Chapter 17
Systoles on Compact Riemann Surfaces with Symbolic Dynamics

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Abstract In this chapter, systolic inequalities are established, precise values are computed, and their behavior is also examined with the variation of the Fenchel–Nielsen coordinates on a compact Riemann surface of genus 2.

17.1 Introduction

The metric and geometric structure of surfaces may be studied by using the closed geodesics spectrum and the Laplace–Beltrami operator spectrum. It is not easy to obtain these spectra and even more difficult is to describe their dependence on the parameters which determine the metric and geometric structure of a surface. The dependence of such spectra dependence is examined using a boundary map when a Riemann surface M of genus 2, thus with negative curvature, is considered.

From a classical point of view the hyperelliptic surfaces are the simplest Riemann surfaces [12]. They can be denned by an algebraic curve $y^2 = F(x)$ where $F(x)$ is a polynomial of degree $2\tau + 1$ or $2\tau + 2$ with distinct roots (τ is the genus of the surface). Hyperelliptic surfaces of genus τ are characterized by the fact that the number of different Weierstrass points is minimal, namely $2\tau + 2$ (the fixed points of the hyperelliptic involution), while the weight of each Weierstrass point is maximal, namely $\frac{1}{2}\tau(\tau - 1)$.

For our purposes, two results for surfaces (see [17]) are significant:

Theorem 17.1 *A closed surface M of genus $t = 2$ is hyperelliptic if and only if M contains $2t - 2$ different simple closed geodesics which all intersect at the same point and mutually intersect in no other point.*

Theorem 17.2 *All closed surfaces of genus 2 are hyperelliptic.*

The systole of a compact Riemann surface is defined as the minimum length of a noncontractile curve. In the 1990s, a number of studies developed this concept: in particular, the article published by Schmutz Schaller (see [17]) that spurred the

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search for maximum surfaces for the systole; also important is the contribution of Bavard (see [1]), which provides a good theoretical framework, modeled on network analysis.

The aim of the present chapter is to provide an understanding of the behavior of systolic quantities on a compact Riemann surface over the geodesic length spectrum of M, endowed with a metric of constant curvature -1. The use of symbolic dynamics, a powerful tool which allows the explicit calculation of the parameters considered, is an essential feature of this study.

The chapter is organized as follows: Sect. 17.2 introduces a geometric description of the surface and defines the Teichmuller space and the global coordinate system, coordinates of Fenchel–Nielsen. The method described in this section involve decomposing a Riemann surface into a set of "pairs of pants" using three simple closed geodesic. The Riemann surface is represented by a quotient space $M = H^2/G$ of the upper half-plane H^2 using a Fuchsian group G which is isomorphic to the fundamental group of M. In Sect. 17.3 a detailed construction of the fundamental domain is given, side-pairing transformations are determined, and the boundary map is obtained. These constructions are the main element of the computations which follow. In Sect. 17.4 the generators of the fundamental group are obtained, and the identification, enumeration, and codification of orbits is carried out. Finally, in Sect. 17.5, explicit values are calculated for the geodesic length spectrum and systolic inequalities are obtained, which is the main objective of this work, reflecting the behavior of the systoles spectrum with the variation of the parameters that characterize the Riemann surface. Applying the main theorems, in the final section, upper and lower limits are determined for the systoles length spectrum under deformation of the surface.

17.2 Geometric Description and Fenchel–Nielsen Coordinates

Let C be the Riemann sphere and $GL_2^+(\mathbb{R})$ the 2×2 matrices group with real entries and a positive determinant. Let us consider the action given by this group of Mobius transformations

$$(g, z) \to gz = \frac{az+b}{cz+d}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $g(-d/c)$ is interpreted as ∞, and $g(\infty)$ as a/c. It is observed that $\frac{(\lambda a)\, z + (\lambda b)}{(\lambda c)\, z + (\lambda d)} = \frac{az+b}{cz+d}$ for any matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\lambda \in \mathbb{R}$, so any Mobius transformation can be realized by an element of $SL_2(\mathbb{R})$. Furthermore, a Mobius transformation

$$gz = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, ad - bc = 1$$

determines the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ less than signal, and $SL_2(\mathbb{R})$ contains the elements $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $-1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which act trivially. Then the quotient group is considered

$$PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm 1\} .$$

Given a surface $\mathcal{M}$ of negative curvature and genus $g = 2$, the universal covering surface of $\mathcal{M}$ is given by the hyperbolic plane which can be represented by the Poincaré disk, $D^2 = \{z \in \mathbb{C} : |z| < 1\}$, with metric $ds^2 = \dfrac{dz \cdot dz}{(1 - |z|^2)^2}$ or upper half-plane, $H^2 = \{z = x + iy : y < 0\}$, with metric $ds^2 = \dfrac{dz \cdot dz}{y^2}$. In both realizations, the isometry group is made up of the linear fractional transformations $h(z) = \dfrac{az + b}{cz + d}$. In the half-plane H^2, the matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$ belong to $SL_2(\mathbb{R})$, the *real unimodular group*. Schematically:

$$Aut(H^2) \cong PSL(2, \mathbb{R}) = SL(2, \mathbb{R}) / \{\pm I\}$$
$$Aut(D^2) \cong PSU(1, 1) = SU(1, 1) / \{\pm I\}$$

The covering group G is a Fuchsian model of $\mathcal{M}$. In this case considering the half-plane H^2, then G is a subgroup of $Aut(H^2)$

The Fuchsian model of a closed Riemann surface of genus $g(\geq 2)$ may thus be characterized exactly. An element of a Fuchsian model of a closed Riemann surface of genus $g(\geq 2)$ is the identity map either a hyperbolic transformation (see [13]).

The pair $(\mathcal{M}, \delta)$ denotes a Riemann surface $\mathcal{M}$ equipped with a conformal structure δ (an equivalence class of metrics). When there is no risk of confusion, it is denoted only by $\mathcal{M}$.

Let $g_i : [0, 1] \to \mathcal{M}$, $i = 1, 2$ be curves, such that $g_1(0) = g_2(0) = p_0$ and $g_1(1) = g_2(1) = p_1$. It can be said that g_1 and g_2 are homotopic if there is a continuous map $g : [0, 1] \times [0, 1] \to \mathcal{M}$ such that $g : \{0\} \times [0, 1] = p_0$, $g : \{1\} \times [0, 1] = p_1$, $g : [0, 1] \times \{0\} = g_1$ e $g : [0, 1] \times \{1\} = g_2$.

For any $p_0 \in \mathcal{M}$, the *fundamental group*$\pi_1(\mathcal{M}, p_0)$ is the group of the homotopy classes $g : [0, 1] \to \mathcal{M}$ such that $g(0) = g(1) = p_0$, i.e., the group of classes of closed paths with p_0 as a starting and terminal point. Often a system of generators $\Sigma_p = \{[A_j], [B_j]\}_{j=1}^{g}$ of a fundamental group $\pi_1(\mathcal{M}, p)$ of a genus 2 closed Riemann surface $\mathcal{M}$ is called a mark in $\mathcal{M}$. Since the choice of base point is irrelevant, the group $\pi_1(\mathcal{M})$ is called the fundamental group of $\mathcal{M}$.

Let us consider the triple$(\mathcal{M}, \delta, f)$, where δ is a conformal structure and $f : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism.

Two triples $(\mathcal{M}, \delta_i, f_i)$, $i = 1, 2$ are considered equivalent if there is a conformal map $k : (\mathcal{M}, \delta_1) \to (\mathcal{M}, \delta_2)$ for which the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f_1} & (\mathcal{M}, \delta_1) \\ & \overset{f_2}{\searrow} & \downarrow k \\ & & (\mathcal{M}, \delta_2) \end{array}$$

commutes by homotopy, i.e., $f_2 \circ f_1^{-1}$ and k are homotopic.

Definition 17.1 The space consisting of those equivalence classes, is called the Teichmüller space and is denoted by $\mathcal{T}_g$ (where g is the genus of $\mathcal{M}$).

In 1940, Teichmüller showed that $\mathcal{T}_g$ $(g \geq 2)$ is homeomorphic to $\mathbb{R}^{6g-6}$. One way to realize this homeomorphism is through F-N coordinates. The Teichmüller space $\mathcal{T}_g$ of genus g is given by a system of generators of the fundamental group of closed Riemann surface $\mathcal{M}$ of genus g. The set of generators is denoted by $\left\{\left[\alpha_j\right], \left[\beta_j\right]\right\}_{j=1}^{g}$. As both an Fuchsian model G of $\mathcal{M}$ and $G' = h^{-1}Gh$, for any $h \in Aut(E)$, are Fuchsian models of $\mathcal{M}$, it is necessary to establish a normalization that allows to define what we call canonical system of generators. The conditions of normalization are:

(i) The generator β_g has the attractive fixed point at ∞ and the repulsive fixed point at 0.
(ii) The generator α_g has the attractive fixed point at 1.
(iii) The axes of these two generators are disjoint.

Such a system of generators satisfies a single fundamental relationship (obtaining the presentation of this discrete group), i.e.:

$$\alpha_1 \circ \beta_1 \circ \alpha_1^{-1} \circ \beta_1^{-1} \circ \ldots \circ \alpha_g \circ \beta_g \circ \alpha_g^{-1} \circ \beta_g^{-1} = id. \tag{17.1}$$

In order to obtain a geometric image of the correspondence between the Riemann surface $\mathcal{M}$ and its Fuchsian model G, let us use the concept of fundamental domain for G.

Definition 17.2 Let $E = D^2, H^2$ and F is an open subset of E. It is said that F is a fundamental domain for G if it satisfies the following conditions:

(i) $g(F) \cap F = \emptyset$ for any $g \in G$, with $g \neq id$.
(ii) If F is the closure of F in E, then $E = \bigcup g(\overline{F})$, $g \in G$.
(iii) The boundary ∂F of F in E is of measure zero (with respect to Lebesgue measure).

The family $\{g(F) : g \in G\}$ is called a pavimentation of E. This means that the Riemann surface $\mathcal{M} = E/G$ is considered to be $\overline{F}$, with points on ∂F identified by group G. In this chapter, H^2 is the universal covering space of $\mathcal{M}$ and the fundamental group G, is a subgroup of $SL_2(\mathbb{R})$.

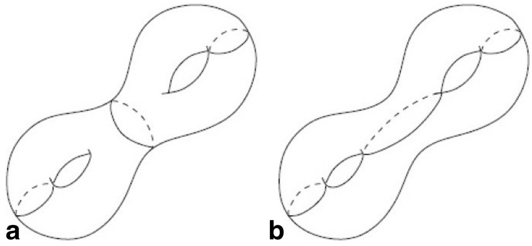

Fig. 17.1 Two pants decomposition

A domain of three circles is homeomorphic to the set

$$P = \left\{ z \in \mathbb{C} : |z| \leq 1, \left| z - \frac{1}{2} \right| \geq \frac{1}{4}, \left| z + \frac{1}{2} \right| \geq \frac{1}{4} \right\},$$

(a sphere with three holes in it), equipped with the hyperbolic metric, where the three components of boundary geodesics are simple. As explained above, P is usually called a "pair of pants." The complex structure of P is uniquely determined by the hyperbolic lengths of the ordered boundary components of P. $\mathcal{M}$ may be decomposed into a union of two "pairs of pants" (surfaces of genus zero with three boundary circles) (see Fig. 17.1).

Figure 17.2 represents an example of decomposition and gluing. This figure is considered $j = 1$, i.e., geodesic L_1.

In this figure c_i are the points of intersection between the geodesic L_1 and the geodesics $D_{1,i}$. The link between these geodesics is forged by means of another geodesic $D_{1,i}$, with minimum length. Denoted by T_1, the arc with the orientation of L_1 between the points c_1 and c_2 and the length of T_1 is denoted τ_1. Then the twist parameter for L_1 is defined by

$$\theta_1 = 2\pi \frac{\tau_1}{l_1}. \tag{17.2}$$

And, similarly, θ_j is well defined, module 2π

$$\theta_j = 2\pi \frac{\tau_j}{l_j}, \quad {}_{j=1,2,3}.$$

Definition 17.3 Let L_1, L_2, L_3 be the oriented decomposition curves. The functions l_j and θ_j, $j = 1, 2, 3$ denote the lengths and angles of torsion (twist) of L_1, L_2, L_3 used for gluing the pieces. This system of coordinates $\{l_j, \theta_j\}_{j=1,2,3}$ is called the Fenchel–Nielsen (F-N) coordinate system.

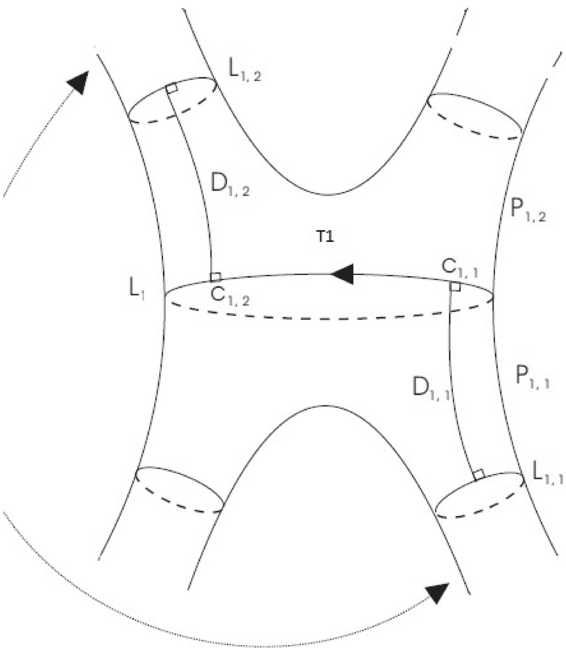

Fig. 17.2 Example of decomposition and gluing

17.3 Construction of Fundamental Domain and Definition of Boundary Map

A *chain* on a surface M is a set of four simple closed nondividing geodesics, labeled $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, where γ_2 intersects γ_1 exactly once; γ_3 intersects γ_2 exactly once and is disjoint from γ_1; γ_4 intersects γ_3 exactly once and is disjoint from both γ_1 and γ_2. It is assumed throughout that these geodesics are directed so that, in terms of the homology intersection number, $\gamma_i \times \gamma_{i+1} = +1$.

Given the chain $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, it can easily be seen that there are unique simple closed geodesics γ_5 and γ_6 so that γ_5 intersects γ_4 exactly once and is disjoint from γ_1, γ_2, and γ_3; and γ_6 intersects both γ_5 and γ_1 exactly once and is disjoint from the other γ_i. As above, it may be assumed that these geodesics are also directed so that, using cyclic ordering, $\gamma_i \times \gamma_{i+1} = +1$. This set of six geodesics is called a *geodesic necklace (*see[14] and Figure 17.3).

If the surface $\mathcal{M}_0$ is cut along the geodesics of a chain, a simply connected subsurface is obtained. It follows that elements A_0, B_0, C_0, D_0 may generate $\pi_1(\mathcal{M}_{00})$, and, conversely, the shortest geodesics in the free homotopy class of loops, corresponding to A_0, B_0, C_0, D_0, are, respectively, γ_1, γ_2, γ_3, γ_4. There are several possible choices for these elements; let us adopt Maskit's choice, which yields to defining relation: $A_0 B_0 D_0 A_0^{-1} C_0^{-1} D_0^{-1} C_0 B_0^{-1} = 1$ (see [13]).

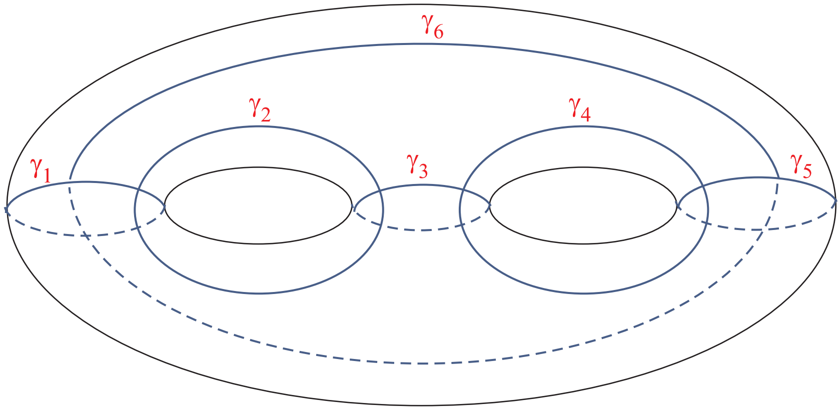

Fig. 17.3 Geodesic necklace

A particular system of generators of a Fuchsian group is obtained, (denoted by G_0) and our surface (nondeformed) $\mathcal{M}_0$ is given by $\mathcal{M}_0 = H^2/G_0$. It is possible to express explicitly the matrices A_0, B_0, C_0, D_0, E_0, $F_0 \in SL(2, \mathbb{R})$ (see [8], [14]).

A point in $\mathcal{T}$ can be regarded as being an equivalence class of orientation-preserving homeomorphisms h of H^2. Two such homeomorphisms are equivalent if the corresponding representations are equivalent; two such representations, A and B are equivalent if there is an element $S \in PSL(2, \mathbb{R})$ so that $SAS^{-1} = B$. When the rule of the decomposition (the way of gluing, see Fig. 17.2) and the lengths of closed geodesics are chosen, the decomposition is determined. The set of lengths of all geodesics used in the decomposition into pants, and the set of so-called twisting angles used to glue the pieces, provide a way of realizing this homeomorphism.

As this group G_0 is discrete, consisting only of hyperbolic elements, the surface, $\mathcal{M}_0$, represented by G_0, is our base surface. But our aim is to study the dependence of dynamic properties on the parameters which implies the variation of these parameters, forcing the consideration of the deformation spaces of the space base defined above. Deformation space $\mathcal{D}$ is defined as the space of representations $\varphi : G_0 \to PSL(2, \mathbb{R})$.

The image of the group $G = \varphi(G_0)$ is a discrete group with $\mathcal{M} = H^2/G$ a closed Riemann surface of genus 2.

Let us consider the chain γ_1, γ_2, γ_3, γ_4 in $\mathcal{M}$ which is decomposed along these geodesics in 4 hexagons triangles. These geodesics are the shortest length of the class of homotopy corresponding to some elements h_i $(i = 1, ..., 6)$ of $\pi_1(\mathcal{M})$, the fundamental group of $\mathcal{M}$. Building up hexagon H_1, as can be seen in Fig. 17.4, whose sides s_i are arcs γ_i, these arcs are contained in the axes of the hyperbolic transformations h_i $(i = 1, ..., 6)$. The translation distance of these axes (measured counterclockwise) along the axes is $2\,l_i$ where l_i denotes the length of $\gamma_i = l(\gamma_i)$.

From this initial hexagon H_1 the fundamental domain can be constructed by reflection and symmetry operations that reflect the decomposition of surface $\mathcal{M}$ along the geodesics (that form the chain γ_1 , γ_2, γ_3, γ_4 chosen). In this construction,

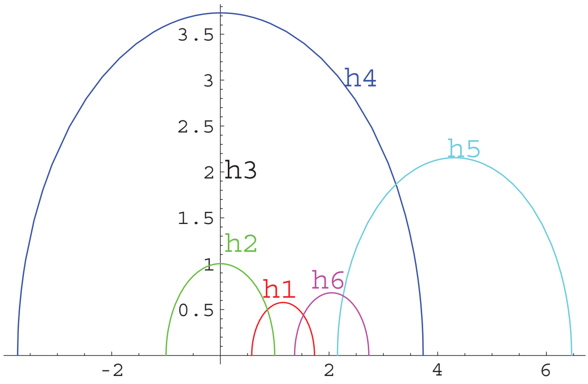

Fig. 17.4 Hexagons H_1

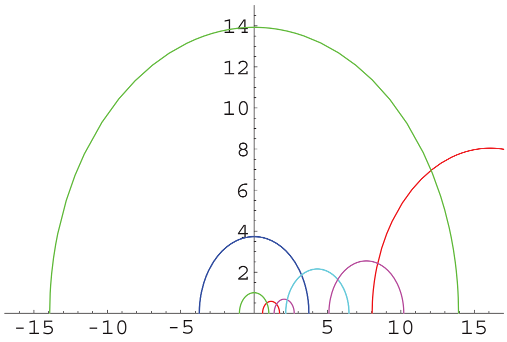

Fig. 17.5 Hexagons $H_1 \cup H_2$

a reflection on the axis of h_4 is considered, while hexagon H_1 is reflected in another hexagon H_2, see Fig. 17.5.

Finally, with the reflection in relation to the imaginary axis (symmetry) the other two hexagons H_3 and H_4 are obtained, building the dodecagon that is the fundamental domain: $F = H_1 \cup H_2 \cup H_3 \cup H_4$, see Fig. 17.6.

If this is not the case, it means that the glue is different and the fundamental domain is different as shown in Fig. 17.7.

This construction depends on the original choice of geodesic chain $\{\gamma_i\}$, and the form of gluing, i.e., the parameters considered.

Definition 17.4 Let us consider a geodesic necklace on a closed Riemann surface $\mathcal{M}$. When all the geodesics have the same length and there is no twist in the collage, it may be said that the regular case for the fundamental domain of a closed Riemann surface $\mathcal{M}$ in the F-N coordinates is being considered.

However, the form of the generators of the Fuchsian group G may be determined (for all cases) ([6], [14]).

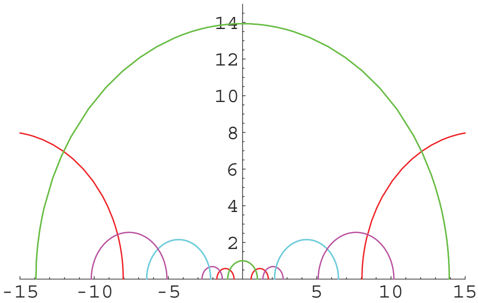

Fig. 17.6 Fundamental domain $F = H_1 \cup H_2 \cup H_3 \cup H_4$

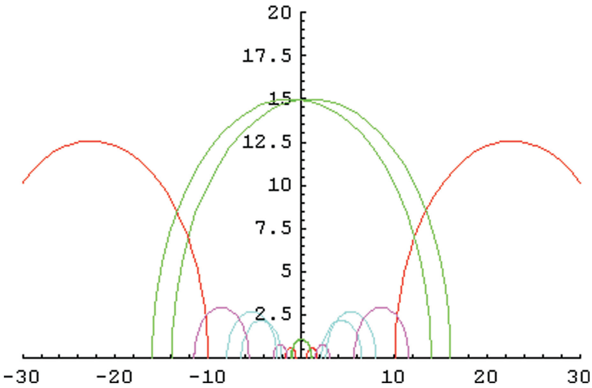

Fig. 17.7 Two dodecagons with twist angles zero ($\alpha_1 = \alpha_2 = \alpha_3 = 0$) and one with angles $\alpha_1, \alpha_2, \alpha_3 \neq 0$

Let us consider once more hexagon H_1. γ, the axis of h, which is also common among the orthogonal axes h_1 and h_3, and is chosen as a reference geodesic segment. Let $\mu' = |\gamma|$, and let us consider the ray joining the origin of the axes to the point of intersection between h and h_1, determining an angle μ between the axis of h_3 and this radius. Then, by using hyperbolic geometry, μ is defined by $\coth \mu = \cosh \mu$ (see [3]).

The point of intersection between h and h_3 is denoted by P and the point of intersection between h_2 and h_3 by P_2. The other parameters are given by the angles of gluing. Then σ is determined by the distance between the intersection of h with h_3 and the intersection of h_2 with h_3. Note that if $h_2 = h$ then σ is equal to zero. The other two parameters τ and ρ are determined by angle θ_2 and θ_3 between $h_2, h_3,$ and h_3, h_4, respectively, see Fig. 17.8. Schematically: $l_1 = l(\gamma_1),\ l_2 = l(\gamma_2),\ l_3 = l(\gamma_3),$ $l_4 = l(\gamma_4),\ \mu = arc\coth(\cosh \mu'),\ \sigma = |P - P_2|,\ \tau = arc\tanh(\cos(\theta_2)),$ and $\rho = arc\tanh(-\cos(\theta_3))$.

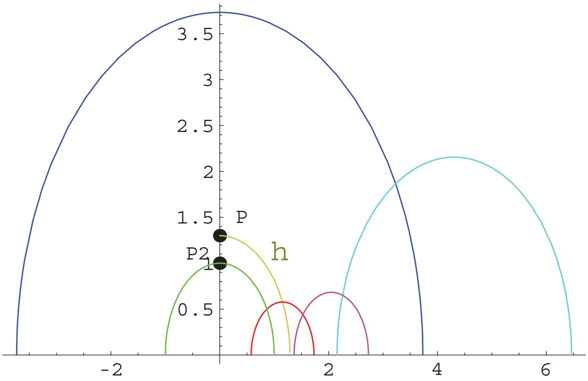

Fig. 17.8 Representation of the parameters in the hexagon

Through the operations of reflection and symmetry (as exemplified above) the following representations (see [9] and [14]) are obtained:

$$A = \frac{1}{\sinh \mu} \begin{pmatrix} \sinh(\mu - \alpha) & \sinh \alpha \\ -\sinh \alpha & \sinh(\mu + \alpha) \end{pmatrix}, C = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix},$$

$$B = \frac{1}{\cosh \tau} \begin{pmatrix} \cosh(\tau + \beta) & e^{\sigma} \sinh \beta \\ e^{-\sigma} \sinh \beta & \cosh(\tau - \beta) \end{pmatrix}, D = \frac{1}{\cosh \rho} \begin{pmatrix} \cosh(\rho - \delta) & -e^{\sigma+\gamma} \sinh \delta \\ -e^{-\sigma-\gamma} \sinh \delta & \cosh(\rho + \delta) \end{pmatrix}.$$

Teichmüller space $\mathcal{T}$ of genus 2 is given by a system of generators of a Fuchsian group, the fundamental group of the surface of the closed Riemann surface $\mathcal{M}$ of genus 2. A particular system of generators determined by following expressions may be considered: $h_1 = B$, $h_2 = A$, $h_3 = F$, $h_4 = E$, $h_5 = BD$, $h_6 = DF^{-1}$. $h_7 = h_1^{-1}, \ldots, h_{12} = h_6^{-1}$ (see Fig.17.9).

Proposition 17.1 *If the twist angles are zero,* $\sigma_1 = \sigma_2 = \sigma_3 = 0$*, then the fundamental domain is a right-angle polygon.*

Proof If $\tau = 0$, then $\theta_2 = \frac{\pi}{2}$ so the axes of B and C are orthogonal; if $\sigma = 0$ then $P = P_2$ so $h_2 \equiv h$. With this equality and the definition of h it may be concluded that the axis of h_2 is orthogonal to the axes of both h_3 and h_1; if $\rho = 0$, then $\theta_3 = \frac{\pi}{2}$, which implies that the axes of h_3 and h_4 are orthogonal. As we have a hyperbolic hexagon with three direct consecutive internal angles it may be concluded that the other three also are right angles and, thus, H_1 is a right-angled hexagon. As the hyperbolic reflection maintains the angles invariant, the resulting polygon F is a right-angled polygon.

This construction depends on the choice of original geodesics γ_i, $i = 1, \ldots, 4$. The chain is then dependent on the parameters $\ell_i = \ell(\gamma_i)$. The sides are obtained by the intersection of the axes, and are geodesic segments. The single point which is the intersection between two consecutive sides is called a vertex.The circular arc

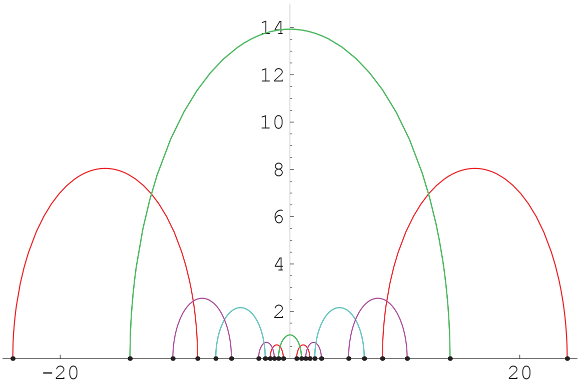

Fig. 17.9 Axes $h_1,...,h_{12}$ and terminal points $\{p_i, q_i\}$, $i = 1, ..., 12$ on $\partial\mathcal{F}$

that contains a side s_i intersects the real axis at two points p_i and q_i (see Fig.17.9). The sides are labeled $s_1, ..., s_{12}$ reading counterclockwise from zero.

It is known (see [3]) that if $\mathcal{F}$ is any locally finite fundamental domain for a Fuchsian group G, then

$$\{g \in G : g(\overline{\mathcal{F}}) \cap \overline{\mathcal{F}} = \phi\}$$

generates G. The fundamental domain $\mathcal{F}$ is a bounded fundamental polygon whose boundary $\partial\mathcal{F}$ consists of the 12 geodesic segments $s_1, ..., s_{12}$. There is a bijection:

$$\Psi : \mathcal{L}_{\mathcal{F}} \to G_{\mathcal{F}}; \qquad \Psi(g) = \overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$$

between the set of the sides of $\mathcal{F}$ and the set of elements g in G for which $\overline{\mathcal{F}} \cap g(\overline{\mathcal{F}})$ is a side of $\mathcal{F}$. These pairing transformations $g : s_i \to s_j$ identifying the sides (side-pairing), elements of $G_{\mathcal{F}}$, i.e., generate the group G.

The identification rule chosen is

$$\begin{aligned} &s_1 \lhd \rhd s_7\ ; \quad s_2 \lhd \rhd s_{12}; \quad s_3 \lhd \ \rhd s_5; \\ &s_4 \lhd \rhd s_{10}; \quad s_6 \lhd \rhd s_8\ \ ; \quad s_9 \lhd \rhd s_{11} \end{aligned} \tag{17.3}$$

Adopting this choice, formulas for the side pairing transformations $g_1, ..., g_{12}$ are calculated and the generators $g_i = g_i(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$, $i = 1, ..., 12$, where $\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3$ are the F-N coordinates, are explicitly obtained. With the linear fractional transformations defined above the boundary map may be obtained: $f_\Gamma : \partial\mathcal{F} \to \partial\mathcal{F}$, defined by piecewise linear fractional transformations in the partition $P = \{\, I_i = [p_i, p_{i+1}),\ i = 1, ..., 11,\ [p_{12}, p_1)\}$, which is orbit equivalent to the action of the fundamental group G on $\partial\mathcal{F}$.

Adopting the identification rule, (17.3) formulas for the side pairing transformations $g_1, ..., g_6$, $g_7 = g_1^{-1}, ..., g_{12} = g_6^{-1}$ may be determined. This means that $s_7 = g_1(s_1)$, ..., $s_9 = g_6(s_{11})$, $s_1 = g_7(s_7)$, ..., $s_{11} = g_{12}(s_9)$.

Let $g_i(z) = (a_i z + b_i)/(c_i z + d_i)$ for $g_i(s_j) = s_k$, with

$$\begin{cases} r_i = (q_i - p_i)/2, \\ c_i = 1/(r_j r_k)^{1/2}, \\ b_i = (a_i d_i - 1)/c_i \end{cases}$$

then the following system of equations is solved

$$\begin{cases} (a_i p_j + b_i)/(c_i p_j + d_i) = q_k, \\ (a_i q_j + b_i)/(c_i q_j + d_i) = p_k \end{cases}$$

and $\{a_i, d_i\}$ is determined, $i = 1, ..., 12$. With $\{a_i, d_i\}$ the generators $g_i = g_i(\ell_1, \ell_2, \ell_3, \sigma_1, \sigma_2, \sigma_3)$, $i = 1, ..., 12$ are computed. The system of generators of G is denoted by G^0.

Let the partition be $\mathcal{Q} = \{I_i = [p_i, p_{i+1}), i = 1, ..., 11, [p_{12}, p_1)\}$. Although this is not a Markov partition it may be refined so as to obtain one that is. One way of doing this is to introduce the lateral limits $p_i^{\pm}$ of the discontinuity points p_i (see [9]).

By means of fractional linear transformations carried out on $\mathcal{Q}$, the boundary map $f_G : \partial\mathcal{F} \to \partial\mathcal{F}$ is defined, represented by:

$$f_G : \bigcup_{i=1,...,12} I_i \to \bigcup_{i=1,...,12} I_i;\ f_G(x)_{|I_i} = g_i(x), i = 1, .., 12.$$

17.4 Fundamental Group

The boundary map f_G determines the associated Markov matrix A. This matrix is given by (see [6])

$$a_{ij} = \begin{cases} 1 & \text{if } I_i \subset f_G(I_i) \\ 0 & \text{otherwise.} \end{cases}$$

The identification, enumeration, and codification of orbits use symbolic dynamics through constructions that involves the geometry of the surface and the algebraic structure of its fundamental group G. The action of this fundamental group on the Poincaré upper half plane boundary is shown to be orbit equivalent to the Markov map, f_G, which has been defined, and codification is obtained by the expansion of the boundary points.

Definition 17.5 *Let*x_{i_0} *be an element of the limit set of* G *. As it belongs to one of the intervals*I_{i_0} *of the Markov partition*$\mathcal{W}$ *the image under* $f_G(x_{i_0}) = g_{i_0}(x_{i_0}) = x_{i_1}$, *is* another boundary pointx_{i_1}. *The point*x_{i_1} *belongs to the intervals*I_{i_1} *so* $f_G(x_{i_1}) = g_{i_2}(x_{i_1}) = x_{i_2}$.This process is repeated successively obtaining the (f_G-expansion) of boundary point x. This sequence is called the word associated with the point x.

Example 17.1

(1) Consider the regular dodecagon: $l_1 = l_2 = l_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$.
$x = 1.5- >$ word $1- >$ 6, 12, 3, 8, 7, 3, 3, 7, 2, 3, 6, 12, 3, 8
$x = 2.0- >$ word $2- >$ 11, 7, 11, 6, 10, 5, 11, 12, 3, 12, 3, 3, 10, 6
(2) Consider the dodecagon: $l_1 = \log(2+\sqrt{3}) + 0.3$, $l_2 = l_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$.
$x = 1.5- >$ word $1- >$ 6, 10, 7, 2, 10, 2, 7, 10, 12, 3, 4, 7, 7, 7
$x = 2.0- >$ word $2- >$ 12, 6, 10, 7, 9, 12, 5, 4, 8, 7, 4, 1, 5, 4.

Proposition 17.2 *Each point of the limit set is associated with a single word. This word is the $f_G-expansion$ boundary point. The admissibility of a given block or word is given by the Markov matrix A.*

Proof The Markov matrix A identifies the possible transitions between states and the associated subshift of finite type, (Σ_A, σ), is identified by the limit set. Thus the occurrences in the limit set are given by the admissibility in subshift of finite type, therefore by A.

The representation (codification) of a geodesic γ in H^2(or D^2) is the juxtaposition of f_G-expansions of its extreme points γ_- and γ_+ on the real axis (or the unit circle Σ).

$$\gamma \leftrightarrows \gamma_-.\gamma_+ \leftrightarrows ...g_{i_2}g_{i_1}.h_{i_1}h_{i_2}... \text{ where } g_{i_2}, g_{i_1}, h_{i_1}, h_{i_2} \in G^0.$$

Definition 17.6 Given $g \in G$ (and $\gamma \in calM$) we define its word length $|g|$ to be the smallest number of elements from G needed in a presentation of g, i.e., $|g| = \inf\{n : g = g_1...g_n \text{ with } g_1, ..., g_n \in G^0\}$; for a closed geodesic γ and associated class $[h]$ in G, we denote $|\gamma| = \inf\{|g| : g \in G \text{ e } [g] = [h]\}$, that is the word length of γ; the geometric length of γ is given by $\ell(\gamma) = \int_\gamma m(z)\,|dz|$ and is dependent on the metric of the surface.

By convention, $|e| = 0$, where e is the identity element in G.

Theorem 17.3 *Admissible geodesics are conjugate underG if and only if the corresponding sequences are shift equivalent.*

Proof See [9].

Theorem 17.4 *On surfaces of negative curvature there is a bijection between closed geodesic, γ, in $\mathcal{M}$ and conjugacy classes of the fundamental group G associated with the fundamental domain F, i.e., each conjugacy class $[g_\gamma]$ to G represents a closed geodesic, γ, in $\mathcal{M}$.*

Proof See [15].

These results establish equivalence between concepts and different mathematical spaces.

Proposition 17.3 *With a finite number of exceptions there is a bijection between the closed geodesic, $\gamma \in \mathcal{M}$ (word length $|\gamma|$ and length $\ell(\gamma)$) and the primitive periodic orbits of period k, $\mathcal{O}(x) = \{x, f(x), f^2(x), ..., f^k(x)\}$.*

Proof See [16].
Schematically:

$$\begin{array}{lcl} \text{Periodic points for} f_G & \leftrightarrow & \text{Admissible sequences for the shift}\sigma \\ \updownarrow & & \updownarrow \\ \text{Primitive closed geodesics in } \mathcal{M} & \longleftrightarrow & \text{Conjugacy classes in} G \end{array}$$

If the elements of group G are viewed as matrices in $SL_2(\mathbb{R})$, then the identification of the matrices g and $-g$ in $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm 1\}$ may be understood automatically. The following proposition allows for the identification of conjugacy classes, by algebraic methods, using the traces of these elements.

Proposition 17.4 *Consider two elements g and h of group G. Its conjugacy classes are equal, if and only if, the squares of its traces are also equal. That is,*

$$[g] = [h] \Leftrightarrow tr^2(g) = tr^2(h).$$

Proof See [3].
The identification of the conjugacy classes is dependent on traces of matrices that represent the elements of the group; but these matrices are, in turn, dependent on the choice of the parameters of F-N. Consequently, the following conclusion may be drawn:

Proposition 17.5 *The conjugacy classes of the fundamental group are dependent of the Fenchel–Nielsen coordinates.*

Example 17.2 If words of length 1 are considered for the case of regular domain ($l_1 = l_2 = l_3 = \log(2 + \sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$) there is only one conjugacy class (Table 17.4), while, for example, for $l_1 = l_2 = \log(2 + \sqrt{3})$, $l_3 = 1.7$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$ 4 distinct classes of conjugation are obtained (Table 17.5).

$$\begin{bmatrix} tr(g_i) & 4 & 4 & -4 & -4 & 4 & 4 & 4 & 4 & 4 & 4 & -4 & -4 \\ tr^2(g_i) & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \end{bmatrix}$$

$$\begin{cases} l_1 = l_2 = l_3 = \log(2 + \sqrt{3}), \sigma_1 = \sigma_2 = \sigma_3 = 0 \\ 1 \text{ distinct class} \end{cases} \tag{17.4}$$

$$\left[\begin{array}{lllllll} tr(g_i) & -4 & 5.65663 & -8.96989 & 2.89726 & -8.96989 & 5.65663 \\ tr^2(g_i) & 16 & 16 & 16 & 16 & 16 & 16 \end{array}\right.$$

$$\begin{bmatrix} \ldots\ldots & 4 & 5.65663 & 8.96989 & 2.89726 & 8.96989 & 5.65663 \\ \ldots\ldots & 16 & 16 & 16 & 16 & 16 & 16 \end{bmatrix}$$

$$\begin{cases} l_1 = l_2 = \log(2+\sqrt{3}), l_3 = 1.7, \sigma_1 = \sigma_2 = \sigma_3 = 0 \\ \text{4 distinct classes} \end{cases} \tag{17.5}$$

17.5 Systolic Inequalities

In order to examine systolic quantities, the spectrum length of the closed geodesics must be determined. This spectrum may be determined by means of the trace of the matrix associated with the boundary map. As g is an element of group G, its length is obtained by the expression (see [3]):

$$l(g) = 2\cosh^{-1}\left[\frac{tr(g)}{2}\right].$$

Consider two examples with the same word length, $|g| = 1$ (examples 17.6, 17.7), but different choices of F-N parameters:

Example 17.3

$$\begin{bmatrix} g_i & g_1 & g_2 & g_3 & \cdots & g_{10} & g_{11} & g_{12} \\ l(g_i) & 2.63392 & 2.63392 & 2.63392 & \ldots & 2.63392. & 2.63392 & 2.63392. \end{bmatrix}$$

$$\begin{cases} l_1 = l_2 = l_3 = \log(2+\sqrt{3}), \sigma_1 = \sigma_2 = \sigma_3 = 0 \\ \text{1 distinct class, word length, } |g| = 1. \end{cases} \tag{17.6}$$

Example 17.4

$$\begin{bmatrix} g_i & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ l(g_i) & 3.4 & 2.63392 & 3.85452 & 1.60608 & 3.85452 & 2.63392 \end{bmatrix}$$

$$\begin{bmatrix} \cdots & g_7 & g_8 & g_9 & g_{10} & g_{11} & g_{12} \\ \ldots & 3.4 & 2.63392 & 3.85452 & 1.60608 & 3.85452 & 2.63392 \end{bmatrix} \tag{17.7}$$

$$\begin{cases} l_1 = l_2 = \log(2+\sqrt{3}), l_3 = 1.7, \sigma_1 = \sigma_2 = \sigma_3 = 0 \\ \text{4 distinct classes, word length, } |g| = 1. \end{cases}$$

Consider two examples with the same Fenchel–Nielsen parameters but different word lengths, $|g| = 1$ and $|g| = 2$ (examples 17.8, 17.9). Let us choose the regular case, $\ell_1 = \ell_2 = \ell_3 = \log(2+\sqrt{3})$ and $\sigma_1 = \sigma_2 = \sigma_3 = 0$.

Example 17.5 If the word length, $|g| = 1$, then there is just one distinct element of the geodesic length spectrum that is, $\ell(g) \simeq 2.63392$.

$\mathbf{g}_i$	$\ell(\mathbf{g}_i) = l_i$
g_1	$l_1 = 2.63392...$
...	...
g_{12}	$l_{12} = 2.63392...$

(17.8)

Example 17.6 If the word length, $|g| = 2$, we have 9 distinct conjugacy classes, then there are 9 distinct values for the length spectrum. Some values:

$\mathbf{g}_1 \cdot \mathbf{g}_i$	$\simeq \ell(\mathbf{g}_1\mathbf{g}_i)$	$\mathbf{g}_2 \cdot \mathbf{g}_i$	$\simeq \ell(\mathbf{g}_2\mathbf{g}_i)$	$\mathbf{g}_3 \cdot \mathbf{g}_i$	$\simeq \ell(\mathbf{g}_3\mathbf{g}_i)$
$g_1 \cdot g_1$	5.26783	$g_2 \cdot g_1$	4.12687	$g_3 \cdot g_1$	5.98645
$g_1 \cdot g_2$	4.12687	$g_2 \cdot g_2$	5.26783	$g_3 \cdot g_2$	6.51323
$g_1 \cdot g_3$	5.98645	$g_2 \cdot g_3$	6.51323	$g_3 \cdot g_3$	5.26783
$g_1 \cdot g_4$	4.12687	$g_2 \cdot g_4$	5.98645	$g_3 \cdot g_4$	4.12687
$g_1 \cdot g_5$	2.63392	$g_2 \cdot g_5$	4.58486	$g_3 \cdot g_5$	0
$g_1 \cdot g_6$	4.12687	$g_2 \cdot g_6$	7.82325	$g_3 \cdot g_6$	4.58486
$g_1 \cdot g_7$	0	$g_2 \cdot g_7$	4.12687	$g_3 \cdot g_7$	2.63392
$g_1 \cdot g_8$	4.12687	$g_2 \cdot g_8$	7.05099	$g_3 \cdot g_8$	6.51323
$g_1 \cdot g_9$	2.63392	$g_2 \cdot g_9$	6.51323	$g_3 \cdot g_9$	7.05099
$g_1 \cdot g_{10}$	4.12687	$g_2 \cdot g_{10}$	2.63392	$g_3 \cdot g_{10}$	4.12687
$g_1 \cdot g_{11}$	5.98645	$g_2 \cdot g_{11}$	4.58486	$g_3 \cdot g_{11}$	7.82325
$g_1 \cdot g_{12}$	4.12687	$g_2 \cdot g_{12}$	0	$g_3 \cdot g_{12}$	4.58486

(17.9)

(...)

Every simple closed curve in the plane satisfies the inequality

$$\frac{A}{\pi} \leq \left(\frac{L}{2\pi}\right)^2$$

where L is the length of the curve and A is the area of the region it bounds. This is a classical *isoperimetric inequality*. In the 1950s, C. Loewer and P. Pu proved a classical *isosystolic inequality*(this result was not published, see [4]).

Let $\mathbb{RP}^2$ be the real projective plane endowed with an arbitrary metric, i.e., an embedding in some $\mathbb{R}^n$. Then

$$\left(\frac{L}{\pi}\right)^2 \leq \frac{A}{2\pi}$$

where A is its total area and L is the length of its shortest noncontractible loop. Similarly, every metric torus $\mathbb{T}^2$ satisfies the inequality

$$L^2 \leq \frac{2}{\sqrt{3}} A$$

In the 1970s, Marcel Berger began studying a new Riemannian invariant, which came to be know as the systole.

The systole of a compact Riemann surface is defined as the minimal length of a noncontractile curve, (by abuse language one employs the same word for the curves carrying out this length). The geometry of systoles was studied by Schmutz within the framework of hyperbolic geometry (see [17] and [18]) and Bavard (see [1]) in the context of abelian manifolds. The notion of systole, in particular, led to the characterization of arithmetic groups using the length spectrum. The study of systoles also provided geometrical answers to the Schottky problem on the Jacobian Riemann surface. This approach was developed by Buser and Sarnak in (to see [5]) Gromov in (to see [10]).

$sys\pi_1(\mathcal{M}, m)$ denoted as the shortest length of a noncontractible loop of M.

$$sys\pi_1(\mathcal{M}, m) = \min_{|\gamma| \neq 0, \gamma \in M} length(\gamma)$$

The systolic ratio SR of $(\mathcal{M}, m)$ is defined as

$$SR(\mathcal{M}, m) = \frac{sys\pi_1(\mathcal{M}, m)^2}{vol(\mathcal{M}, m)}, \tag{17.10}$$

and the optimal systolic ratio of $\mathcal{M}$ as

$$SR(\mathcal{M}) = \sup_m \; SR(\mathcal{M}, m), \tag{17.11}$$

where m runs over the space of all metrics, (see [11]).

The optimal systolic ratio of a genus 2 surface is unknown, but it satisfies the Loewner inequality $SR(\mathcal{M}) \leq 2/\sqrt{3}$, the best available upper bound for the optimal systolic ratio of an arbitrary genus 2 surface, (see [12]). However, the latter ratio is known for the Klein bottle, in addition to the torus, and also the real projective plane. It should be noted that averaging a conformal metric by hyperelliptic involution improves the systolic ratio of the metric. Systolic geometry has recently experienced a period of great development, (see [10], [11]). Thus, a surface is Loewner if $SR(\mathcal{M}) \leq 2/\sqrt{3}$, and in (see [12]) it has recently been shown that the genus 2 surface is Loewner.

In the case of hyperbolic surfaces, a compact surface, is entirely determined by a decomposition into a "pair of pants" and $6g - 6$ real parameters, the F-N coordinates. The regular domain choice corresponds, in geometric terms, to considing the non-deformed surface $\mathcal{M} = H^2/G$. With the explicit computation of geodesic length, explicit inequalities are obtained that show the dependence of systoles on F-N parameters. The variations of F-N coordinates modifies the metric structure of the surface and this implies the alteration of systole length.

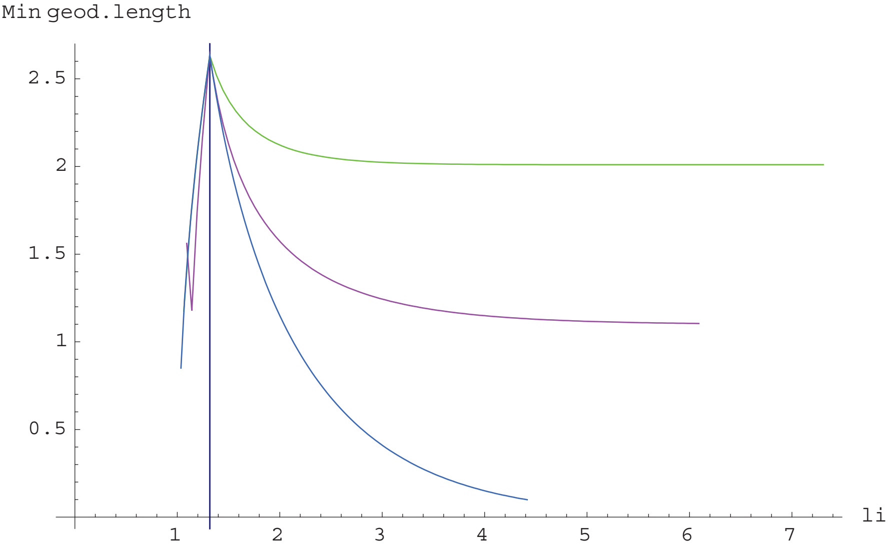

Fig. 17.10 Systoles with Fenchel–Nielsen coordinates variation

Let $\mathcal{M}_0$ be a closed, nondeformed, surface with genus $g = 2$ and $\ell_0 = \max\{\ell(\gamma(t))\}$ the maximum of the shortest closed geodesic lengths on$\mathcal{M}_0$The following result may be introduced:

Theorem 17.5 *Let M be a closed surface with genus $g = 2$. Thus the length $l(\gamma)$ of every systole, γ, verifies the inequality $l(\gamma) \leq \ell_0$.*

It is established that the regular case is a lower bound for systoles with F-N coordinates variation (see Fig. 17.10). Each line represents systole behavior for each F-N parameter: the blue line for ℓ_3; the magenta line for ℓ_3 and the green line for ℓ_1. The fundamental domain that was constructed in the previous section is a dodecagon. In order to maintain this structure in the present study, it was necessary to establish the possible intervals of variation for each of the F-N coordinates.

Computation of the values of systoles lengths was carried out considering these intervals.

Corollary L *et $\mathcal{M}$ be a closed, non-deformed, surface with genus $g = 2$. Then*

$$sys\pi_1(M, m) \leq \log(2 + \sqrt{3})$$

for every systole, $\gamma \in \mathcal{M}$.

However, our aim is a more global (not just two cases), study of this length spectrum with F-N coordinates, in order to provide an understanding of how the geodesic length spectrum behaves under deformation of the surface. Let us recall that these coordinates constitute a system of global coordinates in Teichmüller space $\mathcal{T}$. If considering the variation of the lengths, in function of the coordinates of F-N, also considering the maximum value as the minimum value of these lengths,

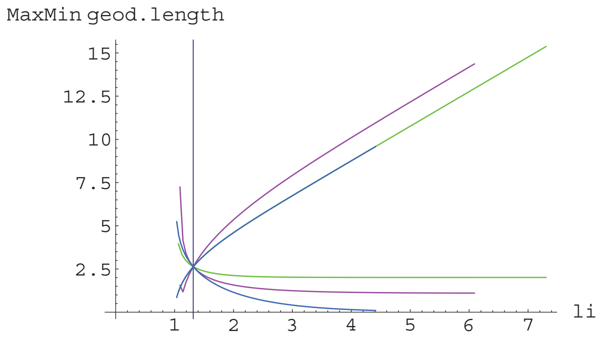

Fig. 17.11 Upper and lower limits of shortest closed geodesic length

the maximum and the minimum, respectively, are obtained, with the regular case. This regular case corresponds, in geometric terms, to surface $\mathcal{M}_0$ being a closed, non-deformed, surface with genus$g = 2$.

Definition 17.7 Let M_0be a closed, non-deformed, surface with genus$g = 2$ and $\ell_0 = \max\{\ell(\gamma(t))\}$ the maximum of the shortest closed geodesic length on$M$

It may be clearly observed (see Fig. 17.11), that it is in the regular case that the upper/lower limits of the shortest closed geodesic length are reached. In the figure, each of the F-N coordinates covers all the possible values (different limits for each ℓ_i) so that the basic domain remains a dodecagon. Thus the following is obtained:

Theorem 17.6 *Let$\mathcal{M}$ be a closed surface with genus$g = 2$. Thus the length $l(\gamma)$ of every systole,γ, verifies the inequality$\ell(\gamma) \leq \ell_0$.*

17.6 Final Considerations

In this chapter new insights have been provided into the study of the systoles length spectrum under deformation of a compact Riemann surface endowed with a metric of constant curvature -1. Rigorous detailed constructions have been achieved using the techniques of symbolic dynamics. Systolic inequalities that determine the upper limits of the systoles lengths spectrum have been established.

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