}{\partial \mu_i}(\overrightarrow{\mu}) = \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(0, \overrightarrow{\mu})$, $i = 0, 1, ..., p-1$. Hence

$$A_1\left(\overrightarrow{\mu}\right) = -1 + \sum_{i=0}^{p-1} \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(\overrightarrow{0})\mu_i.$$

Let $A_1\left(\overrightarrow{\mu}\right) = -1 \pm \beta$ where

$$\begin{cases} \beta = \sum\limits_{i=0}^{p-1} \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(\overrightarrow{0})\mu_i \text{ if } \prod_{i=0}^{p-1} \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(\overrightarrow{0}) > 0 \\ \beta = -\sum\limits_{i=0}^{p-1} \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(\overrightarrow{0})\mu_i \text{ if } \prod_{i=0}^{p-1} \frac{\partial^2 \Psi}{\partial \mu_i \partial z}(\overrightarrow{0}) < 0 \end{cases}.$$

Notice that under this assumption one has necessarily

$$\frac{\partial^2 \Psi}{\partial \mu_0 \partial z}(\overrightarrow{0}) \neq 0, \frac{\partial^2 \Psi}{\partial \mu_1 \partial z}(\overrightarrow{0}) \neq 0, ..., \frac{\partial^2 \Psi}{\partial \mu_{p-1} \partial z}(\overrightarrow{0}) \neq 0.$$

The single map is now

$$v_{n+1} = (-1 \pm \beta)v_n \pm v_n^3. \tag{1.48}$$

The fixed point $v = 0$ of the map representing $v_{n+1} = (-1 + \beta)v_n \pm v_n^3$ is stable if $\beta > 0$ and unstable if $\beta < 0$. Analogously, the fixed point $v = 0$ of the map $v_{n+1} = (-1 - \beta)v_n \pm v_n^3$ is stable if $\beta < 0$ and unstable if $\beta > 0$.

The second iterate map is given by

$$v_{n+2} = (-1 \pm \beta)^2 v_n \pm ((-1 \pm \beta) + (-1 \pm \beta)^3)v_n^3. \tag{1.49}$$

Before simplify this map, we notice that, our main goal here is to study the dynamics of Eq. (1.49) near the origin. Hence, we can rewrite the map (1.49) as

$$v_{n+1} = (-1 \pm \beta)^2 v_n \pm ((-1 \pm \beta) + (-1 \pm \beta)^3)v_n^3, \tag{1.50}$$

because this transformation just shift in one step the sequence of point v_n near the fixed point, but does not change their values, and thus its dynamics is preserved.

Consider the change of variables $h = \sqrt{\left|(-1 \pm \beta) + (-1 \pm \beta)^3\right|}v$. Substituting in (1.50) and simplifying we have

$$h_{n+1} = h_n \pm \gamma h_n \pm h_n^3,$$

where $\gamma = 2\beta + \beta^2$ and $-\gamma = -2\beta + \beta^2$. This is precisely the normal form of the pitchfork bifurcation. In fact, if $G(v,\beta)$ is a map representing (1.50) then at the fixed point $(v,\beta) = (0,0)$ one has

$$\frac{\partial G}{\partial v}(0,0) = ((-1 \pm \beta)^2) \pm 3((-1 \pm \beta) + (-1 \pm \beta)^3)v^2\Big|_{(v,\beta)=(0,0)} = 1,$$

$$\frac{\partial G}{\partial \beta}(0,0) = \pm 2(-1 \pm \beta)v \pm (\pm 1 \pm 3(-1 \pm \beta)^2)v^3\Big|_{(v,\beta)=(0,0)} = 0,$$

$$\frac{\partial^2 G}{\partial\beta\partial v}(0,0) = \pm 2(-1) \pm 3(\pm 1 \pm 3(-1)) \neq 0,$$

and

$$\frac{\partial^2 G}{\partial v^2}(0,0) = 0, \frac{\partial^3 G}{\partial v^3}(0,0) = \mp 12 \neq 0.$$

Conclusion:

If the map $z_{n+1} = \Psi(z_n, \mu_0, \mu_1, ..., \mu_{p-1})$ has a nonhyperbolic fixed point at $\overrightarrow{0} \in \mathbb{R}^{p+1}$ such that

$$\Psi(0, \overrightarrow{\mu}) = 0 \text{ for all } (\mu_0, \mu_1, ..., \mu_{p-1}) \text{ close to } (0,0,...,0), \text{ and } \frac{\partial\Psi}{\partial z}(\overrightarrow{0}) = -1.$$

If

$$\frac{\partial^2\Psi}{\partial\mu_0\partial z}(\overrightarrow{0}) \neq 0, \frac{\partial^2\Psi}{\partial\mu_1\partial z}(\overrightarrow{0}) \neq 0, ..., \frac{\partial^2\Psi}{\partial\mu_{p-1}\partial z}(\overrightarrow{0}) \neq 0 \tag{1.51}$$

and

$$\frac{1}{6}\frac{\partial^3\Psi}{\partial z^3}(\overrightarrow{0}) + \frac{1}{4}\left(\frac{\partial^2\Psi}{\partial z^2}(\overrightarrow{0})\right)^2 \neq 0,$$

then in a neighborhood of the fixed point the map is locally equivalent to one of the following normal forms

$$h_{n+1} = h_n \pm \gamma h_n \pm h_n^3,$$

and the period-doubling bifurcation takes place at $\overrightarrow{0}$.

Since $\frac{\partial\Psi}{\partial y}(\overrightarrow{0}) = -1$, condition $\frac{1}{6}\frac{\partial^3\Psi}{\partial z^3}(\overrightarrow{0}) + \frac{1}{4}\left(\frac{\partial^2\Psi}{\partial z^2}(\overrightarrow{0})\right)^2 \neq 0$ is equivalent to $S\Psi(\overrightarrow{0}) \neq 0$. (The Schwarzian derivative).

Notice that the condition (1.51) must be computed only after shifting the fixed point to the origin. So, in general we cannot apply directly this test for the initial composition operator Φ_s. In order to apply it without any restrictions, we need first to transform our map to the form (1.40). This is needed because this transformation introduces a new variable z which is a function of $\mu_0, \mu_1, ..., \mu_{p-1}$. Therefore the derivatives $\sum_{i=0}^{p-1} \frac{\partial\Psi}{\partial\mu_i}\mu_i$ may be changed. For instance consider the map $g(y, \mu_0, \mu_1) = y - (\mu_0 + \mu_1) + (\mu_0 + \mu_1)^2$. So $\sum_{i=0}^{1} \frac{\partial g}{\partial\mu_i}\mu_i = -(\mu_0 + \mu_1) + 2(\mu_0 + \mu_1)^2$. By the change of variables $z = y - (\mu_0 + \mu_1)$ it follows that $G(z, \mu_0, \mu_1) = z + (\mu_0 + \mu_1)^2$ and therefore $\sum_{i=0}^{1} \frac{\partial G}{\partial\mu_i}\mu_i = 2(\mu_0 + \mu_1)^2$.

Example 29 *Consider the one-dimensional Ricker map given in (1.28). The period-doubling bifurcation equation is given by* $(1 - \overline{x}_0)(1 - \overline{x}_1) = -1$*, or equivalently*

$$(1 - \lambda_0 - \lambda_1 + \overline{x}_0)(1 - \overline{x}_0) = -1. \tag{1.52}$$

Eq. (1.52) has the solutions

$$\overline{x}_{0_\pm} = \overline{x}_\pm^{pd} = \frac{1}{2}\left(\lambda_0 + \lambda_1 \pm \sqrt{(\lambda_0 + \lambda_1)(\lambda_0 + \lambda_1 - 4) + 8}\right).$$

After shifting the fixed point $(\overline{x}_0, \lambda_0, \lambda_1)$ *to the origin one obtains the equation*

$$y_{n+1} = (y_n + \overline{x}_0)e^{\mu_0 + \lambda_0 + \mu_1 + \lambda_1 - (y_n + \overline{x}_0)(1 + e^{\mu_0 + \lambda_0 - (y_n + \overline{x}_0)})} - \overline{x}_0.$$

Bifurcation	$\frac{\partial\Phi_s}{\partial x}$	$\frac{\partial\Phi_s}{\partial\mu_i}$, $i = 0, 1, \dots, p-1$	$\frac{\partial^2\Phi_s}{\partial x^2}$	$\frac{\partial^2\Phi}{\partial\mu_i\partial x}$, $i = 0, 1, \dots, p-1$	$\frac{\partial^3\Phi_s}{\partial x^3}$	$S\Phi_s$
Saddle-node	1	$\neq 0$	$\neq 0$			
Transcritical	1	0	$\neq 0$	$\neq 0$		
Pitchfork	1	0	0	$\neq 0$	$\neq 0$	
Period-doubling	-1			$\neq 0$		$\neq 0$

Table 1.2: Bifurcation conditions for nonhyperbolic periodic points in nonautonomous one-dimensional maps.

By the change of variable $z_n = y_n - Y$, where $Y = \frac{1}{2}(-\mu_1 + (1 - \overline{x}_1)\overline{x}_0\mu_0)$, yields the equation $z_{n+1} = \Psi(z_n)$, where

$$\Psi(z) = (z + Y + \overline{x}_0)e^{\mu_0+\lambda_0+\mu_1+\lambda_1-(z+Y+\overline{x}_0)(1+e^{\mu_0+\lambda_0-(z+Y+\overline{x}_0)})} - Y - \overline{x}_0.$$

Hence

$$\frac{\partial\Psi}{\partial z}(0,0,0) = (1-\overline{x}_0)(1-\overline{x}_1) = (1-\overline{x}_0)(1-\lambda_0-\lambda_1+\overline{x}_0) = -1,$$

$$\frac{\partial^2\Psi}{\partial\mu_0\partial z}(0,0,0) = 4\overline{x}_0^2(-1+\overline{x}_1)^2 - \overline{x}_0^3(-1+\overline{x}_1)^2 + 2\left(1-4\overline{x}_1+\overline{x}_1^2\right) - 2\overline{x}_0\left(2-7\overline{x}_1+3\overline{x}_1^2\right),$$

and

$$\frac{\partial^2\Psi}{\partial\mu_1\partial z}(0,0,0) = \frac{2-4\overline{x}_0+\overline{x}_0^2-(4-8\overline{x}_0+3\overline{x}_0^2)\overline{x}_1+(\overline{x}_0-1)\overline{x}_1^2}{2}.$$

Since $\overline{x}_1 = \lambda_0 + \lambda_1 - \overline{x}_0$ we conclude (via numerical simulations) that $\frac{\partial^2\Psi}{\partial\mu_0\partial z}(0,0,0) > 0$ when $\overline{x}_0 = \overline{x}_{0_+}^{pd}$ and $\frac{\partial^2\Psi}{\partial\mu_0\partial z}(0,0,0) < 0$ when $\overline{x}_0 = \overline{x}_{0_-}^{pd}$, and $\frac{\partial^2\Psi}{\partial\mu_1\partial z}(0,0,0) < 0$ in both situation $\overline{x}_0 = \overline{x}_{0_-}^{pd}$ and $\overline{x}_0 = \overline{x}_{0_+}^{pd}$. Hence

$$\frac{\partial^2\Psi}{\partial\mu_0\partial z}(0,0,0) \neq 0 \text{ and } \frac{\partial^2\Psi}{\partial\mu_1\partial z}(0,0,0) \neq 0.$$

Similar techniques allow us to show that $\frac{1}{6}\frac{\partial^3\Psi}{\partial z^3}(0,0,0) + \frac{1}{4}\left(\frac{\partial^2\Psi}{\partial z^2}(0,0,0)\right)^2 > 0$ in both situation $\overline{x}_0 = \overline{x}_{0_-}^{pd}$ and $\overline{x}_0 = \overline{x}_{0_+}^{pd}$.

Substituting the value of the bifurcation solutions in (1.29) yields two equations of the form

$$\lambda_0 + \lambda_1 - y = y \times e^{\lambda_0 - y}, \tag{1.53}$$

where y is given by $\overline{x}_+^{pd}$ or $\overline{x}_-^{pd}$.

Once again we invoke the implicit function theorem in order to guarantee the existence of solutions of Eq. (1.53). In Fig. 1.5 is represented implicitly, in the parameter space λ_0 and λ_1 these exact solutions (black curves). If the parameters λ_0 and λ_1 are on the black curves, then it take place the period-doubling bifurcation. Note that the period-doubling bifurcation curves intersect the axes at the points (2.53039, 0) *and* (0, 2.53039) *and they intersect them self at the point* (2.52647, 2.52647) = (k_2, k_2). *(The first numbers of the sequence k_i, $i = 1, 2, \dots$ are present in subsection 2.1.4). Moreover, if $\lambda_0 = \lambda_1$, i.e., if the parameters are on the diagonal line, the equation loses it periodicity (in other words, the map is autonomous). Furthermore, the coordinates of the points P_0 and P_1 are $(\tilde{\lambda}, 1)$ and $(1, \tilde{\lambda})$, $\tilde{\lambda} \approx 2.28447$, respectively.*

Before finalize this section we present in Table 1.2 a resume of the bifurcations conditions in one-dimensional nonautonomous systems.

Chapter 2

Competition models

"**Competition** is a contest between individuals, groups, nations, animals, etc. for territory, a niche, or a location of resources. It arises whenever two or more parties strive for a goal which cannot be shared. Competition occurs naturally between living organisms which co-exist in the same environment. Competition can have both beneficial and detrimental effects"[1].

There are two types of competition: (i) **interspecific competition** or (ii) **intraspecific competition**.

(i) "Interspecific competition, in ecology, is a form of competition in which individuals of different species compete for the same resource in an ecosystem (i.e., food or living space). As an example, if a tree species in a dense forest grows taller than surrounding tree species, it is able to absorb more of the incoming sunlight. However, less sunlight is then available for the trees that are shaded by the taller tree".

   In summing, interspecific competition may occur when individuals of two separate species share a limiting resource in the same area. If the resource cannot support both populations, then lowered fecundity, growth, or survival may result in at least one species. Consequently, **interspecific competition has the potential to alter populations, communities and the evolution of interacting species**.

(ii) "Intraspecific competition is a particular form of competition in which members of the same species compete for the same resource in an ecosystem (e.g. food, light, nutrients, space). For example, two trees of the same species growing close together will compete for light, water and nutrients in the same soil. Getting less resources, they will perform more poorly than if they grew by themselves (for example lowered growth rates and fewer seed output). Trees have therefore adapted to grow taller or develop larger root systems through natural selection". Notice that intraspecific competition affects directly the **carrying capacity** of a population (maximum population level supported by the environment).

One of the fundamental tenets of ecology is the **Competitive Exclusion Principle**. "According to this principle too much interspecific competition between two species results in the exclusion of one species" (Cushing et al., [18]). In other words, "the competitive exclusion principle states that two species that use the same resource in the same way in the same space

[1] A relevant part of this introduction is based in the online free Encyclopedia "Wikipedia", namely the biological concepts and the examples.

and time cannot coexist and must diverge from each other over time in order for the two species to coexist. One species will often exhibit an advantage in resource use. This superior competitor will out-compete the other with more efficient use of the limiting resource. As a result, the inferior competitor will suffer a decline in population over time. It will be excluded from the area and replaced by the superior competitor".

Our main objective in Section 2.1 is to study a Ricker competition model of two species. We give a complete analysis of stability and bifurcation and determine the center manifold as well as stable and unstable manifold. It is shown that the autonomous Ricker competition model exhibits subcritical bifurcation, bubbles, period-doubling bifurcation, but no Neimark-Sacker bifurcations. We exhibit the region in the parameter space where the competition exclusion principle applies. We should mention that the results present in this section are accepted for publication [48].

In the next section we study the nonautonomous Ricker competition model. The main contribution here is the proof of existence of the positive solution of the composition map. We also study the stability of the trivial and exclusion cycles and a global result is present for the positive cycle when the carrying capacities of the model are restricted to the unit interval. Finally, we extend to two-dimensional systems the concept of attenuance and resonance and study it to our model.

In Section 2.3 we study a new model called logistic competition model. This model is based in the well known one-dimensional logistic model. Finally, in Section 2.4 we study both the autonomous and the nonautonomous Leslie-Gower competition model.

2.1 An autonomous Ricker competition model

The classical Ricker competition model is given by

$$\begin{cases} u_{n+1} = u_n \exp(K - c_{11}u_n - c_{12}v_n) \\ v_{n+1} = v_n \exp(L - c_{21}u_n - c_{22}v_n) \end{cases},$$

where the parameters K and L are assumed to be positive real numbers and $c_{ij} \in (0,1)$, $1 \leq i, j \leq 2$.

Letting $c_{11}u_n = x_n$ and $c_{22}v_n = y_n$ we get the system

$$\begin{cases} x_{n+1} = x_n \exp(K - x_n - ay_n) \\ y_{n+1} = y_n \exp(L - y - bx_n) \end{cases}, \tag{2.1}$$

where $a = c_{12}/c_{22}$ and $b = c_{21}/c_{11}$. Thus $a, b > 0$. In the language of population dynamics the parameters K and L are known as the carrying capacities of species x and y, respectively, while the parameters a and b are the competition parameters. Eq. (2.1) may represented by the map

$$F(x,y) = (xe^{K-x-ay}, ye^{L-y-bx}).$$

Eq. (2.1) has three fixed points, one extinction fixed point $(0,0)$, and two exclusion fixed points on the axes $(K,0)$ and $(0,L)$. A possible fourth positive coexistence fixed point (x^*, y^*) may be present.

Let us write the map $F = (f,g)$. Then the isoclines are defined as $f(x,y) = x$ and $g(x,y) = y$. These are the lines $ay + x = K$ denoted by s_1 and $y + bx = L$ denoted by s_2 (see Fig. 2.1 A,B). Moreover, the map F takes a point $(x,y) \in \mathbb{R}^2_+$ lying above (below) s_1 to a point with a smaller

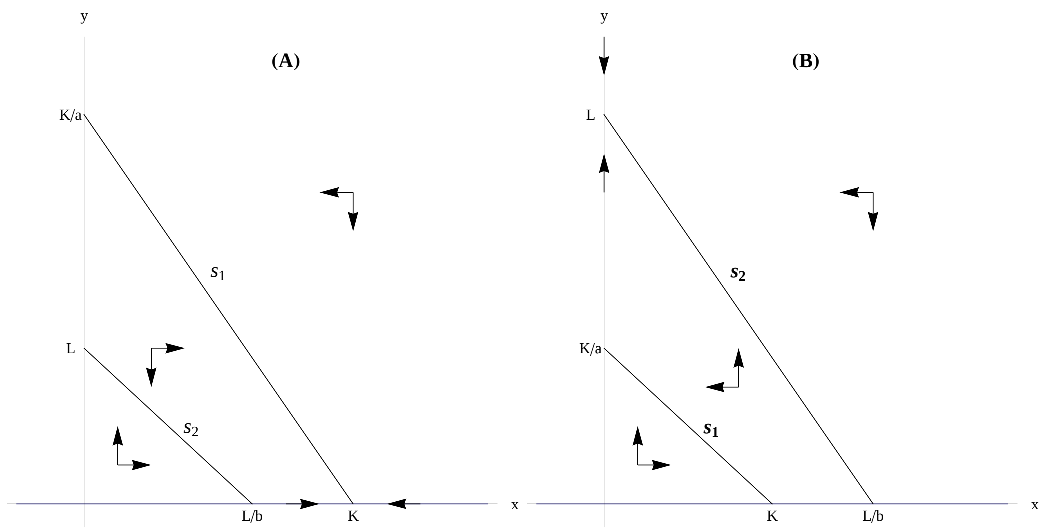

Figure 2.1: The stability of the exclusion fixed point and the validity of the competition exclusion principle. (A) If $0 < K \leq 2$ and $L < bK$, then $(K, 0)$ is locally asymptotically stable and species y goes extinct. (B) If $0 < L \leq 2$ and $L > K/a$, then $(0, L)$ is locally asymptotically stable and species x goes extinct.

(larger) $x-$coordinate. Similarly, the map F takes a point $(x, y) \in \mathbb{R}^2_+$ lying above (below) s_2 to a point with smaller (larger) $y-$coordinate.

Note that on the isocline s_1, the population x has no growth, that is $x_{n+1} = x_n$ and on the isocline s_2 the population y has no growth, that is $y_{n+1} = y_n$.

If the two isoclines s_1 and s_2 intersect in the positive quadrant, we will have the positive fixed point

$$(x^*, y^*) = \left(\frac{K - aL}{1 - ab}, \frac{L - bK}{1 - ab}\right).$$

There are two cases to consider here: (i) $ab < 1$ and (ii) $ab > 1$ (see Fig. 2.2 A,B). The case $ab = 1$ will be discarded since in this case the two isoclines are parallel and no coexistence fixed point is present.

The Jacobian of Eq. (2.1) is given by

$$JF(x, y) = \begin{bmatrix} (1 - x)e^{K-x-ay} & -axe^{K-x-ay} \\ -bye^{L-y-bx} & (1 - y)e^{L-y-bx} \end{bmatrix}.$$

The Jacobians evaluated at the fixed points are

$$J_0 = JF(0, 0) = \begin{bmatrix} e^K & 0 \\ 0 & e^L \end{bmatrix},$$

$$J_K = JF(K, 0) = \begin{bmatrix} 1 - K & -aK \\ 0 & e^{L-bK} \end{bmatrix},$$

$$J_L = JF(0, L) = \begin{bmatrix} e^{K-aL} & 0 \\ -bL & 1 - L \end{bmatrix},$$

and

$$J^* = JF(x^*, y^*) = \begin{bmatrix} 1 - x^* & -ax^* \\ -by^* & 1 - y^* \end{bmatrix}.$$

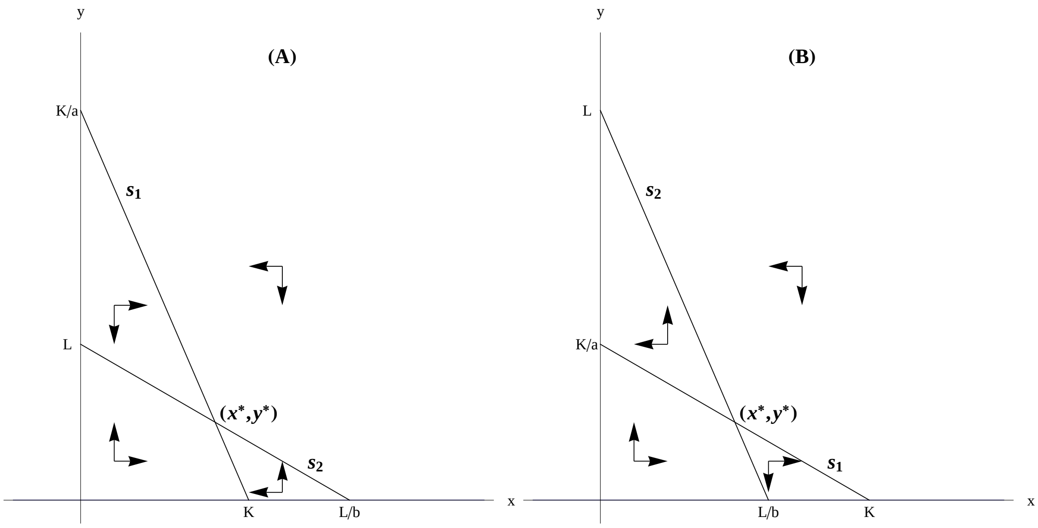

Figure 2.2: Isoclines: (A) The coexistence fixed point of Eq. (2.1) exists if $bK < L < K/a$ and $ab < 1$. (B) The coexistence fixed point of Eq. (2.1) exists if $\frac{K}{a} < L < bK$ and $ab > 1$. In this scenario this equilibrium is a saddle.

Before present the stability of these fixed points we should mention that H. Smith in [73] have been used monotonicity to prove the global stability of the fixed points of the system

$$\begin{cases} u_{n+1} = u_n \exp(r(1 - u_n - Bv_n)) \\ v_{n+1} = v_n \exp(s(1 - Cu_n - v_n)) \end{cases}, \tag{2.2}$$

when $r, s \leq 1$ for which the invariant set is $[0, r^{-1}] \times [0, s^{-1}]$. Notice that by the changes of variables $ru = x$ and $sv = y$, system (2.2) is equivalent to

$$\begin{cases} x_{n+1} = x_n \exp(r - x_n - \frac{Br}{s} y_n) \\ y_{n+1} = y_n \exp(s - y_n - \frac{Cs}{r} x_n) \end{cases}.$$

Consequently, $K = r$, $L = s$, $a = \frac{Br}{s}$ and $b = \frac{Cs}{r}$. Hence, the global results in [73] cover our local analysis when we take the carrying capacities in the unit interval.

In the sequel we study the stability of these fixed points.

2.1.1 Stability of the extinction and exclusion equilibria

The eigenvalues of J_0 are $e^K > 1$ and $e^L > 1$ since $K, L > 0$. Thus $(0, 0)$ is unstable for all $K, L > 0$.

The eigenvalues of J_K are $1 - K$ and e^{L-bK}. Thus $\rho(J_K) < 1$[2] if and only if $0 < K < 2$ and $L < bK$. Thus $(K, 0)$ is asymptotically stable if $0 < K < 2$ and $L < bK$. In the parameter space we call this region R_1 (see Fig. 2.4). Below we will prove that when $K = 2$ this exclusion fixed point is stable and when $L = bK$ it is unstable. Thus one may define the region R_1 as

$$R_1 = \{(K, L) \in \mathbb{R}^2 : 0 < K \leq 2 \wedge L < bK\}.$$

Note that from the inequality $K > L/b$ we obtain $K/a > L/ab$ and consequently $L < L/ab < K/a$. In Fig. 2.1 **(i)** we represent the orientation of the isoclines in the phase-space diagram.

[2] ρ denotes the spectral radius

Similarly, $(0, L)$ is asymptotically stable if $0 < L \leq 2$ and $L > K/a$. The region of stability of $(0, L)$ in the parameter space $K - L$ is denoted by Q_1 (see Fig. 2.4) and is given by

$$Q_1 = \{(K, L) \in \mathbb{R}^2 : 0 < L \leq 2 \wedge L > K/a\}.$$

Note that from the inequality $K/a < L$ it follows that $K < K/ab < L/b$. In Fig. 2.1 **(ii)** we show the orientation of the isoclines in the phase-space diagram.

We now study the stability of the fixed point $(K, 0)$ when $|\rho(J_K)| = 1$. This occurs in two cases, the first is when $K = 2$ and $L < bK$, in which the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 < 1$. The second case is when $0 < K < 2$ and $L = bK$, in which $|\lambda_1| < 1$ and $\lambda_2 = 1$. (The case when $K = 2$ and $L = bK$ at which $\lambda_1 = -1$ and $\lambda_2 = 1$ will not be investigated now due the extension of this work but will be approach in the future). To investigate these cases, we need to use the center manifold theory developed in Section 1.4.

Making the changes of variable $u = x - K$ and $v = y$ in Eq. (2.1) we shift the fixed point $(K, 0)$ to $(0, 0)$. Then the new system is given by

$$\begin{cases} u_{n+1} = (u_n + K)e^{-u_n - av_n} - K \\ v_{n+1} = v_n e^{L - v_n - b(u_n + K)} \end{cases}. \tag{2.3}$$

Let us now consider the first case, i.e. $K = 2$ and $L < bK$. The Jacobian at $(0, 0)$ is now given by

$$\tilde{J}_0 = \begin{bmatrix} -1 & -2a \\ 0 & -e^{L-2b} \end{bmatrix}.$$

Consequently, one may write Eq. (2.3) as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -e^{L-2b} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \tilde{f}(u_n, v_n) \\ \tilde{g}(u_n, v_n) \end{bmatrix}, \tag{2.4}$$

where

$$\tilde{f}(u, v) = (u + 2)e^{-u - av} - 2 + u + 2av,$$

and

$$\tilde{g}(u, v) = ve^{L - v - b(u+2)} + e^{L-2b}v.$$

Let $v = h(u)$ with $h(u) = \alpha u^2 + \beta u^3 + O(|u^4|), \alpha, \beta \in \mathbb{R}$. The map h must satisfy the center manifold equation

$$h(-u - 2ah(u) + \tilde{f}(u, h(u))) + e^{L-2b}h(u) - \tilde{g}(u, h(u)) = 0.$$

By Taylor's series this equation is equivalent to

$$\left(\alpha - e^{-2b+L}\alpha\right)u^2 + \left(be^{-2b+L}\alpha + 4a\alpha^2 - \beta - e^{-2b+L}\beta\right)u^3 + O[u]^4 = 0.$$

Solving the system

$$\begin{cases} \alpha - e^{-2b+L}\alpha = 0 \\ be^{-2b+L}\alpha + 4a\alpha^2 - \beta - e^{-2b+L}\beta = 0 \end{cases}$$

yields the unique solution $\alpha = 0$ and $\beta = 0$. Hence $h(u) = 0$. Consequently, on the center manifold $v = 0$, the new map $\widehat{f}$ is given by

$$\widehat{f}(u) = -u - 2ah(u) + \tilde{f}(u, h(u)) = -2 + e^{-u}(2 + u).$$

The Schwarzian derivative of this map at $u = 0$ is -1. Hence, by Theorem 19 ($p = 1$ and $r = 1$) the exclusion fixed point $(2, 0)$ is asymptotically stable.

We now consider the second case, i.e, $0 < K < 2$ and $L = bK$. After computing the new Jacobian at $(0, 0)$, Eq. (2.3) may be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} 1-K & -aK \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \tilde{\tilde{f}}(u_n, v_n) \\ \tilde{\tilde{g}}(u_n, v_n) \end{bmatrix}, \tag{2.5}$$

where

$$\tilde{\tilde{f}}(u, v) = (u + K)e^{-u-av} - (1 - K)u + (av - 1)K,$$

and

$$\tilde{\tilde{g}}(u, v) = ve^{-v-bu} - v.$$

After some computations the center manifold is given by

$$h(v) = -av - \frac{(1-ab)\,av^2}{K} + \left(\frac{a(-1+ab)(4+a(2+b(-6+K))-K)}{2K^2}\right)v^3. \tag{2.6}$$

So the new map on the center manifold is now

$$\widehat{\widehat{f}}(v) = ve^{-v-bh(v)}. \tag{2.7}$$

Hence, one has $\left(\widehat{\widehat{f}}(v)\right)'_{v=0} = 1$ and $\left(\widehat{\widehat{f}}(v)\right)''_{v=0} = 2(-1 + ab)$. Therefore, by Theorem 17 ($p = 1$ and $r = 1$) the exclusion fixed point on the center manifold $u = h(v)$ is unstable. More precisely, it is a semi-stable fixed point from the right since $2(-1 + ab) < 0$ (see [26, page 31]).

We now summarize these remarks in the following result.

Theorem 30 *For the autonomous Ricker equation (2.1), the following statements hold true:*

1. $(0, 0)$ *is unstable.*
2. $(K, 0)$ *is locally asymptotically stable if* $0 < K \leq 2$ *and* $L < bK$,
3. $(0, L)$ *is locally asymptotically stable if* $0 < L \leq 2$ *and* $L > K/a$.

Before the end of this subsection we note that the competition exclusion principle in Biology is valid for both species. In Fig. 2.1 we presented the two possible scenarios. In Fig. 2.1(i) species y goes extinct while in Fig. 2.1(ii) species x goes extinct.

2.1.2 Stability of the coexistence fixed point: The case $ab < 1$

Recall that $(x^*, y^*) = \left(\frac{K-aL}{1-ab}, \frac{L-bK}{1-ab}\right)$ is a coexistence fixed point if

$$bK < L < K/a \text{ and } ab < 1. \tag{2.8}$$

In this situation the line segments s_1 and s_2 intersect as it is shown in Fig. 2.2A. In order to find the stability region of (x^*, y^*) we need to find the region where

$$|tr(J^*)| - 1 < det(J^*) < 1.$$

(Fore more details about this point see [26, page 200]). This is equivalent to

$$det(J^*) < 1 \wedge det(J^*) > tr(J^*) - 1 \wedge det(J^*) > -tr(J^*) - 1.$$

If at least one of these inequalities is reversed, then (x^*, y^*) is unstable. Now

$$det(J^*) = \frac{ab - 1 + (1-a)L + (1-b)K - (aL - K)(bK - L)}{ab - 1},$$

and

$$tr(J^*) = \frac{2(ab-1) + (1-a)L + (1-b)K}{ab - 1}.$$

Consequently, $det(J^*) < 1$ iff

$$(aL - K)(bK - L) < (1-a)L + (1-b)K, \tag{2.9}$$

$det(J^*) > tr(J^*) - 1$ iff

$$(aL - K)(bK - L) > 0, \tag{2.10}$$

and finally $det(J^*) > -tr(J^*) - 1$ iff

$$(aL - K)(bK - L) > 4(ab-1) + 2(1-a)L + 2(1-b)K. \tag{2.11}$$

Notice that the inequality (2.10) is automatically satisfied when (2.8) holds.

Thus, (x^*, y^*) is locally asymptotically stable if for any fixed $a > 0$ and $b > 0$ with $ab < 1$ the following inequalities hold

$$\begin{cases} (aL - K)(bK - L) < (1-a)L + (1-b)K \\ (aL - K)(bK - L) > 4(ab-1) + 2(1-a)L + 2(1-b)K \end{cases}. \tag{2.12}$$

The solution of this system gives us the region identified by the letter S_1 in the $(K, L)-$plane (see Fig. 2.4). The region S_1 is bounded by the lines $L = K/a$ and $L = bK$ and the curve γ_1 (Points on this curve may be include as it is shown bellow).

Now we show that γ_1 is part of the left branch of the hyperbola defined by the equation

$$(aL - K)(bK - L) = 4(ab-1) + 2(1-a)L + 2(1-b)K.$$

A simple calculation shows that the inequality (2.9) is equivalent to

$$bK^2 + (1-b)K - (1+ab)KL + (1-a)L + aL^2 > 0, \tag{2.13}$$

and the inequality (2.11) is equivalent to

$$bK^2 + 2(1-b)K - (1+ab)KL + 2(1-a)L + aL^2 + 4(ab-1) < 0. \tag{2.14}$$

Before finding the region where these three inequalities are satisfied we give some notes about the following two equations

$$bK^2 + (1-b)K - (1+ab)KL + (1-a)L + aL^2 = 0, \tag{2.15}$$

and

$$bK^2 + 2(1-b)K - (1+ab)KL + 2(1-a)L + aL^2 + 4(ab-1) = 0. \tag{2.16}$$

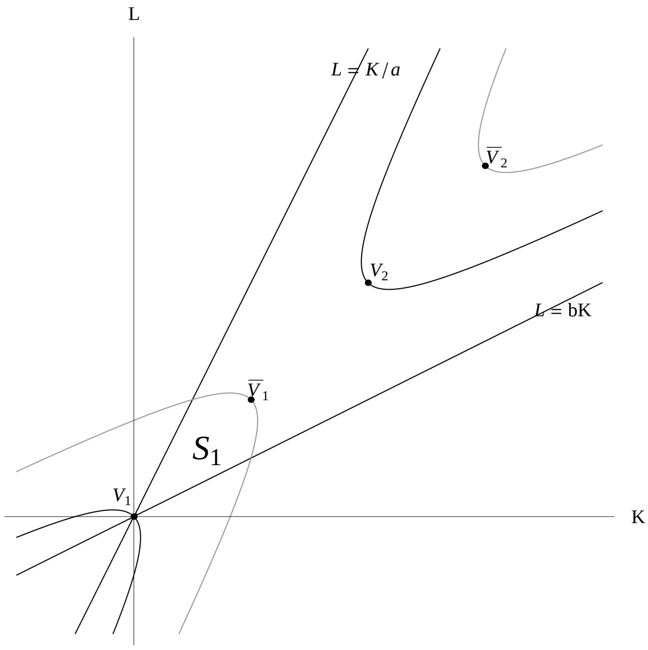

Figure 2.3: The relative position of the hyperbolas $det(J^*) = 1$ and $det(J^*) = -tr(J^*) - 1$ and the lines $L = bK$ and $L = K/a$ when $a = b = 1/2$. The black curves are the implicit solutions of $det(J^*) = 1$ while the grey curves are the implicit solutions of $det(J^*) = -tr(J^*) - 1$. The stability region S_1 is enclosed by the two lies and the left branch of hyperbola $det(J^*) = -tr(J^*) - 1$.

It is easy to show that these two second-degree equations are hyperbolas in the $(K, L)-$plane provided that the constants satisfy the determinant condition

$$D = \begin{vmatrix} b & -\frac{1+ab}{2} \\ -\frac{1+ab}{2} & a \end{vmatrix} = -\frac{1}{4}(1 - ab)^2 < 0.$$

The center (K_c, L_c) of (2.15) is given by $\left(\frac{1+a}{1-ab}, \frac{1+b}{1-ab}\right)$ and the center $(\bar{K}_c, \bar{L}_c)$ of (2.16) is given by $(2K_c, 2L_c)$. The angle of the principal axis of each hyperbola and the positive $K-$axis equals

$$\tan(2\phi) = -\frac{1 + ab}{b - a}.$$

In case of strong symmetry, for example when $a = b = 0.5$, both hyperbolas have the same principal axis, $L = \frac{1+b}{1+a}K$ and the vertices of (2.15) are $V_1 = (0, 0)$ and $V_2 = (2K_c, 2L_c)$ and the vertices of (2.16) are $\bar{V}_1 = (2K_c(1- \sqrt{ab}), 2L_c(1- \sqrt{ab}))$ and $\bar{V}_2 = (2K_c(1+ \sqrt{ab}), 2L_c(1+ \sqrt{ab}))$. It is clear that $V_1 < \bar{V}_1 < V_2 < \bar{V}_2$ since

$$0 < 2K_c(1 - \sqrt{ab}) < 2K_c < 2K_c(1 + \sqrt{ab}),$$

and

$$0 < 2L_c(1 - \sqrt{ab}) < 2L_c < 2L_c(1 + \sqrt{ab}).$$

Knowing these properties and using the implicit function theorem we present in Fig. 2.3, in the $(K, L)-$plane, the solutions of (2.15) and (2.16) when $a = b = 1/2$.

In case of strong asymmetry, i.e., when either $a > 1$ or $b > 1$ such that $ab < 1$ the relative position between these two hyperbolas and the lines is much more involved. However, the origin is always a point of the implicit solutions of Eq. (2.15) (not necessary a vertex). Furthermore,

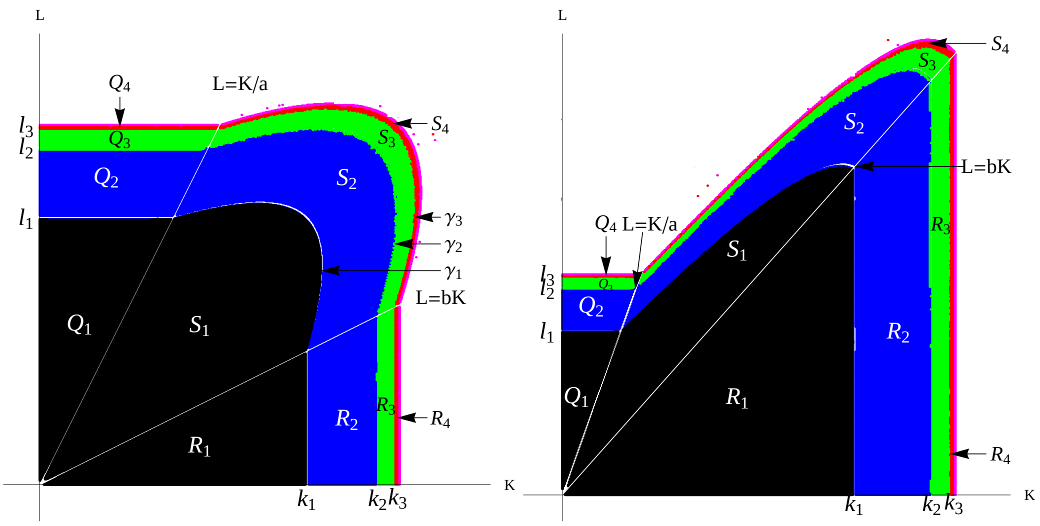

Figure 2.4: The stability regions in the parameter space of the solution of the Ricker competition equation (2.1) when $a > 0$ and $b > 0$ such that $ab < 1$. The plot on the left is when the competition parameters are $a = b = 0.5$ while the plot on the right is obtained when the competition parameters are $a = 0.2$ and $b = 2$.

the left branch of the solutions of Eq. (2.15) intersects the two lines at the origin and it can enter in the first quadrant. However, it does not intersects these two lines in the interior of the first quadrant.

Now we are going to find the region where the inequalities (2.13) and (2.14) hold. If we pick up a point between each branch of the hyperbola either the sign is positive or negative. A good candidate for this test is the center of each hyperbola. We conclude that on (K_c, L_c) the value of the first member of (2.13) is $1 > 0$. Since $K > 0$ and $L > 0$, the inequality (2.13) is verified whenever the values of the carrying capacities K and L are between the positive axes and the right branch of the hyperbola given by (2.15).

Similarly, on $(\bar{K}_c, \bar{L}_c)$ the first member of (2.14) is $4ab > 0$. Hence, the inequality (2.14) is verified whenever the carrying capacities K and L are between the positive axes and the left branch of the hyperbola (2.16) or in the interior of the right branch of the hyperbola (2.16).

From (2.8) it follows that the inequality (2.10) is verified. This corresponds to the points in the $(K, L)-$plane between the lines $L = K/a$ and $L = bK$. The stability region S_1 is intersection of these three regions. (This follows from the two precedent observations and by the relation (2.8)).

In the sequel, we will show that when K and L are on γ_1 the coexistence fixed point is asymptotically stable. This happens when $|\rho(J^*)| = 1$, i.e., $\lambda_1 = -1$ and $\lambda_2 < 1$. Note that on the curve γ_1 one has

$$L = \frac{2(a-1) + (1+ab)K}{2a} \pm \frac{\sqrt{(2(a-1) + (1+ab)K)^2 + 4a(4(1-ab) + 2(b-1)K - bK^2)}}{2a}. \tag{2.17}$$

Making the change of variables $u_n = x_n - x^*$ and $v_n = y_n - y^*$ in Eq. (2.1) we shift the positive fixed point to the origin. Eq. (2.1) is now equivalent to

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \tilde{f}(u_n, v_n) \\ \tilde{g}(u_n, v_n) \end{bmatrix}. \tag{2.18}$$

where

$$\tilde{f}(u,v) = (u+x^*)e^{K-(u+x^*)-a(v+y^*)} - x^* - J_{11}u - J_{12}v,$$
$$\tilde{g}(u,v) = (v+y^*)e^{L-(v+y^*)-b(u+x^*)} - y^* - J_{21}u - J_{22}v,$$

and the Jacobian at $(0,0)$ is given by

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{-1+K-a(-b+L)}{-1+ab} & -\frac{a(-K+aL)}{-1+ab} \\ -\frac{b(bK-L)}{-1+ab} & \frac{-1+ab-bK+L}{-1+ab} \end{bmatrix}.$$

Now we need to diagonalize this matrix. Let us write the diagonal matrix as

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix},$$

where

$$S_{11} = \frac{-(1+b)K+(1+a)L+\nabla}{2b(bK-L)}, S_{12} = \frac{-(1+b)K+(1+a)L-\nabla}{2b(bK-L)},$$
$$\lambda_1 = \frac{2(ab-1)+(1-b)K+(1-a)L-\nabla}{2(ab-1)}, \lambda_2 = \frac{2(ab-1)+(1-b)K+(1-a)L+\nabla}{2(ab-1)},$$
$$\tilde{S}_{11} = \frac{b(bK-L)}{\nabla}, \tilde{S}_{12} = \frac{(1+b)K-(1+a)L+\nabla}{2\nabla},$$
$$\tilde{S}_{21} = \frac{b(-bK+L)}{\nabla}, \tilde{S}_{22} = \frac{-(1+b)K+(1+a)L+\nabla}{2\nabla},$$

with

$$\nabla = \sqrt{\left(1+2b+(1-4a)b^2\right)K^2 + 2\left(a(b-1)-b-1+2a^2b^2\right)KL + \left(1+2a+a^2(1-4b)\right)L^2}.$$

Using again a new change of variables $u = S_{11}z + S_{12}w$ and $v = z + w$, yields the following system

$$\begin{bmatrix} z_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_n \\ w_n \end{bmatrix} + \begin{bmatrix} \tilde{\tilde{f}}(z_n,w_n) \\ \tilde{\tilde{g}}(z_n,w_n) \end{bmatrix}, \tag{2.19}$$

where

$$\begin{bmatrix} \tilde{\tilde{f}}(z_n,w_n) \\ \tilde{\tilde{g}}(z_n,w_n) \end{bmatrix} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix} \begin{bmatrix} \tilde{f}(u_n,v_n) \\ \tilde{g}(u_n,v_n) \end{bmatrix}.$$

Let $z = h(w)$, where $h(w) = \alpha w^2 + \beta w^3 + O(w^4)$. The function h must satisfy the following equation

$$h(\lambda_2 w + \tilde{\tilde{g}}(h(w),w)) - \lambda_1 h(w) - \tilde{\tilde{f}}(h(w),w) = 0. \tag{2.20}$$

After simplifying this equation, we write the Taylor expansion and then we find the values of the constant α and β. Since the computations here are cumbersome we did not find the exact values of α and β. However, we are able to find it numerically. Notice that, the maximum value of the carrying capacity K is given by biggest value on the left branch of hyperbola given by (2.16), namely

$$K_{\max} = \frac{2(1+a-a\sqrt{b})}{1-ab}.$$

Notice that depending of the choice of a and b, $K_{\max}$ can be a very large number namely when $ab \approx 1$.

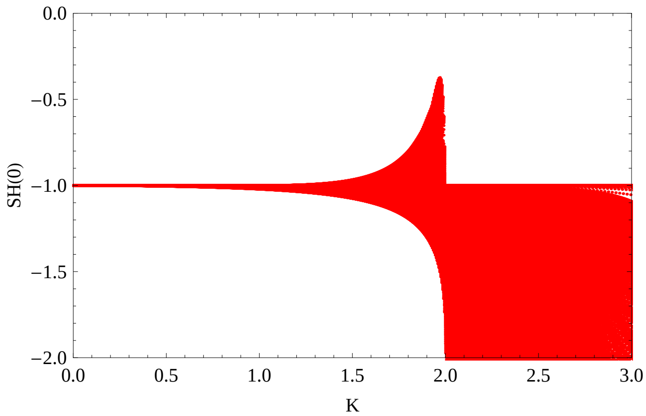

Figure 2.5: Part of the values of the Schwarzian derivative of the new map on the center manifold for the coexistence fixed point. In this simulation we used values of the carrying capacity K in the interval $(0, 3]$ and the competition parameters are in the interval $(0, 2]$ such that $ab < 1$.

We reduce our analysis when the competition parameters belong to the interval $(0, 2]$. Taking randomly the values of a and b in the interval $(0, 2]$ such that $ab < 1$ and using the value of carrying capacity L given in (2.17), we vary the carrying capacity K in the interval $(0, 3]$ and find numerically the values of α and β. After we compute the value of the Schwarzian derivative of the map

$$H(w) = -w + \tilde{\tilde{g}}(h(w), w).$$

From our simulations we conclude that the Schwarzian derivative $SH(0) < 0$ as it is shown in Fig. 2.5. Note that this simulations can be done for a larger values of a and b.

By numerical calculations, we conclude that on the curve γ_1 the coexistence fixed point of Eq. (2.1) must be asymptotically stable. Similar conclusions may be made if we consider the center manifold $w = h(z)$.

We now summarize these conclusions in the following result.

Theorem 31 *Suppose that $ab < 1$ and let $\hat{S} = Int(S_1) \cup \gamma_1$, where $Int(S_1)$ denotes the interior of S_1. Then the coexistence fixed point*

$$(x^*, y^*) = \left(\frac{aL - K}{ab - 1}, \frac{bK - L}{ab - 1} \right)$$

of the Ricker equation (2.1) is locally asymptotically stable if

$$4(ab - 1) + 2(1 - a)L + 2(1 - b)K \leq (aL - K)(bK - L) < (1 - a)L + (1 - b)K.$$

Equivalently, the coexistence fixed point is locally asymptotically stable if $(K, L) \in \hat{S}$.

2.1.3 The stable and unstable manifold

In this section we study (via numerical computations) a celebrated scenario in classic competition theory, the saddle exclusion case (or equivalently, a stable and unstable manifold).

A general two-dimensional map has a stable and an unstable manifold when the following conditions, in the Trace-Determinant plane, are satisfied

$$\begin{cases} det(J^*) < tr(J^*) - 1 \\ det(J^*) > -tr(J^*) - 1 \end{cases} \vee \begin{cases} det(J^*) > tr(J^*) - 1 \\ det(J^*) < -tr(J^*) - 1 \end{cases} \vee$$
$$\begin{cases} det(J^*) < \frac{(tr(J^*))^2}{4} \\ det(J^*) > 1 \\ det(J^*) > -tr(J^*) - 1 \end{cases} \vee \begin{cases} det(J^*) < \frac{(tr(J^*))^2}{4} \\ det(J^*) > tr(J^*) - 1 \\ det(J^*) > 1 \end{cases}. \tag{2.21}$$

For more details about these conditions see [26, page 205]. Hence we have two scenarios to consider: (i) $ab > 1$ in which the winner depends on initial conditions and (ii) $ab < 1$ where we have the presence of both locally asymptotically stable cycles and unstable fixed points.

Case (i): $ab > 1$

In the model (2.1) the saddle scenario occurs when one has a coexistence equilibrium such that

$$aL > K \text{ and } bK > L, \tag{2.22}$$

which implies that

$$ab > 1. \tag{2.23}$$

We now determine, in the parameter space, the region where the relation (2.21) is satisfied. Simplifying the relation $det(J^*) > tr(J^*) - 1$ we get

$$(aL - K)(bK - L) < 0,$$

which is impossible by (2.22). Hence there are two systems in (2.21) that lead to an empty region. Analogously, $det(J^*) < tr(J^*) - 1$ is the region in the $(K, L)-$plane between the two lines $L = K/a$ and $L = bK$, i.e., the assumption (2.22). Notice that by (2.23) one has $b > 1/a$, and consequently $bK > K/a$.

The inequality $det(J^*) > -tr(J^*) - 1$ leads to

$$2(1 - a)L + 2(1 - b)K - (aL - K)(bK - L) + 4(ab - 1) > 0.$$

Notice that by (2.16) the second degree equation

$$2(1 - a)L + 2(1 - b)K - (aL - K)(bK - L) + 4(ab - 1) = 0 \tag{2.24}$$

represents an hyperbola in $(K, L)-$plane (for more details about this hyperbola consult the precedent subsection). Hence the system

$$\begin{cases} det(J^*) < tr(J^*) - 1 \\ det(J^*) > -tr(J^*) - 1 \end{cases} \tag{2.25}$$

is satisfied whenever K and L are between the lines $L = K/a$ and $L = bK$ and the right branch of the hyperbola defined by (2.24).

The relation $det(J^*) > 1$ is equivalent to

$$(1 - a)L + (1 - b)K - (aL - K)(bK - L) > 0.$$

This is the region in the first quadrant outside the right branch of hyperbola defined implicitly by

$$(1-a)L + (1-b)K - (aL-K)(bK-L) = 0 \tag{2.26}$$

which passes in the origin.

The inequality $det(J^*) < \frac{(tr(J^*))^2}{4}$ leads to the following relation

$$(4a^2b-(a+1)^2)L^2 + 2(1+a+b-ab-2(ab)^2)KL + (4ab^2-(b+1)^2)K^2 < 0. \tag{2.27}$$

Notice that the second degree equation

$$(4a^2b-(a+1)^2)L^2 + 2(1+a+b-ab-2(ab)^2)KL + (4ab^2-(b+1)^2)K^2 = 0,$$

represents a conic, in the $(K, L)-$plane, known as a pair of imaginary lines intersecting in a real point (see for instance [8]), provided that the test condition are

$$\left|\begin{pmatrix} 4ab^2-(b+1)^2 & 1+a+b-ab-2(ab)^2 & 0 \\ 1+a+b-ab-2(ab)^2 & 4a^2b-(a+1)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right| = 0$$

and

$$\left|\begin{pmatrix} 4ab^2-(b+1)^2 & 1+a+b-ab-2(ab)^2 \\ 1+a+b-ab-2(ab)^2 & 4a^2b-(a+1)^2 \end{pmatrix}\right| = -4ab(-1+ab)^3 < 0.$$

This point is precisely the origin because the equations of these two lines are $L = m_{\pm}K$ where

$$m_{\pm} = \frac{1+a+b-ab-2a^2b^2 \pm 2\sqrt{ab(-1+ab)^3}}{((1+a)^2-4a^2b)}.$$

Hence, the system

$$\begin{cases} det(J^*) < \frac{(tr(J^*))^2}{4} \\ det(J^*) > 1 \\ det(J^*) > -tr(J^*) - 1 \end{cases} \tag{2.28}$$

represents the region in the $(K, L)-$plane outside both hyperbolas defined by (2.24) and (2.26) and between the two lines $L = m_{\pm}K$.

Using numerical computations it is possible to show that when $a > 1$ and $b > 1$ (which implies $ab > 1$) we have

$$m_+ > b > 1/a > m_-.$$

So assuming this restriction on the competition parameters and under the hypothesis (2.22), the system (2.28) has no solution. Consequently, if $a > 1$ and $b > 1$ the relation (2.21) is equivalent to the system (2.25). Hence, the saddle region is enclosed by the two lines $L = K/a$ and $L = bK$ and the right branch of hyperbola defined by (2.24). In Fig. 2.6 is depicted in the parameter space (K, L) this region when $a = 2$ and $b = 1.5$.

Notice that if we assume that either $a < 1$ or $b < 1$ such that $ab > 1$, then the saddle region is more involved, namely it contains the solution of both systems (2.25) and (2.28).

Now let us take $a > 1$ and $b > 1$ such that $(K, L) \in Z$. Following the same techniques as in section (2.1.2) we find that, locally, the stable manifold of the coexistence fixed point is given by

$$W^s = \{(z, w) \in \mathbb{R}^2 : w = \alpha_1 z^2 + \beta_1 z^3, \alpha_1, \beta_1 \in \mathbb{R}\},$$

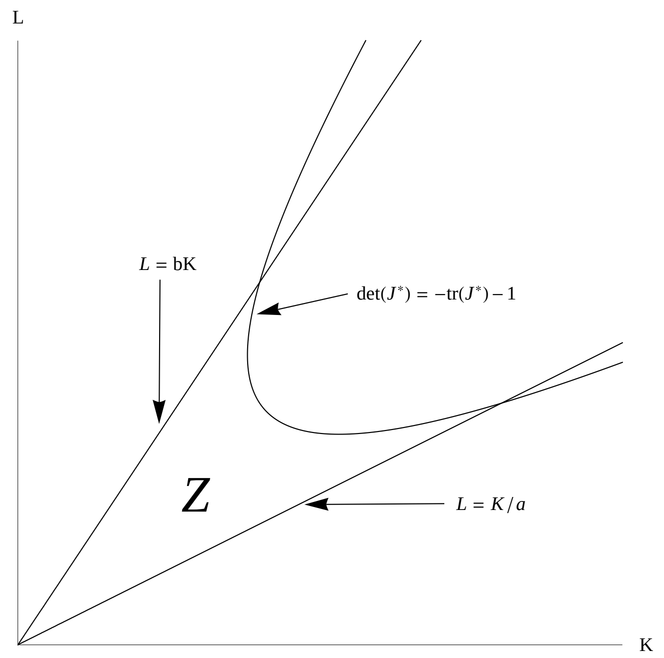

Figure 2.6: The saddle region Z, in the parameter space K and L, when $a = 2$ and $b = 3/2$.

and the unstable manifold is

$$W^u = \{(z, w) \in \mathbb{R}^2 : z = \beta_2 w^2, \beta_2 \in \mathbb{R}\}.$$

We omit the values of α_1, β_1 and β_2 since they are cumbersome and very long. In the original coordinates the values of z and w are given by

$$z = \frac{S_{22}(x - x^*) - S_{12}(y - y^*)}{S_{11}S_{22} - S_{21}S_{12}} \text{ and } w = \frac{S_{11}(y - y^*) - S_{21}(x - x^*)}{S_{11}S_{22} - S_{21}S_{12}},$$

where S_{ij} are the entries of the matrix S determined in the previous subsection.

Case (ii): $ab < 1$

By (2.8) it follows that $det(J^*) < tr(J^*) - 1$ is impossible. Hence the first system in (2.21) leads to an empty region. In the previous subsection we determine the inequality

$$det(J^*) < \frac{(tr(J^*))^2}{4},$$

which leads to the relation (2.27). We claim that when $ab < 1$ the first member of (2.27) is negative. In order to show that, let us assume temporally that $L = mK$ for some $m > 0$. Hence, the relation given in (2.27) is equivalent to $K^2 u(m) < 0$, where

$$u(m) = (4a^2 b - (a + 1)^2)m^2 + 2(1 + a + b - ab - 2(ab)^2)m + 4ab^2 - (b + 1)^2.$$

Solving the equation $u(m) = 0$ one has the following two values

$$m = \frac{(1 + a + b - ab - 2(ab)^2) \pm 2\sqrt{ab(1 - ab)^3}\,i}{(1 + a)^2 + 4a^2 b}, i = \sqrt{-1}.$$

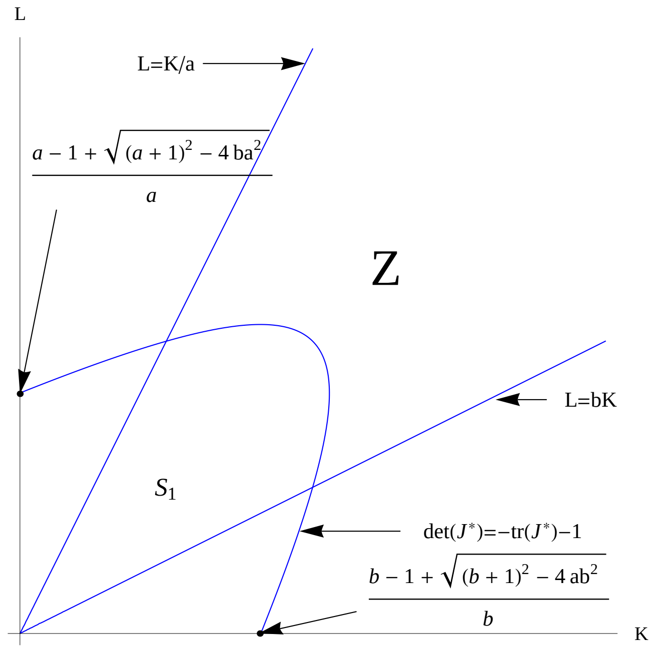

Figure 2.7: The saddle region Z, in the parameter space K and L, when $a = b = 0.5$.

Now we show that the coefficient of m^2 is a negative number. So

$$4a^2b - (a+1)^2 = 4aab - (a+1)^2 < 4a - (a+1)^1 = -(a-1)^2 < 0$$

(since $ab < 1$). Hence, the function u is a negative parabola. Consequently, the relation (2.27) is satisfied.

Because $det(J^*) > tr(J^*) - 1$ is automatically verified and by the fact that $det(J^*) < \frac{(tr(J^*))^2}{4}$ is always true, it follows that the system

$$\begin{cases} det(J^*) < \frac{(tr(J^*))^2}{4} \\ det(J^*) > tr(J^*) - 1 \\ det(J^*) > 1 \end{cases}$$

leads to the same region in the parameter space that

$$\begin{cases} det(J^*) < \frac{(tr(J^*))^2}{4} \\ det(J^*) > -tr(J^*) - 1 \\ det(J^*) > 1 \end{cases} .$$

Therefore, the relation (2.21) gives us

$$\begin{cases} det(J^*) > tr(J^*) - 1 \\ det(J^*) < -tr(J^*) - 1 \end{cases} \vee \begin{cases} det(J^*) > -tr(J^*) - 1 \\ det(J^*) > 1 \end{cases} .$$

This leads to the region Z identified in Fig. 2.7. Note that $Z = \cup_{i \geq 2} S_i$ (in the Section 2.1.4 we will give more details about the regions S_i, $i \geq 2$).

Before the end of this subsection we remark that the computation of the saddle and unstable manifold follows the same guidelines as above.

The exclusion fixed points

We now determine the region, in the parameter space, where the exclusion fixed point $(K, 0)$ of Eq. (2.1) has stable and unstable manifold. This set is given by

$$Z_K = \{(K, L) \in \mathbb{R}^2 : K > 2 \wedge L < bK\}.$$

Let $(K, L) \in Z_K$. Then similar techniques as before lead us to find locally the stable manifold

$$W_K^s = \{(x, y) \in \mathbb{R}^2 : y = 0\}$$

and the unstable manifold

$$W_K^u = \{(x, y) \in \mathbb{R}^2 : x = K\}$$

of the exclusion fixed point $(K, 0)$.

Similarly, in the set

$$Z_L = \{(K, L) \in \mathbb{R}^2 : K > 0 \wedge L > 2 \wedge L > K/a\}$$

the exclusion fixed point $(0, L)$ has the stable manifold (locally)

$$W_L^s = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

and the unstable manifold (locally)

$$W_L^u = \{(x, y) \in \mathbb{R}^2 : y = L\}.$$

2.1.4 Bifurcation scenarios

In the absence of species "y" the dynamics of species "x" is governed by the one-dimensional Ricker equation

$$x_{n+1} = x_n e^{K - x_n}, n \in \mathbb{Z}^+. \tag{2.29}$$

Eq. (2.29) has a globally asymptotically stable fixed point when $0 < K \leq 2$ and its basin of attraction is the positive real line (see appendix A where we present an alternative proof of the enveloping method [12, 13, 14, 15, 47]).

At $K = k_1 = 2$ a period-doubling bifurcation occurs. At the bifurcation point $k_1 = 2$, an asymptotically stable 2−periodic cycle $\{\overline{x}_0, \overline{x}_1\}$ is born. The two points $\overline{x}_0, \overline{x}_1$ satisfy the equations $\overline{x}_1 = \overline{x}_0 e^{K-\overline{x}_0}$ and $\overline{x}_0 = \overline{x}_1 e^{K-\overline{x}_1}$. By the linearization principle the stability of this 2−periodic cycle can be seen from the product of the derivatives of the map given in (2.29) evaluated at $\overline{x}_0$ and $\overline{x}_1$. This product is less than one in absolute value, i.e., $\prod_{i=0}^{1} |1 - \overline{x}_i| < 1$ if $k_1 < K < k_2$, where $k_2 \approx 2.5265$. At $K = k_2$, a new period-doubling bifurcation occurs. Then there exists a k_3 greater than but near k_2 such that a new 4−periodic cycle is asymptotically stable if $k_2 < K < k_3$. This period-doubling scenario continues. So there are two bifurcation points k_j and k_{j+1} for a specific integer j such that the r−periodic cycle $\{\overline{x}_0, \ldots, \overline{x}_{r-1}\}$, where $r = 2^j$, satisfy the relation

$$\prod_{i=0}^{r-1} |1 - \overline{x}_i| < 1. \tag{2.30}$$

The r−periodic cycle $\{\overline{x}_0, \ldots, \overline{x}_{r-1}\}$ yields an exclusion r−periodic cycle

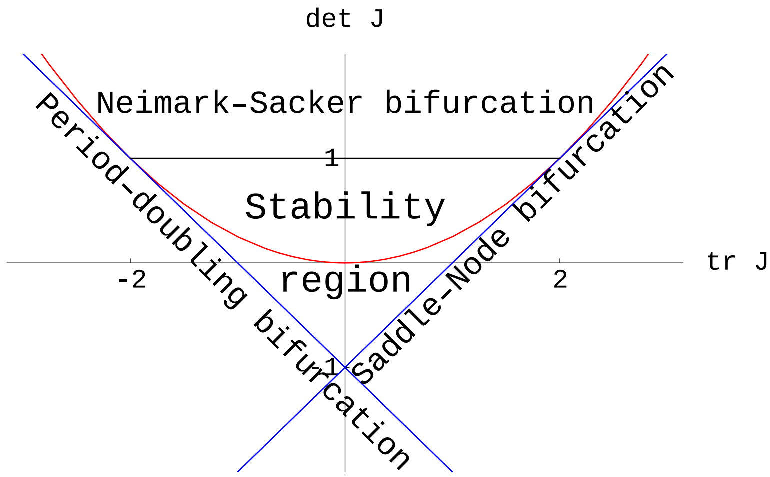

Figure 2.8: The occurrence of the three main types of bifurcation for two-dimensional systems.

$$C_r^x = \{(\overline{x}_0, 0), (\overline{x}_1, 0), \ldots, (\overline{x}_{r-1}, 0)\} \tag{2.31}$$

of the competition Ricker model (2.1).

The Jacobian of C_r^x evaluated along the periodic orbit is given by the following 2×2 matrix

$$\prod_{r-1}^{0} JF(\overline{x}_i, 0) = \begin{pmatrix} \prod_{i=0}^{r-1}(1 - \overline{x}_i) & J_{12} \\ 0 & e^{rL - b\sum_{i=0}^{r-1}\overline{x}_i} \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = \prod_{i=0}^{r-1}(1 - \overline{x}_i)$ and $\lambda_2 = e^{rL-b\sum_{i=0}^{r-1}\overline{x}_i} = e^{r(L-bK)}$. Using the hypothesis $L < bK$ yields $|\lambda_2| = \lambda_2 < 1$ and from (2.30) it follows $|\lambda_1| < 1$. Thus C_r^x is asymptotically stable.

Note that if $L = 0$ one has $\lambda_2 < 1$. This implies that the sequence of parameters $\{k_j\}$ on the $K-$axis follows the one-dimensional case. That is $k_1 = 2$, $k_2 \approx 2.52647$, $k_3 \approx 2.6562$, etc. We think that this sequence deserves further study in order to know if there exists universality.

We now summarize the above discussion

Theorem 32 *Let $0 < L < bK$. Then the periodic cycle C_r^x, defined in (2.31), of Eq. (2.1) is locally asymptotically stable.*

Bifurcation occurs when the eigenvalues cross the unit circle. One can obtain a new parameter family of maps where the fixed point (x^*, y^*) is transformed into the fixed point $(0, 0)$. Let $J = JF(0, 0)$, $T = tr(J)$, and $D = det(J)$. Fig. 2.8 illustrates the three possible bifurcations using the trace-determinant analysis on the $T - D$ plane. The stability region is enclosed in the triangle with sides are $D = 1$, $D = T - 1$, and $D = -T - 1$. Neimark-Sacker bifurcation, period-doubling bifurcation, and either saddle-node bifurcation, transcritical bifurcation or pitchfork bifurcation occur, respectively, when crossing these lines from the stability region. For bifurcation in two-dimensional systems see for instance [26]. For details on bifurcation in higher dimension see for example [77].

Now we are in a position to provide a deeper explanation of Fig. 2.4. Note that the coexistence fixed point (x^*, y^*) is asymptotically stable if $(K, L) \in \hat{S}$. When $L = bK$, the Jacobian of

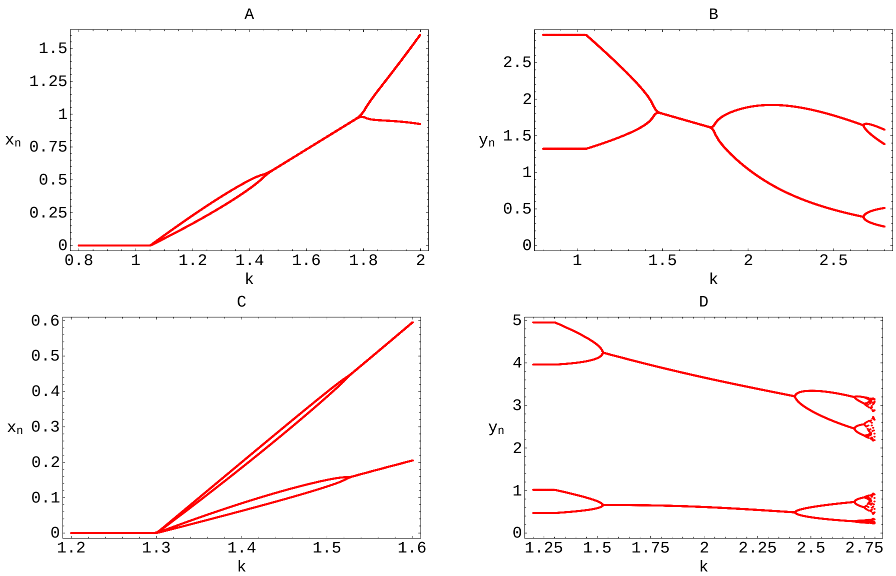

Figure 2.9: The presence of subcritical bifurcation in the autonomous Ricker type competition model (2.1)

Eq. (2.1) has an eigenvalue equal to one. For the map $\widehat{\widehat{f}}$ defined in (2.7) one has

$$\frac{\partial \widehat{\widehat{f}}}{\partial x}(0) = 1, \frac{\partial \widehat{\widehat{f}}}{\partial K}(0) = 0 \text{ and } \frac{\partial^2 \widehat{\widehat{f}}}{\partial x^2}(0) \neq 0.$$

Hence a transcritical bifurcation occurs when $L = bK$, where the coexistence fixed point is $(x^*, y^*) = (K, 0)$, the exclusion fixed point on the $x-$axis. When (K, L) crosses the line $L = bK$ to region R_1, the branch of equilibria (x^*, y^*) transcritically bifurcates with the branch of exclusion equilibria $(K, 0)$, while (x^*, y^*) moves from the first quadrant into the fourth (or second) quadrant, where it becomes ecologically irrelevant. Moreover, stability exchanges from one branch to the other. Similarly, if $L = K/a$, the coexistence fixed point undergoes a transcritical bifurcation.

Eq. (2.1) has a period-doubling bifurcation when we have equality in the relation (2.11). This is represented by the curve γ_1 in Fig. 2.4. Consequently, as K and L passes the curve γ_1 the coexistence fixed point undergoes a period-doubling bifurcation into a coexistence 2−periodic cycle. Thus in region S_2 Eq. (2.1) has one unstable fixed point and one asymptotically stable coexistence 2−periodic cycle.

When K and L passes the line $L = bK$ from region S_2 to region R_2, the coexistence 2−periodic cycle bifurcates (transcritical). This new 2−periodic cycle is in fact an exclusion cycle on the $x-$axis. If, however, we move K and L from R_2 to S_2, then the exclusion 2−periodic cycle undergoes a transcritical bifurcation into a coexistence 2−periodic cycle. Another period-doubling bifurcation appears in the exclusion fixed point if the parameters K and L move from region R_1 to region R_2. Thus if the parameters K and L are in region R_2, Eq. (2.1) possesses an asymptotically stable exclusion 2−periodic cycle on the $x-$axis. Similar analysis can be taken if the parameters are in region Q_2.

The coexistence 2−periodic cycle undergoes a period-doubling bifurcation when the parameters pass the curve γ_2. Thus in region S_3, this coexistence 2−periodic cycle becomes unstable and a new asymptotically stable 4−periodic cycle is born. This new cycle undergoes a transcritical bifurcation to an asymptotically stable exclusion 4−periodic cycle on the $x-$axis whenever the parameters moves from region S_3 to region R_3. We also have a period-doubling bifurcation in the exclusion 2−periodic cycle if we change the parameters from region R_2 to region R_3. Thus in region R_3, Eq. (2.1) has an asymptotically stable exclusion 4−periodic cycle. The same happens in the y−axis if the parameters change from region S_3 to region Q_3. This period-doubling bifurcation route to chaos is reminiscent of the dynamics exhibited by the one-dimensional Ricker-map.

A different scenario appears if the relation between L and K obey the rule $L = \alpha_1 K + \alpha_2$, $\alpha_2 > 0$ and $-\varepsilon < \alpha_1 < \varepsilon$, for a small $\varepsilon > 0$. We call this scenario the "bubble scenario". This occurs if one passes from zone S_{i+1} to the stability region S_i and enter again in the stability region S_{i+1}. In this scenario, if we draw the bifurcation diagram in the (K, x)−plane we find bubbles. In Fig. 2.9 we present two scenarios. In cases A and B we vary K and fix $L = 2.1$, and let $a = b = 1/2$. This results in the presence of one bubble in plot A. This phenomenon happens because for values of $K \lesssim 1.05$, Eq. (2.1) has one attracting 2−periodic cycle on the y−axis (see plot B) . Thus $x_n = 0$ and y_n oscillates between 1.32152 and 2.87848 as n goes to infinity. At $K \approx 1.05$, the exclusion cycle on the y−axis bifurcates (transcritical) and the fixed point 0 bifurcates (period-doubling). This implies that the coexistence 2−periodic cycle in the x−axis is born. Here we see the bubble in plot A and a 2−periodic cycle in plot B. For values $1.45 \lesssim K \lesssim 1.78$, Eq. (2.1) has a coexistence fixed point. This observation implies that at $K \approx 1.45$ the 2−periodic cycle in turn will undergo a bifurcation and return to a stable equilibrium[3]. At $K \approx 1.78$ a new period-doubling bifurcation occurs, and then the coexistence fixed point bifurcates into a coexistence 2−periodic cycle. This is clearly shown in plot A and plot B. In cases C and D we fix $L = 2.6$. For values of $K \approx 1.3$ the equation has an exclusion 4−periodic cycle on the y−axis. As K increases we enter in the zone where we have a coexistence 4−periodic cycle. Here we see two bubbles in plot C and a coexistence 4−periodic cycle in plot D. Both cases lead to a 2−periodic cycle.

The Neimark-Sacker bifurcation starts when

$$det(JF(x^*, y^*)) = 1 \text{ and } -2 < tr(JF(x^*, y^*)) < 2,$$

i.e, when

$$(1 - a)L + (1 - b)K = (aL - K)(bK - L), \tag{2.32}$$

and

$$0 < (1 - a)L + (1 - b)K < 4(1 - ab). \tag{2.33}$$

Inequalities (2.33) are satisfied whenever K and L belongs to the region limited by the positive axes and the line $(1-a)L = -(1-b)K + 4(1-ab)$. One can show that this line does not intersect the hyperbola given by (2.32). On the other hand the vertices $(0, 0)$ and $\left(2\frac{1+a}{1-ab}, 2\frac{1+b}{1-ab}\right)$ of the hyperbola given by (2.32) are outside this triangle. Hence Eq. (2.1) has no Neimark-Sacker bifurcation.

[3]Actually this phenomenon is not a reverse period doubling bifurcation. It is a subcritical bifurcation. For more details about this phenomenon in population models see [16]

2.2 Nonautonomous Ricker competition model

2.2.1 Existence of a positive solution

In this subsection we prove the existence of a solution of the periodic nonautonomous difference equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n e^{K_n - x_n - a_n y_n} \\ y_n e^{L_n - y_n - b_n x_n} \end{pmatrix}, n \in \mathbb{Z}^+, \tag{2.34}$$

where $K_n > 0$ and $L_n > 0$ are the carrying capacities of species x and y, respectively, and $a_n > 0$ and $b_n > 0$ are the competition parameters of species y and x, respectively. Our analysis here will be in the case when $a_n b_n < 1$, for all $n = 0, 1, 2, \ldots$ in which the system (2.34) has a possible stable coexistence cycle. The saddle scenario, i.e., the case when $a_n b_n > 1$ will be omitted.

Let us write $F_n(x, y) = (f_n(x, y), g_n(x, y))$ where

$$f_n(x, y) = xe^{k_n - x - a_n y} \text{ and } g_n(x, y) = ye^{L_n - y - b_n x}.$$

Eq. (2.34) is $p-$periodic whenever $F_n = F_{n \bmod p}$, $\forall n \in \mathbb{Z}^+$, with p the minimal period. This scenario occurs when at least one of the sequence of parameters is periodic, i.e., $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$ or/and $b_n = b_{n \bmod p}$. (There exist 15 different situations for which Eq. (2.34) is $p-$periodic.)

We initiate our investigation proving the existence of periodic solutions of Eq. (2.34) when the parameters are $p-$periodic. To do so we utilize Brouwer's fixed point theorem stated below.

Theorem 33 (Brouwer's fixed point theorem [57]) *Every continuous map from a convex compact subset A of a Euclidean space to itself has a fixed point.*

Remark 34 *For infinite dimensional spaces, Theorem 33 is known as Schauder's fixed point theorem*

Theorem 35 *Let* $(K_n, L_n) \in \widehat{S}_{1,n} = int(S_{1,n}) \cup \gamma_{1,n}$ *such that* $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$, $b_n = b_{n \bmod p}$, *and* $a_n b_n < 1$, *for all* $n \in \mathbb{Z}^+$. *Then the $p-$periodic Ricker competition equation (2.34) has a $p-$periodic cycle in region* $[0, e^{K_{\max}-1}] \times [0, e^{L_{\max}-1}]$, *where* $K_{\max} = \max\{K_n\}$ *and* $L_{\max} = \max\{L_n\}$.

Proof. Consider the map $F_n(x, y) = (f_n(x, y), g_n(x, y))$. Then under the hypothesis of the theorem one has $F_{n+p} = F_n$, for all $n \in \mathbb{Z}^+$. Note that the maps $f_n(x, y)$ and $g_n(x, y)$ attains their maxima at $(1, 0)$ and $(0, 1)$, respectively. Moreover, the maximum of f_n is $e^{K_n - 1}$ and the maximum of g_n is $e^{L_n - 1}$.

Let us define the natural order "$\leq$" by $(x_1, y_1) \leq (x_2, y_2)$ if $x_2 - x_1 \geq 0$ and $y_2 - y_1 \geq 0$. By the above remark, it follows that

$$F_n(x, y) \leq (e^{K_n - 1}, e^{L_n - 1}), \text{for all } x \geq 0 \text{ and } y \geq 0.$$

Let $K_{\max} = \max\{K_n\}$ and $L_{\max} = \max\{L_n\}$ and define the set A as

$$A = [0, e^{K_{\max}-1}] \times [0, e^{L_{\max}-1}].$$

Then A is compact and convex. Moreover A is invariant under the map Φ_p, i.e.,

$$\Phi_p(A) = F_{p-1} \circ \ldots \circ F_1 \circ F_0(A) \subset A.$$

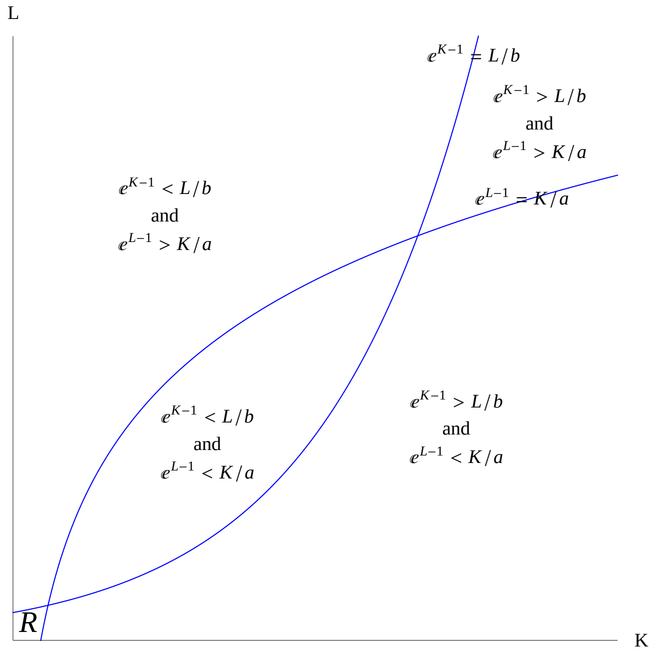

Figure 2.10: Four different scenarios in which an individual Ricker competition map F_i has a maximum value on the axes. The values of the competition parameters are fixed such that $ab < 1$. In these curves $a = b = 1/2$.

Hence, by Theorem 33 the map Φ_p has a fixed point $(\overline{x}, \overline{y})$ in A, which is a $p-$periodic point of the periodic Ricker model (2.34). ■

The above theorem seems to be nice but we find some difficulties. This observation is based in the fact that the set A contains the trivial fixed point (the origin), the exclusion cycles (one in each axis), and a possible positive cycle (coexistence cycle). Hence, we have four candidates for the fixed point of the composition operator Φ_p in the compact and convex region A.

The question that we are going to address is: Is it possible to avoid the trivial and the exclusion cycles? Equivalently, are we able to find a region where Φ_p has a coexistence cycle?

In order to answer this question, it is necessary to find a smaller compact, convex and invariant region such that points in a convenient neighborhood of the axes are excluded. To find a convex, compact and invariant subset A of a Euclidean space we are going to use normal space theory, namely the idea that **each pair of disjoint closed sets can be separated by open sets, in the sense that they have disjoint open neighborhoods** [71, page 132].

Let $F_i(x, y) = (f_i(x, y), g_i(x, y))$, $i = 0, 1, 2, \ldots$ be one of the sequences of maps. Then we have four different scenarios for which the map F_i has the maximum value on the axes as it is shown in Fig. 2.10.

1. $e^{K_i-1} < L_i/b_i$ and $e^{L_i-1} < K_i/a_i$;

2. $e^{K_i-1} > L_i/b_i$ and $e^{L_i-1} > K_i/a_i$;

3. $e^{K_i-1} > L_i/b_i$ and $e^{L_i-1} < K_i/a_i$;

4. $e^{K_i-1} < L_i/b_i$ and $e^{L_i-1} > K_i/a_i$.

Note that the small region R, near the origin, is part of the region defined by the item 2.

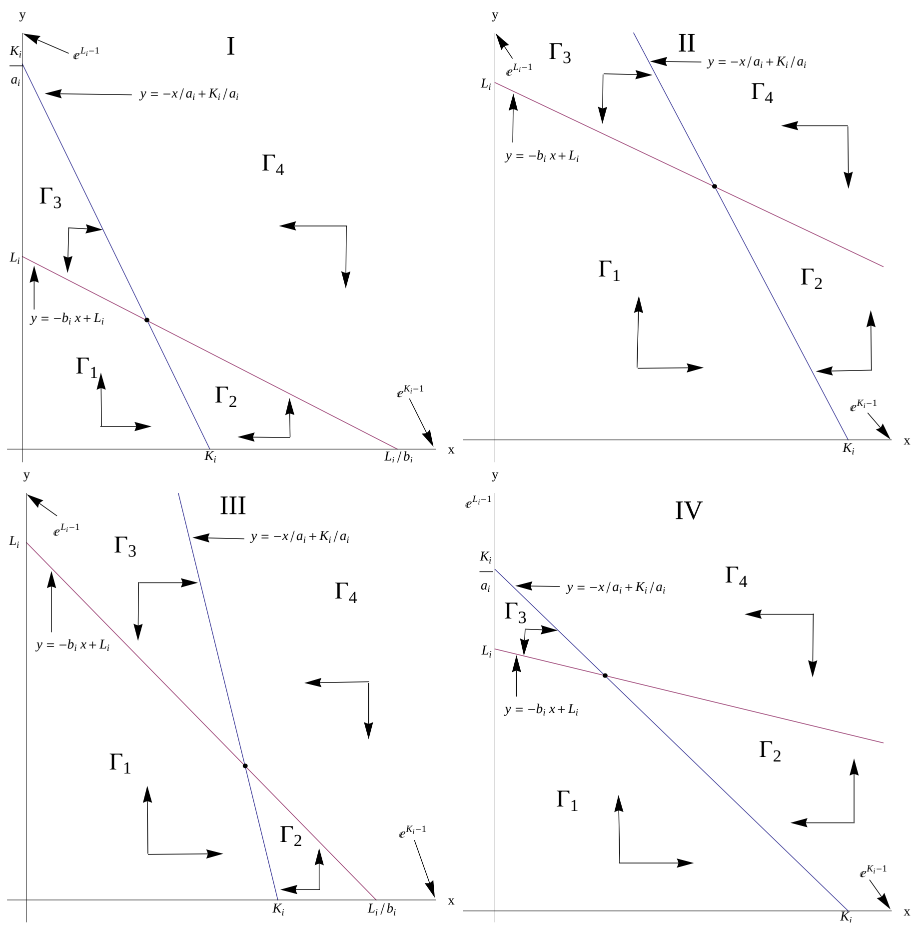

Figure 2.11: The position of the isoclines and the sets Γ_1, Γ_2, Γ_3 and Γ_4, in the first quadrant, for each one of the four situations in which the map F_i has the maximum value on the axes. **I** - $e^{K_i-1} < L_i/b_i$ and $e^{L_i-1} < K_i/a_i$; **II** - $e^{K_i-1} > L_i/b_i$ and $e^{L_i-1} > K_i/a_i$; **III** - $e^{K_i-1} > L_i/b_i$ and $e^{L_i-1} < K_i/a_i$; and **IV** - $e^{K_i-1} < L_i/b_i$ and $e^{L_i-1} > K_i/a_i$.

The isoclines of etch map F_i are given by the equations

$$y = -x/a_i + K_i/a_i \text{ and } y = -b_i x + L_i.$$

(For more details about this point see section 2.1). Note that the intersection of these two lines gives the positive fixed point (x_i^*, y_i^*), where

$$x_i^* = \frac{K_i - a_i L_i}{1 - a_i b_i} \text{ and } y_i^* = \frac{L_i - b_i K_i}{1 - a_i b_i}.$$

Now, for each one of the precedent scenarios, define the sets Γ_1, Γ_2, Γ_3 and Γ_4 (see Fig. 2.11). We now divide our investigation by cases:

Case 1: Let us first focus our attention when $e^{K_i-1} < L_i/b_i$ and $e^{L_i-1} < K_i/a_i$. Then we have three situations:

(i) Consider the sets Γ_4 and the axes. Note that if (x, y) belongs to Γ_4, then $f_i(x, y) < x$ and $g_i(x, y) < y$. Let

$$\delta_{i,1} = \min\{d(f_i(x, y), y - \text{axis}), d(g_i(x, y), x - \text{axis}) : (x, y) \in \Gamma_4\}.$$

(ii) Consider Γ_2 and the $y-$axis. If $(x, y) \in \Gamma_2$ then $f_i(x, y) < x$. Define the number

$$\delta_{i,2} = d((f_i(x, y), y - \text{axis})) \text{ such that } (x, y) \in \Gamma_2.$$

(iii) Consider Γ_3 and the $x-$axis. If $(x, y) \in \Gamma_3$ then $g_i(x, y) < y$. Define the number

$$\delta_{i,3} = d((g_i(x, y), x - \text{axis})) \text{ such that } (x, y) \in \Gamma_3.$$

Finally, let $\delta_i = \min\{\delta_{i,1}, \delta_{i,2}, \delta_{i,3}\}$ and define the set $A_{i,1}$ as

$$A_{i,1} = \Big\{(x, y) \in \mathbb{R}_2^+ : (0 \leq x \leq e^{K_i - 1} \wedge 0 \leq y < \delta_i) \vee \\ (0 \leq y \leq e^{L_i - 1} \wedge 0 \leq x < \delta_i)\Big\}.$$

Consider the set

$$A_i = [0, e^{K_i - 1}] \times [0, e^{L_i - 1}] \setminus A_{i,1}.$$

Then A_i is compact and convex. Moreover $F_i(A_i) \subset A_i$.

For the periodic equation (2.34), generated by the map $\Phi_p = F_{p-1} \circ \ldots \circ F_0$, we consider the values $\delta = \min\{\delta_i\}$, $K_{\max} = \max\{K_i\}$, $L_{\max} = \max\{L_i\}$ and the sets

$$B_1 = \Big\{(x, y) \in \mathbb{R}_2^+ : (0 \leq x \leq e^{K_{\max} - 1} \wedge 0 \leq y < \delta) \vee \\ (0 \leq y \leq e^{L_{\max} - 1} \wedge 0 \leq x < \delta)\Big\}$$

and

$$B = [0, e^{K_{\max} - 1}] \times [0, e^{L_{\max} - 1}] \setminus B_1. \tag{2.37}$$

Consequently, B is compact, convex and invariant under the action of the composition operator Φ_p. Hence, by the Brouwer's fixed point theorem, the map Φ_p has a positive fixed point $(\overline{x}, \overline{y})$ in B, which is a $p-$periodic coexistence cycle of the periodic Ricker model (2.34).

We now summarize these ideas in the following lemma.

Lemma 36 *Let $K_n > 0$, $L_n > 0$, $a_n > 0$ and $b_n > 0$ such that $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$, $b_n = b_{n \bmod p}$, and $a_n b_n < 1$, for all $n \in \mathbb{Z}^+$. Suppose that for each individual map F_i one has $e^{K_i - 1} < L_i/b_i$ and $e^{L_i - 1} < K_i/a_i$, $i = 0, 1, 2, \ldots$. Then, the nonautonomous $p-$periodic Ricker equation (2.34) has a coexistence $p-$periodic cycle in the convex and compact set B, where B is the set defined in (2.37).*

Case 2A: Now let us focus our attention in the case when $e^{K_i - 1} > L_i/b_i$, $e^{L_i - 1} > K_i/a_i$, $K_i \leq 1$, and $L_i \leq 1$. (In general, this restriction on the carrying capacities leads to region R in Fig. 2.10.) It follows that

$$0 < K_i < L_i/b_i < e^{K_i - 1} \leq 1 \text{ and } 0 < L_i < K_i/a_i < e^{L_i - 1} \leq 1.$$

The map $f_i(x, y)$ attains its maximum at $(1, 0)$. We observe that $f_i(x, y)$ is increasing if $0 < x < 1$ (when we fix y). Hence $f_i(x, y)$ will have a maximum value whenever x is a right end point of an interval bounded by 1. Note that $f_i(K_i, y) = K_i e^{-a_i y} < K_i$. Let $x_{m,i} \in (0, K_i)$. Since $x_{m,i} < K_i$ and f_i is increasing it follows that $f_i(x_{m,i}, y) < f_i(K_i, y) < K_i$. On the other hand if $x_{M,i} \in (K_i, L_i/b_i)$ it follows that $f_i(x_{M,i}, y) = x_{M,i} e^{K_i - x_{M,i}} e^{-a_i y} < x_{M,i}$ because $K_i < x_{M,i}$.

Similarly, one can show that $g_i(x, L_i) < L_i$, $g_i(x, y_{m,i}) < L_i$ for any $y_{m,i} \in (0, L_i)$, and $g_i(x, y_{M,i}) < y_{M,i}$ for any $y_{M,i} \in (L_i, K_i/a_i)$. Consequently, the set $[0, x_{M,i}] \times [0, y_{M,i}]$ is invariant under the map F_i.

Now, in the region $[0, x_{M,i}] \times [0, y_{M,i}]$ we construct the sets Γ_1, $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_4$, as it is shown in Fig. 2.12. The dynamics in this new region is similar to the case 1. That is, we follow the

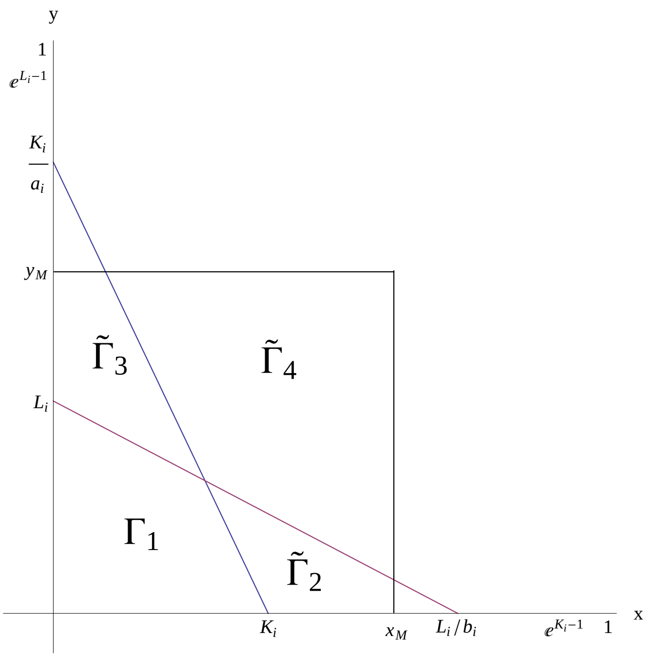

Figure 2.12: The sets and the isoclines when the maximum of the map F_i is less than one

same techniques of Lemma 36 and we show that there exist numbers δ, $x_M = \max\{x_{M,i}\}$, and $y_M = \max\{y_{M,i}\}$, and the sets

$$B_1 = \{(x, y) \in \mathbb{R}_2^+ : (0 \leq x \leq x_M \wedge 0 \leq y < \delta) \vee \\ (0 \leq y \leq y_M \wedge 0 \leq x < \delta)\}$$

and

$$B = [0, x_M] \times [0, y_M] \setminus B_1, \tag{2.39}$$

such that B is invariant under the action of the composition operator Φ_p.

Case 3A: Another case follows when $e^{K_i-1} > L_i/b_i$, $e^{L_i-1} < K_i/a_i$, and $K \leq 1$. In this case we consider mixed techniques as before. More precisely, in the y component we follow the technique presented in case 1 since $e^{L_i-1} < K_i/a_i$, and in the x component we follow the technique present in case 2A since $1 \geq e^{K_i-1} > L_i/b_i$. Henceforth, on the region $[0, x_M] \times [0, e^{L_i-1}]$ we apply a similar technique as in Lemma 36 and find the sets

$$B_1 = \{(x, y) \in \mathbb{R}_2^+ : (0 \leq x \leq x_M \wedge 0 \leq y < \delta) \vee \\ (0 \leq y \leq e^{L_{\max}-1} \wedge 0 \leq x < \delta)\Big\}$$

and

$$B = [0, x_M] \times [0, e^{L_{\max}-1}] \setminus B_1.$$

Case 4A: Similarly to the case 3A one can find the sets B_1 and B when $e^{K_i-1} < L_i/b_i$, $e^{L_i-1} > K_i/a_i$, and $L \leq 1$.

Case 3B: Now we consider the case when $e^{K_i-1} > L_i/b_i$, $e^{L_i-1} < K_i/a_i$, and $K > 1$. Define $\Gamma_4 = \Gamma_{4:1} \cup \Gamma_{4:2}$ where

$$\Gamma_{4:2} = \{(x, y) \in \mathbb{R}_+^2 : L_i/b_i \leq x \leq e^{K_i-1} \wedge 0 < y < e^{L_i-1}\}.$$

Let $(x, y) \in \Gamma_{4:2}$. Since $L_i/b_i > 1$, for any x in the interval $[L_i/b_i, e^{K_i-1}]$ the map $f_i(x, y)$ is decreasing (when we fix y). Thus, the following relation yields

$$f_i(x, y) < f_i\left(\frac{L_i}{b_i}, y\right) = \frac{L_i}{b_i} e^{K_i - \frac{L_i}{b_i}} e^{-a_i y} < \frac{L_i}{b_i} e^{K_i - \frac{L_i}{b_i}}.$$

But $\frac{L_i}{b_i} e^{K_i - \frac{L_i}{b_i}} < \frac{L_i}{b_i}$ because $K_i < \frac{L_i}{b_i}$. Consequently

$$f_i\,(\Gamma_{4:2}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1}.$$

On the region $[0, L_i/b_i)\times[0, e^{L_i-1}]$ we argue as before, namely we apply again a similar technique of Lemma 36 and find the sets

$$B_1 = \{(x, y) \in \mathbb{R}_2^+ : (0 \le x \le L/b \wedge 0 \le y < \delta) \vee \\ (0 \le y \le e^{L_{\max}-1} \wedge 0 \le x < \delta)\Big\}$$

and

$$B = [0, L/b] \times [0, e^{L_{\max}-1}] \setminus B_1,$$

where $L/b = \max\{L_i/b_i\}$.

Case 4B: When $e^{K_i-1} < L_i/b_i$, $e^{L_i-1} > K_i/a_i$ and $L > 1$ we define the set $\Gamma_4 = \Gamma_{4:1} \cup \Gamma_{4:2}$ where

$$\Gamma_{4:2} = \{(x, y) \in \mathbb{R}_+^2 : K_i/a_i \le y \le e^{L_i-1} \wedge 0 < x < e^{K_i-1}\},$$

and show that $g_i\,(\Gamma_{4:2}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1}$. Henceforth, on the region $[0, e^{K_i-1}] \times [0, K_i/a_i)$ we argue as before and find the sets

$$B_1 = \Big\{(x, y) \in \mathbb{R}_2^+ : (0 \le x \le e^{K_{\max}-1} \wedge 0 \le y < \delta) \vee \\ (0 \le y \le K/a \wedge 0 \le x < \delta)\}$$

and

$$B = [0, e^{K_{\max}-1}] \times [0, K/a] \setminus B_1,$$

where $K/a = \max K_i/a_i$.

Case 2B: Another case happens when $e^{K_i-1} > L_i/b_i$, $e^{L_i-1} > K_i/a_i$, $K > 1$ and $L > 1$. In this case we define the set Γ_4 as

$$\Gamma_4 = \Gamma_{4:1} \cup \Gamma_{4:2} \cup \Gamma_{4:3} \cup \Gamma_{4:4},$$

where

$$\Gamma_{4:2} = \{(x, y) \in \mathbb{R}_+^2 : 0 < x < L_i/b_i \wedge K_i/a_i < y < e^{L_i-1}\},$$
$$\Gamma_{4:3} = \{(x, y) \in \mathbb{R}_+^2 : L_i/b_i \le x < e^{K_i-1} \wedge 0 < y < K_i/a_i\}$$

and

$$\Gamma_{4:4} = \{(x, y) \in \mathbb{R}_+^2 : L_i/b_i \le x < e^{K_i-1} \wedge K_i/a_i \le y < e^{L_i-1}\}.$$

Following the same ideas as before, one can show that

$$f_i(\Gamma_{4:3}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1},$$
$$g_i(\Gamma_{4:2}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1},$$
$$f_i(\Gamma_{4:4}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1},$$

and

$$g_i(\Gamma_{4:4}) \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_{4:1}.$$

Hence, we consider the set $[0, L_i/b_i) \times [0, K_i/a_i)$ and arguing as before we find

$$B_1 = \{(x, y) \in \mathbb{R}_2^+ : (0 \le x \le L/b \wedge 0 \le y < \delta) \vee \\ (0 \le y \le K/a \wedge 0 \le x < \delta)\}$$

and

$$B = [0, L/b] \times [0, K/a] \setminus B_1,$$

where $K/a = \max K_i/a_i$ and $L/b = \max\{L_i/b_i\}$.

Cases 2C and 2D: Two cases are remaining. The first is $e^{K_i-1} > L_i/b_i$, $e^{L_i-1} > K_i/a_i$, $K \le 1$ and $L > 1$ and the second $e^{K_i-1} > L_i/b_i$, $e^{L_i-1} > K_i/a_i$, $K > 1$ and $L \le 1$. In both cases we apply the precedent technique in each component and find the invariant set B.

Finally, we summarize these ideas in the following result.

Theorem 37 *Let $(K_n, L_n) \in \widehat{S}_{1,n} = int(S_{1,n}) \cup \gamma_{1,n}$ such that $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$, $b_n = b_{n \bmod p}$ and $a_n b_n < 1$, for all $n \in \mathbb{Z}^+$. Then the $p-$periodic Ricker competition equation (2.34) has a $p-$periodic coexistence cycle (positive solution) in the compact and convex region B.*

2.2.2 The dynamics of the cycle

In this section we present some properties of the $p-$periodic cycle of the model (2.34) when the competition parameters a and b are fixed, i.e., when the individual maps F_i are one-to-one with respect to the parameter vector (K, L).

Let $C_p = \{(\bar{x}_0, \bar{y}_0), (\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_{p-1}, \bar{y}_{p-1}\})$ be a $p-$periodic cycle of equation (2.34). Then we have

$$\Phi_p(\bar{x}_0, \bar{y}_0) = (\bar{x}_0, \bar{y}_0),$$

or equivalently

$$(\bar{x}_0, \bar{y}_0) = \left(\bar{x}_0 e^{\sum_{i=0}^{p-1} K_i - \sum_{i=0}^{p-1} \bar{x}_i - a \sum_{i=0}^{p-1} \bar{y}_i}, \bar{y}_0 e^{\sum_{i=0}^{p-1} L_i - \sum_{i=0}^{p-1} \bar{y}_i - b \sum_{i=0}^{p-1} \bar{x}_i}\right).$$

This is equivalent to

$$\begin{cases} \sum_{i=0}^{p-1} \bar{x}_i + a \sum_{i=0}^{p-1} \bar{y}_i = \sum_{i=0}^{p-1} K_i \\ \\ b \sum_{i=0}^{p-1} \bar{x}_i + \sum_{i=0}^{p-1} \bar{y}_i = \sum_{i=0}^{p-1} L_i \end{cases}. \tag{2.44}$$

Solving this system yields

$$\begin{cases} \sum_{i=0}^{p-1} \bar{x}_i = \frac{1}{1-ab}\left(\sum_{i=0}^{p-1} K_i - a \sum_{i=0}^{p-1} L_i\right) \\ \\ \sum_{i=0}^{p-1} \bar{y}_i = \frac{1}{1-ab}\left(\sum_{i=0}^{p-1} L_i - b \sum_{i=0}^{p-1} K_i\right) \end{cases}. \tag{2.45}$$

Dividing both sides of each equation by $1/p$ one has the following

$$\begin{cases} av(\bar{x}) = \frac{1}{1-ab}(av(K) - aav(L)) \\ \\ av(\bar{y}) = \frac{1}{1-ab}(av(L) - bav(K)) \end{cases}, \tag{2.46}$$

where $av(\cdot)$ is the average of the sequence of points. Thus $av(\overline{x})$ and $av(\overline{y})$ belong to the first quadrant whenever

$$av(K) > a\,av(L) \quad \text{and} \quad av(L) > b\,av(K).$$

Consider now an individual map $F_i(x, y) = (xe^{K_i - x - ay}, ye^{L_i - y - bx})$, $0 \le i \le p-1$.

- In Section 2.1 it is shown that the autonomous difference equation generated by the map $F_i(x, y)$ has a coexistence solution (fixed point or cycle), in the first quadrant, whenever

$$bK_i < L_i < K_i/a, 0 \le i \le p-1, \tag{2.47}$$

  and $ab < 1$. Notice that if we take the sum in the relation given in (2.47) yields

$$b\,av(K) < av(L) < av(K)/a, \tag{2.48}$$

  which is the required condition to have a coexistence cycle of Φ_p in the first quadrant.

- From Section 2.1 the autonomous difference equation generated by the map $F_i(x, y)$ has an exclusion solution on the $x-$axis (fixed point or cycle) whenever $0 < L_i < bK_i, 0 \le i \le p-1$ and $ab < 1$. Adding these relations we have $0 < av(L) < b\,av(K)$. In this case Φ_p has an exclusion cycle on the $x-$axis.

- Similarly, if $0 < K_i/a < L_i, 0 \le i \le p-1$ then F_i has an exclusion solution on the $y-$axis (fixed point or cycle). Taking the sum we have $0 < av(K)/a < av(L)$ and consequently, Φ_p has an exclusion cycle on the $y-$axis.

Now assume that the relation (2.48) is satisfied and $ab < 1$. Under these conditions all the members of the periodic orbit belong to the first quadrant as we can see in the following.

The relation $\Phi_p(\overline{x}_0, \overline{y}_0) = (\overline{x}_0, \overline{y}_0)$ is equivalent to $F_{p-1}(\overline{x}_{p-1}, \overline{y}_{p-1}) = (\overline{x}_0, \overline{y}_0)$, i.e.,

$$\overline{x}_{p-1}e^{K_{p-1} - \overline{x}_{p-1} - a\overline{y}_{p-1}} = \overline{x}_0 \quad \text{and} \quad \overline{y}_{p-1}e^{L_{p-1} - \overline{y}_{p-1} - b\overline{x}_{p-1}} = \overline{y}_0.$$

Adding $\overline{x}_{p-1}$ in both sides of the first equation and $\overline{y}_{p-1}$ in both sides of the second equation we get

$$\overline{x}_{p-1}(1 + e^{u_{p-1}}) = \overline{x}_{p-1} + \overline{x}_0 \quad \text{and} \quad \overline{y}_{p-1}(1 + e^{v_{p-1}}) = \overline{y}_{p-1} + \overline{y}_0,$$

where $u_{p-1} = K_{p-1} - \overline{x}_{p-1} - a\overline{y}_{p-1}$ and $v_{p-1} = L_{p-1} - \overline{y}_{p-1} - b\overline{x}_{p-1}$.

Similarly, we have

$$\overline{x}_{p-2}(1 + e^{u_{p-2}}(1 + e^{u_{p-1}})) = \overline{x}_{p-1} + \overline{x}_{p-2} + \overline{x}_0$$

and

$$\overline{y}_{p-2}(1 + e^{v_{p-2}}(1 + e^{v_{p-1}})) = \overline{y}_{p-1} + \overline{y}_{p-2} + \overline{y}_0,$$

where $u_{p-2} = K_{p-2} - \overline{x}_{p-2} - a\overline{y}_{p-2}$ and $v_{p-2} = L_{p-2} - \overline{y}_{p-2} - b\overline{x}_{p-2}$. Generalizing this process we get

$$\overline{x}_1(1 + e^{u_1}(\ldots(1 + e^{u_p - 1})\ldots)) = \sum_{i=0}^{p-1} \overline{x}_i$$

and

$$\overline{y}_1(1 + e^{v_1}(\ldots(1 + e^{v_p - 1})\ldots)) = \sum_{i=0}^{p-1} \overline{y}_i.$$

It follows that $\overline{x}_1$ and $\overline{y}_1$ are positive numbers since $\sum_{i=0}^{r-1} \overline{x}_i > 0$ and $\sum_{i=0}^{r-1} \overline{y}_i > 0$. Consequently, from the dynamics of the system $\overline{x}_i > 0$ and $\overline{y}_i > 0$, $\forall i \in \{0, \ldots, r-1\}$.

2.2.3 Stability

In this section we study the stability of the solutions of Eq. (2.34) under certain restrictions on the parameters. Before to do that, we give some notes about the Jacobian of the composition operator Φ_p.

Generally speaking, the Jacobian matrix of the composition operator Φ_p is the product of the Jacobian matrices of the individual maps F_i, $i = 0, 1, 2, \ldots$.

Note that if $A_1, A_2, \ldots, A_n$ are $q \times q$ matrices, then $A_1 A_2 \ldots A_n$ and the cyclic permutations $A_{j+1} \ldots A_n A_1 \ldots A_j$ have the same set of eigenvalues [4], where $1 \leq j \leq n$. Thus each of the r matrices of the permuted Jacobian products has the same set of eigenvalues.

Consequently, if $C_p = \{(\overline{x}_0, \overline{y}_0), (\overline{x}_1, \overline{y}_1), \ldots, (\overline{x}_{p-1}, \overline{y}_{p-1}\})$ is a $p-$periodic cycle of Eq. (2.34), then the order of the products in $\prod_{i=p-1}^{0} JF_i(\overline{x}_i, \overline{y}_i)$ is irrelevant for the spectral radius, where JF_i is the Jacobian of the map F_i and it is given by

$$JF_i(x, y) = \begin{bmatrix} (1-x)e^{K_i - x - a_i y} & -a_i x e^{K_i - x - a_i y} \\ -b_i y e^{L_i - y - b_i x} & (1-y)e^{L_i - y - b_i x} \end{bmatrix}.$$

Trivial and exclusion cycles

Now let us study the stability of the trivial and the exclusion cycles of equation (2.34). It is easy to see that $(0, 0)$ is a fixed point of Eq. (2.34) since it is a fixed point of each individual map F_i, $i = 0, 1, 2, \ldots$. Henceforth, the Jacobian at $(0, 0)$ is given by

$$\mathcal{J}_0 = \prod_{i=p-1}^{0} JF_i(0, 0) = \prod_{i=p-1}^{0} \begin{bmatrix} e^{K_i} & 0 \\ 0 & e^{L_i} \end{bmatrix} = \begin{bmatrix} e^{\sum_{i=0}^{p-1} K_i} & 0 \\ 0 & e^{\sum_{i=0}^{p-1} L_i} \end{bmatrix}.$$

The eigenvalues of $\mathcal{J}_0$ are $e^{\sum_{i=0}^{p-1} K_i}$ and $e^{\sum_{i=0}^{p-1} L_i}$. Since $\sum_{i=0}^{p-1} K_i > 0$ and $\sum_{i=0}^{p-1} L_i > 0$, it follows that $(0, 0)$ is always unstable for all $K_i > 0$ and $L_i > 0$, $i = 0, 1, 2, \ldots$.

In the absence of species "y" the dynamics of species "x" is governed by the nonautonomous one-dimensional Ricker equation

$$x_{n+1} = x_n e^{K_n - x_n}, n \in \mathbb{Z}^+, \tag{2.49}$$

where $K_n = K_{n \bmod p}$. In [66] the author proved that Eq. (2.49) has a globally asymptotically stable $p-$periodic cycle

$$\{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{p-1}\}$$

whenever $0 < K_i < 2$, $i = 0, 1, 2 \ldots$. In other words,

$$\prod_{i=0}^{p-1} |1 - \overline{x}_i| < 1. \tag{2.50}$$

Moreover, neither attenuance nor resonance occurs[4], i.e,

$$\sum_{i=0}^{p-1} K_i = \sum_{i=0}^{p-1} \overline{x}_i. \tag{2.51}$$

[4]A $p-$periodic system is called attenuant (resonant) if the average of the stable periodic orbit is less (greater) than the average of the fixed points of each individual map.

The $p-$periodic cycle $\{\overline{x}_0, \ldots, \overline{x}_{p-1}\}$ yields an exclusion $p-$periodic cycle

$$C_p^x = \left\{(\overline{x}_0, 0), (\overline{x}_1, 0), \ldots, (\overline{x}_{p-1}, 0)\right\} \tag{2.52}$$

of the competition Ricker model (2.34).

Notice that, in the parameter space, the map $F_i(x, y), 0 \leq i \leq p-1$ has an exclusion fixed point on the $x-$axis whenever the carrying capacities are in the region

$$R = \{(K_i, L_i)\mathbb{R}_2^+ : 0 < L_i < b_{min} K_i \wedge K_i < 2\}, \tag{2.53}$$

such that $a_i b_i < 1$, $i = 0, 1, 2, \ldots$. Henceforth, we assume that, for each map F_i the vector parameter $(K_i, L_i) \in R$, which implies that $0 < av(L) < bav(K)$.

The Jacobian of Φ_p, evaluated along the periodic cycle C_p^x, is given by the following triangular matrix

$$\mathcal{J}_{C_p^x} = \prod_{i=p-1}^{0} JF_i(\overline{x}_i, 0) = \begin{pmatrix} e^{\sum_{i=0}^{p-1} K_i - \sum_{i=0}^{p-1} \overline{x}_i} \prod_{i=0}^{p-1}(1-\overline{x}_i) & J_{12} \\ 0 & e^{\sum_{i=0}^{p-1} L_i - \sum_{i=0}^{p-1} b_i \overline{x}_i} \end{pmatrix}.$$

The eigenvalues of $\mathcal{J}_{C_p^x}$ are $\lambda_1 = \prod_{i=0}^{p-1}(1-\overline{x}_i)$ and $\lambda_2 = e^{\sum_{i=0}^{p-1} L_i - \sum_{i=0}^{p-1} b_i K_i}$. Clearly, $0 < \lambda_2 < 1$, since $\sum_{i=0}^{p-1} L_i < \sum_{i=0}^{p-1} b_i K_i$. By (2.50) and (2.51) it follows that $|\lambda_1| < 1$. Consequently, C_p^x is asymptotically stable.

We now summarize this conclusion in the following result.

Theorem 38 *Let $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$, and $b_n = b_{n \bmod p}$, $\forall n \in \mathbb{Z}^+$ such that $(K_i, L_i) \in R, 0 \leq i \leq p-1$ and $a_i b_i < 1, 0 \leq i \leq p-1$. Then C_p^x defined in (2.52) is an asymptotically stable $p-$periodic exclusion cycle of Eq. (2.34).*

Using the same reasonings we get

Theorem 39 *Let $K_n = K_{n \bmod p}$, $L_n = L_{n \bmod p}$, $a_n = a_{n \bmod p}$, and $b_n = b_{n \bmod p}$, $\forall n \in \mathbb{Z}^+$ such that $(K_i, L_i) \in Q_i, 0 \leq i \leq p-1$, where*

$$Q_i = \{(K_i, L_i)\mathbb{R}_2^+ : K_i/a_i < L_i \wedge K_i > 0\},$$

and $a_i b_i < 1, 0 \leq i \leq p-1$. Then Eq. (2.34) has an asymptotically stable $p-$periodic exclusion cycle on the $y-$axis.

Coexistence cycle

For each $i = 0, 1, 2, \ldots$ suppose that the sequence of competition parameters satisfy the relation $a_i b_i < 1$. Rewrite the stability region $\hat{S}_i$ of each individual map F_i as

$$\hat{S}_i = Int(S_{1,i}) \cup \gamma_{1,i},$$

where $Int(S_{1,i})$ denotes the interior of the stability region $S_{1,i}$ of each individual map. Or equivalently

$$\hat{S}_i = \{(K_i, L_i)\mathbb{R}_+^2 : 4(a_i b_i - 1) + 2(1-a_i)L_i + 2(1-b_i)K_i \leq (a_i L_i - K_i)(b_i K_i - L_i) < (1-a_i)L_i + (1-b_i)K_i.\}$$

Now define the set

$$\Lambda_i = \{(K_i, L_i) \in \hat{S}_i : K_i \leq 1 \wedge L_i \leq 1\}.$$

In [73] Smith proved that the map F_i generates a discrete monotone dynamical system if the carrying capacities are in Λ_i. Moreover, F_i has a globally asymptotically stable fixed point.

Let $(K_i, L_i) \in \Lambda_i$, $i = 0, 1, 2, \ldots, p-1$. Since the composition of monotone maps is monotone it follows that the composition operator Φ_p is monotone. Moreover, in the compact and convex set B defined in the previous subsection, the map Φ_p has a globally asymptotically stable coexistence fixed point which is a globally asymptotically stable $p-$periodic coexistence cycle of the map (2.34).

We now summarize the above remarks.

Theorem 40 *Let $(K_i, L_i) \in \Lambda_i$, $i = 0, 1, 2, \ldots, p-1$ and assume the periodicity of the sequence of parameters. Then, the $p-$periodic nonautonomous difference equation (2.34) has a globally asymptotically stable $p-$periodic coexistence cycle in the convex and compact set B.*

2.2.4 A bifurcation scenario from computational simulations

As remarked earlier, in the absence of species "y" the dynamics of species "x" is governed by the nonautonomous one-dimensional Ricker equation

$$x_{n+1} = x_n e^{\lambda_n - x_n}, n \in \mathbb{Z}^+, \tag{2.54}$$

where $\lambda_n = \lambda_{n \bmod p}$. Write Eq. (2.54) as $x_{n+1} = R_n(x_n)$ where $R_n(x) = xe^{\lambda_n - x}$.

In [66] the author proved that Eq. (2.54) has a globally asymptotically stable $p-$periodic cycle

$$\{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_{p-1}\}$$

whenever $0 < \lambda_i < 2$, $i = 0, 1, 2 \ldots$. In other words,

$$\prod_{i=0}^{p-1} |1 - \overline{x}_i| < 1.$$

However, the author did not study the existence of bifurcation of C_p. In other words what happens when

$$\prod_{i=0}^{p-1} (1 - \overline{x}_i) = 1 \quad \text{or} \quad \prod_{i=0}^{p-1} (1 - \overline{x}_i) = -1?$$

Since each map $R_n = R_{n \bmod p}$, $n \in \mathbb{Z}^+$ is one to one with respect to the parameter $\lambda_n = \lambda_{n \bmod p} > 0$, it follows by Theorem 8 that the only possible cycles for Eq. (2.54) are cycles with minimal period r, $r = pt$, $t = 1, 2, \ldots$.

Let $C_p = \left\{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{p-1}\right\}$ be a nontrivial $p-$periodic cycle of the $p-$periodic equation (2.54), i.e. a fixed point of the map $\Phi_p(x)$. (The existence is assured by the Brouwer's fixed point theorem). Thus one has $\overline{x}_0 = \Phi_p(\overline{x}_0)$. Simplifying this equation one obtains

$$\lambda_0 + \lambda_1 + \ldots \lambda_{p-1} - \overline{x}_0 - \overline{x}_1 - \ldots - \overline{x}_{p-2} = \overline{x}_{p-1}, \tag{2.55}$$

or equivalently

$$\overline{x}_{p-1} = \lambda_{p-1} + \sum_{i=0}^{p-2} (\lambda_i - \Phi_i(\overline{x}_0)). \tag{2.56}$$

A period-doubling bifurcation occurs when $\frac{\partial}{\partial t}\left(\Phi_p(t)\right)|_{t=\overline{x}_0} = -1$. This leads to the equation $\prod_{i=0}^{p-1}(1-\overline{x}_i) = -1$, or equivalently

$$\left(1 - \lambda_{p-1} - \sum_{i=0}^{p-2}(\lambda_i - \Phi_i(\overline{x}_0))\right)\prod_{i=0}^{p-2}(1 - \Phi_i(\overline{x}_0)) = -1. \tag{2.57}$$

In general, is not possible to find analytically (except when $p = 2$) the solutions of Eq. (2.57). However one can see that all the solutions of Eq. (2.57) are of the form

$$\overline{x}_0 = \psi(\lambda_0, \dots, \lambda_{p-1}),$$

where ψ is a function such that $\psi : \mathbb{R}^p \to \mathbb{R}$. We call these solutions the period-doubling bifurcation solutions.

Let $\psi = \psi(\lambda_0, \dots, \lambda_{p-1})$ such that $\psi(\lambda_0, \dots, \lambda_{p-1}) \in \mathbb{R}^+$. Substituting ψ in Eq. (2.56) yields the equation

$$\sum_{i=0}^{p-1}\lambda_i = \sum_{i=0}^{p-1}\Phi_i(\psi). \tag{2.58}$$

One can find implicitly, in the parameter space, all the solutions of Eq. (2.58). This process will give us a bifurcation surface in the parameter space $(K_0, \dots, K_{p-1})$. Note that the other case of bifurcation ($\frac{\partial}{\partial t}\left(\Phi_p(t)\right)|_{t=\overline{x}_0} = 1$) can be investigated in a similar way.

Remember that in Section 1.5 we computed explicitly the period-doubling bifurcation curves and the saddle-node bifurcation curves when $p = 2$.

The derivative along the 2−periodic cycle C_2 is given by

$$\Phi_2'(\overline{x}_0) = (1 - \overline{x}_1)(1 - \overline{x}_0) = (1 - \lambda_0 - \lambda_1 + \overline{x}_0)(1 - \overline{x}_0).$$

Hence, C_2 is asymptotically stable if

$$|(1 - \lambda_0 - \lambda_1 + \overline{x}_0)(1 - \overline{x}_0)| < 1.$$

These two inequalities are equivalent to the following system of inequalities

$$\begin{cases} \overline{x}_0^2 - (\lambda_0 + \lambda_1)\overline{x}_0 + \lambda_0 + \lambda_1 > 0 \\ \overline{x}_0^2 - (\lambda_0 + \lambda_1)\overline{x}_0 + \lambda_0 + \lambda_1 - 2 < 0 \end{cases}. \tag{2.59}$$

The solution of the system given in (2.59) is given by the following cases:

1. $\lambda_0 + \lambda_1 > 4$. Under this assumption it follows that

$$0 < \overline{x}_-^{pd} < \overline{x}_-^{sn} < \overline{x}_+^{sn} < \overline{x}_+^{pd}.$$

Note that $\overline{x}_-^{pd} > 0$ if $\lambda_0 + \lambda_1 > 2$. Thus, the system (2.59) can be solved whenever

$$\overline{x}_0 \in]\overline{x}_-^{pd}, \overline{x}_-^{sn}[\cup]\overline{x}_+^{sn}, \overline{x}_+^{pd}[.$$

Since $\overline{x}_1 = \lambda_0 + \lambda_1 - \overline{x}_0$ it follows that

$$\overline{x}_1 \in]\lambda_0 + \lambda_1 - \overline{x}_+^{pd}, \lambda_0 + \lambda_1 - \overline{x}_+^{sn}[\cup]\lambda_0 + \lambda_1 - \overline{x}_-^{sn}, \lambda_0 + \lambda_1 - \overline{x}_-^{pd}[.$$

We can show with direct computations that $\overline{x}_-^{pd} = \lambda_0 + \lambda_1 - \overline{x}_+^{pd}$, $\overline{x}_-^{sn} = \lambda_0 + \lambda_1 - \overline{x}_+^{sn}$, $\overline{x}_+^{sn} = \lambda_0 + \lambda_1 - \overline{x}_-^{sn}$, and $\overline{x}_+^{pd} = \lambda_0 + \lambda_1 - \overline{x}_-^{pd}$. Hence C_2 is asymptotically stable if

$$\overline{x}_0, \overline{x}_1 \in]\overline{x}_-^{pd}, \overline{x}_-^{sn}[\cup]\overline{x}_+^{sn}, \overline{x}_+^{pd}[.$$

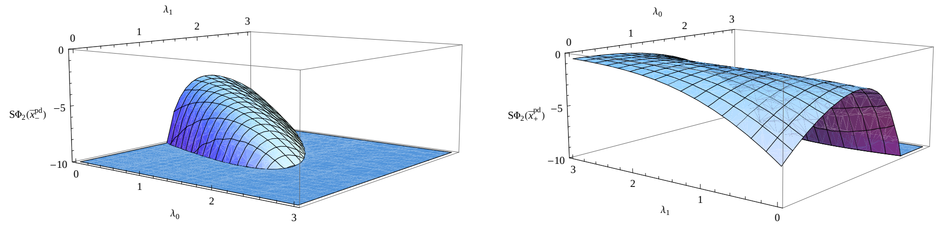

Figure 2.13: The progress of the Schwarzian derivative of the two-periodic one-dimensional Ricker map Φ_2 at $(\overline{x}_0)$ when the parameters λ_0 and λ_1 are on the period doubling bifurcation curves.

2. $\lambda_0 + \lambda_1 = 4$. It follows that $\overline{x}_-^{sn} = \overline{x}_+^{sn} = 2$. Hence C_2 is asymptotically stable if

$$\overline{x}_0, \overline{x}_1 \in]2 - \sqrt{2}, 2 + \sqrt{2}[\setminus\{2\}.$$

3. $\lambda_0 + \lambda_1 < 4$. It follows that $\overline{x}_-^{sn}$ and $\overline{x}_+^{sn}$ are complex numbers and the inequality

$$\overline{x}_0^2 - (\lambda_0 + \lambda_1)\overline{x}_0 + \lambda_0 + \lambda_1 > 0$$

is verified. We now subdivide this case depending on $\overline{x}_-^{pd}$ being greater, equal or less than 0.

   (a) If $2 < \lambda_0 + \lambda_1 < 4$, then C_2 is asymptotically stable whenever $\overline{x}_0, \overline{x}_1 \in]\overline{x}_-^{pd}, \overline{x}_+^{pd}[$.

   (b) If $\lambda_0 + \lambda_1 = 2$, then C_2 is asymptotically stable whenever $\overline{x}_0, \overline{x}_1 \in]0, 2[$.

   (c) If $0 < \lambda_0 + \lambda_1 < 2$ it follows that $\overline{x}_-^{pd} < 0$ and $0 < \overline{x}_+^{pd} < 2$. Hence C_2 is asymptotically stable if $\overline{x}_0, \overline{x}_1 \in]0, \overline{x}_+^{pd}[$.

We now study the stability of C_2 when the parameters are on the curves where occur the bifurcations, i.e, if $\overline{x}_0$ and $\overline{x}_1$ are the end points of the intervals shown before.

A preliminary analysis on the parameter space (example 29) shows that for the period doubling bifurcation curves the values of the parameters are less than 3. Hence is enough to study the values of $S\Phi_2(\overline{x}_+^{pd})$ and $S\Phi_2(\overline{x}_-^{pd})$ in the region $D =]0, 3[\times]0, 3[$. Since the analytical computations is very cumbersome we use numerical simulations. We find that the maximum of $S\Phi_2(\overline{x}_-^{pd})$ on D is ≈ -3.95264 and the maximum of $S\Phi_2(\overline{x}_+^{pd})$ on D is ≈ -0.537015. In Fig. 2.13 are presented the values of the Schwarzian derivative on D in both situations. Hence, by Theorem 19 the $2-$cycle C_2 is asymptotically stable when the parameters λ_0 and λ_1 are on the period-doubling bifurcation curves.

We now study the stability of C_2 when the parameters are on the saddle-node bifurcation curves. The saddle-node bifurcation exists if $\lambda_0 + \lambda_1 \geq 4$. From the implicit solutions one can see that $\lambda_0 + \lambda_1 = 4$ iff $\lambda_0 = \lambda_1 = 2$. Under this scenario $\overline{x}_-^{sn} = \overline{x}_+^{sn} = 2$, $\Phi_2''(2) = 0$ and $\Phi_2'''(2) = -2 < 0$. Hence by Theorem 17 C_2 is asymptotically stable.

Now consider that $\lambda_0 + \lambda_1 > 4$ such that $\lambda_0 > 2$ and $\lambda_1 > 2$. As it is shown in Example 26 $\Phi_2''(\overline{x}_-^{sn}) > 0$ and $\Phi_2''(\overline{x}_+^{sn}) < 0$ for all $\lambda_0, \lambda_1 \in (2, 3) \times (2, 3)$. Hence $\Phi_2''(\overline{x}_0) \neq 0$. Consequently, the periodic cycle C_2 is unstable when λ_0 and λ_1 are on the saddle-node bifurcation curves.

We now summarize these ideas in the following result.

Proposition 41 *Let $p = 2$ be the period of Eq. (2.54) and $C_2 = \{\overline{x}_0, \overline{x}_1\}$ be a non trivial $2-$periodic cycle of the $2-$periodic equation (2.54). Then*

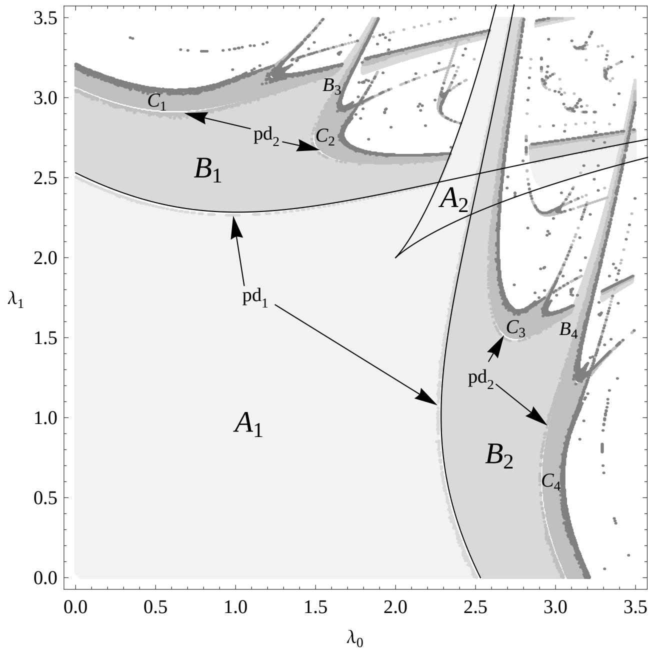

Figure 2.14: The bifurcation scenario in the parameter space of the map $\Phi_2 = R_1 \circ R_0$.

1. *If* $\lambda_0 + \lambda_1 > 4$ *then* C_2 *is asymptotically stable whenever*
$$\overline{x}_0, \overline{x}_1 \in [\overline{x}_-^{pd}, \overline{x}_-^{sn}[\cup]\overline{x}_+^{sn}, \overline{x}_+^{pd}].$$
2. *If* $\lambda_0 + \lambda_1 = 4$ *then* C_2 *is asymptotically stable whenever*
$$\overline{x}_0, \overline{x}_1 \in [2 - \sqrt{2}, 2 + \sqrt{2}].$$
3. *If* $2 < \lambda_0 + \lambda_1 < 4$ *then* C_2 *is asymptotically stable whenever* $\overline{x}_0, \overline{x}_1 \in [\overline{x}_-^{pd}, \overline{x}_+^{pd}]$.
4. *If* $0 < \lambda_0 + \lambda_1 \leq 2$ *then* C_2 *is asymptotically stable whenever* $\overline{x}_0, \overline{x}_1 \in (0, \overline{x}_+^{pd})$.

In Fig. 2.14 is presented the bifurcation scenario of C_2 in the parameter space (λ_0, λ_1). As mentioned earlier in region A_1, Eq. (2.54) has an asymptotically stable 2−periodic cycle. The period-doubling bifurcation curves are represented by pd_1. Hence, as λ_0 and λ_1 passes these two curves undergoes a period-doubling bifurcation. The 2−periodic cycle become unstable and a new asymptotically stable 4−periodic cycle is born. This new 4−periodic cycle is always asymptotically stable whenever the parameters are in the region $B_1 \cup B_2$. This 4−periodic cycle losses it stability whenever the parameters passes the curves pd_2. On these curves undergoes a period-doubling bifurcation. Consequently, the 4−periodic cycle becomes unstable and a new asymptotically stable 8−periodic cycle is created. The stability region of the 8−periodic cycle is given by $\cup_{i=1}^{4} C_i$. Again, this cycle losses it stability whenever the parameters passes the curves pd_3 in which a period-doubling bifurcation occurs. This period-doubling scenario continuous route to chaos.

If the parameters are in region A_2, then Eq. (2.54) has three 2−periodic cycles. Two of them are asymptotically stable and the third is unstable. When the parameters λ_0 and λ_1 leave this region, we enter in a zone where Eq. (2.54) has a coexistence of multiple attractors.

In region $B_3 \cup B_4$, Eq. (2.54) has two asymptotically stable 4−periodic cycles and one unstable 4−periodic cycle. When the parameters leave this region we enter again in a region where one has a coexistence of multiple attractors.

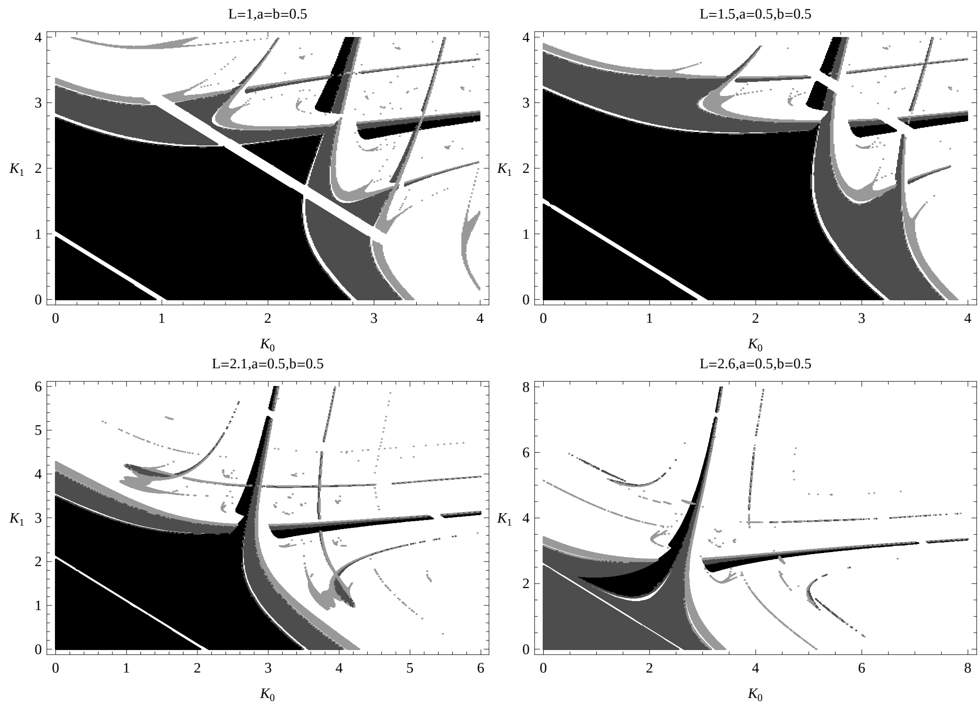

Figure 2.15: The bifurcation scenario of the 2−periodic Ricker competition equation in the parameter space (K_0, K_1) when the carrying capacity L and the competition parameters a and b are fixed.

We now discus a bifurcation scenario, via computational manipulations, when Eq. (2.34) is 2−periodic.

As first scenario we suppose that $K_n = K_{n \bmod 2}$, $L_n = L$, $a_n = a$ and $b_n = b$, $\forall n \in \mathbb{Z}^+$. This leads to a 2−periodic equation. Let $C_2 = \{(\overline{x}_0, \overline{y}_0), (\overline{x}_1, \overline{y}_1)\}$ be a 2−periodic solution of Eq. (2.34), i.e., $\Phi_2(\overline{x}_0, \overline{y}_0) = (\overline{x}_0, \overline{y}_0)$. Using computer simulations we plot in the parameter space (K_0, K_1) the region where C_2 is asymptotically stable (see Fig. 2.15, black color).

By the relation (2.48) C_2 is a coexistence cycle whenever $b\, av(K) < av(L) < av(K)/a$, i.e., $b(K_0 + K_1) < 2L < (K_0 + K_1)/2$. Define the sets

$$
\begin{aligned}
Q &= \{(K_0, K_1) \in \mathbb{R}^2_+ : K_1 < -K_0 + 2aL\},\\
R &= \{(K_0, K_1) \in \mathbb{R}^2_+ : K_1 > -K_0 + 2L/b\},\\
S &= \{(K_0, K_1) \in \mathbb{R}^2_+ : -K_0 + 2aL < K_1 < -K_0 + 2L/b\}.
\end{aligned}
$$

If $(K_0, K_1) \in Q$, then one has a stable solution on the y−axis, if $(K_0, K_1) \in R$, then we have stable solutions on the x−axis and finally, if $(K_0, K_1) \in S$, then C_2 is a stable coexistence cycle. Crossing the lines $K_1 = -K_0 + 2aL$ and $K_1 = -K_0 + 2aL/b$, the system undergoes a saddle-node bifurcation. The exclusion cycle becomes unstable and a new asymptotically stable coexistence cycle is born.

A period-doubling bifurcation occurs whenever the parameters K_0 and K_1 move from the black region to the dark grey region. Thus the 2−periodic cycle C_2 becomes unstable and a new asymptotically stable 4−periodic cycle is born. Again a new period-doubling bifurcation occurs when the parameters change from the dark grey region to the light grey region.

Now we turn our attention to the parameter space (K_0, L_0) when the parameters K_1, L_1, a

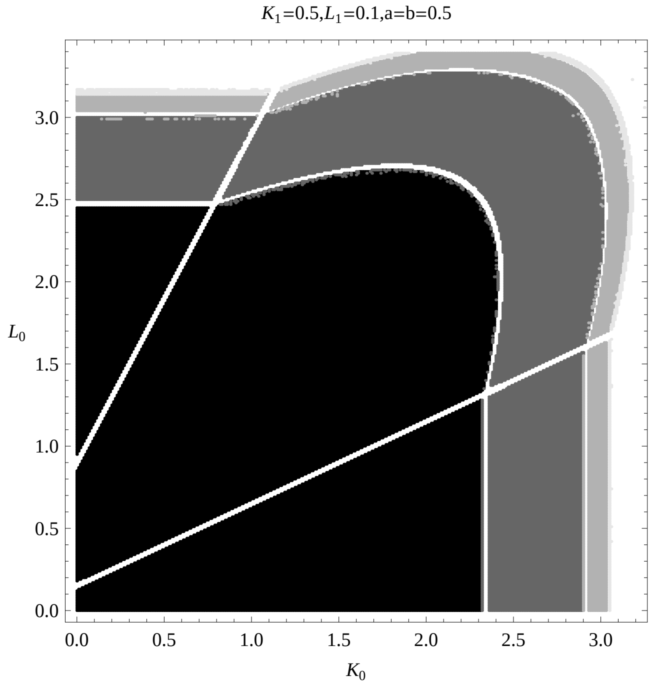

Figure 2.16: The bifurcation scenario of Eq. (2.34) in the parameters space (K_0, L_0) when $K_1 = 1/2$, $L_1 = 1/10$, and $a = b = 1/2$.

and b are fixed. First assume that $K_1 = 1/2$, $L_1 = 1/10$, and $a = b = 1/2$. Define the sets

$$\begin{aligned} Q &= \{(K_0, L_0) \in \mathbb{R}^2_+ : L_0 > 2K_0 + .9\}, \\ R &= \{(K_0, L_0) \in \mathbb{R}^2_+ : L_0 < .5K_0 + .15\}, \\ S &= \{(K_0, L_0) \in \mathbb{R}^2_+ : .5K_0 + .15 < L_0 < 2K_0 + .9\}. \end{aligned}$$

In Fig. 2.16 is presented the dynamics of the 2−periodic cycle. In region Q, Eq. (2.34) has an exclusion cycle on the y−axis. If $(K_0, L_0) \in S$ then Eq. (2.34) has a coexistence cycle in the interior of the first quadrant. Finally, if the parameters K_0 and L_0 are in region R, Eq. (2.34) has an exclusion cycle on the x−axis. The black region is the region where one has an asymptotically stable 2−periodic cycle. Eq. (2.34) undergoes a period-doubling bifurcation when the parameters K_0 and L_0 moves from the black region to the grey region. Hence, on grey region Eq. (2.34) has an asymptotically stable 4−periodic cycle. Again, a period-doubling bifurcation occurs when the parameters moves from the grey region to the light-grey region. This scenario of bifurcation continues.

In Fig. 2.17 we present another examples of the dynamics of Eq. (2.34). Notice that one can define the sets Q, P and S as follows

$$\begin{aligned} Q &= \{(K_0, L_0) \in \mathbb{R}^2_+ : L_0 > \frac{K_0}{a} + \Big(\frac{K_1}{a} - L_1\Big)\}, \\ R &= \{(K_0, L_0) \in \mathbb{R}^2_+ : L_0 < bK_0 + (bK_1 - L_1)\}, \\ S &= \{(K_0, L_0) \in \mathbb{R}^2_+ : bK_0 + (bK_1 - L_1) < L_0 < \frac{K_0}{a} + \Big(\frac{K_1}{a} - L_1\Big)\}. \end{aligned}$$

The dynamics of the cycle is similar as the precedent example.

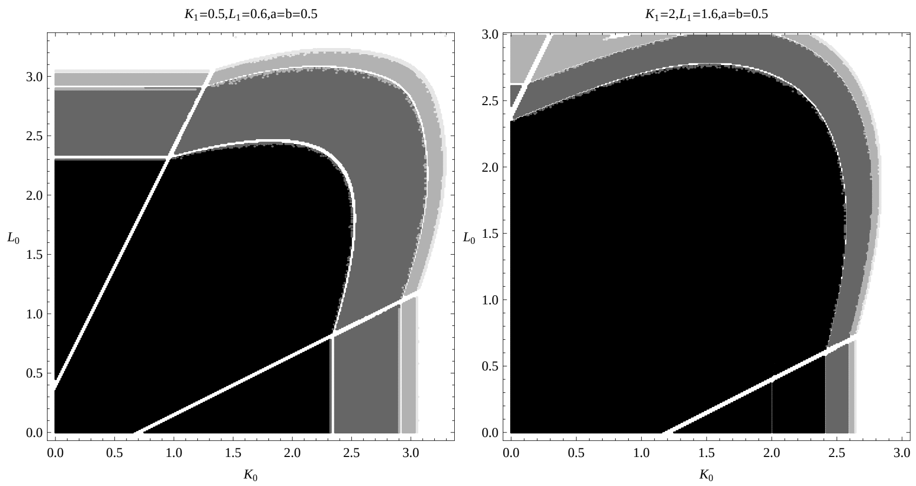

Figure 2.17: Some examples of the bifurcation scenario of Eq. (2.34) in the parameter space (K_0, L_0).

2.2.5 Attenuance and resonance

In this section we extend the notions of attenuance and resonance to two-dimensional systems. (For more details about this concept in one-dimensional nonautonomous models we refer [17, 28, 29, 31, 42, 41, 66, 75]).

Let $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be a parameter family of maps such that

$$F_n(x, y) = (f_n(x, y), g_n(x, y)), n \in \mathbb{Z}^+.$$

We now make the following assumptions:

1. The point $(K_n, 0)$ is a fixed point of the map on the $x-$axis. In population dynamics this fixed point is known as an exclusion equilibrium and K_n the carrying capacity of the map f_n,

2. The point $(0, L_n)$ is an exclusion equilibrium of the map on the $y-$axis. L_n is the carrying capacity of the map g_i,

3. The point (x^*_n, y^*_n) is the positive fixed point of the map $F_n(x, y)$. In population dynamics this fixed point is known as a coexistence equilibrium.

Let $C_r = \{(\overline{x}_0, \overline{y}_0), (\overline{x}_1, \overline{y}_1), \ldots, (\overline{x}_{r-1}, \overline{y}_{r-1})\}$ be an asymptotically stable $r-$periodic cycle of the $p-$periodic nonautonomous difference equation

$$(x_{n+1}, y_{n+1}) = F_n(x_n, y_n), \tag{2.60}$$

where $F_n = F_{n \bmod p}$, $\forall n \in \mathbb{Z}^+$. In population dynamics C_r is known as a coexistence cycle.

In the sequel, we will use the following notation to facilitate the new definition of attenuance and resonance

$$av(\overline{x}) = \frac{1}{r}\sum_{i=0}^{r-1}\overline{x}_i, \quad av(\overline{y}) = \frac{1}{r}\sum_{i=0}^{r-1}\overline{y}_i, \quad av(x^*) = \frac{1}{p}\sum_{i=0}^{p-1}x^*_i, \quad \text{and} \quad av(y^*) = \frac{1}{p}\sum_{i=0}^{p-1}y^*_i.$$

Definition 42 *Eq. (2.60) is said to be*

1. *Attenuant - Attenuant relative to* C_r *if* $av(\overline{x}) < av(x^*)$ *and* $av(\overline{y}) < av(y^*)$,

2. *Attenuant - Resonant relative to* C_r *if* $av(\overline{x}) < av(x^*)$ *and* $av(\overline{y}) > av(y^*)$,

3. *Resonant - Attenuant relative to* C_r *if* $av(\overline{x}) > av(x^*)$ *and* $av(\overline{y}) < av(y^*)$,

4. *Resonant - Resonant relative to* C_r *if* $av(\overline{x}) > av(x^*)$ *and* $av(\overline{y}) > av(y^*)$.

If this is true for all the attracting periodic orbits, then we remove the phrase "relative to C_r*".*

Now suppose that Eq. (2.34) is $p-$periodic such that the competition parameters a and b are fixed. The coexistence fixed point for each individual map F_i, $0 \leq i \leq p-1$ is given by $x_i^* = \frac{aL_i - K_i}{ab-1}$ and $y_i^* = \frac{bK_i - L_i}{ab-1}$. Then

$$av(x^*) = \frac{1}{p}\sum_{i=0}^{p-1} x_i^* = \frac{1}{p(ab-1)}\sum_{i=0}^{p-1}(aL_i - K_i) = \frac{1}{ab-1}\left(a\, av(L) - av(K)\right).$$

Using the relation (2.46) it follows $av(x^*) = av(\overline{x})$. Similarly, one has $av(y^*) = av(\overline{y})$. Thus neither attenuance nor resonance prevails. Notice that a similar result for the one-dimensional Ricker map was obtained in [66].

To this end we have been studying the properties of a periodic cycle when the carrying capacities are periodic and the competition parameters are fixed. Now we are going to study the attenuance and resonance for the other situations in which Eq. (2.34) has a periodic orbit. When $p > 2$ the dynamics of the equation involves horrendous computations. So we focus our attention on the attenuance and resonance when $p = 2$.

Let K_n, L_n and a_n be 2−periodic sequences such that $(K_n, L_n) \in S_{1,n}$, $\forall n \in \mathbb{Z}^+$ and assume that the parameter b is fixed. This produces an asymptotically stable coexistence 2−periodic cycle of Eq. (2.34). Let $C_2 = \{(\overline{x}_0, \overline{y}_0), (\overline{x}_1, \overline{y}_1)\}$ be this periodic cycle. After simple computations one can show that

$$\overline{x}_0 + \overline{x}_1 = K_0 + K_1 - a_0\overline{y}_0 - a_1\overline{y}_1 \text{ and } \overline{y}_0 + \overline{y}_1 = L_0 + L_1 - b(\overline{x}_0 + \overline{x}_1). \tag{2.61}$$

Substituting the value of $\overline{y}_0$ in the first equation of (2.61) yields

$$K_0 + K_1 - a_0(L_0 + L_1) + (a_0 - a_1)\overline{y}_1 = (1 - ba_0)(\overline{x}_0 + \overline{x}_1),$$

or equivalently

$$x_0^* + x_1^* + \frac{K_1 - a_0L_1}{1 - ba_0} - \frac{K_1 - a_1L_1}{1 - ba_1} + \frac{a_0 - a_1}{1 - ba_0}\overline{y}_1 = \overline{x}_0 + \overline{x}_1, \tag{2.62}$$

where $x_i^* = \frac{K_i - a_iL_i}{1 - ba_i}$, $i = 0, 1$. Similarly, one obtain

$$x_0^* + x_1^* + \frac{K_0 - a_1L_0}{1 - ba_1} - \frac{K_0 - a_0L_0}{1 - ba_0} + \frac{a_1 - a_0}{1 - ba_1}\overline{y}_0 = \overline{x}_0 + \overline{x}_1. \tag{2.63}$$

Adding Eq. (2.62) and Eq. (2.63) we get

$$av(x^*) + \epsilon_x = av(\overline{x}), \tag{2.64}$$

where $av(x^*) = \frac{x_0^*+x_1^*}{2}$, $av(\overline{x}) = \frac{\overline{x}_0+\overline{x}_1}{2}$, and

$$\epsilon_x = \frac{K_1 - K_0 + a_0(L_0 - L_1) + (a_0 - a_1)\overline{y}_1}{4(1 - ba_0)} + \frac{K_0 - K_1 + a_1(L_1 - L_0) + (a_1 - a_0)\overline{y}_0}{4(1 - ba_1)}.$$

Using similar computations we can show that

$$av(y^*) + \epsilon_y = av(\overline{y}), \tag{2.65}$$

where $av(y^*) = \frac{y_0^*+y_1^*}{2}$, $y_i^* = \frac{L_i - bK_i}{1-ba_i}$, $i = 0, 1$, $av(\overline{y}) = \frac{\overline{y}_0+\overline{y}_1}{2}$, and

$$\epsilon_y = \frac{L_0 - bK_0 - (1 - ba_0)\overline{y}_0}{2(1 - ba_1)} + \frac{L_1 - bK_1 - (1 - ba_1)\overline{y}_1}{2(1 - ba_0)}.$$

Therefore one may conclude the following:

- if $\epsilon_x = \epsilon_y = 0$, then neither attenuance nor resonance is present;
- if $\epsilon_x > 0$ and $\epsilon_y > 0$, then Eq. (2.34) is resonance - resonance relative to C_2;
- if $\epsilon_x > 0$ and $\epsilon_y < 0$, then Eq. (2.34) is resonance - attenuance relative to C_2;
- if $\epsilon_x < 0$ and $\epsilon_y > 0$, then Eq. (2.34) is attenuance - resonance relative to C_2;
- if $\epsilon_x < 0$ and $\epsilon_y < 0$, then Eq. (2.34) is attenuance - attenuance relative to C_2.

Before the end of this subsection we note that similar conclusions can be taken in other situations for which Eq. (2.34) has an asymptotically stable 2−periodic cycle. For example, fixing a and varying b or fixing the carrying capacities and varying the competition parameters, etc.

Thus in the case of periodicity of the competition parameters attenuance and resonance are present in the periodic solution of the nonautonomous equation given in (2.34).

2.3 Logistic competition model

In [35] we introduced a two-dimensional model called **logistic competition model**. The dynamics of this new model is similar to the autonomous Ricker competition model given in section 2.1. Hence, in this section we present the necessary steps to develop the model and the principal results without details.

We developed a competition model based on sound biological assumptions of intraspecific and interspecific competitions.

In developing this model, it is assumed that, without interspecific competition, each species is modeled by the logistic map. The logistic map is used to model species with non overlapping generations under the assumption that the fitness function decreases when the population density (size) increases. Let z_n be the density of species z at time period n. Then the fitness function is defined as

$$u(z) = \frac{z_{n+1}}{z_n}.$$

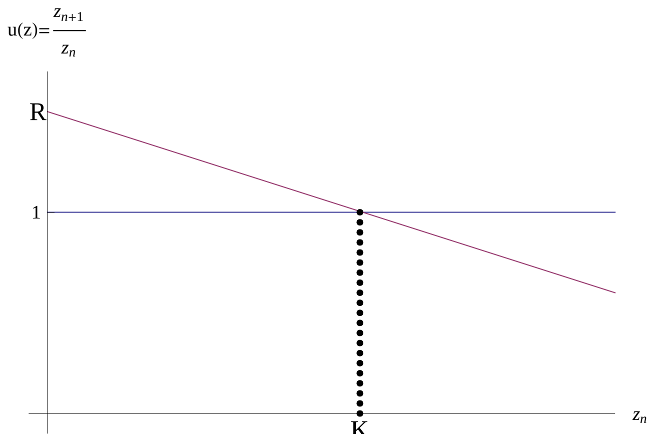

Figure 2.18: The fitness function.

When the population is sufficiently small (close to zero), the intraspecific competition (competition among individuals of species z) is negligible and consequently,

$$u(z) = \frac{z_{n+1}}{z_n} = R,$$

where $R > 1$ is a constant, commonly called, the intrinsic growth rate of the population. When the population grows, the fitness function decreases due to significant intraspecific competition and reaches the value 1 when the population density reaches the carrying capacity K.

Fig. 2.18 depicts a typical fitness function. Assuming that the decrease in the fitness function is linear, then the model is obtained by finding the equation of the line connecting the points $(0, R)$ and $(K, 1)$. The equation of this line is given by

$$\frac{z_{n+1}}{z_n} = -\frac{R-1}{K} z_n + R,$$

where $R > 1$. Using the change of variables, $x_n = \frac{R-1}{RK} z_n$ we obtain the celebrated logistic difference equation

$$x_{n+1} = R x_n (1 - x_n).$$

Introducing a new species y to compete with species x, interspecific competition (competition between two different species) would negatively affect the growth of species x and vice-versa. We propose the following new competition model

$$\begin{cases} x_{n+1} = \frac{a x_n (1-x_n)}{1+c y_n} \\ y_{n+1} = \frac{b y_n (1-y_n)}{1+d x_n} \end{cases}, \tag{2.66}$$

where $a, b > 0$ and $c, d \in (0, 1)$. The map associated with Eq. (2.66) is given by

$$F(x, y) = \left(\frac{ax(1-x)}{1+cy}, \frac{by(1-y)}{1+dx} \right).$$

To insure that the range of this map lies in the first quadrant, we make the following two assumptions:

1. x and y are in $[0, 1]$,

2. a and b are in $(0, 4]$[5].

These two assumptions guarantee that nonnegative points are mapped to nonnegative points and specifically the map F maps $[0, 1] \times [0, 1]$ into $[0, 1] \times [0, 1]$. To show this we note that the maximum of the $x-$component of the image of the point (x, y) is $a/4$ and occurs at $x = 1/2$ and $y = 0$ and the maximum of the $y-$component of the image of the point (x, y) is $b/4$ and occurs at $x = 0$ and $y = 1/2$.

In model (2.66) the parameters a and b are the intrinsic growth rates of species x and y, respectively, and the parameters c and d are the competition parameters of species y and x, respectively.

The map F has one extinction fixed point $(0, 0)$, two exclusion fixed points $(\frac{a-1}{a}, 0)$, $(0, \frac{b-1}{b})$, and one coexistence fixed point

$$(x^*, y^*) = \left(\frac{-cb + ab - b + c}{ab - cd}, \frac{-da + ab - a + d}{ab - cd}\right).$$

In the next result we give sufficient conditions for the stability of the extinction fixed point.

Lemma 43 *Let (x_n, y_n) denote the solution of the Logistic competition model (2.66) with an initial condition $(x_0, y_0) \in (0, 1) \times (0, 1)$. If $a \in (0, 1]$ then $\lim_{n\to\infty} x_n = 0$. If $b \in (0, 1]$ then $\lim_{n\to\infty} y_n = 0$. Moreover, if $a, b \in (0, 1]$, then $\lim_{n\to\infty}(x_n, y_n) = (0, 0)$.*

Proof. The inequality $x_{n+1} \le a x_n$, $\forall n \in \mathbb{Z}^+$ holds since

$$0 \le x_{n+1} = \frac{a x_n(1 - x_n)}{1 + c y_n} \le a x_n - a x_n^2 \le a x_n.$$

Let $a \in (0, 1)$. Then by induction we show that $x_n \le x_0 a^n$ for all $n \in \mathbb{Z}^+$. Thus $\lim_{n\to\infty} x_n = 0$.

When $a = 1$ one has $x_{n+1} < x_n$, $\forall n \in \mathbb{Z}^+$. Thus x_n is a decreasing sequence of numbers that is bounded above by 1 and bounded bellow by 0 which implies the convergence of x_n as n goes to infinity. Let L be this limit. Then $0 \le L \le x_n < 1$, $\forall n \in \mathbb{Z}^+$. Note that from $x_{n+1} \le x_n(1 - x_n)$, $\forall n \in \mathbb{Z}^+$ it follows that

$$x_n \le x_0 \prod_{i=0}^{n-1}(1 - x_i), \forall n \ge 1.$$

By the fact that $1 - x_i \le 1 - x_{i+1}$, $i \in \mathbb{Z}^+$, by induction we show that

$$\prod_{i=0}^{n-1}(1 - x_i) \le (1 - x_{n-1})^n, \forall n \ge 1.$$

Using this last relation we obtain

$$x_n \le \frac{x_0}{1 - x_n} \prod_{i=0}^{n}(1 - x_i) \le \frac{x_0}{1 - x_n}(1 - x_n)^{n+1} = (1 - x_n)^n.$$

But the relation $L \le x_n < 1$ implies that $0 < 1 - x_n \le 1 - L$, $\forall n \in \mathbb{Z}^+$ and consequently one has

$$x_n \le (1 - x_n)^n \le (1 - L)^n \underset{n\to\infty}{\longrightarrow} 0.$$

[5]In fact this restriction on the values of a and b may be slightly relaxed as it is shown in the stability regions of the coexistence fixed point

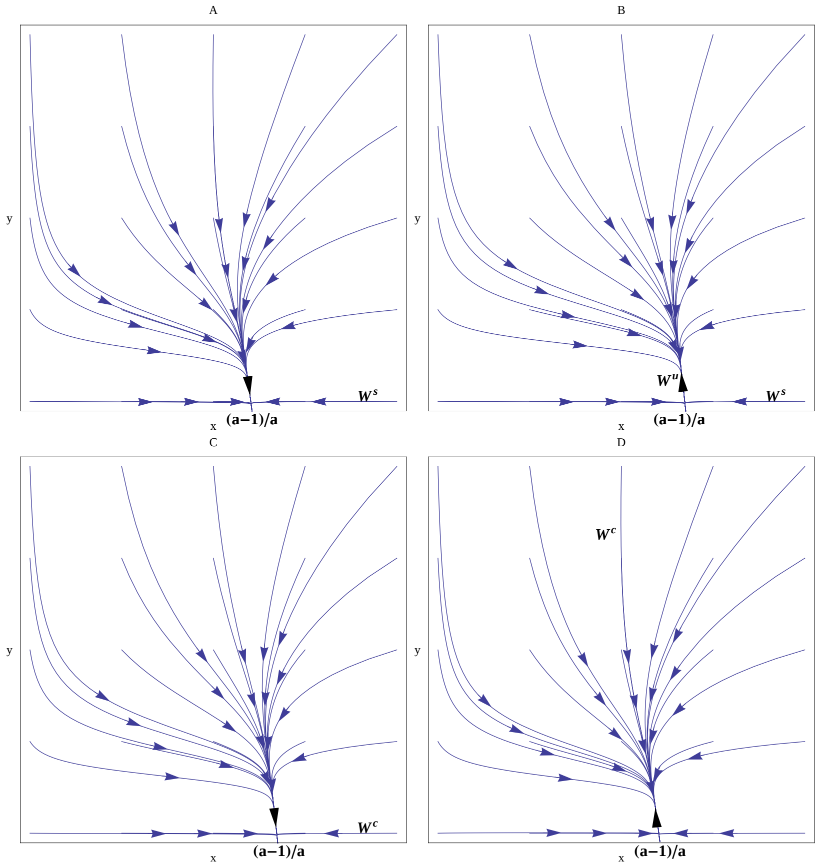

Figure 2.19: The phase-space diagram for the exclusion fixed point $(\frac{a-1}{a}, 0)$ of the logistic competition model. A - The exclusion fixed point is asymptotically stable: $a = 2.5$, $b = 1.01$, $c = 0.3$, $d = 0.1$. B - The exclusion fixed point is a saddle: $a = 2$, $b = 1.1$, $c = 0.3$, $d = 0.1$. C - The exclusion fixed point is asymptotically stable: $a = 3$, $b = 1.01$, $c = 0.3$, $d = 0.1$, where the center manifold is stable on the $x-$axis. D - The exclusion fixed point is a saddle: $a = 2.5$, $b = 1.06$, $c = 0.3$, $d = 0.1$ where the center manifold is unstable on the $x-$axis and in the interior of the first quadrant.

A similar argument proves the assertion when $b \in (0, 1]$. ■

Hence, If $a \leq 1$ and $b \leq 1$, then by Lemma 43 the extinction fixed point $(0, 0)$ is globally asymptotically stable. Clearly, when $a > 1$ or $b > 1$, the fixed point $(0, 0)$ becomes unstable.

For the fixed points $(\frac{a-1}{a}, 0)$ and $(0, \frac{b-1}{b})$, we have the following result:

Theorem 44 *[35] The following statements holds true:*

1. *The fixed point* $(\frac{a-1}{a}, 0)$ *of Eq. (2.66) is locally asymptotically stable if* $1 < a \leq 3$ *and* $1 < b < 1 + \frac{d(a-1)}{a}$ *and is unstable if* $1 < a < 3$ *and* $b = 1 + \frac{d(a-1)}{a}$*,*

2. *The fixed point* $(0, \frac{b-1}{b})$ *of Eq. (2.66) is locally asymptotically stable if* $1 < b \leq 3$ *and* $1 < a < 1 + \frac{c(b-1)}{b}$ *and is unstable if* $1 < b < 3$ *and* $a = 1 + \frac{c(b-1)}{b}$.

In Figures 2.19 and 2.20 we present the phase-space diagram for the exclusion fixed point $(\frac{a-1}{a}, 0)$. In Fig. 2.19A both eigenvalues are inside the unit circle, i.e., $1 < a < 3$ and $1 < b < 1 + d\left(\frac{a-1}{a}\right)$ and the stable manifold is on the $x-$axis. In Fig. 2.19B we can see the unstable and stable manifold when $1 < a < 3$ and $b > 1 + d\left(\frac{a-1}{a}\right)$, i.e., the first eigenvalue is inside the unit circle and the second is outside the unit circle. In Fig. 2.19C the stable center manifold is on the

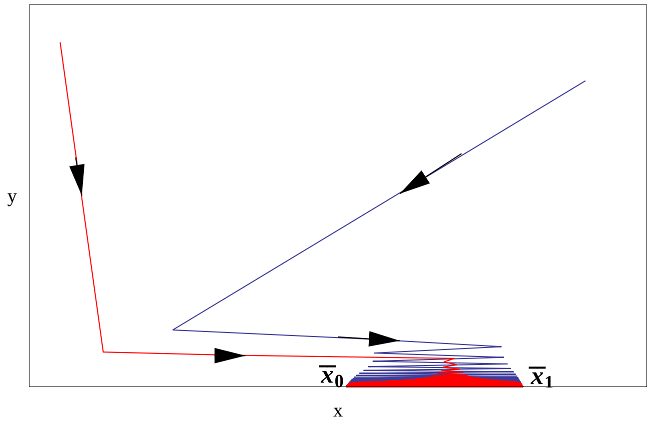

Figure 2.20: The existence of an exclusion asymptotically stable 2−periodic cycle on the x−axis of the logistic competition model when one eigenvalue is outside the unit circle and the second eigenvalue is inside the unit circle.

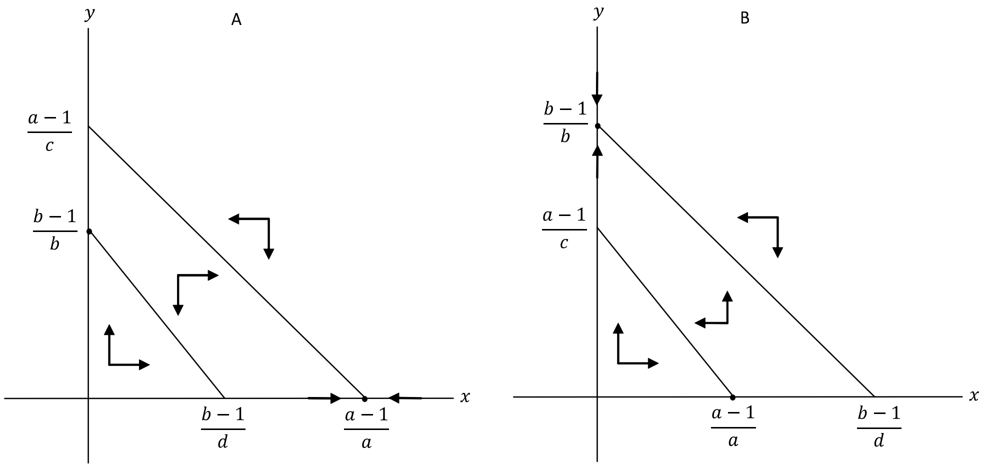

Figure 2.21: The stability of the exclusion fixed points and the validity of the exclusion principle in the logistic competition model. A - If $1 < a \leq 3$ and $b < 1 + d\left(\frac{a-1}{a}\right)$ (or $\frac{b-1}{d} < \frac{a-1}{a}$), then $(\frac{a-1}{a}, 0)$ is asymptotically stable and species y goes extinct. B - If $1 < b \leq 3$ and $b > \frac{c}{1+c-a}$ (or $\frac{b-1}{b} > \frac{a-1}{c}$), then $(0, \frac{b-1}{b})$ is asymptotically stable and species x goes extinct.

x−axis when $a = 3$ and $1 < b < 1+d\left(\frac{a-1}{a}\right)$. In this case the first eigenvalue is -1 and the second eigenvalue is inside the unit circle. Fig. 2.19D we see the unstable center manifold when the first eigenvalue is inside the unit circle and the second is on the unit circle. This happens when $1 < a < 3$ and $b = 1 + d\left(\frac{a-1}{a}\right)$.

When $a > 3$ and $1 < b < 1 + d\left(\frac{a-1}{a}\right)$ the exclusion fixed point $(\frac{a-1}{a}, 0)$ becomes unstable and 2−periodic orbit is created (Fig. 2.20).

We note that the exclusion principle in Biology is valid for both species. In Fig. 2.21 is presented the two possible scenarios. In Fig. 2.21A species y goes extinct while in Fig. 2.21B species x goes extinct.

For the coexistence fixed point we make the assumption that

$$b > 1 + \frac{d(a-1)}{a} \text{ and } a > 1 + \frac{c(b-1)}{b}. \tag{2.67}$$

Relation (2.67) insures that the positive fixed point (x^*, y^*) of the logistic competition model lies in the positive first quadrant.

Theorem 45 *[35] The coexistence fixed point (x^*, y^*) of the Logistic competition equation (2.66) is locally asymptotically stable if the following conditions hold:*

$$\frac{-c(b-c+bc)d^2 + a^3b^2(2-b+2d) + a(b-c+bc)d(3b+c+cd)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} + \frac{a^2b\left(2b^2(1+c)+3c(1+d)-b(3+5d+c(5+4d))\right)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} < 0, \tag{2.68}$$

and

$$\frac{-c(b-c+bc)d^2 + a^3b^2(3-b+3d) - ad\left(-b^2\left(9+14c+5c^2\right)+c^2(1+d)\right)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} - \frac{adbc\,(8+4c+4d+3cd)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} + \frac{a^2b\left(3b^2(1+c)+c\left(9+14d+5d^2\right)-3b(3+4d+4c(1+d))\right)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} > 0. \tag{2.69}$$

Note that inequality (21) in [35], i.e.

$$\frac{(b(-1+a-c)+c)(a(-1+b-d)+d)(ab-cd)}{ab(-(1+c)d+a(1+d))(-b(1+c)+c(1+d))} < 0, \tag{2.70}$$

holds true under condition (2.67). This observation has not been stated in [35].

Equivalently, the positive fixed point (x^*, y^*) of Eq. (2.66) is asymptotically stable if $(a, b) \in Int(S_1)$, where S_1 is the region depicted in Fig. 2.22. Note that the curves τ_1 and τ_2 in Fig. 2.22 are define as

$$\tau_1 = \{(a, b) \in \mathbb{R}^2_+ : b = 1 + \tfrac{d(a-1)}{a}\} \text{ and } \tau_2 = \{(a, b) \in \mathbb{R}^2_+ : a = 1 + \tfrac{c(b-1)}{b}\}.$$

In Fig. 2.23 we present the phase-space diagram when this coexistence fixed point is asymptotically stable.

2.4 Leslie-Gower model

2.4.1 The dynamics of the autonomous model

The Leslie-Gower competition model used by Park and Leslie in their studies [44] is given by the modified autonomous difference equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{aKx_n}{K+(a-1)x_n+cy_n} \\ \frac{bLy_n}{L+(b-1)y_n+dx_n} \end{pmatrix}, n \in \mathbb{Z}^+, \tag{2.71}$$

where x_n and y_n are the population size of species x and y, respectively, at time n, $K > 0$ and $L > 0$ are the carrying capacities of species x and y, respectively, $a > 0$ and $b > 0$ are the intrinsic growth rate of species x and y, respectively, and $c, d \in (0, 1)$ are the competition coefficients.

Before to study the local stability of the fixed pints of the system (2.71) we give some notes about the dynamics of (2.71) when $a \in (0, 1)$ and $b \in (0, 1)$. Since in this case $a - 1$ and $b - 1$

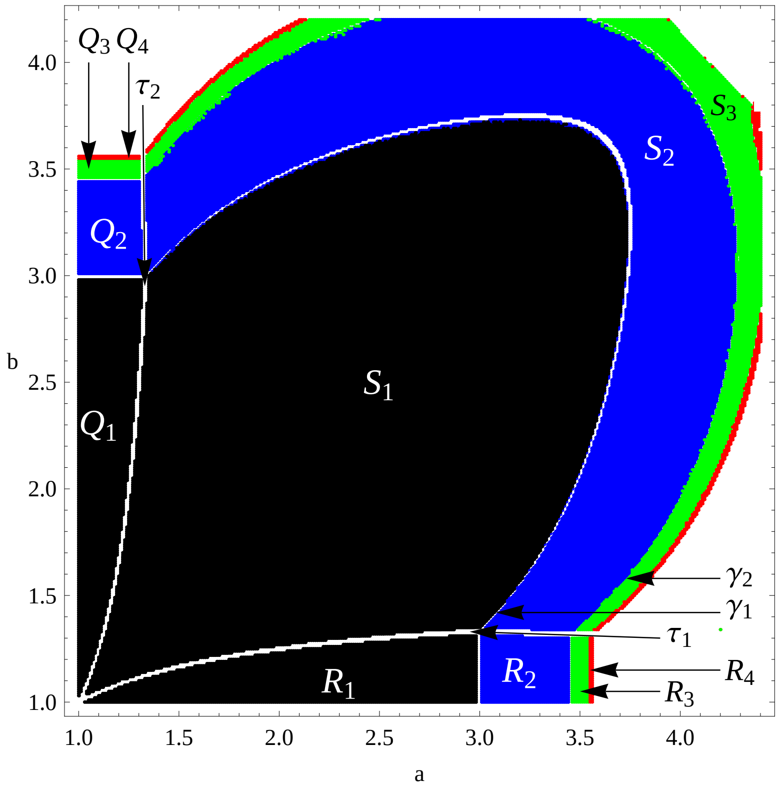

Figure 2.22: The stability regions and the bifurcation scenario of the logistic competition model in the parameter space $a - b$.

are negative numbers we need to consider a special initial conditions in order to unsure that $x_n > 0$ and $y_n > 0$, for all $n = 0, 1, 2, \ldots$. This observation is motived by the fact we have some difficulties in population dynamics assuming that $a > 0$ and $b > 0$. For instance if we consider that $a = 0.1$, $K = 0.5$ and $c = 0.2$, there exists a large region in the plane such that the function $f(x, y) = \frac{aKx}{K+(a-1)x+cy}$ is negative.

Let (x_0, y_0) be a point in the interior of the first quadrant such that when $a \in (0, 1)$ we have

$$y_0 > -\frac{K}{c} + \frac{1-a}{c} x_0 \tag{2.72}$$

and when $b \in (0, 1)$ we have

$$y_0 < \frac{L}{1-b} + \frac{d}{1-b} x_0. \tag{2.73}$$

Notice that when $a \geq 1$ and $b \geq 1$ we have $x_n > 0$ and $y_n > 0$, $n = 0, 1, 2 \ldots$ for any positive initial conditions.

We consider three cases.

1. Suppose that the condition (2.72) holds and assume that $b \geq 1$ (which implies that $y_n > 0$ for all $n = 0, 1, 2, \ldots$). By the fact that $0 < K$ it follows that

$$(a-1)x_0 + cy_0 < K + (a-1)x_0 + cy_0.$$

Consequently

$$0 < \frac{(a-1)x_0 + cy_0}{K + (a-1)x_0 + cy_0} < 1. \tag{2.74}$$

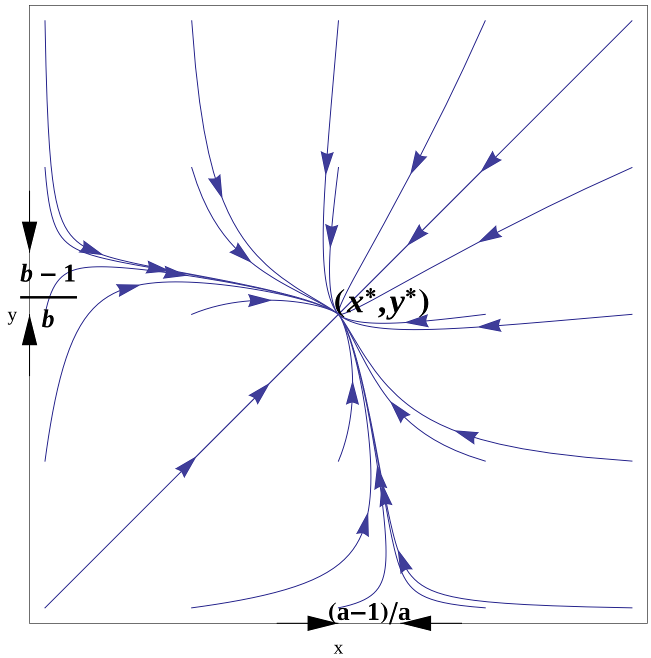

Figure 2.23: Phase-space diagram for the coexistence fixed point of the logistic competition model.

Hence

$$0 < \frac{K}{K + (a-1)x_0 + cy_0} = 1 - \frac{(a-1)x_0 + cy_0}{K + (a-1)x_0 + cy_0} < 1. \tag{2.75}$$

Thus $0 < x_1 < ax_0$. Since $y_1 > 0$ it follows that (x_1, y_1) is a point that belongs to the region defined by (2.72) and thus it satisfy the relation $y_1 > -\frac{K}{c} + \frac{1-a}{c}x_1$. Analogously we show that $0 < x_2 < ax_1$. Continuing this process, by induction we prove that $0 < x_n < a^n x_0$. Thus one has $x_n < a^n x_0 \to 0$ as n goes to infinity.

2. Suppose that the condition (2.73) holds and assume that $a \geq 1$ (which implies that $x_n > 0$ for all $n = 0, 1, 2, \ldots$). Following the same reasoning as the item 1. we can show that $y_n < b^n y_0 \to 0$ as n goes to infinity.

3. Suppose that the conditions (2.72) and (2.73) hold. Hence there exists (x_0, y_0) such that

$$-\frac{K}{c} + \frac{1-a}{c}x_0 < y_0 < \frac{L}{1-b} + \frac{d}{1-b}x_0. \tag{2.76}$$

By a similar argument we can show that $0 < x_1 < ax_0$ and $0 < y_1 < by_0$. Hence (x_1, y_1) is a point that belongs to the region defined by the relation (2.76). Thus the following relation yields

$$-\frac{K}{c} + \frac{1-a}{c}x_1 < y_1 < \frac{L}{1-b} + \frac{d}{1-b}x_1.$$

Consequently, by induction we can establish that $x_n < a^n x_0$ and $y_n < b^n y_0$. Therefore we have $(x_n, y_n) \to (0, 0)$ as n goes to infinity.

If $a = 1$, then $x_{n+1} = \frac{Kx_n}{K+cy_n}$. But $\frac{K}{K+cy_n} < 1$, $\forall n \in \mathbb{Z}^+$. Let $M = \max\{\frac{K}{K+cy_n}, n \in \mathbb{Z}^+\}$. Then $x_{n+1} < Mx_n$, $\forall n \in \mathbb{Z}^+$ and therefore $x_n < M^n x_0$ for any positive initial condition x_0. This implies that $x_n \to 0$ as n goes to ∞. Analogously, when $b = 1$ one can show that $y_n \to 0$ as n goes to ∞.

He now summarize these ideas in the following lemma.

Lemma 46 *Let (x_n, y_n) denotes the solution of the Leslie-Gower competition model (2.71). If $a \in (0,1]$ such that the condition (2.72) holds, then population x will go to extinct. If $b \in (0,1]$ such that the condition (2.73) holds, then population y is going to extinction.*

In the absence of species y, the dynamics of species x is governed by the difference equation

$$x_{n+1} = \frac{aKx_n}{K + (a-1)x_n}, n \in \mathbb{Z}^+,$$

which is the well known Beverton-Holt model for one species. Thus one can consider the model (2.71) as an extension of the Beverton-Holt equation.

In order to avoid the extinction of both species at the same time and the trouble region in the first octant we assume that

$$a > 1 \quad \text{and} \quad b > 1.$$

Let $F(x, y) = (f(x, y), g(x, y))$ where

$$f(x, y) = \frac{aKx}{K + (a-1)x + cy} \text{ and } g(x, y) = \frac{bLy}{L + (b-1)y + dx}.$$

The map F takes $(0, \infty) \times (0, \infty)$ into itself. The same happens for the coordinate axes $[0, +\infty) \times \{0\}$ and $\{0\} \times [0, +\infty)$.

The origin is a trivial fixed point of the map F while the two points $(K, 0)$ and $(0, L)$ are two exclusion equilibria of the Leslie-Gower model (2.71) on the axes. The only other fixed point of F is

$$(x^*, y^*) = \left(\frac{(a-1)(b-1)K - c(b-1)L}{(a-1)(b-1) - dc}, \frac{(a-1)(b-1)L - d(a-1)K}{(a-1)(b-1) - dc} \right).$$

This is the coexistence equilibrium.

The Jacobian of the map representing Eq. (2.71) is given by the following matrix

$$JF(x, y) = \begin{pmatrix} \frac{aK(K+cy)}{(K+(a-1)x+cy)^2} & -\frac{acKx}{(K+(a-1)x+cy)^2} \\ -\frac{bdLy}{(L+(b-1)y+dx)^2} & \frac{bL(L+dx)}{(L+(b-1)y+dx)^2} \end{pmatrix}. \tag{2.77}$$

The Jacobians evaluated at the fixed points are

$$J_0 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, J_K = \begin{pmatrix} \frac{1}{a} & -\frac{c}{a} \\ 0 & \frac{bL}{dK+L} \end{pmatrix}, J_L = \begin{pmatrix} \frac{aK}{K+cL} & 0 \\ -\frac{d}{b} & \frac{1}{b} \end{pmatrix},$$

and

$$J^* = \begin{pmatrix} \frac{(a-1)(b-1)(K+cL)-acdK}{ak((a-1)(b-1)-cd)} & \frac{c(b-1)(cL-(a-1)K)}{ak((a-1)(b-1)-cd)} \\ \frac{d(a-1)(dK-(b-1)L)}{bL((a-1)(b-1)-cd)} & \frac{(a-1)(b-1)(L+dK)-bcdL}{bL((a-1)(b-1)-cd)} \end{pmatrix}.$$

The eigenvalues of J_0 are $a > 1$ and $b > 1$. Hence the fixed point $(0, 0)$ is always unstable.

Let us now consider the exclusion fixed point $(K, 0)$. The eigenvalues of J_K are $1/a$ and $bL/(dK + L)$. Since $1/a < 1$ we have the following.

i. If $(b-1)L > dK$ then $(K, 0)$ is unstable.

ii. If $(b-1)L < dK$ then $(K, 0)$ is locally asymptotically stable.

iii. If $(b-1)L = dK$ then $tr(J_K) = 1 + det(J_K)$ and consequently one eigenvalue is 1 and the other is less than 1 in absolute value. We use the equivalent of the Liapunov's second method to two-dimensional difference equations see [26, section 4.9] or [25, section 4.5]. Notice that $1 < b = 1 + \frac{dK}{L}$. Shifting the fixed point $(K, 0)$ to the origin we get the system

$$\begin{cases} u_{n+1} = \frac{K(u_n - cv_n)}{aK+(a-1)u_n+cv_n} \\ v_{n+1} = \frac{L(dK+L)v_n}{L(L+dK)+dLu_n+dKv_n} \end{cases}. \tag{2.78}$$

Hence it follows that

$$u_{n+1} < \frac{Ku_n}{aK + (a-1)u_n + cv_n} = \frac{u_n}{a + \frac{(a-1)u_n + cv_n}{K}},$$

and by the fact that $a > 1$ we obtain

$$u_{n+1} < \frac{u_n}{a} < u_n.$$

Similarly

$$v_{n+1} = \frac{v_n}{1 + \frac{dLu_n + dKv_n}{dKL + L^2}} < v_n.$$

Clearly, the function $V(u_n, v_n) = u_n + v_n$ is continuous and

$$\Delta V(u_n, v_n) = \frac{u_n}{a + \frac{(a-1)u_n + cv_n}{K}} + \frac{v_n}{1 + \frac{dLu_n + dKv_n}{dKL + L^2}} - v_n - u_n - v_n \leq 0.$$

Thus V is a Liapunov function. Consequently, by Theorem 4.6 in [26, page 209], $(0, 0)$ is asymptotically stable and consequently, the fixed point $(K, 0)$ of (2.71) is locally asymptotically stable.

Hence we have the following result

Theorem 47 *Let $K > 0$, $L > 0$, $a > 1$, $b > 1$, $c \in (0, 1)$ and $d \in (0, 1)$. The exclusion fixed point $(K, 0)$ of the Leslie-Gower competition model (2.71) is locally asymptotically stable if $(b-1)L \leq dK$ and it is unstable if $(b-1)L > dK$.*

Analogously,

Theorem 48 *Let $K > 0$, $L > 0$, $a > 1$, $b > 1$, $c \in (0, 1)$ and $d \in (0, 1)$. The exclusion fixed point $(0, L)$ of the Leslie-Gower competition model (2.71) is locally asymptotically stable if $(a-1)K \leq cL$ and it is unstable if $(a-1)K > cL$.*

We now turn our attention to the positive fixed point (x^*, y^*) which is the solution of the system

$$\begin{cases} (a-1)x^* + cy^* = (a-1)K \\ dx^* + (b-1)y^* = (b-1)L \end{cases}. \tag{2.79}$$

From [46] it follows

$$\frac{a-1}{d} > \frac{(a-1)K}{(b-1)L} > \frac{c}{b-1},$$

or equivalently

$$(a-1)(b-1)L - (a-1)dK > 0 \text{ and } (a-1)(b-1)K - (b-1)cL > 0. \tag{2.80}$$

Simplifying these two inequalities we get

$$(b-1)L > dK \text{ and } (a-1)K > cL. \tag{2.81}$$

Consequently, the following relation yields

$$(a-1)(b-1) > cd. \tag{2.82}$$

The stability region of the coexistence fixed point is given by the following conditions:

1. $det(JF(x^*, y^*)) < 1$;
2. $det(JF(x^*, y^*)) > tr(JF(x^*, y^*)) - 1$;
3. $det(JF(x^*, y^*)) > -tr(JF(x^*, y^*)) - 1$.

The inequality given in 1. is equivalent to

$$\frac{(a-1)(b-1)(dK(K+cL) + L((1-ab)K + cL))}{abKL((a-1)(b-1) - dc)} < 0.$$

From the relation (2.82) this inequality holds if and only if

$$dK(K+cL) + L((1-ab)K + cL) < 0. \tag{2.83}$$

Using the relation (2.81) one has $dK < (b-1)L$ and $(1-a)K < -cL$. Thus,

$$\begin{aligned} dK(K+cL) + L((1-ab)K + cL) &< L((b-1)(K+cL) + K + cL - abK) \\ &= L(Kb(1-a) + bcL) \\ &< L(-bcL + bcL) = 0. \end{aligned}$$

Consequently, under the relations (2.81) and (2.82) we have $det(JF(x^*, y^*)) < 1$.

The condition of item 2 is equivalent to

$$-\frac{(a-1)(b-1)(dK + (1-b)L)((a-1)K - cL)}{abKL((a-1)(b-1) - dc)} > 0,$$

which is analogous to prove that $(dK + (1-b)L)((a-1)K - cL) < 0$. Using (2.80) this last inequality holds for all values of the parameters. Thus one has $det(JF(x^*, y^*)) > tr(JF(x^*, y^*)) - 1$ whenever $(b-1)L > dK$ and $(a-1)K > cL$.

The condition of item 3 is equivalent to

$$\frac{(a-1)(b^2-1)L((a+1)K + cL) + dK((a^2-1)(b-1)K - (-1 + a + b + 3ab)cL)}{abKL((a-1)(b-1) - dc)} > 0.$$

Using the relation $(b-1)L > dK$ it follows

$$\begin{aligned} (a-1)(b^2-1)L((a+1)K + cL) &> dK(a-1)(b+1)((a+1)K + cL) \\ &> dK(a-1)(b-1)((a+1)K + cL) \\ &= dK((a^2-1)(b-1)K + (a-1)(b-1)cL). \end{aligned}$$

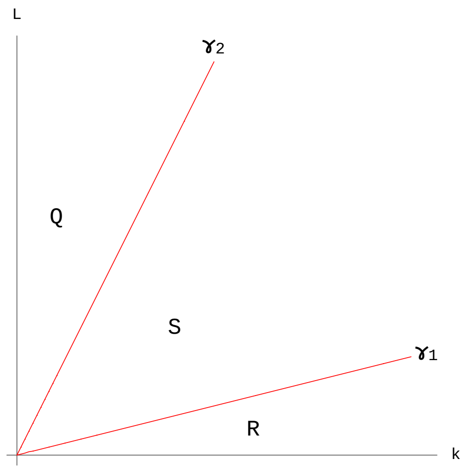

Figure 2.24: If the carrying capacities K and L are in region S, then the coexistence fixed point of the autonomous Leslie-Gower model is globally asymptotically stable. If the parameters are in region R and Q, then the exclusion fixed point is globally asymptotically stable.

But $(a-1)(b-1) > -(-1+a+b+3ab)$ and thus

$$(a-1)(b^2-1)L((a+1)K+cL) > dK((a^2-1)(b-1)K-(-1+a+b+3ab)cL),$$

which implies that $det(JF(x^*,y^*)) > -tr(JF(x^*,y^*))-1$ is verified for all values of the parameters whenever $(b-1)L > dK$ and $(a-1)K > cL$.

Consequently, the inequalities $|tr(JF(x^*,y^*))|-1 < det(JF(x^*,y^*)) < 1$ are automatically verified under the hypothesis (2.81). In Fig. 2.24 we represent in the $(K,L)-$plane the region where the relation (2.81) is satisfied. The two lines γ_1 and γ_2 represent, respectively, $(b-1)L = dK$ and $cL = (a-1)K$ and the regions identified by the letters P, Q, and S are defined as follows

$$\begin{aligned} S &= \{(K,L) \in \mathbb{R}_+^2 : (b-1)L > dK \text{ and } cL < (a-1)K\}, \\ R &= \{(K,L) \in \mathbb{R}_+^2 : (b-1)L < dK \text{ and } L > 0\}, \\ Q &= \{(K,L) \in \mathbb{R}_+^2 : K > 0 \text{ and } cL > (a-1)K\}. \end{aligned}$$

If K and L belong to the region S then (x^*,y^*) is locally asymptotically stable. We now summarize these ideas.

Theorem 49 *Let $K > 0$, $L > 0$, $a > 1$, $b > 1$, $c \in (0,1)$ and $d \in (0,1)$. Then the coexistence fixed point (x^*,y^*) of the Leslie-Gower competition model (2.71) is locally asymptotically stable if*

$$(b-1)L > dK \text{ and } (a-1)K > cL.$$

In [46] the authors proved that the map F generate a discrete monotone system on $\mathbb{R}_+^2$ in the sense of "competitive ordering", i.e., a mapping $T = (T_1,T_2) : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ is said to be competitive if and only if

$$x_1 \geq x_2 \text{ and } y_1 \leq y_2,$$

implies

$$T_1(x_1,y_1) \geq T_1(x_2,y_2) \text{ and } T_2(x_1,y_1) \leq T_2(x_2,y_2).$$

For general theory of monotone maps see [72]. From now on we use this definition of monotonicity in this section. Thus the monotonicity of the map F is sufficient to prove the convergence of the solutions of the corresponding difference equation (2.71).

Making the change of variables $x = \frac{Ku}{a-1}$ and $y = \frac{Lv}{b-1}$ one can show that Eq. (2.71) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{au_n}{1+u_n+Cv_n} \\ \frac{bv_n}{1+v_n+Du_n} \end{pmatrix}, n \in \mathbb{Z}^+, \tag{2.84}$$

where $C = \frac{cL}{K(b-1)} > 0$ and $D = \frac{dK}{L(a-1)} > 0$. By the fact that for any positive U and V one has

$$\frac{au}{1+u+Cv} < a\frac{u}{1+u} < a \quad \text{and} \quad \frac{bv}{1+v+Du} < b\frac{v}{1+v} < b,$$

it follows that all the iterations of the map

$$G(x,y) = \left(\frac{au}{1+u+Cv}, \frac{bv}{1+v+Du}\right), (u,v) \in \mathbb{R}^2_+,$$

are bounded above. Then $G^n(x,y)$ monotonically converges to a limit point in $\mathbb{R}^2_+$. This means that the map G is dissipative[6] in $\mathbb{R}^2_+$. Moreover, in [18] it is shown that the map $G : \mathbb{R}^2_+ \to S \doteq [0,a) \times [0,b)$ is invertible and bicontinuous.

Thus it follows that the map F is dissipative in $\mathbb{R}^2_+$. Then the limit set of F is compact and contains only the fixed point of F in $\mathbb{R}^2_+$, i.e., all the solutions of Eq. (2.71) in $\mathbb{R}^2_+$ converge to an equilibrium as $n \to +\infty$.

If the carrying capacities belong to the region S (Fig. 2.24) then all the iterations are attracted by the positive fixed point. In this region this fixed point of Eq. (2.71) is globally asymptotically stable (Theorem 4 in [46]). If K and L belong R (Fig. 2.24) all the iterations for any positive initial point (x_0, y_0) converge to $(K, 0)$. Here $(K, 0)$ is a globally asymptotically stable fixed point of Eq. (2.71) (Theorem 4 in [46]). If K and L are on the line γ_1 then $(K, 0)$ is asymptotically stable. Similar conclusions can be taken for the fixed point $(0, L)$.

We now summarize the above discussion in the following theorem.

Theorem 50 *Let $a > 1$, $b > 1$ and $0 < c, d < 1$. Then Eq. (2.71) has a*

1. *globally asymptotically stable coexistence fixed point if $K, L \in S$;*
2. *globally asymptotically stable exclusion fixed point on the $x-$axis if $K, L \in R$;*
3. *globally asymptotically stable exclusion fixed point on the $y-$axis if $K, L \in Q$.*

2.4.2 The dynamics of the nonautonomous model

Now we are going to study the nonautonomous Leslie-Gower competition model given by the following difference equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{a_nK_nx_n}{K_n+(a_n-1)x_n+c_ny_n} \\ \frac{b_nL_ny_n}{L_n+(b_n-1)y_n+d_nx_n} \end{pmatrix}, n \in \mathbb{Z}^+, \tag{2.85}$$

where the sequence of parameters are $p-$periodic, i.e

$$(K_n, L_n, a_n, b_n, c_n, d_n) = (K_{n \bmod p}, L_{n \bmod p}, a_{n \bmod p}, b_{n \bmod p}, c_{n \bmod p}, d_{n \bmod p}),$$

[6]If all the solutions of a difference equation are bounded above, then we say the corresponding map is dissipative.

for all $n = 0, 1, 2, \ldots$. Let

$$F_n(x, y) = \left(\frac{a_n K_n x}{K_n + (a_n - 1)x + c_n y}, \frac{b_n L_n y}{L_n + (b_n - 1)y + d_n x}\right), n \in \mathbb{Z}^+.$$

Since the sequence of the parameters is $p-$periodic one has $F_n = F_{n \bmod p}$, for all $n \in \mathbb{Z}^+$. Each one of the members of the set

$$\mathcal{F} = \{F_0, F_1, \ldots, F_{p-1}\}$$

is monotone. Since the composition of monotone maps is monotone, it follows that the composition operator

$$\Phi_p^i = F_{p+i-1} \circ \ldots \circ F_{i+1} \circ F_i, i \in \{0, \ldots, p-1\}$$

is monotone.

Since for each $i \in \{0, \ldots, p-1\}$ one has

$$\frac{a_i K_i x}{K_i + (a_i - 1)x + c_i y} < \frac{a_i K_i}{a_i - 1} \text{ and } \frac{b_i L_i y}{L_i + (b_i - 1)y + d_i x} < \frac{b_i L_i}{b_i - 1},$$

it follows that all the members of $\mathcal{F}$ are dissipative maps. Using again the composition argument it follows that Φ_p^i is dissipative. Thus for Eq. (2.85) one has the following result.

Theorem 51 *Suppose that all the parameters of Eq. (2.85) are $p-$periodic such that $K_n > 0$, $L_n > 0$, $a_n > 1$, $b_n > 1$, $c_n \in (0, 1)$ and $d_n \in (0, 1)$, $\forall n \in \mathbb{Z}^+$ and let $i = 0, 1, 2, \ldots, p-1$. Then the $p-$periodic nonautonomous difference equation (2.85) has a globally asymptotically stable $p-$periodic cycle. Moreover, it is*

1. *a coexistence cycle if $K_i, L_i \in S_i$, where*

$$S_i = \{(K_i, L_i) \in \mathbb{R}_+^2 : (b_i - 1)L_i > d_i K_i \text{ and } c_i L_i < (a_i - 1)K_i\};$$

2. *an exclusion cycle on the $x-$axis if $K_i, L_i \in R_i$, where*

$$R_i = \{(K, L) \in \mathbb{R}_+^2 : (b - 1)L < dK \text{ and } L > 0\};$$

3. *an exclusion cycle on the $y-$axis if $K_i, L_i \in Q_i$, where*

$$Q_i = \{(K_i, L_i) \in \mathbb{R}_+^2 : K_i > 0 \text{ and } c_i L_i > (a_i - 1)K_i\}.$$

Proof. Since the map Φ_p^i, $i \in \{0, \ldots, p-1\}$ is monotone and dissipative, then the difference equation

$$(x_{n+1}, y_{n+1}) = \Phi_p^i(x_n, y_n), n \in \mathbb{Z}^+, \tag{2.86}$$

has a globally asymptotically stable fixed point. Let $(\overline{x}_i, \overline{y}_i)$ be this fixed point. Then the fixed point $(\overline{x}_i, \overline{y}_i)$ of the autonomous equation (2.86) is a periodic point, with period p, of the nonautonomous $p-$periodic difference equation (2.85).

Now suppose that the carrying capacities $(K_i, L_i) \in S_i$. Hence $(\overline{x}_i, \overline{y}_i)$ is a coexistence fixed point which generates a coexistence $p-$periodic cycle of Eq. (2.85) if $K_i, L_i \in S_i$.

If the carrying capacities $(K_i, L_i) \in R_i$, then $(\overline{x}_i, \overline{y}_i)$ is an exclusion fixed point on the $x-$axis that it generates an exclusion $p-$periodic cycle of Eq. (2.85) on the $x-$axis. Similarly if $(K_i, L_i) \in Q_i$, $(\overline{x}_i, \overline{y}_i)$ is an exclusion fixed point on the $y-$axis that generates an exclusion cycle of Eq. (2.85) on the $y-$axis. ∎

2.4.3 Attenuance and the resonance

In this section we give some notes about the attenuance and the resonance of equation (2.85). First we turn our attention to the following example.

Example 52 *Let* $K_0 = 1.8$, $K_1 = 2$, $K_2 = 3$, $K_3 = 4$, $L_0 = 2$, $L_1 = 3$, $L_2 = 1.5$, $L_3 = 5$, $a_0 = \ldots = a_3 = 2$, $b_0 = \ldots = b_3 = 3$, $c_0 = \ldots = c_3 = 0.5$, *and* $d_0 = \ldots = d_3 = 0.5$. *With these values of the parameters Eq. (2.85) is* 4−*periodic. This leads to the following globally asymptotically stable* 4−*periodic cycle*

$$C_4 = \left\{ \begin{pmatrix} \overline{x}_0 \\ \overline{y}_0 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{x}_3 \\ \overline{y}_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1.36197 \\ 1.5461 \end{pmatrix}, \begin{pmatrix} 1.776 \\ 2.64345 \end{pmatrix}, \begin{pmatrix} 1.30542 \\ 1.94017 \end{pmatrix}, \begin{pmatrix} 1.2213 \\ 2.31799 \end{pmatrix} \right\}.$$

For these values of the parameters the positive fixed point of the individual maps F_0, F_1, F_2, *and* F_3 *are*

i	0	1	2	3
x_i^*	0.914286	0.571429	2.57143	*1.71429*
y_i^*	1.77143	2.85714	0.857143	*4.57143*

Thus $av(\overline{x}) \approx 1.41617$, $av(\overline{y}) \approx 2.111928$, $av(x^*) \approx 1.44285$, *and* $av(y^*) \approx 2.51425$. *Consequently,* C_4 *is* A/A *(see the definition in section 2.2.5).*

Several examples like this one suggest that when the carrying capacities of each individual map belong to the stability region S_i such that the competition parameters and the intrinsic growth rates are fixed then if

$$C_r = \left\{ \begin{pmatrix} \overline{x}_0 \\ \overline{y}_0 \end{pmatrix}, \begin{pmatrix} \overline{x}_1 \\ \overline{y}_1 \end{pmatrix}, \ldots, \begin{pmatrix} \overline{x}_{p-1} \\ \overline{y}_{p-1} \end{pmatrix} \right\}$$

is a globally asymptotically stable coexistence cycle of Eq. (2.71), then the following relations must be true

$$av(\overline{x}) < \frac{(a-1)(b-1)av(K) - c(b-1)av(L)}{(a-1)(b-1) - dc} = av(x^*)$$

and

$$av(\overline{y}) < \frac{(a-1)(b-1)av(L) - d(a-1)av(K)}{(a-1)(b-1) - dc} = av(y^*),$$

i.e., the coexistence cycle C_r must be attenuant/attenuant.

Notice that this conjecture is analogous to the one-dimensional case, the famous Cushing-Henson conjecture [17]. Cushing and Henson conjectured that a nonautonomous p−periodic Beverton-Holt equation

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n}, n \in \mathbb{Z}^+, \tag{2.87}$$

with periodically varying carrying capacity must be attenuant. This means that if

$$C_p = \{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{p-1}\}$$

is its p−periodic cycle, and K_i, $0 \le i \le p-1$ are the carrying capacities, then

$$\frac{1}{p}\sum_{i=0}^{p-1} \overline{x}_i < \frac{1}{p}\sum_{i=0}^{p-1} K_i. \tag{2.88}$$

Since the periodic cycle C_p is globally asymptotically stable on $(0, \infty)$, it follows that for any initial population density x_0, the time average of the population density x_n is eventually less than the average of the carrying capacities, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} x_i < \frac{1}{p} \sum_{i=0}^{p-1} K_i. \tag{2.89}$$

Relation (2.89) gives a justification for the use of the word "attenuance" to describe the phenomenon in which a periodically fluctuation carrying capacity of the Beverton-Holt equation has a deleterious effect on the population. This conjecture was first proved by Elaydi and Sacker in [28, 29, 31] and independently by Kon [42], Kocic [41] and Stevo [75].

Kocic [41], however gave the most elegant proof for the presence of attenuance. Utilizing effectively the Jensen's inequality, he was able to give a more general result.

If the growth rates and the competition parameters of Eq. (2.85) are periodic then the phenomenon is drastically different. For instance, consider that the following parameters generate the equation given in (2.85).

i	K_i	L_i	a_i	b_i	c_i	d_i
0	1	2	3	3	0.5	0.5
1	6/17	12/17	4	4	0.6	0.5
2	2	3.9	2	2	0.5	0.5
3	4/11	8/11	5	5	0.6	0.5

With these parameters we have a 4−periodic difference equation. In the following table are presented the values of the 4−periodic cycle and the respective positive fixed point for each individual map.

i	$\overline{x}_i$	$\overline{y}_i$	x_i^*	y_i^*
0	0.26274	0.766905	8/15	28/15
1	0.412911	1.25544	0.219067	0.669371
2	0.248592	0.757648	1/15	58/15
3	0.378458	1.23583	0.259409	0.694847
sum	1.302701	4.015823	1.078476	7.0975513

In this situation we have $av(\overline{x}) > av(x^*)$ and $av(\overline{y}) < av(y^*)$, i.e, the cycle is R/A.

If, however, we permute the values of the carrying capacities in $i = 0$ and $i = 2$ and keep the rest of the parameters we obtain the following values.

i	$\overline{x}_i$	$\overline{y}_i$	x_i^*	y_i^*
0	0.321055	0.717422	28/15	8/15
1	0.641935	0.829271	0.219067	0.669371
2	0.326427	0.666201	58/15	1/15
3	0.558419	0.941821	0.259409	0.694847
sum	1.847836	3.154715	6.21181	1.964218

Under this scenario we have A/R since $av(\overline{x}) < av(x^*)$ and $av(\overline{y}) > av(y^*)$. This is precisely the opposite of the previous case.

This is an expected scenario because in the case of the periodic Beverton-Holt equation (2.87) in which both parameters μ_n and K_n are periodic of common period p, Eq. (2.87) may be attenuant or resonant. In fact, when $p = 2$, Elaydi and Sacker [29] showed that

$$\overline{x} = \overline{K} + \sigma\frac{K_0 - K_1}{2} - \Delta\frac{(\mu_0 - 1)(\mu_1 - 1)}{2(\mu_0\mu_1 - 1)}(K_0 - K_1)^2 \tag{2.90}$$

where

$$\overline{x} = \frac{\overline{x}_0 + \overline{x}_1}{2} \text{ and } \overline{K} = \frac{K_0 + K_1}{2},$$

$$\sigma = \frac{\mu_1 - \mu_0}{\mu_0\mu_1 - 1}, 0 \leq |\sigma| < 1,$$

and

$$\Delta = \frac{\mu_0(\mu_1^2 - 1)K_0 + \mu_1(\mu_0^2 - 1)K_1}{\mu_0(\mu_1 - 1)^2K_0^2 + (\mu_0 - 1)(\mu_1 - 1)(\mu_0\mu_1 + 1)K_0K_1 + \mu_1(\mu_0 - 1)^2K_1^2} > 0.$$

It follows that attenuance is present if either $(\mu_1 - \mu_0)(K_0 - K_1) < 0$ (out of phase) or the algebraic sum of the last two terms in Eq. (2.90) is negative. On the other hand, resonance is present if the algebraic sum of the last two terms in Eq. (2.90) is positive.

Notice that if $\mu_0 = \mu_1 = \mu$ with $p = 2$, then we have

$$\frac{1}{p}\sum_{i=0}^{p-1}\overline{x}_i = \frac{1}{p}\sum_{i=0}^{p-1}K_i - \frac{\mu(K_0 + K_1)(K_1 - K_0)^2}{2\left[\mu K_0^2 + (\mu^2 + 1)K_0K_1 + \mu K_1^2\right]},$$

which gives an exact expression for the difference in the averages.

Chapter 3

The Allee effect

In the last decade, there has been a renewed interest in a biological phenomenon called the **Allee effect**. A biology book [11], published recently, which is solely dedicated to the study of this phenomenon and hundreds of papers dealing with the Allee effect were cited in this book. Surprisingly, the literature on mathematical modelling of the Allee effect is lagging behind. Some of the relevant work may be found in Yakubu [78, 79], Jang [38, 74], Li, Song, and Wang [45], Elaydi and Sacker [67], Allen, Fagan, Hognas, and Fagerholm [3], Schreiber [69], Dennis [23], Cushing [19, 20], Anazawa [7], Eskola and Parvinen [32], Fowler and Ruxton [33] and Henson [36]

But what is the Allee effect? The Allee effect is a phenomenon in population dynamics attributed to the biologist Wander Claude Allee [1, 2]. Allee proposed that the **per capita birth rate declines at low density or population sizes**. In the languages of dynamical systems or difference equations, a map representing the Allee effect must have tree fixed points, an asymptotically stable zero fixed point, a small unstable fixed point, called the threshold point, and a bigger positive fixed point, called the carrying capacity, that is asymptotically stable at least for smaller values of the parameters.

Our main interest in this chapter is to study nonautonomous periodic difference equations/discrete dynamical systems in which the maps of the system are "Allee maps". Such systems model population with fluctuating habitat and they are commonly called periodically forced systems.

In [50] we develop the theory in population dynamics. A new class of maps called **unimodal Allee maps** is introduced. The properties and stability of the three fixed points are studied in the setting of nonautonomous periodic dynamical systems or difference equations. Finally the bifurcation of periodic systems/difference equations when the system consists of two unimodal Allee maps is investigated.

In [49, 51, 58] we formulate a mathematical models based on **Marx theory of economics**. The profit rate r is considered as a function of both the exploitation rate e and the organic composition of the capital k . This model possesses the Allee effect, in which the profit rate declines to zero if it falls below a certain threshold. It is represented by the difference equation $r_{n+1} = f_a(r_n)$, which is a family of unimodal maps depending on the parameter a, where a measures the relative growth of the exploitation rate when the profit rate is zero. Moreover, the model predicts a period-doubling bifurcation scenario as the parameter a increases. Finally, we allow the parameter a to fluctuate periodically, which leads to a periodic nonautonomous difference equations.

3.1 Population dynamics

3.1.1 Preliminaries

We start this subsection by the definition of unimodal Allee map.

Definition 53 *Let $I = [0, b] \subset \mathbb{R}^+$. A continuous function $f : I \to I$ is called an* ***Allee map*** *if the following conditions hold:*

- $f(0) = 0$;
- *there are positive points A_f and K_f such that $f(x) < x$ for $x \in \left(0, A_f\right) \bigcup \left(K_f, b\right)$ and $f(x) > x$ for $x \in \left(A_f, K_f\right)$.*

If, in addition, the map is unimodal, then it is called an ***unimodal Allee map****. Explicitly, we require the following:*

- $f(b) = 0$ *when b is finite or $\lim\limits_{x\to+\infty} f(x) = 0$ otherwise;*
- *there exists a unique critical point C_f of f, where $f(x)$ is strictly increasing on $[0, C_f)$ and strictly decreasing on (C_f, b) (or $(C_f, +\infty)$ when $b = +\infty$).*

Fig. 3.1 depicts a prototype of unimodal Allee maps.

From Definition 53, it follows that A_f and K_f are positive fixed points. We call the smaller positive fixed point the threshold point A_f of the map f, and the greater positive fixed point K_f the carrying capacity of the map f. It is easy to verify that $x^* = 0$ is an attracting fixed point and

$$[0, A_f) \cup \left(\tilde{A}_f, b\right) \subset \mathcal{B}_f(0),$$

where $\mathcal{B}_f(0)$ is the basin of attraction of zero and $\tilde{A}_f = f^{-1}(A_f)$, i.e. $f\left(\tilde{A}_f\right) = A_f$, with $\tilde{A}_f > K_f$. Note that the threshold point A_f is always repelling while the carrying capacity K_f may be (or not) stable.

We now define a unimodal Allee map f by using two maps, a left map f_l and a right map f_r. Thus we have

$$f(x) = \begin{cases} f_l(x) & \text{if } 0 \le x \le A_f \\ f_r(x) & \text{if } A_f < x \le b \end{cases}. \tag{3.1}$$

It follows that $f(0) = f_l(0) = f_r(b) = 0$ (or $\lim\limits_{x\to\infty} f_r(x) = 0$). Since f is continuous in $\mathbb{R}^+$, it follows that $f\left(A_f\right) = f_l\left(A_f\right) = f_r\left(A_f\right) = A_f$ and $f\left(K_f\right) = f_r\left(K_f\right) = K_f$.

To facilitate our study we introduce two zones, the threshold zone and the carrying capacity zone.

Definition 54

1. *The square that contains the origin and the points $\left(A_f, 0\right)$, $\left(0, A_f\right)$ and $\left(A_f, A_f\right)$ will be called the* ***threshold zone****.*
2. *The rectangle that contains the points $\left(A_f, A_f\right)$, $\left(A_f, f\left(C_f\right)\right)$, $\left(\tilde{A}_f, A_f\right)$ and $\left(\tilde{A}_f, f\left(C_f\right)\right)$ will be called the* ***carrying capacity zone****.*

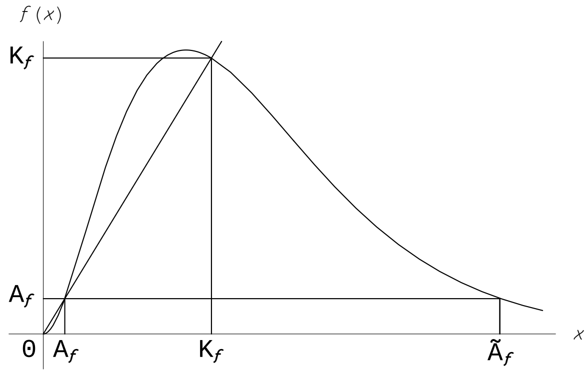

Figure 3.1: An instance of one unimodal Allee map f.

Let $\mathcal{F} = \{f, g\}$ be a periodic system formed by two unimodal Allee maps such that

$$f(x) > g(x) \text{ for all } x \text{ on } (0, b).$$

Hence, if f and g are two maps in $\mathcal{F}$, then we have

$$0 < A_f < A_g < K_g < K_f.$$

Henceforth we assume that the right end point b of I is fixed for all the unimodal Allee maps.

The composition map $f \circ g$ may be written as follows

$$f \circ g = \begin{cases} f(g_l(x)) \text{ if } 0 \leq x \leq A_g \\ f(g_r(x)) \text{ if } A_g < x \leq b \end{cases}. \tag{3.2}$$

The first branch of (3.2) may be written as

$$f(g_l(x)) = \begin{cases} f_l(g_l(x)) \text{ if } 0 \leq g_1(x) \leq A_f \wedge 0 \leq x \leq A_g \\ f_r(g_l(x)) \text{ if } A_f \leq g_1(x) < b \wedge 0 \leq x \leq A_g \end{cases}.$$

Hence

$$f(g_l(x)) = \begin{cases} f_l(g_l(x)) \text{ if } 0 \leq x \leq A_f^- \\ f_r(g_l(x)) \text{ if } A_f^- < x \leq A_g \end{cases}, \tag{3.3}$$

where A_f^- represents the left preimage of A_f under the map g, i.e., $g\left(A_f^-\right) = A_f$, or equivalently

$$f_l\left(g_l\left(A_f^-\right)\right) = f_r\left(g_l\left(A_f^-\right)\right) = f\left(A_f\right) = A_f.$$

The second branch of (3.2) may be written as

$$f(g_r(X)) = \begin{cases} f_r(g_r(x)) \text{ if } A_g < x < A_f^+ \\ f_l(g_r(x)) \text{ if } A_f^+ \leq x \leq b \end{cases}, \tag{3.4}$$

where A_f^+ represents the right preimage of A_f under the map g, i.e., $g\left(A_f^+\right) = A_f$, or equivalently

$$f_r\left(g_r\left(A_f^+\right)\right) = f_l\left(g_r\left(A_f^+\right)\right) = f\left(A_f\right) = A_f.$$

From (3.3) and (3.4) we obtain

$$f \circ g = \begin{cases} f_l(g_l(x)) \text{ if } 0 \le x \le A_f^- \\ f_r(g_l(x)) \text{ if } A_i^- < x \le A_g \\ f_r(g_r(x)) \text{ if } A_g < x < A_f^+ \\ f_l(g_l(x)) \text{ if } A_f^+ \le x \le b \end{cases} . \tag{3.5}$$

Similarly

$$g \circ f = \begin{cases} g_l(f_l(x)) \text{ if } 0 \le x \le A_f \\ g_l(f_r(x)) \text{ if } A_f < x < A_g^- \\ g_r(f_r(x)) \text{ if } A_g^- \le x \le A_g^+ \\ g_l(f_r(x)) \text{ if } A_g^+ < x \le b \end{cases} , \tag{3.6}$$

where A_g^- and A_g^+ represents the left and the right preimages of A_g under the map f, respectively, that is, $f\left(A_g^-\right) = f\left(A_g^+\right) = A_g$. In other words we have

$$g_l\left(f_r\left(A_g^-\right)\right) = g_r\left(f_r\left(A_g^-\right)\right) = g\left(A_g\right) = A_g,$$

and

$$g_r\left(f_r\left(A_g^+\right)\right) = g_l\left(f_r\left(A_g^+\right)\right) = g\left(A_g\right) = A_g.$$

Fig. 3.2 summarizes the above remarks.

Lemma 55 *Let $f, g \in \mathcal{W}$. If $C_f > A_g$, where C_f is the unique critical point of f and A_g is the threshold point of g, then f and g, both, are homeomorphisms on $\left[0, A_g\right]$.*

3.1.2 Threshold points of the composition map

In this subsection we prove the existence of the fixed points, called threshold points, of the composition map. In addition we establish an order relation between these fixed points.

From here until the end of this subsection we assume that A_f and A_g are the threshold points of the unimodal Allee maps f and g, respectively. We also assume that A_f^- and A_g^- are, respectively, the first preimage of A_f by the map g and the first preimage of A_g by the map f.

Theorem 56 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Then both $f \circ g$ and $g \circ f$, have threshold points, that we denote by A_{fg} and A_{gf}, respectively. Moreover $A_f^- < A_{fg} < A_g$ and $A_f < A_{gf} < A_g^-$.*

Proof. Assume that f and g satisfy the hypothesis of the theorem. First let us prove the existence of A_{fg} and A_{gf}. We know that $g_l\left(A_f\right) < A_f$ and f is increasing on $\left[0, A_f\right]$. This implies that $f_l\left(g_l\left(A_f\right)\right) < f_l\left(A_f\right) = A_f$. On the other hand $f_r\left(g_l\left(A_g\right)\right) = f_r\left(A_g\right) > g_l\left(A_g\right) = A_g$. Hence the function $f \circ g(x) - x$ changes sign on (A_f, A_g). Then there exits $x \in (A_f, A_g)$ such that $f \circ g(x) = x$, i.e. $A_f < A_{fg} < A_g$. In the same way we prove that $A_f < A_{gf} < A_g$.

To proof that $A_{fg} \in (A_f^-, A_g)$ first we will proof that $A_{fg} \notin \left[A_f, A_f^-\right]$. Let $x \in \left[A_f, A_f^-\right]$. We know that $f \circ g\left(A_f\right) < A_f$ and $f \circ g\left(A_f^-\right) = A_f < A_f^-$. If $x \in (A_f, A_f^-)$ we have that $g_l(x) < A_f$ and so $f_l(g_l(x)) < A_f < x$. Therefore $A_{fg} \notin \left[A_f, A_f^-\right]$.

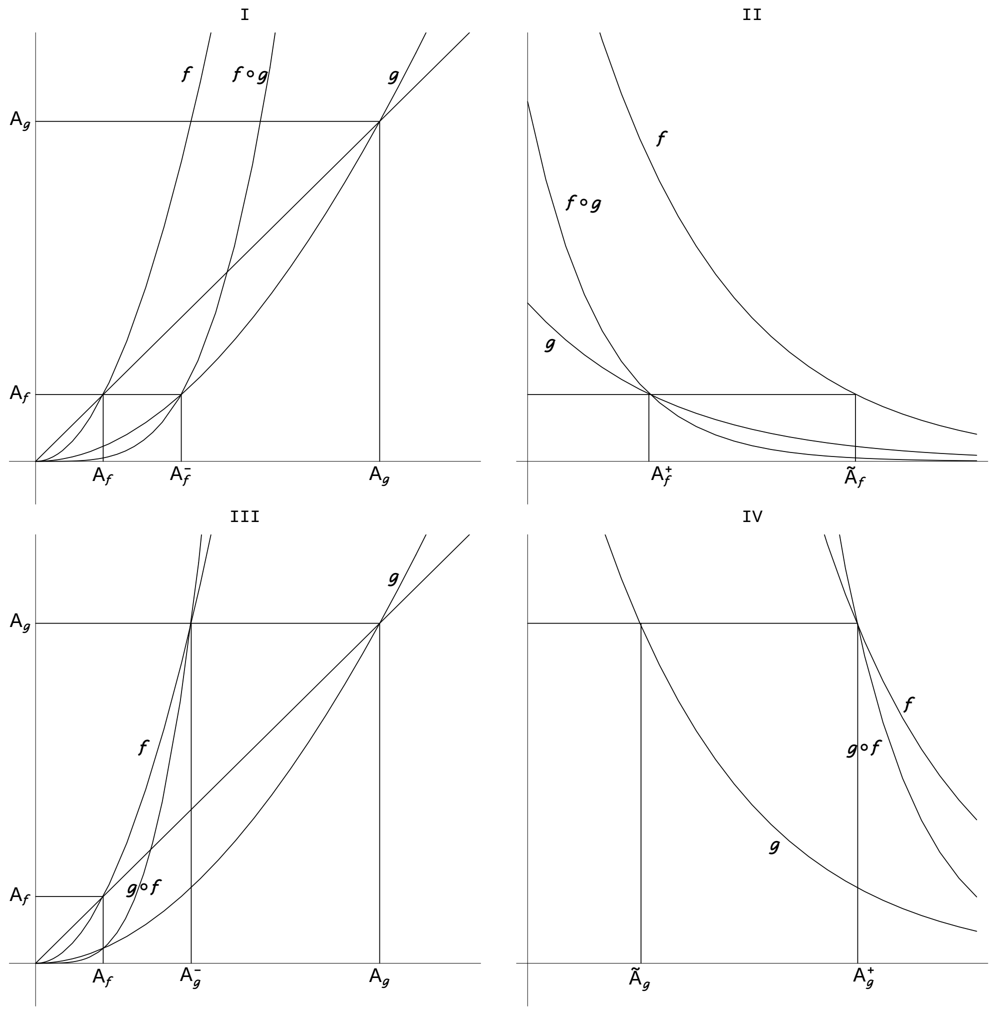

Figure 3.2: Parts I and II depicts the left and the right preimages of A_f under g while parts III and IV depicts the left and the right preimages of A_g under the map f.

Now let $x \in (A_f^-, A_g)$. By one side we have $f \circ g\left(A_g\right) = f\left(A_g\right) > A_g > x$ and on the other side $f \circ g\left(A_f^-\right) = A_f < x$, consequently, there exits $y \in (A_f^-, A_g)$ such that $f \circ g(y) = y$, that is, $A_{fg} \in (A_f^-, A_g)$. Following the same reasoning we prove $A_f < A_{gf} < A_g^-$. ■

Next we establish an order relation between these two threshold points of $f \circ g$ and $g \circ f$, respectively.

Theorem 57 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Suppose that in the threshold region, i.e., on $J = \left[0, A_g\right]$, these two maps are convex, f is increasing and $f'(x) > g'(x)$, $\forall x \in J$. Suppose also that*

$$f'\left(A_g\right) + g'\left(A_g\right) \leq f'\left(A_f\right) g'\left(A_g\right). \tag{3.7}$$

Then $A_g^- \leq A_f^-$. Moreover $A_{gf} < A_{fg}$.

Proof. By hypothesis we have $A_g - A_f = \varepsilon > 0$ and $\forall x \in J = \left[0, A_g\right]$, $f(x) - g(x) = \delta(x) > 0$ such that $\delta(x)$ is increasing. We need to prove that the first preimage of A_f and A_g, both, satisfy the relation $A_g^- \leq A_f^-$ or $f^{-1}\left(A_g\right) \leq g^{-1}\left(A_f\right)$, that is equivalent to proof that

$$A_g \leq f \circ g^{-1}\left(A_f\right). \tag{3.8}$$

By the Taylor's series we know that

$$\begin{aligned} g^{-1}\left(A_f\right) &= g^{-1}\left(A_g - \varepsilon\right) \\ &= g^{-1}\left(A_g\right) - g^{-1\prime}\left(A_g\right)\varepsilon + O\left(\varepsilon^2\right) \\ &= A_g - \frac{\varepsilon}{g'\left(A_g\right)} + O\left(\varepsilon^2\right). \end{aligned}$$

Substituting the previous relation in (3.8) we get

$$A_g \leq f\left(A_g - \frac{\varepsilon}{g'\left(A_g\right)} + O\left(\varepsilon^2\right)\right).$$

Again by Taylor's series we have that

$$A_g = f\left(A_g\right) - \frac{f'\left(A_g\right)}{g'\left(A_g\right)}\varepsilon + O\left(\varepsilon^2\right),$$

that is

$$\frac{f'\left(A_g\right)}{g'\left(A_g\right)}\varepsilon \leq f\left(A_g\right) - A_g + O\left(\varepsilon^2\right).$$

Once that $f\left(A_g\right) - A_g = f\left(A_g\right) - g\left(A_g\right) = \delta$ we get

$$\frac{f'\left(A_g\right)}{g'\left(A_g\right)}\varepsilon \leq \delta + O\left(\varepsilon^2\right) \approx \delta. \tag{3.9}$$

So relation (3.8) is equivalent to relation (3.9).

the map f is convex and therefore $f'\left(A_f\right) < \frac{f\left(A_g\right) - f\left(A_f\right)}{A_g - A_f} = f'(M) < f'\left(A_g\right)$, where $M \in \left]A_f, A_g\right[$. So $f'(M) = \frac{f\left(A_g\right) - A_g + A_g - A_f}{\varepsilon}$, and therefore $f'(M) = \frac{\delta + \varepsilon}{\varepsilon}$. By hypothesis we have $f'\left(A_g\right) + g'\left(A_g\right) \leq f'\left(A_f\right) g'\left(A_g\right)$, that is equivalent to

$$\frac{f'\left(A_g\right)}{g'\left(A_g\right)} + 1 \leq f'\left(A_f\right).$$

But $f'\left(A_f\right) < f'(M)$ and so

$$\frac{f'\left(A_g\right)}{g'\left(A_g\right)} + 1 \leq f'(M) = \frac{\delta + \varepsilon}{\varepsilon}.$$

Multiplying by ε both of the members of the last relation we get

$$\frac{f'\left(A_g\right)}{g'\left(A_g\right)}\varepsilon \leq \delta,$$

which is equivalent to relation (3.9), and so this part of the theorem is proved.

From Theorem 56 and by the fact that $A_g^- \leq A_f^-$ it follows $A_{gf} < A_{fg}$. ■

Hypothesis (3.7) requires that f and g stay sufficiently far apart to avoid the collapse of the interval where belong the threshold points of $f \circ g$ and $g \circ f$.

3.1.3 The carrying capacity of the composition map

In this subsection we study the existence, the location and the properties of the carrying capacity of the composition map.

Note that if f and g are two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$, then the critical points of $f \circ g$ are the solutions of the equation $f'(g(x))\, g'(x) = 0$. This implies that C_g is a critical point of both g and $f \circ g$. The equation $f'(g(x)) = 0$ has a solution if and only if the equation $g(x) = C_f$ has a solution. Thus either $g^{-1}(C_f) = \emptyset$ or $g^{-1}(C_f)$ consists of two points one on the left side of C_g and the other on the right side of C_g. Let us represent these points by C_{fg}^- (resp. C_{fg}^+), the critical point of the composition map $f \circ g$ on the left (resp. on the right) side of C_g.

So if C_{fg}^- and C_{fg}^+ exists then the composition map $f \circ g$ has four intervals of monotonicity (otherwise $f \circ g$ has two intervals of monotonicity). Explicitly, $f \circ g$ is strictly increasing on $\left[0, C_{fg}^-\right] \bigcup \left[C_g, C_{fg}^+\right]$ and is strictly decreasing on $\left[C_{fg}^-, C_g\right] \bigcup \left[C_{fg}^+, b\right]$. The same analysis can be made for the map $g \circ f$.

Note that the threshold point of the composition map $f \circ g$ (resp. $g \circ f$) belongs always to the first interval where the composition map is increasing.

Recall from the previous subsections that K_f and K_g are the carrying capacities of f and g, respectively, and A_f^+ (resp. A_g^+) the right positive preimage of A_f (resp. A_g) under the map g (resp. f).

Theorem 58 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Then both $f \circ g$ and $g \circ f$, have carrying capacities, that we denote by K_{fg} and K_{gf}, respectively. Moreover $K_g < K_{fg} < A_f^+$ and $A_g^- < K_{gf} < A_g^+$.*

Proof. It is clear that $0 < A_f < A_g < K_g < K_f$, $A_f^+ > K_f$ and $A_g^- < A_g < K_g < A_g^+$.

We can see that $f \circ g\left(K_g\right) = f\left(K_g\right) > g\left(K_g\right) = K_g$ and $f \circ g\left(A_f^+\right) = f\left(A_f\right) = A_f < A_f^+$. Therefore the map $f \circ g(x) - x$ changes sign on (K_g, A_f^+). Hence there exists $x \in (K_g, A_f^+)$ such that $f \circ g(x) = x$.

Note that $C_{fg}^+ < A_f^+$. To see this fact suppose by contradiction that $C_{fg}^+ \geq A_f^+$ or equivalently $g^{-1}(C_f) \geq g^{-1}(A_f)$. We know that $C_{fg}^+, A_f^+ > C_g$ and g is decreasing on (C_g, b). Consequently, applying g in both sides of the last inequality we get $C_f \leq A_f$ that is impossible. Similarly we prove $C_{gf}^+ < A_g^+$.

Since $C_{fg}^+ < A_f^+$ the carrying capacity of $f \circ g$, K_{fg}, is the greater root of $f \circ g(x) = x$ on (K_g, A_f^+). We also can see that $g \circ f\left(A_g^+\right) = g\left(A_g\right) < A_g^+$ and $g \circ f\left(A_g^-\right) = A_g > A_g^-$. So the map $g \circ f(x) - x$ changes sign on (A_g^-, A_g^+) and therefore there exists $K_{gf} \in (A_g^-, A_g^+)$ such that $g \circ f\left(K_{gf}\right) = K_{gf}$ since $C_{gf}^+ < A_g^+$. ■

Remark 59 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. If $f \circ g\left(C_g\right) > C_g$ (resp. $g \circ f\left(C_f\right) > C_f$) then the map $f \circ g$ (resp. $g \circ f$) has exactly two positive fixed points, the threshold point and the carrying capacity.*

Corollary 60 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. If $C_g, C_f > K_f$ then $f \circ g(x) > g \circ f(x)$, $\forall x \in \left[K_g, K_f\right]$. Moreover, $K_g < K_{gf} < K_{fg} < K_f$.*

Proof. If $C_g, C_f > K_f$ we have that f and g are increasing on $\left[K_g, K_f\right]$. The composition of increasing maps is an increasing map. The interval $\left[K_g, K_f\right]$ is invariant under composition because $f \circ g\left(K_g\right) > K_g$, $f \circ g\left(K_f\right) < K_f$ and $g \circ f\left(K_f\right) < K_f$, $g \circ f\left(K_g\right) > K_g$. So the map $f \circ g(x) - x$ (resp. $g \circ f(x) - x$) changes sign on $\left[K_g, K_f\right]$. We know that $f\left(K_g\right) > K_g$ and therefore $g \circ f\left(K_g\right) < f\left(K_g\right) = f \circ g\left(K_g\right)$ ($g(x) < x$, $\forall x > K_g$). On other hand we know that $g\left(K_f\right) < K_f$ so $f \circ g\left(K_f\right) > g\left(K_f\right) = g \circ f\left(K_f\right)$ ($f(x) > x$, $\forall x \in \left]A_f, K_f\right[$). Consequently, $f \circ g(x) > g \circ f(x)$, $\forall x \in \left[K_g, K_f\right]$. Once that $f \circ g(C_g) < C_g$ (resp. $g \circ f(C_f) < C_f$) from Remark 59 it follows that K_{fg} (resp. K_{gf}) is the unique fixed point of $f \circ g$ (resp. $g \circ f$) on $\left[K_g, K_f\right]$. The order relation between K_{fg} and K_{gf} is an immediate consequence of the order relation between the composition maps. ■

It is possible, in certain cases, to establish an order relation between the two carrying capacities K_{fg} and K_{gf} of the composition maps $f \circ g$ and $g \circ f$. In particular we are interested in this order when the fixed points are between the carrying capacities of the individual maps. The next result provides this information.

Theorem 61 *Let f and g be two unimodal Allee maps such that $f(x) > g(x)$ for all x on $(0, b)$. Suppose that $C_f < K_f$, $C_g < K_g$, $f \circ g\left(K_f\right) < K_f$, $f \circ g\left(C^+_{fg}\right) < C^+_{fg}$, $g \circ f\left(K_g\right) > K_g$ and $g \circ f\left(C^+_{gf}\right) > C^+_{gf}$. Let $y \in J = \left[k_g, k_f\right]$ and suppose that $g(y) > K^-_f$, $\forall y \in J$, where K^-_f is the left preimage of K_f by the map f. Then $g \circ f(y) < f \circ g(y)$, $\forall y \in J$. Moreover, $K_g < K_{gf} < K_{fg} < K_f$.*

Proof. Let K^-_f be the left preimage of K_f by the map f, i.e., $f\left(K^-_f\right) = K_f$. Then $A_f < K^-_f < K_f$. Note that g is decreasing on $J = \left[k_g, k_f\right]$, $g(y) < y$, $\forall y \in J$ and $f(y) > y$, $\forall y \in J$.

From the hypothesis we have $g(y) > K^-_f$ and therefore $f \circ g(y) > K_f$, $\forall y \in J$. On the other hand $f(y) > K_f > y > K_g$, and then $g \circ f(y) < f(y) < K_g$, $\forall y \in J$. Consequently, $g \circ f(y) < f \circ g(y)$, $\forall y \in J$.

From the hypothesis and by Remark 59 the theorem is established. ■

3.1.4 Stability and bifurcation

Our aim in this subsection is to study, in the parameter space, the stability region and the bifurcation curves of a 2−periodic system formed by unimodal Allee maps. Before do that we make the following observations.

Let $\mathcal{F}$ be a set formed by unimodal Allee maps. Hence $x^* = 0$ is a fixed point of each individual map in $\mathcal{F}$ and so $\Phi_i(x^*) = x^*$, for all $i \geq 1$. Since this fixed point is asymptotically stable for each map we have $|\Phi'_i(0)| < 1$, for all $i \geq 1$ and thus from Remark 12 $x^* = 0$ is an asymptotically stable fixed point of $\mathcal{F}$.

Now suppose that $\mathcal{F}$ is a periodic set formed by unimodal Allee maps such that $f_i < f_{i+1}$, $\forall i \in \{0, 1, ..., p\}$. The threshold point A_{Φ_p} of Φ_p is unstable since the map $\Phi_p(x)$ is increasing on $[0, A_{\Phi_p}]$ and $\Phi_p(x) < x$, $\forall x \in (0, A_{\Phi_p})$.

Hence neither the zero fixed point nor the threshold point can contribute to bifurcations, since the former is always asymptotically stable and the latter is always unstable. Hence bifurcation may only occur at the carrying capacity of $\mathcal{F}$.

Now we are going to study the bifurcation of the system $\mathcal{W} = \{f_0, f_1\}$, where $f_i(x) = a_i x^2(1 - x)$, $i = 0, 1$, $x \in [0, 1]$ and $a_i > 0$.

For an individual map f_i, the dynamics is interesting but predictable. For $a_i < 4$ we have a globally asymptotically stable zero fixed point and no other fixed point. A new unstable fixed point born at $a_i = 4$ after which f_i becomes a unimodal map with an Allee effect.

Since 0 is the only fixed point under the system $\mathcal{W}$, we focus our attention on 2−periodic cycles $\{\overline{x}_0, \overline{x}_1\}$ with $f_0(\overline{x}_0) = \overline{x}_1$, and $f_1(\overline{x}_1) = \overline{x}_0$.

To determine the two main types of bifurcation, we solve the equations

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(f_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = 1 \end{cases} \tag{3.10}$$

and

$$\begin{cases} \overline{x}_0 = f_1\left(f_0\left(\overline{x}_0\right)\right) \\ f_1'\left(f_0\left(\overline{x}_0\right)\right) f_0'\left(\overline{x}_0\right) = -1 \end{cases} . \tag{3.11}$$

Eliminating[1] the variable $\overline{x}_0$ in both systems, Eq. (3.10) yields

$$16777216 + 16384a_0a_1 - 576000a_0^2a_1 + 84375a_0^3a_1 - 576000a_0a_1^2 + 914a_0^2a_1^2 - 350a_0^3a_1^2 + 84375a_0a_1^3 - 350a_0^2a_1^3 + 19827a_0^3a_1^3 - 2916a_0^4a_1^3 - 2916a_0^3a_1^4 + 432a_0^4a_1^4 = 0,$$

while Eq. 3.11 yields

$$10^8 - 120000a_0a_1 - 2998800a_0^2a_1 + 453789a_0^3a_1 - 2998800a_0a_1^2 - 4598a_0^2a_1^2 + 2702a_0^3a_1^2 + 453789a_0a_1^3 + 2702a_0^2a_1^3 + 89765a_0^3a_1^3 - 13500a_0^4a_1^3 - 13500a_0^3a_1^4 + 2000a_0^4a_1^4 = 0.$$

For each one of these two equations we invoke the implicit function theorem to plot, in the (a_0, a_1)−plane, the bifurcation curves (see Fig. 3.3).

The black curves are the solution of the former equation at which saddle-node bifurcation occurs, while the grey curves are the solution of the latter equation at which period-doubling bifurcations occurs. In the singular point of the black cusp we have a pitchfork bifurcation. For more details about this kind of bifurcation in 2−periodic one-dimensional (logistic) systems we cite a recent work done by E. D'Anniello and H. Oliveira [21].

In the regions identified by letters one can conclude the following.

- If $a_0, a_1 \in A$ then the fixed point $x^* = 0$ is globally asymptotically stable.
- If $a_0, a_1 \in B \backslash D$ then there are two 2−periodic cycles, one attracting and one unstable.
- If $a_0, a_1 \in D$ then there are an attracting 2−periodic cycle (from the pitchfork bifurcation) and two unstable 2−periodic cycles.

[1] We use the helpful of a computer.

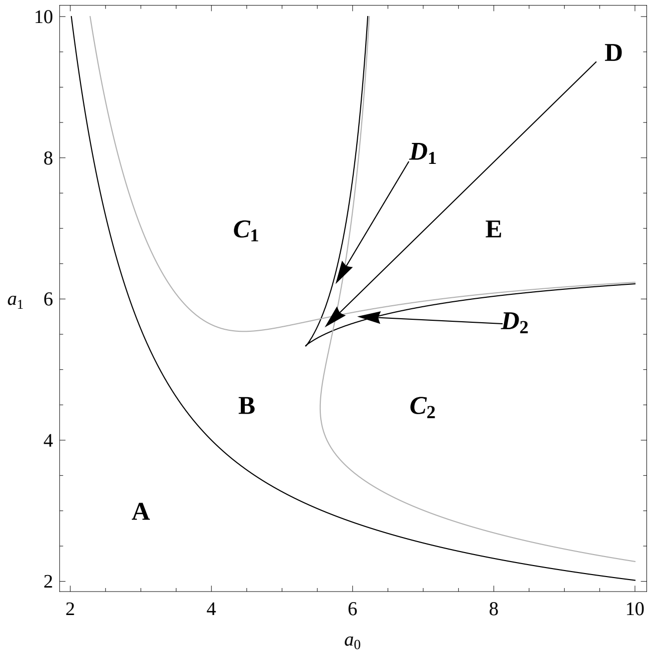

Figure 3.3: Bifurcations curves for the 2−periodic nonautonomous difference equation with Allee effects $x_{n+1} = a_n x_n^2(1 - x_n)$, $a_{n+2} = a_n$ and $x_{n+2} = x_n$ in the (a_0, a_1)−plane.

- If $a_0, a_1 \in (C_1 \cup C_2)\backslash(D_1 \cup D_2)$ then there are an attracting 4−periodic cycle (from the period doubling bifurcation) and two unstable 2−periodic cycles.

- If $a_0, a_1 \in D_1 \cup D_2$ then there are an attracting 2−periodic cycle (from pitchfork bifurcation) and two unstable 2−periodic cycles.

- If $a_0, a_1 \in E$ then there are an attracting 4−periodic cycle (from period doubling bifurcation), and three unstable 2−periodic cycles.

Note that the bifurcation curves for the system $\mathcal{W}$ in Fig. 3.3 are incomplete. If we want to draw more bifurcation curves in the space of the parameters we must do the same for 4−periodic cycles, 8−periodic cycles, and so on. Finding the implicit solutions of these new equations involve horrendous computations. The symbolic computations that we used does not produce answers after certain values of the degree of the polynomial. So, for the system $\mathcal{W}$, unfortunately we are unable to draw these curves for the 4−periodic cycle.

It should be noted that AlSharawi and Angelos [5] have used similar techniques to investigate the bifurcations of the periodically forced logistic map, and they were able to draw these curves for the 4−periodic cycles of the 2−periodic system. Moreover, these authors drew the bifurcation surfaces for the 3−periodic cycle of the 3−periodic system in the three dimensional space of the parameters.

Finally, we should mention that Grinfeld et al [34] studied the bifurcation in 2−periodic logistic equations.

3.2 Economics

3.2.1 Model formulation

Marx defined the *"organic composition of capital k"* as the ratio of what he called *constant capital to variable capital*. It's important to verify that constant capital is *not* what we today call fixed capital, but circulating capital, such as raw materials. Marx's *"variable capital"* is defined as advances to labor, that is, total wage payments, or heuristically, $v = wL$ (where w is wages and L is labor employed). From the definition we have

$$k = \frac{c}{v}, \tag{3.12}$$

where c is the constant capital and v is the variable capital.

The *profit rate*, according to Marx, is defined as

$$r = \frac{s}{v + c}, \tag{3.13}$$

where r represent the profit, s is the surplus, and $v + c$ are the total advances (constant and variable). The surplus, s, is the amount of total output produced above total advances, or $s = y - (v + c)$, where y is the total output. It is important to note that, for Marx, only labor produces surplus value. Marx called the ratio of surplus to variable capital the "exploitation rate e", that is

$$e = \frac{s}{v}, \tag{3.14}$$

(surplus produced for every dollar/euro spent on labor).

Dividing the numerator and the denominator of (3.13) by v we get the equation

$$r = \frac{e}{1 + k}. \tag{3.15}$$

To study the profit rate using the discrete dynamical system approach, let r_n be the profit rate at the time unit n. We assume that the exploitation rate and the organic composition of the capital at time $n + 1$ depend on the profit rate at time n, that is

$$e_{n+1} = E(r_n), k_{n+1} = K(r_n). \tag{3.16}$$

In this case the profit rate depends also on the profit rate of the previous time unit, that is

$$r_{n+1} = \frac{E(r_n)}{1 + K(r_n)}. \tag{3.17}$$

The specific model that we propose here is based on the following assumptions:

1. We first assume that the economy will not suffer any losses, that is, a negative profit rate doesn't occur. In fact, we think that in all of the economic activity there exists a nonnegative balance (possibly zero) of the profit. On the other hand, the profit rate can't be unlimited because that contradicts the fact that the total quantity of "money that exists" in the planet is finite.

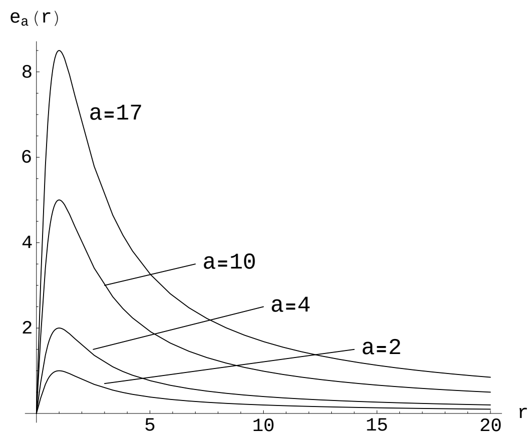

Figure 3.4: Progress of the exploitation rate.

2. When the profit rate is low, the exploitation rate tends to increase. On the other hand, when the profit rate is high, the pressure on workers tends to decrease and therefore the exploitation rate tends to decrease to values near zero. This leads to the equation

$$E_a(r_n) = \frac{ar_n}{1 + r_n^2} \tag{3.18}$$

as a model for the exploitation rate, where $a > 0$ is constant. Note that the function E_a has the following properties

$$E_a(0) = E_a(\infty) = 0, E_a'(0) = a. \tag{3.19}$$

Hence $a > 0$ measures the relative growth of the exploitation rate when the profits are low.

3. When the profit is low we have a relative amount for investment that decreases as long as the profit increases. If the economic system has a low profit rate, the tendency will be to incorporate more capital (invest) and, on the other hand, to decrease the human capital, through dismissals, which will lead to an increase of the organic composition of the capital. If the profit rate is very high, the tendency will be to reinvest in the capital: human capital (technical formation) and constant capital (technological innovation). A model that describes this reality is given by an incorporation of the capital function depending on the former profit rate, that is,

$$K_{b,d}(r_n) = \frac{e^{r_n - b}}{(r_n + d)^2}, \tag{3.20}$$

where $b, d > 0$ are constant. Note that the function $K_{b,d}$ has the following properties

$$K_{b,d}(0) = \frac{1}{d^2 e^b} > 0, K_{b,d}(\infty) = \infty, K_{b,d}'(0) = \frac{d-2}{d^3 e^b}. \tag{3.21}$$

This means that $\frac{1}{d^2 e^b}$ gives the initial amount, the organic composition of the capital as a function of the profit, grows unlimitedly, which isn't realistic, but it is used to make the model approachable and $\frac{d-2}{d^3 e^b}$ measures the growth for the organic composition of the capital when the profits are low.

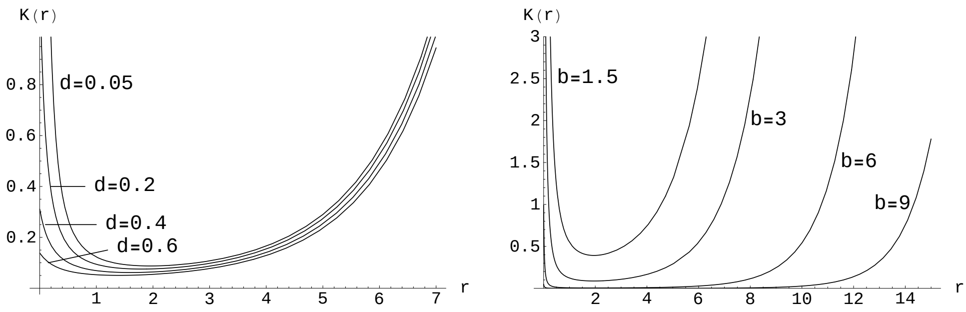

Figure 3.5: The progression of the organic composition of the capital in the model (3.20). In the left plot we fix the parameter b and increase d, while in the right plot we fix d and increase b. In both cases, in general, the behaviour of the organic composition of the capital is similar. As according b or d increases, for low profits the tendency is decreases the organic composition of the capital, for high profits, the tendency is increases the organic composition of the capital.

Using equations (3.18) and (3.20), the model (3.17) for the profit rate becomes

$$r_{n+1} = \frac{ar_n (r_n + d)^2}{(1 + r_n^2)\left[(r_n + d)^2 + e^{r_n - b}\right]}. \tag{3.22}$$

Now we are going to study the effect of varying the parameters b and d on K in equation (3.20).

When we fix the parameter b and increase d, we observe that the qualitative behavior of the organic composition of the capital is similar for various values of d. However, the variation of d leads to a decrease of the organic composition of the capital, when the profit rate is low. This means that the main impact of the parameter d is on the initial value of the organic composition of the capital, that is, the initial investment (see the left plot in Fig. 3.5).

On the other hand, when we fix the parameter d and increase b we observe that the graphs of the corresponding organic composition of the capital in model (3.20) are similar. Moreover, as the parameter b varies, the organic composition of the capital decreases for low values of the profits, until it stabilizes during a specific profits period after which it increases as profits increase. Note that, when the profits are low, this variation of b, leads to a decrease of the initial amount of the organic composition of the capital (see the right plot in Fig. 3.5).

Hence, for a concrete study we need to choose specific values of b and d. It should be noted that the essence of our results will not change if one chooses other values of b and d. Accordingly, we select $b = 3$ and $d = 0.01$. Note that, the choice of these two values is conditioned by the initial amount of the organic composition of the capital.

We rewrite the model (3.22) as

$$r_{n+1} = \frac{ar_n \left(r_n + \frac{1}{100}\right)^2}{(1 + r_n^2)\left[\left(r_n + \frac{1}{100}\right)^2 + e^{r_n - 3}\right]} = f_a (r_n). \tag{3.23}$$

We observe that the function $f_a (r)$ is continuous, for all $a > 0$ and $r \in [0, +\infty)$. Moreover, $f_a (r) \geq 0$, $f_a (0) = f_a (\infty) = 0$. The function $f_a' (r)$ is also continuous and has only one root for all $r \geq 0$. Therefore $f_a (r)$ attains its maximum at the positive critical point $c \approx 1.10042$ and thus has an upper limit. Furthermore, $f_a' (0) = pa$, where $p \approx 0.00200453$. Thus, the origin is a locally asymptotically stable fixed point if $a < \frac{1}{p}$ and unstable if $a > \frac{1}{p}$. When $a = \frac{1}{p}$ we have $f_a'' (0) \approx 397.202$ and consequently the origin is unstable .

3.2.2 Existence of positive fixed points and their stability

In this subsection we investigate the existence of positive fixed points of the model (3.23) and their stability, if they exist.

Theorem 62 *For the model* (3.23)*, there are two positive fixed points if* $a > a_c$ *where* $a_c \approx 1.64271$*, a unique positive fixed point if* $a = a_c$ *and no positive fixed points if* $a < a_c$*.*

Proof. To find the positive fixed point of (3.23) we consider the equation

$$\frac{a\left(r+\frac{1}{100}\right)^2}{(1+r^2)\left[\left(r+\frac{1}{100}\right)^2+e^{r-3}\right]}=1, \tag{3.24}$$

or, equivalently,

$$\ln\left(1+r^2\right)+\ln\left(\left(r+\frac{1}{100}\right)^2+e^{r-3}\right)-2\ln\left(r+\frac{1}{100}\right)=\ln a. \tag{3.25}$$

Let $g(r)=\ln\left(1+r^2\right)+\ln\left(\left(r+\frac{1}{100}\right)^2+e^{r-3}\right)-2\ln\left(r+\frac{1}{100}\right)$. We have

$$g'(r)=\frac{2r}{1+r^2}-\frac{2}{r+\frac{1}{100}}+\frac{2\left(r+\frac{1}{100}\right)+e^{r-3}}{\left(r+\frac{1}{100}\right)^2+e^{r-3}}. \tag{3.26}$$

Solving $g'(r)=0$, we have a unique positive critical point $r_c \approx 0.478625$ of $g(r)$.

Observe that $g(r) > 0$ for all $r \geq 0$. Moreover, for $a \approx 1.64271$, we have the following results:

1. when $g(r_c) = \ln a$, Eq. (3.25) has a unique positive solution;
2. if $g(r_c) < \ln a$, there exists two positive solution of (3.25);
3. if $g(r_c) > \ln a$, Eq. (3.25) has no positive solutions.

■

Fig. 3.6 exhibits these observations.

Remark 63 *If* r_1 *and* r_2 *are two positive fixed points of* (3.23) *such that* $r_1 < r_2$*, then* $r_1 < r_c < r_2$ *and* $g'(r_1) < g'(r_c) < g'(r_2)$*, with* $g'(r_c) = 0$*. When* $a > a_c$ *this kind of maps exhibits the Allee effect.*

To investigate the stability of the positive fixed points for the model (3.23), if they exist, we observe that

$$\begin{aligned} f_a'(r) \quad = \quad & \frac{a\left(r+\frac{1}{100}\right)}{(1+r^2)\left(e^{r-3}+\left(\frac{1}{100}+r\right)^2\right)} \\ & \left[-r\left(r+\frac{1}{100}\right)\left[\frac{e^{r-3}+2\left(\frac{1}{100}+r\right)}{\left(e^{r-3}+\left(\frac{1}{100}+r\right)^2\right)}+\frac{2r}{1+r^2}\right]+3r+\frac{1}{100}\right]. \end{aligned}$$

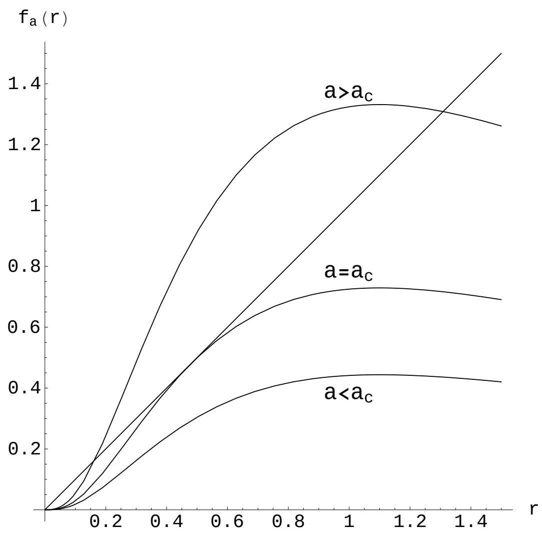

Figure 3.6: Zero, one and two positive fixed points for the profit rate when a is respectively, less than a_c, equal a_c and greater than a_c, where $a_c \approx 1.64271$.

From (3.24) we know that, if r is a positive fixed point, then it satisfies

$$\frac{a\left(r+\frac{1}{100}\right)^2}{1+r^2} = \left(r+\frac{1}{100}\right)^2 + e^{r-3}. \tag{3.27}$$

So it follows that

$$f_a'(r) = 1 - r\left[\frac{2r}{1+r^2} - \frac{2}{r+\frac{1}{100}} + \frac{1+r^2}{a}\left(\frac{2}{r+\frac{1}{100}} - 1\right) + 1\right] = 1 - rg'(r). \tag{3.28}$$

Therefore r is locally asymptotic stable if

$$0 < rg'(r) < 2 \tag{3.29}$$

and is unstable if

$$rg'(r) < 0 \text{ or } rg'(r) > 2. \tag{3.30}$$

To study the stability of each positive fixed point, if they exist, we consider the following two cases:

1. Suppose that the model (3.23) has only one positive fixed point $r = r_c$. Then from (3.28) we have $f_a'(r_c) = 1$. Since $f_a''(r_c) \approx -2.36074 \neq 0$ it follows by Theorem 17 that $r = r_c$ is unstable. More precisely, $r = r_c$ is semi-stable from the right side, if the initial value $r_0 \in \,]r_c, r_3[$, where $f_a(r_3) = r_1$ and $r_3 > c$ where c is the critical point of f_a determined in section 3.2.1 and $r = r_c$ is unstable if the initial value $r_0 \in \,]0, r_c[\bigcup]r_3, \infty[$ (see Figure 3.7).

2. Let us now assume that there are two positive fixed points for the profit rate r_1 and r_2 such that $r_1 < r_2$. It follows from remark 63 that $r_1 g'(r_1) < 0$ and therefore r_1 is always unstable (note that $r_1 \in \,]0, r_c[$). For r_2 we know that $g'(r_2) > 0$, and therefore $r_2 g'(r_2) > 0$. We have two situations:

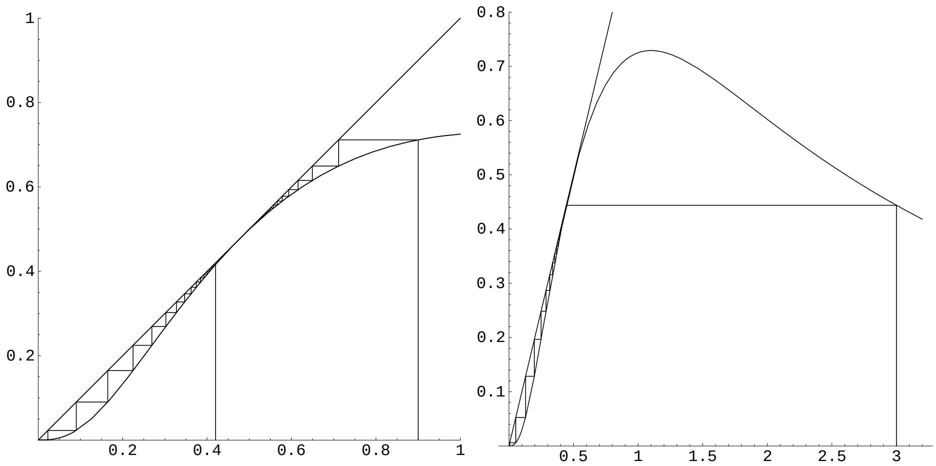

Figure 3.7: Stability of the unique positive fixed point. In the first case $r = r_c$ is semi-stable on the right side and unstable on the left side ($r_c \approx 0.478625$). In the second case $r = r_c$ is always unstable.

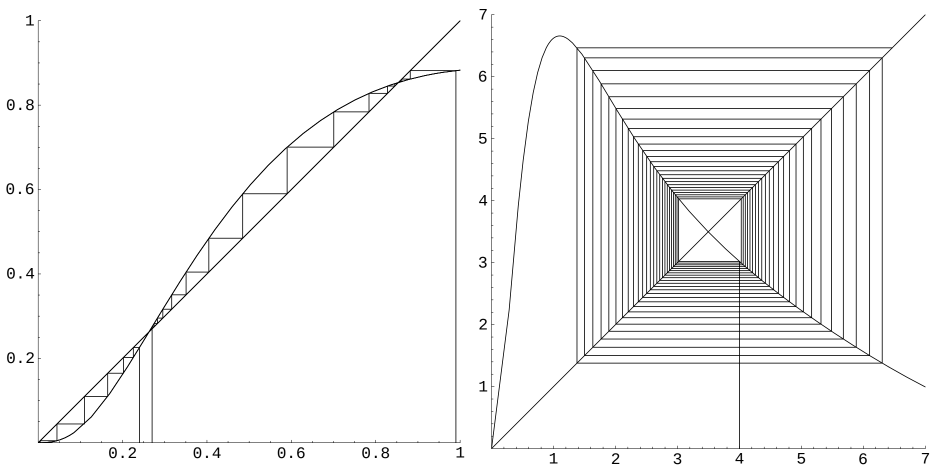

Figure 3.8: Stability of the two positive fixed points. In the first case the 1st positive fixed point is unstable and the 2nd positive fixed point is locally asymptotic stable. In the second case the 2nd positive fixed point is unstable (the 1st positive fixed point is not visible because it is small, but it exists).

- $0 < r_2 g'(r_2) < 2$ if and only if $r_2 \in]r_c, r_s[$, where $r_s \approx 3.3976$ and consequently r_2 is locally asymptotic stable,
- $r_2 g'(r_2) > 2$ if and only if $r_2 \in]r_s, +\infty[$ and therefore r_2 is unstable.

In Fig. 3.8 we illustrate these ideas.

The following theorem summarizes the above discussion.

Theorem 64 *Consider r_c and a_c in the same conditions of the proof of Theorem 62. For $a = a_c$ there exists a unique positive unstable fixed point $r = r_c$. For $a > a_c$ there exists two positive fixed points r_1 and r_2 such that $r_1 < r_2$, where r_1 is always unstable and r_2 is locally asymptotic stable if $r_2 \in]r_c, r_s[$, and is unstable if $r_2 \in]r_s, +\infty[$, where $r_s \approx 3.3976$.*

3.2.3 Chaos in the profit rate

In this subsection we use the notions of Liapunov exponents and topological entropy to establish the existence of chaos in the profit rate for certain values of the parameter a.

When we increase the parameter a, the model (3.23) exhibits a typical period-doubling bifurcation.

For values $0 < a < 12.5$, the model (3.23) has a stable positive equilibrium point. The profit rate tends to approach the bigger positive equilibrium point. When $12.5 < a < 17.09957$, we have a two-cycle. As the parameter a increases we have more cycles of the profit rate with period-doubling scenario 4, 8, 16, ... In particular, for $a = 21.94$, we have a stable period-five orbit. The stable period-three window happens for values $23.082565 < a < 23.217854$.(This means that we have a triennial repetition of the profit rate.) This depicts the behavior of a typical unimodal map satisfying the Sharkovsky ordering [70].

A Liapunov exponent is a mathematical indicator of the exponential degree of the velocity according to which two arbitrary nearby orbits grow further apart as the number of iterations increases. We can define it as follows.

Definition 65 *The Liapunov exponent $\lambda(r_0)$ for a point r_0 is given by*

$$\lambda(r_0) = \lim_{n\to+\infty} sup \frac{1}{n} \sum_{k=0}^{n-1} \ln | f_a^{'}(r_k) |, \tag{3.31}$$

where $r_k = f_a^k(r_0)$.

In practice, to calculate experimentally the value of the Liapunov exponent we can use the formula

$$\lambda(r_0) = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln | f_a^{'}(r_k) |, \tag{3.32}$$

that is equivalent to

$$\lambda(r_0) = \ln \lim_{n\to+\infty} \sqrt[n]{| (f_a^n(r_0))' |}. \tag{3.33}$$

If the absolute value of $f_a^{'}(r_k)$ is greater than one, then the Liapunov exponent is positive, which implies that the system possesses sensitive dependence on initial conditions [26, page 130].

In Fig. 3.9 we can see the progression of the Liapunov exponent of the function f_a when varying a. We observe that if a is less than ≈ 18.245 the system has no sensitive dependence on initial profit rates. However, when a exceeds ≈ 18.245, the system starts to posses sensitive dependence on initial profit rates and we enter in the chaotic region.

Another mathematical invariant which detect the existence of chaos is topological entropy. To introduce this concept we will use kneading theory [54]. For this, which is a typical procedure, we define the growth number s for unimodal maps.

Definition 66 *The lap number, $l(f_a^n)$, is the number of maximal intervals of monotonicity of f_a^n (f_a^n is piecewise-monotone). The growth number s is defined as*

$$s(f_a) = \lim_{n\to+\infty} (l(f_a^n))^{\frac{1}{n}}. \tag{3.34}$$

When the growth of the laps is small (polynomial with the number of iterates) we do not have chaos, but when the growth of the lap number is exponential we have chaos. This happens when the growth number is greater than 1. When the growth number is equal to 1, we can have (or not) chaos, but this invariant does not give us a precise answer.

To calculate the topological entropy, we define the parity function for the turning point.

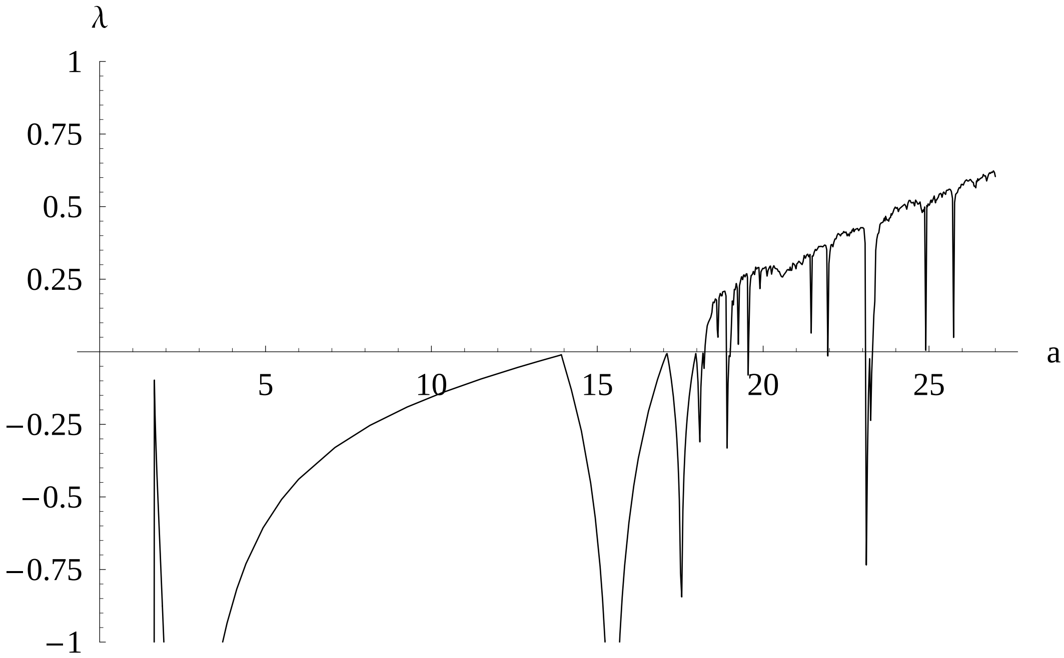

Figure 3.9: The Liapunov exponent λ of the function f_a under the parameter a applied to the profit rate.

Definition 67 *The parity function for the turning point c is given by*

$$\varepsilon(x) = \begin{cases} 1 \text{ if } x < c \\ 0 \text{ if } x = c \\ -1 \text{ if } x > c \end{cases}. \quad (3.35)$$

We now give a few classical results in this issue.

Theorem 68 *[54]*

1. *The kneading determinant is a formal power series in t given by*

$$Z(t,a) = 1 + \sum_{n=1}^{\infty} \left(\prod_{j=1}^{n} \varepsilon\left(f_a^n(c)\right) t^n \right). \quad (3.36)$$

2. *In the case of periodic orbits of c, $Z(t,a)$ is a polynomial of degree $(n-1)$. The inverse of the smallest root of $Z(t,a)$ in $[0,1]$ is the growth number of f_a, usually denoted by s.*

3. *The topological entropy h_t is given by the relation $h_t = log_2(s)$.*

Our function has two intervals of monotonicity, that is, f_a increases in $]0,c[$ and it decreases in $]c,+\infty[$. When $a = 25$ we have that, the first terms of the kneading determinant are

$$1 - t - t^2 - t^3 + t^4 - t^5 + t^6 - t^7 + t^8 - t^9 + t^{10} - t^{11} + \ldots.$$

The smallest real root of this determinant belonging in the interval $[0,1]$ is approximately equal to 0.562781. The topological entropy therefore is given by

$$h_t = log_2\left(0.562781^{-1}\right)0.83064.$$

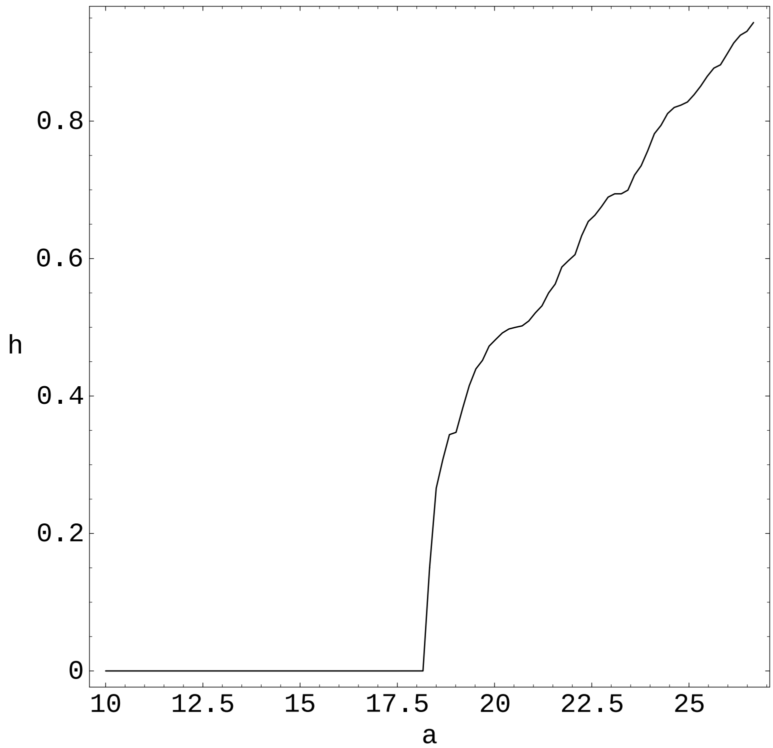

Figure 3.10: Progress of the topological entropy h_t for the model (3.23). According the growth of the parameter a the topological entropy h_t increases for values near 1.

In Fig. 3.10 we can see the evolution of the topological entropy values for the profit rate.

The fact that the topological entropy increases and is greater than zero (in addition to the fact that the Liapunov exponent is positive) means that the dynamical system becomes more complex as the parameter a increases. For values of $a > 18.245$, the model exhibits chaos, which may be seen in the bifurcation diagram. One can see clearly in this diagram an aperiodic band. From the economic point of view this situation would also result in a huge complexity and instability of the system. This happens when the exploitation rate is very high and the profit rate is very low.

In this model, trying to compensate for low profits at the expense of high exploitation rate leads inevitably to instability and chaos, both from the mathematical point of view and from the common sense point of view (economic chaos). For values of $a > 18.245$ our model can be unrealistic in the short term, because in those circumstances, after a high profit, we will have a low profit. This variation of the profit rate can be explained by some extraordinary factors that happen in an economy as a whole, like wars, natural catastrophes, pressures of various agents, etc. If it were possible to introduce an exploitation rate so high that we would be led to this situation, that would mean that the system would react by presenting a reasonable profit rate from five to five or three to three units of time (values of a close to 21.94, in the first case, and 23.1 in the second case).

Hence, in order to maintain the stability of the system, the exploitation rate must be kept below a certain threshold value for which $a = 12.5$.

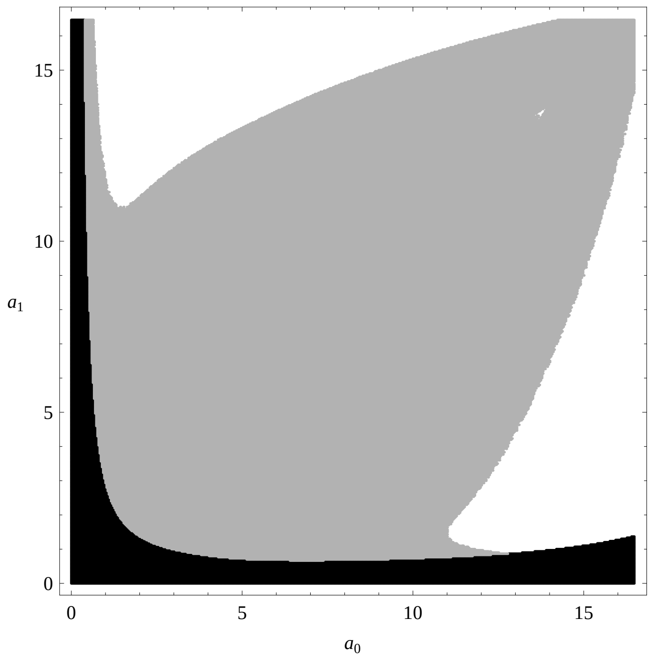

Figure 3.11: The stability region of the 2-periodic cycle for the nonautonomous 2-periodic system in the (a_0, a_1)-plane.

3.2.4 Periodic fluctuation in seasons

So far, we have limited our analysis to the autonomous case in which the parameter a is assumed to be independent of time. This means we have assumed that the exploitation rate E, when the profit rate is close to zero, is fixed for all seasons.

It is, however, possible to assume that a varies with varying seasons. In this scenario, a depends on seasons. This will lead to assuming that $a = a_n$, $n = 0, 1, 2, \ldots$, and a_n is periodic of minimal period p. Under this assumption we have the nonautonomous periodic equation

$$x_{n+1} = f_n(x_n), n \in \mathbb{Z}^+, \tag{3.37}$$

where $f_{n+p} = f_n$, $\forall n \in \mathbb{Z}^+$.

In this subsection we focus our attention on the case when $p = 2$, that is, we have only two maps $\{f_0, f_1\}$.

Let us consider an economic system modeled by Eq. (3.23). We assume that the unit of time between iterations is six months. For instance, the initial condition x_0 will be taken in the 1st of January, x_1 will be taken in the 1st of July and so on. Model (3.23) may be written as

$$x_{n+1} = ag(x_n).$$

Now, suppose that due to political decisions, i.e., periodic variation of taxes, or the intrinsic nature of the economic model, the behavior of the economy is different in the first half of the year from the second half of the year. In this case we will have a periodic model with period 2 where the model changes every six months. This leads to the nonautonomous 2-periodic system

$$x_{n+1} = f_n(x_n), \quad \text{with } f_0 = a_0 g, f_1 = a_1 g, \tag{3.38}$$

where $f_{2n+1} = f_1$ and $f_{2n} = f_0$, $n \in \mathbb{Z}^+$ and a_0 will be used for the first half of the year and a_1 for the second half. Since $a_0 = a_1$ reduces Eq. (3.38) to the autonomous case, we assume that $a_0 \neq a_1$. Observe that $a_0 < a_1$ and $a_0 > a_1$ exhibit the same dynamics.

The 2-periodic solution of the difference equation (3.23) is given by

$$x_0 = f_1 (f_0 (x_0)).$$

This equation is equivalent to

$$x_0 = a_1 g (a_0 g (x_0)). \tag{3.39}$$

Rewriting Eq. (3.39) we have

$$G (a_0, a_1, x_0) = a_1 g (a_0 g (x_0)) - x_0 = 0. \tag{3.40}$$

We find numerically the solutions of Eq. (3.40). The existence of these solutions is guaranteed by the Implicit Function Theorem. In Fig. 3.11 we represent in the (a_0, a_1)-plane the region of stability of the 2−periodic cycle of the nonautonomous system (3.38). The black region represents the zone of the atractivity of the zero fixed point while the grey region is the zone in the parameter space where the 2−periodic cycle is asymptotically stable.

Hence, if a_0 and a_1 are in the black region then model (3.40) exhibits null profit. If however, these parameters are in the grey region, model (3.40) has stable double variation in the profit.

One of the most interesting problems is to determine whether periodic fluctuations in a dynamical system produces boom or bust in the profit rate. In other words, we would like to know the impact of introducing artificial or natural oscillations in the model and whether this periodical forcing has a deleterious or a booming effect on the system.

We are now going to give precise definitions for boom and bust in the profit rate. Let

$$C_p = \left\{\overline{x}_0, \overline{x}_1, ..., \overline{x}_{p-1}\right\}$$

be a p−periodic cycle of Eq. (3.37) and let $\mathcal{K}_p = \left\{K_0, K_1, ..., K_{p-1}\right\}$ be the carrying capacities (rightmost fixed points) of the individual maps f_n, $n = 0, 1, 2, \ldots, p - 1$, respectively. Let

$$av\left(C_p\right) = \frac{1}{p}\sum_{i=0}^{p-1} \overline{x}_i \text{ and } av\left(\mathcal{K}_p\right) = \frac{1}{p}\sum_{i=0}^{p-1} K_i,$$

where $av\,(.)$ denotes the average function.

Definition 69 *Eq. (3.37) is said to be*

1. *bust if* $av\left(C_p\right) < av\left(\mathcal{K}_p\right)$*;*
2. *boom if* $av\left(C_p\right) > av\left(\mathcal{K}_p\right)$*;*
3. *indolent if* $av\left(C_p\right) = av\left(\mathcal{K}_p\right)$*.*

Notice that if the system is bust, then periodic forcing of a system has a deleterious effect on the profit rate, while if the system is boom, periodic forcing leads to a booming effect on the profit rate.

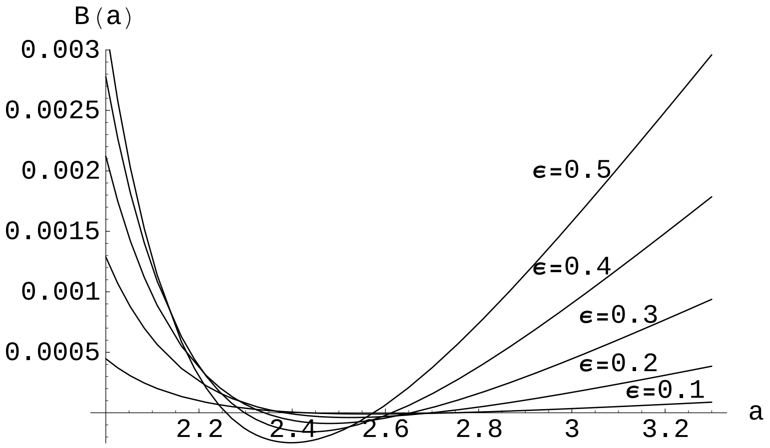

Figure 3.12: The booming function as a function of the parameter a for different values of the parameter ε.

In our model we must compare the average of the periodic orbit and the average of the carrying capacities, that is,

$$av(C_2) = \frac{\overline{x}_0 + \overline{x}_1}{2} \text{ and } av(\mathcal{K}_2) = \frac{K_0 + K_1}{2}.$$

Since the explicit computation of these averages are prohibitively difficult, we will estimate them numerically.

Using Mathematica software with working precision 10^{-15}, we computed the fixed points of $f_0 = ag$, and $f_1 = (a+\varepsilon)g$ and the fixed points of the two compositions $f_0 \circ f_1$ and $f_1 \circ f_0$. Surprisingly, both boom and bust occur for values of a, $a \in [2.0, 3.3]$ and with $b = a+\varepsilon$. In Fig. 3.12, with different values of ε, we see the graph of the booming function $B(a)$, as a function of the parameter a

$$B(a) = av(C_2)(a) - av(\mathcal{K}_2)(a) = \frac{\overline{x}_0(a) + \overline{x}_1(a)}{2} - \frac{K_0(a) + K_1(a)}{2},$$

The function $B(a)$ is positive when we have boom and negative when we have bust. In our system both cases occur. The zeros of the booming function are the indolent points.

The presence of both bust and boom in our model has been rarely observed in the literature. It is one of the main characteristics of our model.

3.2.5 Kondratiev waves in the profit rate

If the time unit is long (a decade, a quarter of century, etc), model (3.23) exhibits economic cycles that are similar to the Kondratiev waves [43].

According to George Modelski [55], "Kondratiev waves may be defined as a pattern of regularity characteristic of structural change in the modern world economy in which a cycle is about 50 years long. It consists of an alternation of periods of high growth with others, start-up periods of slower growth. The study of this pattern helps to trace the evolution of the global economy, and aids in politic and economic prediction".

A Kondratiev wave consists of four distinct phases, dramatic mood changes, which one determines the actions of individuals involved in the economy. The awareness of these characteristics allows for the anticipation of the change in the economy and the psychological mood

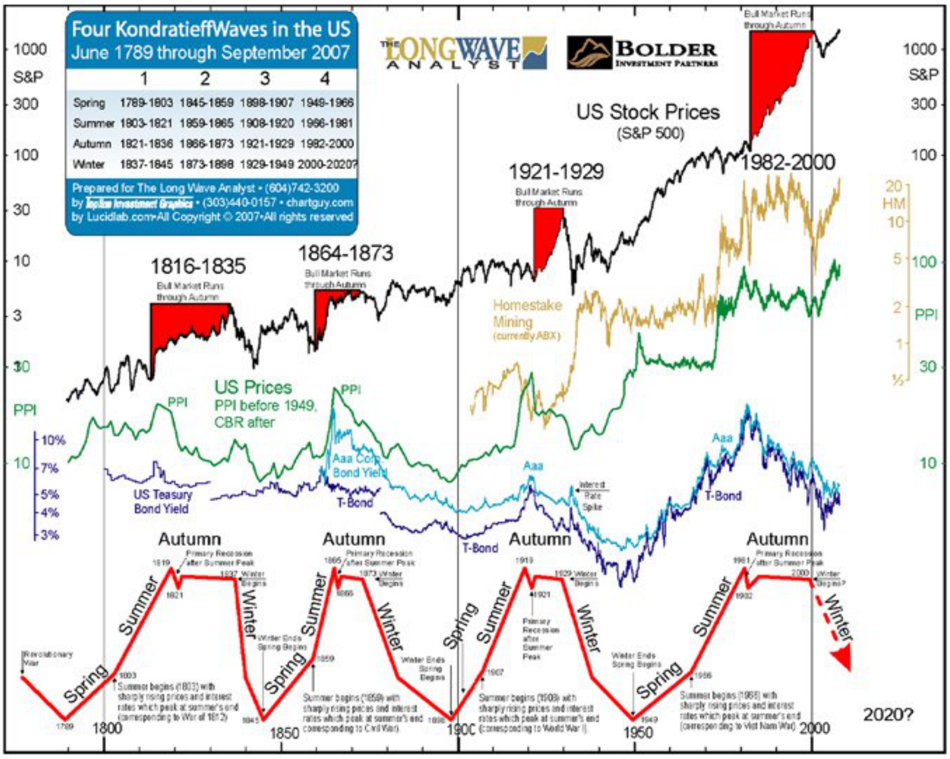

Figure 3.13: Illustration taken from "http://www.thelongwaveanalyst.ca" that shows the Kondratieff Wave periods along with US stocks market, US prices and T-Bond rates.

that will prevail. Some analysts compared these four distinct phases in the Kondratiev waves to the seasons: spring (inflationary growth, expansion), summer (stagflation, recession), autumn (deflationary growth, plateau) and winter (depression), that changes each fifty years. In Fig. 3.13 is presented a schema of the four Kondratieff waves in the US.

For example, when $a = 1.35$ model (3.23) exhibits the same characteristics as those of the Kondratiev waves (Fig. 3.14). However the Autumn's situation does not permit us to infer any conclusions, because we have first a decline in the profit rate and after that an accelerated growth until the beginning of the winter. This situation is not similar to the Kondratiev waves because there, after the first decline, is a moderate growth rate and then a slow decline until the winter. We believe that this situation in our model is caused by the fact that when the exploitation rate grows, it has a strong influence on the profit rate. That leads to an abrupt variation in the autumn. But this last scenario, in a half of century, represents at least, two or three years. Therefore the system (3.23) will adjust itself in the long term.

3.2.6 Discussion

The problem of the decrease of the profit rate with time has been the subject of numerous discussions in the last 150 years.

In this study, we use the equation that arises from economics theory, to develop a dynamical system model describing the evolution of the profit rate. This model is a one-dimensional unimodal discrete system which provides a relationship between the profit rate at two consecutive

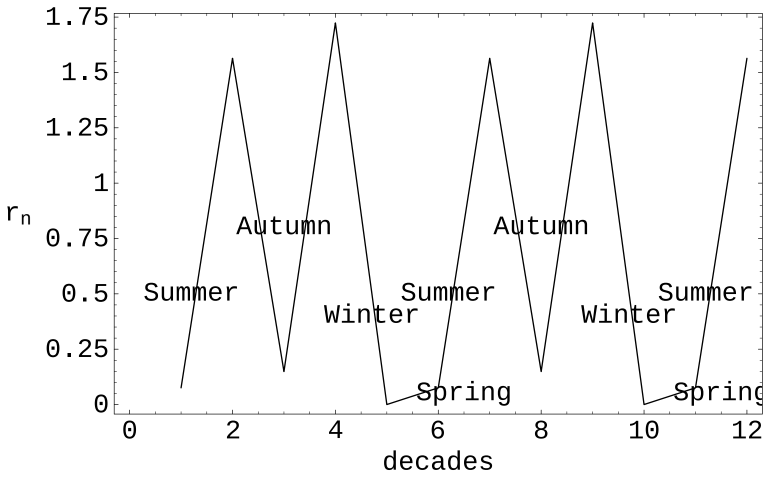

Figure 3.14: For $a = 1.35$ model (3.23) can be interpreted as the Kondratiev waves. The progression of the profit rate can be viewed in four seasons that repeat each five decades.

cycles.

When the system does not react desperately to low profits in the previous cycle, increasing the exploitation rate in these circumstances forces the system to approach an equilibrium profit value. This equilibrium value doesn't vary much, with the variation of the exploitation rate as a response to a null profit. Such conclusion contradicts Marx hypothesis of the decrease of the profit rate in time.

When the exploitation rate at null profit exceeds a critical value (in our case $a > 12.5$), we have a bifurcation, and the system starts to exhibit a cyclic oscillation between two profit rate values. As the parameter a increases, we obtain a period-doubling bifurcation cascade. For certain values of a, the Liapunov exponents, computed numerically (Eq. (3.32)), become positive, indicating the presence of chaos and unstable orbits. When the exploitation rate with at null profit rate, is too high, the system starts to exhibit again stable orbits of odd periods (periods three and five, etc.).

In this economic model we saw that boom in the profit rate is more predominant than bust in the profit rate. From the economic point of view, this means that the periodic fluctuation of the exploitation rate when the profit are low has, in general, a booming effect on the profit rate. Hence, it is generally beneficial for an enterprise micro or macro to vary the exploitation rate.

Finally, if the scale is a decade, it is very interesting to observe that, our unimodal model permit to adjust the long term cycles in economy.

Chapter 4

Future work

In writing this thesis, we recognized that these are still many open problems and conjectures that need to be tackled. We list them here, not in chronological order, as part of this thesis to formalize our future research plans.

- We would like to extend some topological notions such as topological entropies and Lypunov exponents to the setting of skew-product discrete dynamical systems. The objective here is to use this scheme to investigate nonautonomous systems, particularly periodic difference equations.
- The question of when local stability implies global stability has been addressed successfully in one dimension (see P. Cull [12, 13, 14, 15] and E. Liz [47]). It would be interesting to extend these results to higher-dimensional maps.
- We would like to extend the theory of bifurcation to periodic discrete systems (p-periodic difference equations). In particular, we will apply these results to study the bifurcation of p-periodic competition models.
- For larger values of the parameters, the autonomous logistic competition model and the autonomous Ricker competition model exhibit what looks like strange attractors. We would like to delve deeper into this phenomenon in future work (see Figures 4.1 and 4.2).
- The question of attenuance and resonance in higher dimensional nonautonomous systems is still unresolved and only partial results have been obtained. In our future works, we would like to revisit this question in two-dimensional competition models.
- Systems with Allee effects have been investigated in one-dimensional maps. We plan to investigate the stability and bifurcation of higher dimensional systems with Allee effects.

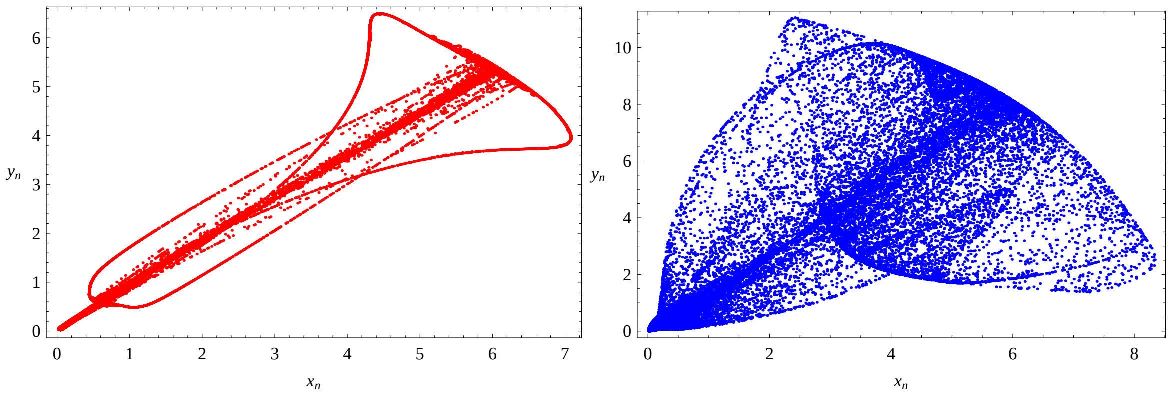

Figure 4.1: The presence of strange attractors in the autonomous Ricker competition model when $a = b = 0.5$. In plot on the left $K = 3.2$ and $L = 3.1$ while in plot on the right $K = 3.2$ and $L = 3.5$.

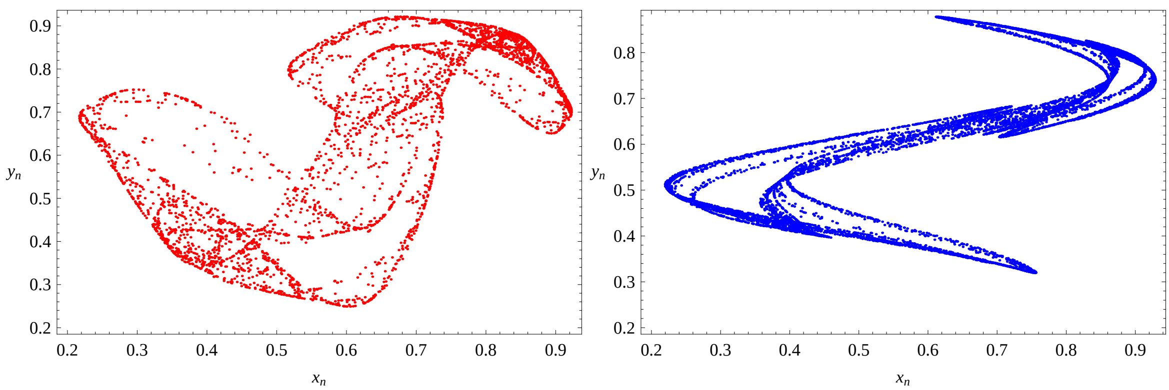

Figure 4.2: The presence of strange attractors in the autonomous logistic competition model when $a = 4.3$ and $b = 3.9$. In plot on the left $c = 0.6$ and $d = 0.2$ while in plot on the right $c = 0.4$ and $d = 0.5$.

Appendix A

Local stability implies global stability of the one-dimensional autonomous Ricker map

One of the biological models widely used (see for example [53, 56, 63]) is the autonomous Ricker map given by

$$x_{n+1} = x_n e^{p-x_n}, p > 0, x_n > 0, n \in \mathbb{Z}^+. \tag{A.1}$$

It is well known that $0 < p \leq 2$ is the necessary condition for local stability of the unique positive fixed point $x^* = p$.

Let us write Eq. (A.1) as $x_{n+1} = R_p(x_n), n \in \mathbb{Z}^+$, where $R_p(x) = xe^{p-x}$. First we assume that $0 < p < 2$. Notice that $R_p'(x) = (1-x)e^{p-x}$. Hence the maximum value of $R_p(x)$ occurs at $x_c = 1$ and is equal to e^{p-1}. Moreover, $R_p'(x) = 1$ has the unique solution $x_w = 1 - Lw(e^{1-p})$, where $Lw(x)$ is the Lambert-W function. Since $e^{1-p} \leq e$, $0 < Lw(e^{1-p}) < Lw(e) = 1$. This implies that $0 < x_w < 1$.

The function $R_p'(x)$ is decreasing on $(0, 2)$ and increasing on $(2, \infty)$. Furthermore, $|R_p'(2)| = |-e^{p-2}| < 1$ for $0 < p < 2$. Moreover, $\lim_{x\to\infty} R_p'(x) = 0$. Hence, $|R_p'(x)| < 1$ for all $x \in (x_w, \infty)$.

Since the fixed point $x^* = p$ is locally stable, there is an open interval $(p - \delta, p + \delta)$ in which $|R_p'(x)| < 1$. It follows that $x_w < p$.

Claim that the interval $J = [x_w, \tilde{x}_w]$ is an invariant set, where $\tilde{x}_w = \max\{R_p^{-1}(x_w)\}$. To show this we first notice that $R_p(x_w) < R_p(1) = e^{p-1}$. Next we will show that $\tilde{x}_w > R_p(1)$. For $0 < p \leq 1$, it is clear that $\tilde{x}_w > 1 > e^{p-1}$. Consider now that $1 < p < 2$. Let $h(p) = R_p(e^{p-1})$. Then $h(p) = e^{2p-1-e^{p-1}}$ and $h'(p) = (2 - e^{p-1})e^{2p-1-e^{p-1}}$. Moreover, $h'(p) > 0$ if $1 < p < 1 + \ln 2$, $h'(p) < 0$ if $p > 1 + \ln 2$, and $h'(p) = 0$ if $p = 1 + \ln 2$. Since $h(1) = 1$, $h(p) > 1$ for $p \in (1, 1 + \ln 2)$ and reaches it maximum value $h(1 + \ln 2) = 4e^{-1}$, after which it decreases to the minimum value $h(2) = e^{3-e} > 1$. Since R_p is decreasing monotonically on $(1, \infty)$ and $R_p(e^{p-1}) > 1 > R_p(\tilde{x}_w) = x_w$, it follows that $e^{p-1} < \tilde{x}_w$.

Let δ be sufficiently small such that $z_w = x_w + \delta < 1$ and $\tilde{z}_w = \max\{R_p^{-1}(z_w)\} > e^{p-1}$. Then the set $[z_w, \tilde{z}_w]$ is invariant and $|R_p'(x)| \leq M < 1$, for all $x \in [z_w, \tilde{z}_w]$. Moreover, $R_p(z_w) < R_p(1) = e^{p-1} < \tilde{z}_w$. It follows that the interval $[z_w, \tilde{z}_w]$ is in basin of attraction of the fixed point $x^* = p$. It is clear that, after one iteration, all points in the interval $(\tilde{z}_w, \infty)$ will be in $(0, z_w)$. Moreover, all points in $(0, z_w)$ will eventually be in $[z_w, \tilde{z}_w]$ after a sufficient number of iterations. Hence $x^* = p$ is globally asymptotically stable on $(0, \infty)$ if $0 < p < 2$.

When $p = 2$, $R_p'(2) = -1$, $R_p''(2) = 0$ and $R_p'''(2) = e^{-1}$. Hence the schwarzian derivative

$SR_p(2) = -e^{-1}$ and thus by Theorem 19, $x^* = 2$ is locally asymptotically stable with a basin of attraction that contains an interval $(2-\epsilon, 2+\epsilon)$ for some small $\epsilon > 0$. The previous argument will be applied on the set $[z_w, \tilde{z}_w]\backslash(2-\epsilon, 2+\epsilon)$ to show that in this case ($p = 2$), $x^* = 2$ is also globally asymptotically stable.

We now summarize these ideas:

Theorem 70 *In the autonomous Ricker map* $R_p(x) = xe^{p-x}$*, the fixed point* $x^* = p$ *is globally asymptotically stable on* $(0, \infty)$ *for* $0 < p \leq 2$.

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