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SOLVABILITY OF HIGHER-ORDER BVPS IN THE HALF-LINE WITH UNBOUNDED NONLINEARITIES*

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Abstract. This work presents sufficient conditions for the existence of unbounded solutions of a Sturm-Liouville type boundary value problem on the half-line. One-sided Nagumo condition plays a special role because it allows an asymmetric unbounded behavior on the nonlinearity. The arguments are based on fixed point theory and lower and upper solutions method. An example is given to show the applicability of our results.

1. Introduction.

This paper is concerned with the study of a general Sturm-Liouville type boundary value problem composed by a third-order differential equation defined on the half line

$$u'''(t) = f\left(t, u(t), u'(t), u''(t)\right), t \in [0, +\infty) \tag{1}$$

and

$$u(0) = A,\ au'(0) + bu''(0) = B,\ u''(+\infty) = C. \tag{2}$$

with $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ a L^1- Carathéodory function, $a > 0, b < 0, A, B, C \in \mathbb{R}$.

Higher order boundary value problems on infinite intervals appear in several real phenomena such as the gas pressure in a semi-infinite porous medium, or theoretical results as, for example, the study of radially symmetric solutions of nonlinear elliptic equations. For these and other applications see, for example, [1].

Third-order differential equations, in general, arise in many areas, such as the deflection of a curved beam having a constant or a varying cross-section, three layer beam, electromagnetic waves or gravity-driven flows, ([9]). In infinite intervals, third order boundary value problems can describe the evolution of physical phenomena, for example some draining or coating fluid-flow problems, (see [4, 15, 16]).

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By the non-compactness of the interval, the discussion about sufficient conditions for the solvability of boundary value problems on the half-line is more delicate. In the existent literature the main methods to obtain existence results are the extension of continuous solutions on the corresponding finite intervals under a diagonalization process and fixed point theorems in special Banach spaces (see [2, 3, 11, 17] and the references therein.)

Lower and upper solutions method is an useful technique to deal with boundary value problems as it provides not only the existence of solution but also its localization and, from that, it can be obtained some qualitative data about solutions variation and behavior (see [5, 8, 12, 13, 14]). An important tool of this technique is the Nagumo condition, useful to obtain *a priori* estimates on some derivatives of the solution. As it can be seen in the references above, the usual growth condition of the Nagumo type is a bilateral one. However the same estimations hold with a similar one-sided assumption, allowing that the boundary value problems can include unbounded nonlinearities. In this way it generalizes the two-sided condition, as it is proved in [7, 10].

The paper is organized as it follows: In Section 2 is defined the space, the weighted norms and the unilateral Nagumo conditions to be used. As far as we know, it is the first time where such conditions are assumed for boundary value problems defined on the half-line.

Section 3 contains the main result: an existence and localization theorem, where it is proved the existence of a solution, and some bounds on the first and second derivatives as well.

Finally, an example shows the applicability of the theorem and, moreover, it is emphasized that the nonlinearity considered verifies the one-sided Nagumo condition but not the bilateral one.

2. **Definitions and auxiliary results.** In this section it is proved an *a priori* estimate on the derivative u'' under an unilateral Nagumo-type condition. With that aim, let us introduce the following definitions:

Let $w_i(t) = 1 + t^{2-i}, i = 0, 1, 2$ and define the space

$$X = \left\{ x \in C^2[0, +\infty) : \lim_{t \to +\infty} \frac{x^{(i)}(t)}{w_i(t)} \in \mathbb{R},\ i = 0, 1, 2 \right\}$$

with the norm $\|x\| = \max \{\|x\|_0, \|x'\|_1, \|x''\|_2\}$, where

$$\|y\|_i = \sup_{0 \le t < +\infty} \left| \frac{y(t)}{w_i(t)} \right|, \text{ for } i = 0, 1, 2.$$

Therefore $(X, \|.\|)$ is a Banach space.

Definition 2.1. A function $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is L^1- Carathéodory if it verifies

(i) for each $(x, y, z)) \in \mathbb{R}^3$, $t \mapsto f(t, x, y, z)$ is measurable on $[0, +\infty)$;

(ii) for almost every $t \in [0, +\infty), (x, y, z) \mapsto f(t, x, y, z)$ is continuous in $\mathbb{R}^3$;

(iii) for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1[0, +\infty)$ such that, for $(x(t), y(t), z(t)) \in \mathbb{R}^3$ with $\max \{\|x\|_0, \|y\|_1, \|z\|_2\} < \rho$,

$$|f(t, x, y, z)| \le \varphi_\rho(t),\ a.e.\ t \in [0, +\infty).$$

Let $\gamma, \Gamma \in X$ be such that $\gamma(t) \le \Gamma(t), \gamma'(t) \le \Gamma'(t), \forall t \in [0, +\infty)$ and consider the set

$$E = \left\{ (t, x, y, z) \in [0, +\infty) \times \mathbb{R}^3, \gamma(t) \le x \le \Gamma(t), \gamma'(t) \le y \le \Gamma'(t) \right\}.$$

The following one-sided condition generalizes the usual bilateral Nagumo condition and allows some *a priori* bounds on the second derivative of the solution.

Definition 2.2. A function $f : E \to \mathbb{R}$ is said to satisfy one-sided Nagumo-type growth condition in E if, for some positive continuous functions ψ, h and some $\nu > 1$, such that

$$\int_0^{+\infty} \psi(s)ds < +\infty, \sup_{0\le t<+\infty} \psi(t)(1+t)^{\nu} < +\infty, \int_0^{+\infty} \frac{s}{h(s)}ds = +\infty, \quad (3)$$

it verifies either

$$f(t,x,y,z) \le \psi(t)h(|z|), \forall (t,x,y,z) \in E, \quad (4)$$

or

$$f(t,x,y,z) \ge -\psi(t)h(|z|), \forall (t,x,y,z) \in E. \quad (5)$$

Lemma 2.3. *Let $f : [0,+\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function satisfying (3) and (4), or (5), in E. Then for every $r > 0$ there exists $R > 0$ (not depending on u) such that every u solution of (1),(2) satisfying*

$$\gamma(t) \le u(t) \le \Gamma(t), \gamma'(t) \le u'(t) \le \Gamma'(t), \forall t \in [0,+\infty), \quad (6)$$

verifies $\|u''\|_2 < R$.

Proof. Let u be a solution of (1),(2) verifying (6). Consider $r > 0$ such that

$$r > \max\left\{\left|\frac{B - a\Gamma'(0)}{b}\right|, \left|\frac{B - a\gamma'(0)}{b}\right|, |C|\right\}. \quad (7)$$

By the previous inequality we cannot have $|u''(t)| > r, \forall t \in [0,+\infty)$, because

$$|u''(0)| = \left|\frac{B - au'(0)}{b}\right| \le \max\left\{\left|\frac{B - a\Gamma'(0)}{b}\right|, \left|\frac{B - a\gamma'(0)}{b}\right|\right\} < r.$$

If $|u''(t)| \le r, \forall t \in [0,+\infty)$, taking $R > \frac{r}{2}$ the proof is complete as

$$\|u''\|_2 = \sup_{0\le t<+\infty} \left|\frac{u''(t)}{2}\right| \le \frac{r}{2} < R.$$

In the following it will be proved that even when there exists $t \in (0,+\infty)$ such that $|u''(t)| > r$, the norm $\|u''\|_2$ remains bounded, in all possible cases, either f verifies (4) or (5).

Suppose there exists $t_0 \in (0,+\infty)$ such that $|u''(t)| > r$, that is $u''(t) > r$ or $u''(t) < -r$. In the first case, by (3), we can take $R > r$ such that

$$\int_r^R \frac{s}{h(s)}ds > M \max\left\{M_1 + \sup_{0\le t<+\infty} \frac{\Gamma'(t)}{1+t}\frac{\nu}{\nu-1}, M_1 - \inf_{0\le t<+\infty} \frac{\gamma'(t)}{1+t}\frac{\nu}{\nu-1}\right\}$$

with $M := \sup_{0\le t<+\infty} \psi(t)(1+t)^{\nu}$ and $M_1 := \sup_{0\le t<+\infty} \frac{\Gamma'(t)}{(1+t)^{\nu}} - \inf_{0\le t<+\infty} \frac{\gamma'(t)}{(1+t)^{\nu}}$.

If condition (4) holds, then by (7) there are $t_*, t_+ \in [0,+\infty)$ such that $t_* < t_+, u''(t_*) = r$ and $u''(t) > r, \forall t \in (t_*, t_+]$. Therefore

$$\begin{aligned}\int_{u''(t_*)}^{u''(t_+)} \frac{s}{h(s)}ds &= \int_{t_*}^{t_+} \frac{u''(s)}{h(u''(s))}u'''(s)ds \leq \int_{t_*}^{t_+} \psi(s)u''(s)ds \\ &\leq M\int_{t_*}^{t_+} \frac{u''(s)}{(1+s)^{\nu}}ds = M\int_{t_*}^{t_+}\left[\left(\frac{u'(s)}{(1+s)^{\nu}}\right)' + \frac{\nu u'(s)}{(1+s)^{1+\nu}}\right]ds \\ &\leq M\left(M_1 + \sup_{0\leq t<+\infty}\frac{\Gamma'(t)}{1+t}\int_0^{+\infty}\frac{\nu}{(1+s)^{\nu}}ds\right) < \int_r^R \frac{s}{h(s)}ds.\end{aligned}$$

So $u''(t_+) < R$ and as t_* and t_+ are arbitrary in $(0,+\infty)$, we have $u''(t) < R, \forall t \in [0,+\infty)$. Similarly, it can be proved the case where there are $t_-, t_* \in [0,+\infty)$ such that $t_- < t_*$ and $u''(t_*) = -r, u''(t) < -r, \forall t \in [t_-, t_*)$.

Therefore $\|u''\|_2 < \frac{R}{2} < R, \forall t \in [0,+\infty)$.

Now consider that f verifies (5). By (7), consider that there are $t_-, t_* \in [0,+\infty)$ such that $t_- < t_*$ and $u''(t_*) = r, u''(t) > r, \forall t \in [t_-, t_*)$. Therefore, following similar steps as before

$$\begin{aligned}\int_{u''(t_*)}^{u''(t_-)} \frac{s}{h(s)}ds &= \int_{t_*}^{t_-} \frac{u''(s)}{h(u''(s))}u'''(s)ds \leq \int_{t_-}^{t_*} \psi(s)u''(s)ds \leq M\int_{t_-}^{t_*}\frac{u''(s)}{(1+s)^{\nu}}ds \\ &= M\left(M_1 + \sup_{0\leq t<+\infty}\frac{\Gamma'(t)}{1+t}\frac{\nu}{\nu-1}\right) < \int_r^R \frac{s}{h(s)}ds.\end{aligned}$$

So $u''(t_-) < R$ and by the arbitrariness of t_- and t_* in $[0,+\infty)$, we have $u''(t) < R, \forall t \in [0,+\infty)$.The case where there are $t_*, t_+ \in [0,+\infty)$, with $t_* < t_+$, such that $u''(t_*) = -r, u''(t) < -r, \forall t \in (t_*, t_+]$ is proved in the same way. □

The exact solution for the associated linear problem can be obtained by simple calculations:

Lemma 2.4. *If $e \in L^1[0,+\infty)$, then the BVP*

$$\begin{cases} u'''(t) + e(t) = 0, & t \in (0,+\infty), \\ u(0) = A,\ au'(0) + bu''(0) = B,\ u''(+\infty) = C. \end{cases} \tag{8}$$

has a unique solution in X. Moreover, this solution can be expressed as

$$u(t) = g(t) + \int_0^{+\infty} G(t,s)e(s)ds \tag{9}$$

where

$$g(t) = \frac{C}{2}t^2 + \frac{B-bC}{a}t + A, \quad G(t,s) = \begin{cases} -\frac{b}{a}t + st - \frac{1}{2}s^2, & 0 \leq s \leq t \\ \frac{1}{2}t^2 - \frac{b}{a}t, & t \leq s \leq +\infty. \end{cases}$$

and $u'(t) = g'(t) + \int_0^{+\infty} G_1(t,s)e(s)ds$ with

$$G_1(t,s) = \begin{cases} -\frac{b}{a} + s, & 0 \leq s \leq t \\ -\frac{b}{a} + t, & t \leq s \leq +\infty. \end{cases} \tag{10}$$

The lack of compactness is overcome by the following lemma which gives a general criterium for relative compactness, (see [1]) :

Lemma 2.5. *A set $M \subset X$ is relatively compact if the following conditions hold:*
(i) *all functions from M are uniformly bounded;*
(ii) *all functions from M are equicontinuous on any compact interval of $[0, +\infty)$;*
(iii) *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that*

$$\left| \frac{u^{(i)}(t)}{w_i(t)} - \frac{u^{(i)}(+\infty)}{w_i(+\infty)} \right| < \epsilon,$$

for all $t > t_\epsilon$, $u \in M$ and $i = 0, 1, 2$.

3. Main Result.

In this section we prove the existence of at least one solution for the problem (1),(2) applying lower and upper solutions method and, moreover, some data on its behavior and variation are given.

First we define the usual lower and upper functions:

Definition 3.1. Given $a > 0, b < 0$, and $A, B, C \in \mathbb{R}$, a function $\alpha \in C^3[0, +\infty) \cap X$ is a lower solution of problem (1),(2) if

$$\begin{cases} \alpha'''(t) \geq f\left(t, \alpha(t), \alpha'(t), \alpha''(t)\right), t \in [0, +\infty), \\ \alpha(0) \leq A,\ a\alpha'(0) + b\alpha''(0) \leq B,\ \alpha''(+\infty) < C. \end{cases}$$

A function $\beta \in C^3[0, +\infty) \cap X$ is an upper solution if it satisfies the reversed inequalities.

The existence and localization result is given by next theorem:

Theorem 3.2. *Let $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ be a L^1- Carathéodory function. Suppose there are $\alpha, \beta \in C^3[0, +\infty) \cap X$ lower and upper solutions of the problem (1),(2), respectively, such that*

$$\alpha'(t) \leq \beta'(t), \forall t \in [0, +\infty). \tag{11}$$

If f verifies the one-sided Nagumo condition (4), or (5), in the set

$$E_* = \left\{(t, x, y, z) \in [0, +\infty) \times \mathbb{R}^3, \alpha(t) \leq x \leq \beta(t), \alpha'(t) \leq y \leq \beta'(t)\right\},$$

and

$$f\left(t, \alpha(t), y, z\right) \geq f(t, x, y, z) \geq f\left(t, \beta(t), y, z\right), \tag{12}$$

for (t, y, z) fixed and $\alpha(t) \leq x \leq \beta(t)$, then the problem (1),(2) has at least one solution $u \in C^3(0, +\infty) \cap X$ and there exists $R > 0$ such that

$$\alpha(t) \leq u(t) \leq \beta(t), \qquad \alpha'(t) \leq u'(t) \leq \beta'(t), \qquad \|u''\|_2 < R,\ \forall t \in [0, +\infty).$$

Proof. Let $\alpha, \beta \in C^3[0, +\infty) \cap X$ be, respectively, lower and upper solutions of (1),(2) verifying (11).

Consider the modified equation

$$u'''(t) = f\left(t, \delta_0\left(t, u(t)\right), \delta_1\left(t, u'(t)\right), u''(t)\right) + \frac{1}{1+t^2} \frac{u'(t) - \delta_1\left(t, u'(t)\right)}{1 + |u'(t) - \delta_1\left(t, u'(t)\right)|}, \tag{13}$$

for $t \in [0, +\infty)$, where the functions $\delta_j : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, j = 0, 1,$ are given by

$$\delta_j(t, u(t)) = \begin{cases} \beta^{(j)}(t) & , u^{(j)}(t) > \beta^{(j)}(t) \\ u^{(j)}(t) & , \alpha^{(j)}(t) \leq u^{(j)}(t) \leq \beta^{(j)}(t) \\ \alpha^{(j)}(t) & , u^{(j)}(t) < \alpha^{(j)}(t). \end{cases} \tag{14}$$

Notice that the relation $\alpha(t) \leq \beta(t)$ is obtained by integration from (11), by the boundary conditions (2) and by Definition 3.1.

The proof will include three steps:

- *STEP 1: If u is a solution of problem (13), (2), then*

$$\alpha(t) \leq u(t) \leq \beta(t),\ \alpha'(t) \leq u'(t) \leq \beta'(t),\ \forall t \in [0,+\infty).$$

Suppose, by contradiction, that there exists $t \in [0,+\infty)$ that $\alpha'(t) > u'(t)$ and define

$$\inf_{0\leq t<+\infty}(u'(t)-\alpha'(t)) = u'(t_*)-\alpha'(t_*) < 0.$$

If $t_* \in (0,+\infty)$ then $u''(t_*) = \alpha''(t_*)$ and $u'''(t_*)-\alpha'''(t_*) \geq 0$. Therefore, by (12) and Definition 3.1, we get the following contradiction

$$\begin{aligned}
0 &\leq u'''(t_*)-\alpha'''(t_*) \\
&= f(t_*,\delta_0,\delta_1,u''(t_*)) + \frac{1}{1+t_*^2}\frac{u'(t_*)-\alpha'(t_*)}{1+|u'(t_*)-\alpha'(t_*)|} - \alpha'''(t_*) \\
&\leq f(t_*,\alpha(t_*),\alpha'(t_*),\alpha''(t_*)) + \frac{1}{1+t_*^2}\frac{u'(t_*)-\alpha'(t_*)}{1+|u'(t_*)-\alpha'(t_*)|} - \alpha'''(t_*) \\
&\leq \frac{1}{1+t_*^2}\frac{u'(t_*)-\alpha'(t_*)}{1+|u'(t_*)-\alpha'(t_*)|} < 0.
\end{aligned}$$

- If $t_* = 0$ we have

$$\min_{0\leq t<+\infty}(u'(t)-\alpha'(t)) := u'(0)-\alpha'(0) < 0$$

and

$$u''(0)-\alpha''(0) \geq 0.$$

By Definition 3.1 and since $a>0, b<0$, it yields the contradiction

$$0 \geq bu''(0)-b\alpha''(0) \geq B-au'(0)-B+a\alpha'(0) = a(\alpha'(0)-u'(0)) > 0.$$

- If $t_* = +\infty$

$$\inf_{0\leq t<+\infty}(u'(t)-\alpha'(t)) := u'(+\infty)-\alpha'(+\infty) < 0,$$

$$u''(+\infty)-\alpha''(+\infty) \leq 0$$

and the following contradiction holds

$$0 \geq u''(+\infty)-\alpha''(+\infty) > C-C = 0.$$

So $\alpha'(t) \leq u'(t), \forall t \in [0,+\infty)$. In a similar way we can prove that $\beta'(t) \geq u'(t), \forall t \in [0,+\infty)$.

Integrating $\alpha'(t) \leq u'(t) \leq \beta'(t)$, by (2) and Definition 3.1 we can prove that $\alpha(t) \leq u(t) \leq \beta(t), \forall t \in [0,+\infty)$.

STEP 2: If u is a solution of the modified problem (13),(2) then there exists $R>0$, not depending on u, such that

$$\|u''\|_2 < R.$$

By the previous step, all solutions of equation (13) are solutions of (1), and as f verifies the one-sided Nagumo condition (4), or (5), this claim is a direct application of Lemma 2.3.

STEP 3: Problem (13),(2) has at least one solution.

Take $\rho > \max\left\{\|\alpha\|_0, \|\beta\|_0, \|\alpha'\|_1, \|\beta'\|_1, R\right\}$.

Define the operator $T : X \to X$

$$Tu(t) = g(t) + \int_0^{+\infty} G(t,s)F(u(s))ds,$$

with

$$g(t) := \frac{C}{2}t^2 + \frac{B - bC}{a}t + A$$

and

$$F(u(s)) := f\left(s, \delta_0\left(s, u(s)\right), \delta_1\left(s, u'(s)\right), u''(s)\right) + \frac{1}{1+s^2}\frac{u'(s) - \delta_1\left(t, u'(s)\right)}{1 + \left|u'(s) - \delta_1\left(s, u'(s)\right)\right|}.$$

As f is a L^1-Carathéodory function, for any $u \in X$ with $\|u\| < \rho$ then $F \in L^1$ because

$$\begin{aligned}\int_0^{+\infty} |F(u(s))|\,ds &\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2}\frac{u'(s) - \delta_1\left(t, u'(s)\right)}{1 + \left|u'(s) - \delta_1\left(s, u'(s)\right)\right|}ds \\ &\leq \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2}ds < +\infty.\end{aligned}$$

By Lemma 2.4, the fixed points of T are solutions of problem (13),(2). So it is enough to prove that T has a fixed point.

Claim 1: *$T : X \to X$ is well defined.*

By Lebesgue dominated theorem and Lemma 2.4,

$$\lim_{t\to+\infty} \frac{(Tu)(t)}{1+t^2} = \frac{C}{2} + \frac{1}{2}\int_0^{+\infty} F(u(s))ds < +\infty.$$

Analogously, by (10),

$$\begin{aligned}\lim_{t\to+\infty} \frac{(Tu)'(t)}{1+t} &= \lim_{t\to+\infty} \frac{g'(t)}{1+t} + \int_0^{+\infty} \lim_{t\to+\infty} \frac{G_1(t,s)}{1+t}F(u(s))ds \\ &= C + \int_0^{+\infty} F(u(s))ds < +\infty,\end{aligned}$$

and

$$\lim_{t\to+\infty} \frac{(Tu)''(t)}{2} = \lim_{t\to+\infty} \frac{g''(t)}{2} + \frac{1}{2}\lim_{t\to+\infty}\int_t^{+\infty} F(u(s))ds = \frac{C}{2} < +\infty.$$

Therefore $Tu \in X$.

Claim 2: *T is continuous.*

Consider a convergent sequence $u_n \to u$ in X. Then there exists $r_1 > 0$ such that $\|u_n\| < r_1$ and

$$\begin{aligned}\|Tu_n - Tu\| &\leq \int_0^{+\infty} \max\left\{\begin{array}{c}\sup\limits_{0\leq t<+\infty}\left|\frac{G(t,s)}{1+t^2}\right|, \\ \sup\limits_{0\leq t<+\infty}\left|\frac{G_1(t,s)}{1+t}\right|, \frac{1}{2}\end{array}\right\}|F(u_n(s)) - F(u(s))|\,ds \\ &\leq \int_0^{+\infty} |F(u_n(s)) - F(u(s))|\,ds \longrightarrow 0\,, \quad as \quad n \to +\infty.\end{aligned}$$

Claim 3: *T is compact.*

Let $B \subset X$ be any bounded subset. Therefore there is $R > 0$ such that $\|u\| < R, \forall u \in B$.

Claim 3.1: *TB is uniformly bounded.*

For any $u \in B$, as $\|\alpha\|_0 \leq \|\delta_0\|_0 \leq \|\beta\|_0$, $\|\alpha'\|_1 \leq \|\delta_1\|_1 \leq \|\beta'\|_1$, by (4) one has

$$\begin{aligned}
\|Tu\|_0 &= \sup_{0\leq t<+\infty} \frac{|Tu(t)|}{1+t^2} \leq \sup_{0\leq t<+\infty} \frac{|g(t)|}{1+t^2} + \int_0^{+\infty} \sup_{0\leq t<+\infty} \frac{|G(t,s)|}{1+t^2} |F(u(s))|\, ds \\
&\leq \frac{|C|}{2} + \frac{1}{2}\int_0^{+\infty} \left(\varphi_\rho(s) + \frac{1}{1+s^2}\right) ds < +\infty,
\end{aligned}$$

$$\begin{aligned}
\|Tu\|_1 &= \sup_{0\leq t<+\infty} \frac{|(Tu)'(t)|}{1+t} \leq \sup_{0\leq t<+\infty} \frac{|g'(t)|}{1+t} + \int_0^{+\infty} \sup_{0\leq t<+\infty} \frac{|G_1(t,s)|}{1+t} |F(u(s))|\, ds \\
&\leq |C| + \int_0^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2} ds < +\infty,
\end{aligned}$$

and

$$\|Tu\|_2 = \sup_{0\leq t<+\infty} \frac{|(Tu)''(t)|}{2} \leq \sup_{0\leq t<+\infty} \frac{|g''(t)|}{2} \leq |C| < +\infty.$$

Claim 3.2: *TB is equicontinuous.*

For $T>0$ and $t_1, t_2 \in [0,T]$, we have

$$\begin{aligned}
\left|\frac{Tu(t_1)}{1+t_1^2} - \frac{Tu(t_2)}{1+t_2^2}\right| &\leq \left|\frac{g(t_1)}{1+t_1^2} - \frac{g(t_2)}{1+t_2^2}\right| \\
&\quad + \int_0^{+\infty} \left|\frac{G(t_1,s)}{1+t_1^2} - \frac{G(t_2,s)}{1+t_2^2}\right| |F(u(s))|\, ds \longrightarrow 0, \ as\ t_1 \to t_2.
\end{aligned}$$

Analogously

$$\begin{aligned}
\left|\frac{(Tu)'(t_1)}{1+t_1} - \frac{(Tu)'(t_2)}{1+t_2}\right| &= \left|\frac{g'(t_1)}{1+t_1} - \frac{g'(t_2)}{1+t_2}\right| \\
&\quad + \int_0^{+\infty} \left|\frac{G_1(t_1,s)}{1+t_1} - \frac{G_1(t_2,s)}{1+t_2}\right| |F(u(s))|\, ds \longrightarrow 0, \ as\ t_1 \to t_2
\end{aligned}$$

and

$$\left|\frac{(Tu)''(t_1)}{2} - \frac{(Tu)''(t_2)}{2}\right| = \left|\int_{t_1}^{t_2} F(s) ds\right| \leq \int_{t_1}^{t_2} \varphi_\rho(s) + \frac{1}{1+s^2} ds \longrightarrow 0, \ as\ t_1 \to t_2.$$

Claim 3.3: *TB is equiconvergent at infinity.*

Indeed,

$$\begin{aligned}
\left|\frac{Tu(t)}{1+t^2} - \lim_{t\to+\infty} \frac{Tu(t)}{1+t^2}\right| &\leq \left|\frac{g(t)}{1+t^2} - \frac{C}{2}\right| \\
&\quad + \int_0^{+\infty} \left|\frac{G(t,s)}{1+t^2} - \frac{1}{2}\right| |F(u(s))|\, ds \longrightarrow 0, \quad as \quad t \to +\infty,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{(Tu)'(t)}{1+t} - \lim_{t\to+\infty} \frac{(Tu)'(t)}{1+t}\right| &= \left|\frac{g'(t)}{1+t} - C\right| \\
&\quad + \int_0^{+\infty} \left(\frac{G_1(t,s)}{1+t} - 1\right) |F(u(s))|\, ds \longrightarrow 0, \quad as \quad t \to +\infty,
\end{aligned}$$

and

$$\left|\frac{(Tu)''(t)}{2}\right| = \left|\int_t^{+\infty} F(u(s)) ds\right| \leq \int_t^{+\infty} \varphi_\rho(s) + \frac{1}{1+s^2} ds \longrightarrow 0, \ as \quad t \to +\infty.$$

So, by Lemma 2.5, TB is relatively compact.

As T is completely continuous then by Schauder Fixed Point Theorem, T has at least one fixed point $u \in X$. $\square$

4. **Example.** Consider the problem composed by the third order differential equation

$$u'''(t) = \frac{1}{(t+1)^2}\left(-\arctan\left(u(t)\right) - 10\left|u''(t)\right| e^{u''(t)}\right), t \in [0, +\infty), \tag{15}$$

and the boundary conditions

$$u(0) = A,\ au'(0) + b''(0) = B,\ u''(+\infty) = C, \tag{16}$$

with $A \in [-1, 0]$, $a > 0$, $b < 0$ such that $-2(a+b) \leq B \leq 0$ and $C \in (-2, 0)$. Define

$$E_0 = \left\{(t, x, y, z) \in [0, +\infty) \times \mathbb{R}^3 : -(t+1)^2 \leq x \leq 0, -2t - 2 \leq y \leq 0\right\}.$$

The function $f : [0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(t, x, y, z) := \frac{1}{(t+1)^2}\left(-\arctan x - 10\left|z\right| e^{z}\right),$$

verifies on E_0 the inequality $|f| \leq \frac{K_\rho}{(t+1)^2} := \varphi_\rho(t)$, for some $K_\rho > 0$ and ρ such that $\max\left\{2, \|z\|_2\right\} < \rho$. Therefore f is L^1- Carathéodory.

Functions $\alpha(t) = -(t+1)^2$ and $\beta(t) \equiv 0$ are, respectively, lower and upper solutions of problem (15),(16) with $\alpha(t) \leq \beta(t)$ and $\alpha'(t) \leq \beta'(t)$, $\forall t \in [0, +\infty)$, verifying (12).

As

$$f(t, x, y, z) \leq \frac{1}{(t+1)^2}\frac{\pi}{2},$$

the one-sided Nagumo-type growth condition (4) holds in E_0 with

$$\psi(t) := \frac{1}{(t+1)^2},\ \nu \in (1, 2),\ \text{and } h(|z|) := \frac{\pi}{2}.$$

Therefore, by Theorem 3.2, there is at least a solution u of (15),(16) with

$$-(t+1)^2 \leq u(t) \leq 0, -2t - 2 \leq u'(t) \leq 0, \forall t \in [0, +\infty).$$

Moreover, from the localization part of the theorem, we can precise some qualitative properties of this solution: it is nonpositive, nonincreasing and, as $C \neq 0$, this solution is unbounded.

Notice that the nonlinearity f does not satisfy the usual two-sided Nagumo-type condition. In fact, if there exist $\psi_0, h_0 \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ satisfying

$$|f(t, x, y, z)| \leq \psi_0(t)\ h_0(|z|), \forall (t, x, y, z) \in E_0,$$

with $\displaystyle\int_0^{+\infty} \frac{s}{h_0(s)} ds = +\infty$, then, in particular,

$$-f(t, x, y, z) \leq \psi_0(t)\ h_0(|z|),\ \forall (t, x, y, z) \in E_0.$$

So, for $x = 0$, y, $z \in \mathbb{R}$, we have

$$-f(t, 0, y, z) = \frac{10}{\left(t+1\right)^2}\left|z\right| e^{z} \leq \psi_0(t)\ h_0(|z|).$$

Considering $\psi_0(t) := \frac{1}{(t+1)^2}$, the following contradiction holds:

$$+\infty > \int_0^{+\infty} \frac{s}{10se^s} ds \geq \int_0^{+\infty} \frac{s}{h_0(s)} ds = +\infty.$$

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